WOODS HOLE OCEANOGRAPHIC INSTITUTION

Woods Hole, Massachusetts

Reference No. 62-38

The 1962 Summer Program in Geophysical Fluid Dynamics

Contains lectures by Mostel

by

Melvin Stern

Final Report

Submitted to the National Science Foundation
under Research Grant NSF 22332

November, 1962

APPROVED FOR DISTRIBUTION ______________________
Paul M. Fye, Director

A. Description of the program:

The following participants of the Geophysical Fluid Dynamics program have been wholly or partially supported by National Science Foundation funds:

Peter J. Bryant, Emmanuel College, Cambridge, England
Arnt Eliassen, University of Oslo, Blindern, Norway
Raymond Hide, Massachusetts Institute of Technology
James Holton, Massachusetts Institute of Technology
Robert H. Kraichnan, Institute for Mathematical Sciences, New York
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H. Thomas Rossby, Royal Technical University, Stockholm, Sweden
Pierre Souffrin, Institut d'Astrophysique, Paris, France
Edward A. Spiegel, Institute for Mathematical Sciences, New York
Alar Toomre, Massachusetts Institute of Technology
Roger Williams, University of Southern California at Los Angeles

The six pre-doctoral students included in this list were expected, through their discussions with the senior participants, to formulate a research problem of their own and to communicate this to their colleagues in a one-hour lecture at the end of the ten-week period. The printed manuscripts have been included herein. These are by no means finished products but in almost all cases we believe the students will either continue these investigations or else call on the tools acquired in their current research.

In addition we have had two formal lecture series, each one given once a week for a period of an hour and a half. Dr. Robert Kraichnan discussed his work on Thermal Turbulence. (See the preprint "Mixing-length Analyses of Turbulent Thermal Convection at Arbitrary Prandtl Number" - R. Kraichnan (1962). N.Y.U. Research Report No. HSN-6.)

Some of the problems of large scale circulations in the sun were discussed by Dr. Leon Mestel in his lecture series. This departure from our standard scope was found to be stimulating by those on our staff interested in the general circulation of our own atmosphere. Copies of Dr. Mestel's notes have been included in this report.

B. Other Seminars.

In addition to the talks mentioned previously, we have had occasional seminars by staff participants and visitors as listed below:

1. "The Effect of Slight Incompressibility on a Highly Rotating Fluid" by Dr. Alar Toomre.
2. "Boussinesq Approximation and Energy Integral in Convection" by Dr. W.V.R. Malkus.
3. "Discrete Vortex Models of the Ocean Circulation" by Prof. Henry Stommel.
4. "Wave Propagation in an Anisotropic Atmosphere" by Dr. Derek Moore.
5. "The Hydrodynamics of Jupiter's Atmosphere" by Prof. Raymond Hide.
6. "Generalization of Mixing Length Theory of Turbulent Convection" by Dr. Edward A. Spiegel.
7. "Two Year Oscillation in the Atmospheric Circulation" by Prof. Glen Brier, M.I.T.
8. "Observations of a Wake Vortex" by Dr. Eric Mollo-Christensen.
9. "Solar Convection" by Dr. George Veronis.

C. Plans for the future.

Dr. George Veronis of this institution plans to submit a proposal to the NSF for continuation of the Geophysical Hydrodynamics Program in 1963.

ASTROPHYSICAL APPLICATIONS OF MAGNETOHYDRODYNAMICS

by

L. Mestel
Visiting Astronomer
from
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Internal Memorandum

Lectures delivered
at Stanford University
(Microwave Laboratory and Electronics Research Laboratory)
July - August 1961

M. L. 857

October 1961

Microwave Laboratory
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CHAPTER I

INTRODUCTORY SURVEY

The theoretical equipment required for this course consists of Maxwell's equations to the electromagnetic field, and simple approximations describing the flow of fully and partially ionized gases. Before proceeding, we have to estimate the conditions of validity of the "magnetohydrodynamic" approximation - defined by the use of a nearly isotropic pressure, possibly with small viscous terms included, to represent the material stress tensor. Well-known ideas of magnetohydrodynamics may be taken over: for example, the "freezing of the field" into highly conducting matter is likely to hold far better in most astrophysical than in most terrestrial situations. As a consequence, the dynamical coupling between the velocity and magnetic fields will often be well described by Alfvén or magnetohydrodynamic waves.

However, the infinite conductivity approximation, though often adequate, is not the only case we shall treat. Finite resistivity implies a steady loss of energy from a magnetic field. If perfect conductivity were always a good approximation, there would be no difficulty in supplying this energy from the kinetic energy of the gas, which would in turn be supplied from nonmagnetic sources - e.g., from thermal convection. But Cowling's "anti-dynamo theorem" shows that a magnetic field of simple symmetry - e.g., a dipole-type field - has singular regions in which the perfect conductivity approximation always breaks down, unless the conductivity is literally infinite, or unless there is a source of matter in the singular region. Such fields cannot be maintained by dynamo action against spontaneous decay. However, fields of lower symmetry do not suffer from this defect, and after recent work - both analytical and numerical - we can feel confident that dynamo action can and probably does maintain the earth's magnetic field with its strong dipole component, against Ohmic dissipation.

But while sufficiently complicated motions can prevent decay of an existing field, dynamo action is in itself incapable of generating a field from nothing: magnetic field lines cannot be convected if none exist.

We must therefore study the spontaneous generation of "seed" magnetic fields by the cosmical analogs of technological batteries. It turns out that the plasma equations used do contain a term corresponding to flux generation, so that it is plausible to expect dynamo action to start in the earth's liquid core and in stellar convective regions. However, the theoretical dynamos discussed in detail employ laminar motions - nonuniform rotation and large-scale circulation. The interaction of a random, turbulent velocity field with a seed magnetic field, in the absence of strong large-scale motions, is a more difficult problem, still unresolved. It has possible important applications to the generation of a galactic magnetic field. The different authorities are divided as to whether the field is always ultimately increased by the turbulence, whether the dissipative mechanisms that occur at small wave-lengths are importantor not, and whether, in the cases for which spontaneous growth occurs, the magnetic and kinetic fields reach equipartition of energy, or whether the magnetic field is always substantially weaker than the kinetic.

Given a stellar magnetic field, one can study its interaction with the stellar rotation field, under simple symmetry assumptions. If the only motion is one of rotation, the freezing of the field at once demands that in a steady state the angular velocity be uniform along a poloidal field line - otherwise a torsional hydromagnetic oscillation will result. This is known as Ferraro's law of isorotation. However, the sum is observed to rotate more rapidly at the equator than at the poles. Further, the external general poloidal magnetic field is limited to the poles: in the region of sunspot activity, where we have some evidence for internal large-scale circulation of matter, any general poloidal field lines seem to be confined beneath the surface, presumably by the circulation.

If we postulate a meridian circulation of matter in the sunspot zones, then we can look for the steady-state generalisations of the law of isorotation. For a certain parameter-range, the theory does predict equatorial acceleration. However, the solar cycle demands a periodic rather than a steady solution of the equations; further, the recently discovered reversal of the sun's general field suggests that a periodic dynamo process is at work. Qualitatively, it does not seem impossible that a nearly steady equatorial acceleration may be associated with fluctuating poloidal and toroidal fields; but the mathematical treatment is likely to be difficult, even in the axially symmetric approximation, supplemented by a dynamo hypothesis.

The thermal generation of magnetic fields - the "battery" process referred to above - can, if left to operate, build up very large toroidal magnetic fields inside rotating stars. Biermann estimated that in the sun, with its low rotation rate, fields of about 1,000 gauss (comparable with sunspot fields) could be built up during the solar lifetime; in rapidly rotating O and B stars, much stronger fields should arise. Unfortunately, if the star has even a very weak poloidal field - e.g. a slowly decaying primeval field - then it can be shown that Biermann's field is reduced far below his estimate. Further, if there is a meridian circulation of matter, which necessarily generates nonuniform rotation and hence a toroidal field, it is found that the battery term plays an insignificant role in determining the asymptotic structure of the magnetic and rotation fields. Except as an initial creator of flux, the battery term is probably ignorable.

If a magnetic field within a star is not too strong, the poloidal forces it exerts can be balanced by non-spherical perturbations to the density and temperature fields. In a convective zone, however, an externally generated strong magnetic field may in addition seriously interfere with the transport of energy. The existence of sunspots is plausibly explained in this way. A more-or-less vertical magnetic field cuts down the convective transport, so lowering the temperature locally. The thermal pressure outside the spot-region thus compresses the field, increasing its strength, and so also increasing the magnetic inhibition of convection. An approximately steady state is ultimately reached, with internal thermal pressure and lateral magnetic force balancing external thermal pressure. We have tentative explanations of the sharp change between umbra and penumbra, in terms of differing gas motions; but there remains a large phenomenological element in the theory, because of our lack of an adequate theory of convection, with or without a magnetic field.

Magnetic fields in regions of low pressure play a much more active role than deep inside stars, where the magnetic energy must be rather less than the thermal. If the nonmagnetic forces are weak, the field may exert strong pinching forces, leading to violent discharges, perhaps to be associated with flares. If the pinched state is hydromagnetically unstable, or if there is an unreplenished loss of the pinched matter (e.g., through plasma recombination) the field may adjust itself to a partially force-free state, with currents flowing parallel to the field lines in the low-density region. Sometimes, the reverse process may occur. Thus a contracting cloud with a strong frozen-in field may attempt to break contact with the galactic magnetic field, by formation of a neutral ring; but the gravitational field of the cloud may distort the field until the neutral ring is pulled into the cloud, with a zone of strongly pinched matter outside, maintained by gravitational flow from the galactic matter.

Magnetic forces probably play some role in galactic gas dynamics. Chandrasekhar and Fermi suggested that a longitudinal field along a spiral arm stabilised the arm against its self-gravitation. However, it is by no means certain that a spiral arm field is of this structure - recent work suggests that it is in fact helical rather than longitudinal, implying a pinching rather than a disruptive force.

In fact, "spiral arms" are observed by different techniques, and it is not obvious that unique results should be obtained. The neutral hydrogen, observed by its radio emission near 21 cms, may very well define spiral arms different from the bright stars with their zones of hot hydrogen; and the nonthermal radio emission, associated with supernova remnants, may define still another set of arms. Observations are not yet conclusive, and different results are claimed by different observers. Should significant differences be obtained, one may get valuable information on the galactic magnetic field. For only the gas feels the magnetic forces; once stars form, they break loose and move under gravitational and centrifugal forces only. During its main-sequence life, a bright star defines the "stellar" spiral arms. If and when it has become a supernova remnant, it has moved still further from the gaseous arm, though still a small fraction of an orbit in the galactic gravitational field.

The radial gas streaming observed by the radio astronomers invites theoretical explanation by magnetohydrodynamics. Twisting of a galactic field by nonuniform rotation generates a toroidal force, which transports angular momentum from one part of the galaxy to another; gas which gains angular momentum tends to move outwards. The difficulty is to locate the gas which is losing angular momentum, and so moving in. It may be that there is a strong enough toroidal gravitational interaction between gaseous arms and stellar arms for a significant amount of angular momentum to be exchanged - something that cannot happen in a smoothed out, axially symmetric galaxy. But although it seems unlikely that magnetic forces are the only important factor , it is a plausible hypothesis that they play a vital role. However, little convincing quantitative work seems to have been done on the problem.

Finally we have the effect of a strong galactic magnetic field on the problem of star formation. It is found that the field puts a high lower limit on the mass of a diffuse galactic cloud that can be gravitationally bound; further, if the field is frozen-in, this limit is hardly reduced if the cloud contracts nearly isotropically. There are probably good dynamical reasons for non-isotropic contraction sometimes to occur: e.g., the nonspherical gravitational field due to the presence of the magnetic field itself, or the effect of a centrifugal field. Alternatively, the difficulty may be got over by relaxing the freezing-in condition. This demands study of the magnetohydrodynamics of a lightly ionized gas.

Besides being a hindrance, the magnetic field may be a help, in that it may act as an efficient means of transporting angular momentum from a contracting cloud. However, this requires detailed study of the structure of the field, in particular study of the tendency of distorted field lines to break off from the galactic field: it is not certain that the process will be efficient enough.

Not all these problems are discussed in detail in Part III; but it is hoped that enough are treated to convince the reader of the relevance of magnetohydrodynamics for astrophysics, and of the importance of attempting to find mathematical solutions of fairly precise models, rather than being frightened by the genuine difficulties into being content with a merely qualitative treatment.

CHAPTER 2 THEORETICAL BASIS

Magnetohydrodynamics (or, for brevity, "hydromagnetics") is based on Maxwell's equations to the electromagnetic field, and the equations to the flow of fully or partially ionized gases. Study of the lightly ionized case is postponed until later. Maxwell's equations are assumed correct -- no fundamental modification is proposed, e.g., to link up magnetism and rotation by a new law. We are concerned with nonrelativistic problems, so that as usual in a medium of high conductivity, the displacement current may be dropped in low frequency problems.

The coupling between the electromagnetic and dynamical fields is two-way: the flow of the gas is affected by the electromagnetic forces while flow of gas across a magnetic field generates current, changes the magnetic field, and so induces an electric field. The very large dimensions of cosmical bodies makes self-induction especially important, and is the main reason why it is difficult to simulate cosmical situations in the laboratory.

In this chapter we first discuss Maxwell's equations as seen in different frames, and the approximations that may be made. Then we survey the general properties of a cosmical plasma, being especially concerned to justify our using the simple magnetohydrodynamic approximation.

A. MAXWELL'S EQUATIONS

In a given frame of reference, the vectors $\underset{\sim}{E}$ and $\underset{\sim}{H}$ satisfy

$$\nabla \cdot \underset{\sim}{E} = 4\pi\rho_e \quad , \tag{2.1}$$

$$\nabla \cdot \underset{\sim}{H} = 0 \quad , \tag{2.2}$$

$$\nabla \wedge \underset{\sim}{E} = -\frac{1}{c}\frac{\partial \underset{\sim}{H}}{\partial t} \quad , \tag{2.3}$$

$$\nabla \wedge \underset{\sim}{H} = \frac{4\pi}{c}\,\underset{\sim}{j} + \frac{1}{c}\frac{\partial \underset{\sim}{E}}{\partial t} \quad . \tag{2.4}$$

where ρ_e and $\underset{\sim}{j}$ are the charge and current density. $\underset{\sim}{E}$, $\underset{\sim}{H}$, ρ_e and $\underset{\sim}{j}$ are "collective," "large-scale" vectors, averaged over regions large compared with the scale of the random fluctuations present.

Under a Lorentz transformation to a frame moving with velocity $\underset{\sim}{v}$ relative to the original, we have

$$\underset{\sim}{E}' = \frac{\underset{\sim}{E} + \dfrac{\underset{\sim}{v} \wedge \underset{\sim}{H}}{c}}{\sqrt{1 - \dfrac{v^2}{c^2}}} \tag{2.5}$$

$$\underset{\sim}{H}' = \frac{\underset{\sim}{H} - \dfrac{1}{c} \underset{\sim}{v} \wedge \underset{\sim}{E}}{\sqrt{1 - \dfrac{v^2}{c^2}}} \tag{2.6}$$

$$\rho_e' = \frac{\rho_e - \dfrac{\underset{\sim}{v} \cdot \underset{\sim}{j}}{c^2}}{\sqrt{1 - \dfrac{v^2}{c^2}}} \tag{2.7}$$

$$\underset{\sim}{j}' = \frac{\underset{\sim}{j} - \rho_e \underset{\sim}{v}}{\sqrt{1 - \dfrac{v^2}{c^2}}} \quad . \tag{2.8}$$

We are concerned only with nonrelativistic speeds, so that terms of order $(v/c)^2$ are dropped. Further, we anticipate a basic result, that the electric field in a frame moving with the bulk velocity of the gas is "small", or

$$\underset{\sim}{E} + \frac{\underset{\sim}{v} \wedge \underset{\sim}{H}}{c} \approx 0 \quad ; \tag{2.9}$$

if $\underset{\sim}{v}$ is the local gas velocity, the terms on the right-hand side are small compared with $|\underset{\sim}{E}|$ and $|\underset{\sim}{v} \wedge \underset{\sim}{H}/c|$.

Thus, if we drop terms of order $(v/c)^2$, we have from (2.6) and (2.5)

$$\underset{\sim}{H}' = \underset{\sim}{H} , \tag{2.10}$$

$$\underset{\sim}{E}' = \underset{\sim}{E} + \frac{\underset{\sim}{v} \wedge \underset{\sim}{H}}{c} . \tag{2.11}$$

We may talk about a magnetic field without, however, specifying the frame in which it is measured; the electric field is not invariant, but transforms according to (2.11).

If D is a characteristic scale of variation of the vectors, the charge density, by (2.1) and (2.9), is

$$\rho_e \simeq - \nabla \cdot \left(\frac{\underset{\sim}{v} \wedge \underset{\sim}{H}}{4\pi c}\right) \simeq \frac{v}{c}\left(\frac{H}{4\pi D}\right) . \tag{2.12}$$

The convection of this charge density by the fluid yields a contribution to the total current-density:

$$|\rho_e \underset{\sim}{v}| \sim \left(\frac{v}{c}\right)^2 \left(\frac{cH}{4\pi D}\right) \sim \left(\frac{v}{c}\right)^2 \left(\frac{c|\nabla \wedge \underset{\sim}{H}|}{4\pi}\right) . \tag{2.13}$$

If the displacement current can be dropped from (2.4), then the total current $\underset{\sim}{j}$ is given by $c/4\pi \, \nabla \wedge \underset{\sim}{H}$, and is therefore of the order $cH/4\pi D$. Thus (2.13) implies that the convection of the net charge density makes a negligible contribution to the total current, which must be due mainly to a "conduction current" -- the drift of electrons relative to the ions. Further, we have

$$\frac{\rho_e}{n_e e} \sim \frac{\frac{vH}{c}}{4\pi D n_e e} = \left(\frac{v}{c^2}\right) \frac{j}{n_e e} = \left(\frac{v}{c}\right)\left(\frac{V}{c}\right) , \tag{2.14}$$

where V is the drift speed of electrons through the ions. In most cases, we find $V \ll v$, so that in a nonrelativistic theory the charge separation is small. We shall see later that the electric force density is smaller than the magnetic force density by the same factor $(v/c)^2$.

We must now justify neglect of the displacement-current. If τ is a characteristic time of variation of the field quantities, this requires

$$\frac{1}{c}\left|\frac{\partial \underset{\sim}{E}}{\partial t}\right| \sim \frac{E}{c\tau} \ll \left|\nabla \wedge \underset{\sim}{H}\right| \sim \frac{H}{D} \tag{2.15}$$

or, by (2.9),

$$\tau \gg \left(\frac{v}{c}\right)\left(\frac{D}{c}\right) . \tag{2.16}$$

The usual condition for the "quasi-static" approximation is that $\tau \gg D/c$ -- electromagnetic waves cross a region D in a time short compared with the time of variation of the field quantities. In the present work, the condition (2.16) introduces an extra small factor (v/c) .

Our truncated Maxwell equations are then

$$\nabla . \underset{\sim}{E} = 4\pi\rho_e \quad , \tag{2.17}$$

$$\nabla . \underset{\sim}{H} = 0 \quad , \tag{2.18}$$

$$\nabla \wedge \underset{\sim}{H} = \frac{4\pi}{c} \underset{\sim}{j} \quad , \tag{2.19}$$

$$\nabla \wedge \underset{\sim}{E} = -\frac{1}{c}\frac{\partial \underset{\sim}{H}}{\partial t} . \tag{2.20}$$

These equations are invariant to the __Galilean transformation__

$$\underset{\sim}{r}' = \underset{\sim}{r} - \underset{\sim}{v}t ,$$
$$t' = t \quad , \tag{2.21}$$

provided $\underset{\sim}{E}$ and $\underset{\sim}{H}$ transform according to (2.10) and (2.11), and, by (2.7) and (2.8), ρ_e and $\underset{\sim}{j}$ transform like

$$\rho_e' = \rho_e - \frac{\underset{\sim}{v} . \underset{\sim}{j}}{c^2} \tag{2.22}$$

and

$$\underset{\sim}{j}' = \underset{\sim}{j} .$$

The dropping of the term $\rho_e \mathbf{v}$ compared with $\mathbf{j}$ in (2.23) is forced on us, because of our neglect of all terms of order $(v/c)^2$; but in general $\mathbf{v} \cdot \mathbf{j}/c^2$ is of the same order as ρ_e, and so must be retained in (2.22) -- a curious relic of relativistic theory in an essentially nonrelativistic approximation.

Our truncated Maxwell equations (2.17-2.20) contain only one time-derivative -- $\partial \mathbf{H}/\partial t$. If all the field quantities are known at any time, then (2.20) yields $\mathbf{H}$ an instant later, and the "Ampère equation" (2.19) gives the associated current density. Only one more equation is needed for the computation of the electromagnetic quantities to proceed. In a simple stationary conductor, Ohm's law

$$\mathbf{j} = \sigma \mathbf{E} \qquad (2.24)$$

relates $\mathbf{E}$ to $\mathbf{j}$ and completes the system. In our problems the analog of Ohm's law will be found to involve the velocity and pressure of the fluid, while the motion of the fluid is affected by electromagnetic forces -- the coupling between dynamics and electromagnetism already referred to. For the moment, we wish to remark that neglect of the displacement current implies that in a time dependent problem Maxwell's equations fix $\partial \mathbf{H}/\partial t$ in terms of $\mathbf{E}$, which must be expressed from the analog of Ohm's law in terms of the other field vectors-- $\mathbf{j}$, $\mathbf{H}$, etc. -- at the instant considered. The current at a later instant is then given by the "Ampère equation" applied at the later instant -- when $\mathbf{H}$ "determines" $\mathbf{j}$, rather than $\mathbf{j}$ "determining" $\mathbf{H}$. Only in problems where the change in $\mathbf{H}$ is at most a small perturbation may one determine $\mathbf{j}$ directly without regard to Ampère's law, verifying afterwards that the charge in $\mathbf{H}$ due to the new $\mathbf{j}$ is negligible. This will be further discussed in Chapter 5. Equally we shall find that the charge density will be determined by reading Poisson's equation (2.17) from right to left. This procedure is familiar, for example, in wave-guide theory. Maxwell's complete equations are solved for the dielectric interior, subject to suitable boundary conditions; the charge-current field on the boundary surface is then determined from the integrated forms of (2.17) and (2.19).

B. PROPERTIES OF COSMICAL PLASMA

The electric and magnetic fields that appear in Eqs. (2.17) and (2.20) are, in the terminology of current plasma physics, "collective," as compared with the random fields present in a thermal plasma. If the gas is near thermodynamic equilibrium, the maximum length scale over which thermal fluctuations cause substantial separation of positive and negative charges is of the order of the Debye shielding length:

$$\lambda_D = \left(\frac{kT}{4\pi n_e e^2}\right)^{1/2} \sim 7 \left(\frac{T}{n_e}\right)^{1/2} , \qquad (2.25)$$

where e is the electronic charge, k is Boltzmann's constant, T the temperature, and n_e the electron density. For problems involving length scales $\gg \lambda_D$, we may divide the electromagnetic field into two parts, with length-scales respectively greater than or less than λ_D. The large-scale field appears in Maxwell's equations, and is related to the charge-current field as determined by thaking the zero-order and first-order moments of the particle distribution functions. By the relation (2.9) the large-scale magnetic field has much more energy than the electric field. The small-scale field is mainly the unshielded Coulomb field of individual particles; it is a random field, which tends to restore a general distribution function to the Maxwellian form, and to inhibit the drift of electrons relative to ions.

For a collective description of phenomena to be valid, the Debye length must be small compared with the length scales of the problem. Consider then a typical interstellar cloud of neutral hydrogen (an "HI" cloud), of temperature $\sim 100^{\circ}$ K , and number density $n_H = 10/cm^3$. Intermingled with it is a plasma component (metals with low ionization potentials) of density $\sim 10^{-4}\, n_H$. The Debye length is then $\sim 2 \times 10^3$ cm , as compared with the cloud radius -- say 10 parsecs or 3×10^{19} cm (1parsec $\simeq$ 3 light years). Hydrogen clouds near hot stars are largely ionized by the photo-electric effect, and kept at a temperature near $10^{4\circ}$ K; but again, $\lambda_D \sim 700/n_e^{1/2}$ is far smaller than normal macroscopic lengths. Inside stars we find $\lambda_D < 10^{-6}$ cm, as compared with scale heights which are not less than 10^8 cm.

We conclude that, in normal astrophysical contexts, charge-separation is small, and it is legitimate to think of the plasma as defining average, macroscopic quantities that can be fed into Maxwell's equations. However problems can and do rise in which the region of interest is one with random growing space-charge fields -- e.g., the unstable region between two colliding gas clouds, studied by Kahn. The zero-order state is violently non-Maxwellian, and work based on assumed proximity to local thermodynamic equilibrium is inapplicable.

In discussing the effect of random, small-scale electric fields on the particle distribution function, it is customary to introduce a mean free path

$$\lambda = \frac{1}{n\pi b^2} , \qquad (2.26)$$

where n is the number density of the scattering particles, and b the "collision radius," by analogy with elementary kinetic theory. In fact, under Coulomb scattering it is the large number of slight deflections rather than the few large-angle deflections that make the largest contribution to the total cross section: -- in fact the Rutherford cross section diverges logarithmically unless the integration is cut off at a maximum impact parameter. In Cowling's original work this was taken to be the mean inter-particle distance; however, Spitzer pointed out that a better choice is the Debye length, beyond which the Coulomb field of the scattering particle is effectively shielded. For ion-electron collisions, b is found to be $\simeq 3/2\left(Ze^2/kT\right)$, where Ze is the ionic charge -- somewhat larger than the distance where the kinetic energy relative to the mass-center is equal to the mutual potential energy. Thus we have (taking $Z = 1$ -- ionized hydrogen),

$$\lambda_{i,e} \simeq 6 \times 10^{+4} \frac{T^2}{n} , \qquad (2.27)$$

and similar lengths for ion-ion and electron-electron scattering. Again we find λ small compared with the macroscopic lengths of interest in many problems (but not, for example, in the theory of solar-terrestrial relations, where λ is of the order of the radius of the earth's orbit).

The derivation of macroscopic equations of motion from the Boltzmann equation for a nonionized gas depends on λ being small compared with macroscopic length-scales, or alternatively, on the interval between collisions being short compared with frequencies imposed at the boundaries. If this is so, then the distribution function for the molecules will always be close to the Maxwellian, with small departures, proportional to the mean free path, that yield transport phenomena (viscous force, thermal conduction, diffusion) as corrections to the adiabatic, nondissipative equations to the motion of a perfect gas. A problem such as, for example, the effect of striking the wall of a container with a frequency greater than the collision frequency v_T/λ (v_T being the thermal speed) cannot be treated by these equations, but requires one to return to the Boltzmann equation; but otherwise the usual macroscopic equations are adequate.

In a fully ionized gas with a magnetic field, the situation is complicated by the anisotropy introduced by the field. A single particle of charge Ze and mass m gyrates in a uniform field $\underset{\sim}{H}$ with angular frequency

$$\omega = \frac{ZeH}{mc} . \tag{2.28}$$

For ions of mass Am_H, we have

$$\omega_i = 10^4 \left(\frac{Z}{A} \right) H ; \tag{2.29}$$

for electrons,

$$\omega_e = 2 \times 10^7 H . \tag{2.30}$$

Nonuniformities in the field, of a scale much greater than the mean gyration radius v_T/ω, superimpose on the gyration a slow drift. A component of force parallel to $\underset{\sim}{H}$ continually accelerates a charge, without any magnetic interference, but a component $\underset{\sim}{F}_\perp$ across $\underset{\sim}{H}$ yields a drift

$$\frac{c}{e} \frac{\underset{\sim}{F}_\perp \wedge \underset{\sim}{H}}{H^2} . \tag{2.31}$$

For the moment we ignore the drifts, and note that the field introduces as a new length-scale the mean gyration radius (v_T/ω), where v_T is the thermal speed of the species considered, which must be compared with the mean free path λ; or alternatively, a time-scale $1/\omega$, to be compared with the mean collision time $\tau = \lambda/v_T$. If

$$\omega\tau > 1 \tag{2.32}$$

then the field introduces a microscopic anisotropy: a particle performs a number of turns of a spiral before "colliding" with another particle, and acquiring a random direction of motion. For electrons, the condition (2.32) is

$$\omega_e \tau_e \simeq 1.75 \times 10^6 \frac{HT^{3/2}}{n} >> 1 ; \tag{2.33}$$

for ions, we have an extra factor $\sqrt{m_e/m_i}$:

$$\omega_i \tau_i \approx 4.2 \times 10^4 \frac{HT}{n}^{3/2} >> 1 , \qquad (2.34)$$

where for simplicity we have put $Z = A = 1$ (pure hydrogen), so that λ is approximately the same for ion-ion, ion-electron and electron-electron encounters.

Inside main sequence stars it is found that if the field is not to be too strong for mechanical equilibrium, $\omega_i \tau_i$ is rather less than unity, while $\omega_e \tau_e$ approaches unity only at the upper limit for H. (It is not impossible that in the diffuse envelopes of giant stars conditions (2.33) and (2.34) may be more easily satisfied.) By contrast, in diffuse gas clouds -- e.g., a region of ionized hydrogen near a bright star, with $T = 10^4 {}^\circ K$ and $n = 10$, say, $\omega\tau >> 1$ for both species if $H > 10^{-9}$ -- whereas a conservative estimate for the galactic field is 10^{-6}. (The cross section for collisions between a charged and a neutral particle is much smaller than that for two charged particles.)

Thus in cosmical gas clouds we may expect thereto be microscopic anisotropy due to the magnetic field. Suppose then that the plasma has macroscopic nonuniformities -- in density, magnetic field, electric field, etc. -- of a scale small compared with λ, but larger than the gyration radius for either species; or alternatively suppose that the externally driven time variations are more rapid than $1/\tau$, but less than ω. Then as seen from a frame moving with the local mean velocity of either species, the distriubtion function will in general not be nearly Maxwellian if one averages over a volume of linear dimensions $<< \lambda$. In particular, there is no guarantee that the pressure perpendicular to $\underset{\approx}{H}$ -- averaged over a gyration period -- will be the same as that parallel to $\underset{\approx}{H}$ (although it is known that too great a difference leads to hydromagnetic instabilities that presumably tend towards isotropy). Further, the confining effect of the magnetic field for particle motion across it implies a much shorter "quasi-mean free path" across $\underset{\sim}{H}$ than parallel to $\underset{\sim}{H}$: equivalent to a much greater "viscosity" along $\underset{\sim}{H}$.

If we are interested in properties of our system over such small length-scales with such short time- scales , then the "magnetohydrodynamic approximation" -- involving a nearly isotropic stress -- is dubious, (unless we can show that instabilities occur so as to restore systematically approximate isotropy). Such problems should be treated by the Boltzmann equation. But if we can be content with a fairly coarse-grained theory, we can average all quantities over volumes with linear scales of the order of a mean free path. The theory should then be adequate, but strictly in-applicable to such problems as the interaction of the tenuous solar wind with the earth's field (though even there it is probably a useful first approximation). Equally, though the theory will predict ordinary hydromagnetic shocks, it will be unable to deal with the steepening of a compression wave within a region of scale much less than λ . Some workers have suggested that the steepening will be halted when the thickness of the pulse is of the order of the ion gyration radius. Hydromagnetics is no more capable of discussing the adjustment of the particle distribution function as gas flows through such a shock, than ordinary gas dynamics is of dealing with the structure of an ordinary shock; both problems require treatment from the particle point of view.

The microscopic anisotropy shows itself up when we discuss the analog of Ohm's law -- the equation for the drift of electrons relative to the ions, under the influence of macroscopic fields, averaged over a mean free path. The resulting anisotropic conductivity has been the source of some confusion in the past, it does not in fact give rise to any real difficulties.

The simplification gained by taking the mean free path as the basic length is enormous : instead of having to deal with the Boltzmann equation, depending on three velocity components as well as space and time variables, we have essentially modified equations of gas dynamics. However, we wish to eat our cake and have it; as well as wanting a moderately short mean free path, to keep the stress tensor nearly isotropic, we want a longish mean free path so that the resistance to the flow of current is not too great -- the electrons can travel a good distance before having their drift velocity randomized into heat. We shall see later that these requirements are not mutually inconsistent -- the criterion for high conductivity involves not only the mean free path, but macroscopic quantities as well.

CHAPTER 3 MACROSCOPIC EQUATIONS FOR COSMICAL PLASMA

There are two ways of deriving macroscopic equations. One can solve the microscopic equations for individual particles, under the assumption that all quantities vary slowly over a gyration radius and in a gyration period, so that the motion of each particle is expressed as a gyration round the field, with drifts superimposed. Macroscopic equations -- e.g., the current density under given electric and magnetic fields -- can be derived by averaging over all the particles present. The difficulty is to be sure that one has a correct density in both physical and velocity space, especially in view of the complicated trochoidal trajectories of particles in nonuniform magnetic fields. For example, it is possible to "show" that a nonuniform magnetic field causes motions of charged particles in a direction opposite to that required by Ampère's law. The fallacy lies in neglecting the nonuniform density implied by the particle motions in the nonuniform field.

Again, from (2.31) we see that a uniform electric field $\underset{\sim}{E}$, applied across a uniform $\underset{\sim}{H}$, yields a drift $c\underset{\sim}{E} \wedge \underset{\sim}{H}/H^2$ independent of charge, superimposed on the gyrations (which have opposite senses for positive and negative charges). But in a neutral uniform plasma, there can be no net macroscopic electric field, and no mean motion of the gas. It is in fact found that a correct averaging over the orbits of the two species does yield a vanishing mean velocity at each point of space.

However, the resolution of these paradoxes requires rather subtle argument, and it is far simpler to derive macroscopic equations by summing the corresponding microscopic <u>equations</u>, instead of first finding solutions of the equations of motion of individual particles, and then summing. By taking moments of the Boltzmann equation for the distribution function of each species, one arrives at a set of equations in physical space. In general the system is not closed; but in the magnetohydrodynamic approximation the distribution function is nearly isotropic, and so odd moments are small. Each species can be described by an isotropic pressure with small transport terms in the dynamical and thermal equations. Collisions between like particles do not change the mean drift of the component, but merely isotropises the stress; but collisions between electrons and ions impede the flow of current, and account for electrical resistance.

A. EQUATIONS TO THE FLOW OF THE WHOLE GAS

The mutual friction between electrons and ions disappears when we consider the flow of the whole gas. Except when discussing turbulence in Chapter 9, viscous force and dissipation will be ignored in what follows. The electromagnetic force density is

$$n_i Z_e \left(E + \frac{\underset{\sim}{v_i} \wedge \underset{\sim}{H}}{c} \right) - n_e e \left(\underset{\sim}{E} + \frac{\underset{\sim}{v_e} \wedge \underset{\sim}{H}}{c} \right) = \frac{\underset{\sim}{j} \wedge \underset{\sim}{H}}{c} , \qquad (3.1)$$

where n_i, n_e, $\underset{\sim}{v_i}$, $\underset{\sim}{v_e}$ are densities and mean drift velocities of each species.

The quasi-neutrality of the plasma is assumed here, so that $(n_i Ze - n_e e) \underset{\sim}{E} = \rho_e \underset{\sim}{E} \cong 0$. Strictly, there is an electrical force density, of order

$$\left(\frac{vH}{c} \right) \left(\frac{vH}{4\pi cD} \right) = \left(\frac{v}{c} \right)^2 \left(\frac{E^2}{4\pi D} \right) << \left| \frac{j \wedge H}{c} \right|$$

in the notation of Chapter 2; again it must be neglected in the nonrelativisitc approximation.

The equation of motion to the gas as a whole is then

$$\rho \frac{d}{dt} \underset{\sim}{v} = -\nabla p + \rho g + \frac{\underset{\sim}{j} \wedge \underset{\sim}{H}}{c} \qquad (3.2)$$

where $\underset{\sim}{v}$ = mean velocity of gas $\simeq \underset{\sim}{v}_i$ } because of $m_e \ll m_i$

ρ = mass density $\simeq n_i m_i$

p = total pressure $= p_i + p_e$,

$\underset{\sim}{g}$ = gravitational field ,

$\frac{d}{dt}$ = total time derivative .

We have also the conservation of mass (in a nonrecombining plasma):

$$\frac{d\rho}{dt} + \rho \nabla \cdot \underset{\sim}{v} = \frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \underset{\sim}{v}) = 0 \quad . \tag{3.3}$$

The thermal equation is sometimes required. For rapid motions inside opaque stars, the adiabatic approximation is good enough. In the interstellar medium e.g., the heating of the gas by galactic radiation and its cooling by collisional ionization followed by recombination, by radiation from dust grains, and by molecular excitation and radiation -- often dominates over compressional, viscous and shock heating, so that the isothermal approximation is a better one.

B. THE ANALOG OF OHM'S LAW

To complete the system of equations, we need to consider the drift of electrons through the ions under the influence of the macroscopic fields $\underset{\sim}{E}$ and $\underset{\sim}{H}$, and impeded by collisions with the ions. The following treatment follows closely that given by Cowling.

Let $\underset{\sim}{v}$ be the mean velocity of the gas -- essentially that of the ions -- and $(\underset{\sim}{v} + \underset{\sim}{V})$ the electron mean velocity. In accordance with the basic assumptions discussed above, we again represent the effect of the electron random velocities by an isotropic pressure p_e . Then the forces acting on the electrons per unit volume are:

(i) the partial pressure gradient $-\nabla p_e$.

(ii) the gravitational force density -- neglected, because of the small electronic mass.

(iii) the electromagnetic force density

$$- n_e e \left[\underset{\sim}{E} + (\underset{\sim}{v} + \underset{\sim}{V}) \wedge \underset{\sim}{H}/c \right] .$$

(iv) the frictional drag due to ion-electron encounters.

Having introduced a "collision-time" τ_e -- computed from the theory of non-uniform gases -- we may idealize this force by considering the drift velocity $\underset{\sim}{V}$ of an electron to be completely randomized at intervals τ_e : thus the mean rate of loss per cm^3 of electron momentum is $+ n_e m_e \underset{\sim}{V}/\tau_e$.

The equation to be motion of the electron is therefore

$$-\nabla p_e - n_e e \left(\underset{\sim}{E} + \frac{\underset{\sim}{v} \wedge \underset{\sim}{H}}{c} \right) - \frac{n_e e \underset{\sim}{V} \wedge \underset{\sim}{H}}{c} - \frac{n_e m_e \underset{\sim}{V}}{\tau_e} = 0 , \qquad (3.4)$$

where we ignore the electron inertia. This amounts to dropping the gyration in the field $\underset{\sim}{H}$, and assuming the macroscopic time-scales are long compared with the plasma oscillation period (see following) .

Following Cowling's notation, we introduce the "equivalent electric field,"

$$\underset{\sim}{E}' = \underset{\sim}{E} + \frac{\underset{\sim}{v} \wedge \underset{\sim}{H}}{c} + \frac{\nabla p_e}{n_e e} , \qquad (3.5)$$

and write (3.4) in the form

$$\underset{\sim}{E}' = \frac{\underset{\sim}{j}}{\sigma} + \frac{\underset{\sim}{j} \wedge \underset{\sim}{H}}{c n_e e} , \qquad (3.6)$$

since by definition $\underset{\sim}{j} = - n_e e \underset{\sim}{V}$. The scalar

$$\sigma = \frac{n_e e^2 \tau_e}{m_e} \qquad (3.7)$$

is called the conductivity -- sometimes the "unreduced" conductivity. Equivalently, (3.6) may be written

$$\underset{\sim}{j} = \sigma \left[\underset{\sim}{E} + \frac{\underset{\sim}{v} \wedge \underset{\sim}{H}}{c} + \frac{\left(\nabla p_e - \frac{\underset{\sim}{j} \wedge \underset{\sim}{H}}{c} \right)}{n_e e} \right] . \tag{3.8}$$

By definition, $\underset{\sim}{E}'$ is the electric field as seen in a frame moving with the local bulk velocity of the gas, plus the extra term $\nabla p_e / n_e e$ (which we shall call the "battery term.") If $\underset{\sim}{E}'$ is parallel to $\underset{\sim}{H}$, (3.6) reduces to

$$\underset{\sim}{j} = \sigma \underset{\sim}{E}' , \tag{3.9}$$

a familiar form of Ohm's law. The current flows parallel to the equivalent electric field; the accelerating force on the electrons $(\underset{\sim}{E}')$ is just balanced by the decelerating frictional force $\underset{\sim}{j}/\sigma$. If $\underset{\sim}{E}'$ is perpendicular to $\underset{\sim}{H}$, one can introduce Cowling's anisotropic conductivity, writing (3.6) as

$$\underset{\sim}{j} = \sigma^{I} \underset{\sim}{E}' + \sigma^{II} \frac{\underset{\sim}{H} \wedge \underset{\sim}{E}'}{H} \tag{3.10}$$

where

$$\sigma^{I} = \frac{\sigma}{1 + \left(\frac{\sigma H}{n_e e c} \right)^2} = \frac{\sigma}{1 + (\omega_e \tau_e)^2} \tag{3.11}$$

and

$$\sigma^{II} = \frac{\sigma H}{n_e e c} \sigma^{I} = \frac{(\omega_e \tau_e) \sigma}{1 + (\omega_e \tau_e)^2} . \tag{3.12}$$

If $\omega_e \tau_e \ll 1$, then we have $\sigma^{I} \approx \sigma$, and $\sigma^{II} \ll \sigma$, so that (3.10) reduces to (3.9), whatever the direction of $\underset{\sim}{E}'$. This is the case for high density and weak field: the electrons do not have time to take cognizance of the anisotropy of the medium (due to the presence of the field $\underset{\sim}{H}$), because a collision occurs before the electron has traveled far along its circle of gyration. The field $\underset{\sim}{H}$ has an effect only as part of the definition of $\underset{\sim}{E}'$.

By contrast, if $\left|\underset{\sim}{j} \wedge \underset{\sim}{H}/cn_e e\right| \gg \left|\underset{\sim}{j}/\sigma\right|$ -- so that $\sigma H/cn_e e = \omega_e \tau_e \gg 1$ -- then we have $\sigma^{I} \ll \sigma^{II} \ll \sigma$. This is the case of "free spiralling of electrons between collisions;" the drift of the electrons across the field is strongly impeded. In the limit $\tau_e \to \infty$, $\sigma \to \infty$, $\sigma_I \to 0$, and $\sigma^{II} \to n_e ec/H$; the current across $\underset{\sim}{H}$ becomes

$$\underset{\sim}{j} = (-n_e e)\left(-c\,\frac{\underset{\sim}{H} \wedge \underset{\sim}{E}'}{H^2}\right) \qquad (3.13)$$

-- just the current due to electrons drifting across the field with the velocity $-c(\underset{\sim}{H} \wedge \underset{\sim}{E}'/H^2)$. In the absence of friction a component of $\underset{\sim}{E}'$ parallel to $\underset{\sim}{H}$ continually accelerates electrons, and yields an infinite parallel conductivity; whereas a component $\underset{\sim}{E}'_{\perp}$ yields a finite drift velocity perpendicular to both $\underset{\sim}{E}'_{\perp}$ and $\underset{\sim}{H}$. When $(\omega_e \tau_e) \ll 1$, then $\underset{\sim}{j}$ and $\underset{\sim}{E}'$ are parallel; but when $(\omega_e \tau_e) \gg 1$, $\underset{\sim}{j}_{\perp}$ is perpendicular to both $\underset{\sim}{E}'_{\perp}$ and $\underset{\sim}{H}$, and $\sigma^{II}/\sigma \ll 1$. As $\underset{\sim}{H}$ is increased at constant density and temperature, σ^{II}/σ decreases.

However, it should be emphasised that this reduced conductivity does not imply increased dissipation. To the order of the present theory, the number of ion-electron collisions is unaffected by the presence of $\underset{\sim}{H}$, and the volume rate of dissipation is given by j^2/σ , where $\underset{\sim}{j}$ is again $(c/4\pi)\,\nabla \wedge \underset{\sim}{H}$, and σ the unreduced conductivity (3.7). A more accurate theory based on Boltzmann equation does yield an increase in dissipation up to a factor two for currents $\underset{\sim}{j}_{\perp}$, as compared with $\underset{\sim}{j}_{\parallel}$. This is due to the difference in the distribution functions, and is far smaller than the enormous increases that would be given by writing σ^{I} or σ^{II} instead of σ in j^2/σ .

Schlüter pointed out that a redefinition of the equivalent electric field enables Ohm's law to be written in isotropic form, whatever the value of $\omega_e \tau_e$. For on eliminating the nonlinear term $\underset{\sim}{j} \wedge \underset{\sim}{H}/c$ between (3.8) and the equation of motion (3.2), we have

$$\underset{\sim}{j} = \sigma \underset{\sim}{E}'' \ , \tag{3.14}$$

where

$$\underset{\sim}{E}'' = \left[\underset{\sim}{E} + \frac{\underset{\sim}{v} \wedge \underset{\sim}{H}}{c} + \frac{1}{n_e e} \left(\rho \underset{\sim}{g} - \nabla p_i - \rho \frac{d\underset{\sim}{v}}{dt} \right) \right], \tag{3.15}$$

where now $\underset{\sim}{E}''$ does not involve $\underset{\sim}{j}$. This is sometimes a more convenient form to use than (3.8); but as emphasised by Cowling, it is derived from (3.2) and (3.8), and so when used in any particular problem with the equation of motion (3.2) it must yield precisely similar results. Equation (3.15) involves the unreduced conductivity, even for components perpendicular to $\underset{\sim}{H}$; then $(\underset{\sim}{E}'')_\perp$ must be less than $(\underset{\sim}{E}')_\perp$, and rotated relative to it.

Schlüter remarks that his equation (3.15), with $\underset{\sim}{j}/\sigma$ substituted for $\underset{\sim}{E}''$, cannot be usefully regarded as an alternative to (3.2), determining the motion of the gas, even though it can be re-written so that $\rho(d\underset{\sim}{v}/dt)$ is given in terms of field vectors. The electric field $\underset{\sim}{E}$ is not known *a priori*; it consists of an irrotational part, related to the distribution of space charges, and a solenoidal part, related to the change in the magnetic field. Even in problems of "low magnetic Reynolds number" (see following), where the change in the magnetic field can be ignored, in general there will be a space-charge field. On the contrary, it is (3.8) or (3.15) that we must use to find $\nabla \cdot \underset{\sim}{E} = 4\pi\rho_e$ -- just as we used the approximation (2.9) to estimate ρ_e (2.12). In a conducting medium one cannot prescribe a charge distribution and so determine the associated electric field from Poisson's equation; charge is mobile, and adjusts itself to satisfy the analog of Ohm's law. [In elementary electrostatics, one is accustomed to reading Poisson's equation from left to right (for insulators) and from right to left (conductors).]

By neglecting the displacement current and electron inertia, we imply that the space-charge field associated with a varying $\underset{\sim}{E}$ is built up instantaneously. If we retain the linear electron inertia term, Ohm's law becomes

$$\frac{m_e}{n_e e^2} \frac{\partial \underset{\sim}{j}}{\partial t} = \underset{\sim}{E} + \left(\frac{\underset{\sim}{v} \wedge \underset{\sim}{H}}{c} + \frac{\nabla p_e}{n_e e} - \frac{\underset{\sim}{j} \wedge \underset{\sim}{H}}{c n_e e} \right) - \frac{\underset{\sim}{j}}{\sigma} . \tag{3.16}$$

Consider for simplicity the case of n_e and σ uniform. Taking the divergence of (3.16) yields

$$-\frac{m_e}{n_e e^2} \ddot{\rho}_e = 4\pi\rho_e + \frac{\dot{\rho}_e}{\sigma} + \nabla \cdot \left[\frac{\underset{\sim}{v} \wedge \underset{\sim}{H}}{c} + \frac{\nabla p_e}{n_e e} - \frac{\underset{\sim}{j} \wedge \underset{\sim}{H}}{c n_e e} \right] , \tag{3.17}$$

where as usual the inclusion of the displacement current in (2.4) allows a nonvanishing $\nabla \cdot \underset{\sim}{j}$ to be consistent with Poisson's equation and conservation of charge. Equation (3.17) implies that the charge density oscillates with the plasma frequency $\omega_p = (4\pi n_e e^2/m_e)^{1/2}$ for a time of order $8\pi\sigma/\omega_p^2$ (σ in esu), after which electron-ion collisions damp out the oscillations, and the charge density is given by the divergence of the normal form (3.8) of Ohm's law. Although $8\pi\sigma/\omega_p^2$ is long compared with the plasma frequency (see below), it is far shorter than macroscopic time-scales. Subtle effects such as Landau damping can be predicted only by detailed consideration of the non-Maxwellian nature of the electron distribution function: the explicit use of an isotropic pressure automatically removes this effect. Equally, electrostatic instabilities -- the feeding in of macroscopic kinetic energy into random space-charge waves -- cannot be included in our theory. Such ffects, however, require electron drifts of the order of the thermal velocity, and so would not normally be expected.

The "unreduced" conductivity σ is by (3.7) and (2.27),

$$\frac{n_e e^2}{m_e}\left(\frac{1}{\pi\sqrt{\frac{3kT}{m_e}}\left(\frac{3Ze_e^2}{2kT}\right)^2}\right) \approx \frac{2 \times 10^7 T^{3/2}}{Z} . \qquad (3.18)$$

The density disappears, as the number of carriers of current n_e increases proportionately to the number of resisters n_i. Inside a star where $T \sim 10^6\,^\circ K$, the conductivity is like that of copper; in the photosphere of the sun, where $T \sim 6000^\circ K$, it is lower, being like that of sea water. The ratio of the decay time of plasma oscillations to their period is

$$\frac{\frac{8\pi\sigma}{\omega_p^2}}{\frac{2\pi}{\omega_p}} = \frac{4\sigma}{\omega_p} \sim \left(\frac{1}{n_e^{1/3} b}\right)^{3/2} ,$$

where b is again the collision radius $\sim e^2/kT$. As b is always much smaller than the inter-particle distance $n_e^{-1/3}$ in an uncondensed plasma, we see that the description of the process in terms of slowly damped plasma oscillation is valid.

C. THE ENERGY EQUATION FOR A FULLY IONIZED GAS

The scalar product of "Ohm's law" in the form (3.8), with the current density $\underset{\sim}{j}$, yields

$$\underset{\sim}{j} \cdot \underset{\sim}{E} = \frac{\underset{\sim}{j}^2}{\sigma} - \frac{\nabla p_e}{n_e e} \cdot \underset{\sim}{j} \quad \frac{\underset{\sim}{j} \wedge \underset{\sim}{H}}{c} \cdot \underset{\sim}{v} \qquad (3.19)$$

Now $\underset{\sim}{j} \cdot \underset{\sim}{E}$ is the rate of working per cm^3 of the electromagnetic forces:

$$- n_e e \left(\underset{\sim}{E} + \frac{(\underset{\sim}{v} + \underset{\sim}{V})}{c} \wedge \underset{\sim}{H} \right) . (\underset{\sim}{v} + \underset{\sim}{V}) + n_i Ze \left(\underset{\sim}{E} + \frac{\underset{\sim}{v} \wedge \underset{\sim}{H}}{c} \right) . \underset{\sim}{v}$$

$$= \left[n_i Ze\underset{\sim}{v} - n_e e(\underset{\sim}{v} + \underset{\sim}{V}) \right] . \underset{\sim}{E} = \underset{\sim}{j} \cdot \underset{\sim}{E}$$

(This well-known result does not depend on the quasi-neutrality of the plasma.) In fact, the truncated Maxwell equations (2.17-2.20) yield

$$\int_\tau \underset{\sim}{j} \cdot \underset{\sim}{E} d\tau = \int_\tau \frac{c}{4\pi} (\nabla \wedge \underset{\sim}{H}) \cdot \underset{\sim}{E} d\tau$$

$$= \int_\tau - \nabla \cdot (\tfrac{c}{4\pi} \underset{\sim}{E} \wedge \underset{\sim}{H}) \, d\tau + \int_\tau \tfrac{c}{4\pi} (\underset{\sim}{H} \cdot \nabla \wedge \underset{\sim}{E}) \, d\tau$$

$$= - \int_S (\tfrac{c}{4\pi} \underset{\sim}{E} \wedge \underset{\sim}{H}) \cdot \underset{\sim}{n} dS - \frac{\partial}{\partial t} \int \frac{H^2}{8\pi} d\tau \quad , \qquad (3.20)$$

where S is the surface bounding the volume τ, and $\underset{\sim}{n}$ is the outward-drawn normal. Thus $\int - (\underset{\sim}{j} \cdot \underset{\sim}{E}) d\tau$ is equal to the rate of increase of the magnetic energy in τ together with the outward Poynting flux. Because of our neglect of the displacement current there is no electrical energy density. Also, the magnetic field due to a current distribution of finite dimensions falls off according to the inverse square at infinity, and so yields a vanishing Poynting integral at infinity -- the system does not lose any energy by radiation.

Thus (3.19) gives the energy balance for the field: the rate of working per cm^3 of the electromagnetic forces is the sum of three terms:

(i) $$\frac{\underset{\sim}{j}^2}{\sigma} = \frac{(-n_e e\underset{\sim}{V})^2}{\frac{n_e e^2 \tau_e}{m_e}} = - \underset{\sim}{V} \cdot \left[\frac{-n_e m_e \underset{\sim}{V}}{\tau_e} \right] .$$

The frictional drag density on the electrons drifting through the ions is $-n_e m_e \underset{\sim}{V}/\tau_e$, so that the loss of energy per cm^3 per sec, due to randomization of the electron drift velocity, is $\left(-n_e m_e/\tau_e\right)\underset{\sim}{V} \cdot \underset{\sim}{V} = j^2/\sigma$. This is the familiar Joule heat term. Note that σ is the ordinary "unreduced" conductivity; in fact the Hall field -- which we saw yielded the anisotropic conductivity -- has disappeared from (3.19), because it is perpendicular to $\underset{\sim}{j}$. Thus given a magnetic field $\underset{\sim}{H}$, "maintained" by a current density $\underset{\sim}{j} = (c/4\pi)\nabla_\wedge \underset{\sim}{H}$, the volume rate of dissipation of its energy into heat by electron-ion collisions is j^2/σ.

$$\text{(ii)} \quad -\left(\frac{\nabla p_e}{n_e e}\right) \cdot \underset{\sim}{j}$$

-- minus the rate of working per cm^3 of the "battery field" -- sometimes called the "impressed electric field." If $\nabla p_e/n_e e$ is essentially parallel to $\underset{\sim}{j}$, it pumps energy into the system; if anti-parallel, energy is going out of the magnetic field -- the "battery is being charged."

$$\text{(iii)} \quad \left(\frac{\underset{\sim}{j} \wedge \underset{\sim}{H}}{c}\right) \cdot \underset{\sim}{v}$$

-- the rate of working of the magnetic body force density on the fluid moving with bulk velocity $\underset{\sim}{v}$. This is positive if $(\underset{\sim}{j} \wedge \underset{\sim}{H}/c)$ is essentially parallel to $\underset{\sim}{v}$; the magnetic force drives the fluid against the other forces acting -- pressure, gravity, viscous force, etc. Some of the work done may be dissipated by viscosity or by nonadiabatic compression, but we may expect some to contribute to the kinetic and potential energies of the gas. This case is the analog of the electric motor. On the other hand, if $\left(\underset{\sim}{j} \wedge \underset{\sim}{H}/c\right) \cdot \underset{\sim}{v}$ is negative, then the nonmagnetic forces acting on the fluid are driving it against the magnetic force: energy is pumped into the field at the expense of the energy sources of the forces driving the fluid -- the analog of the dynamo.

In considering the equation to the heating of the gas, not only the normal viscous and compressional heating should be included but also the Joule term (i), and the battery term (ii).

CHAPTER 4

COUPLING OF OHM'S LAW AND MAXWELL'S EQUATIONS

For the moment we ignore the thermal and Hall fields, and use the simple form of Ohm's law

$$\frac{\underset{\sim}{j}}{\sigma} = \underset{\sim}{E} + \frac{\underset{\sim}{v} \wedge \underset{\sim}{H}}{c} \quad . \tag{4.1}$$

This is the invariant form that replaces the equation

$$\frac{\underset{\sim}{j}}{\sigma} = \underset{\sim}{E} \quad , \tag{4.2}$$

valid at a point of the fluid at rest in the frame in which $\underset{\sim}{E}$ is measured. Equation (4.2) must be used also when dealing with moving <u>solid</u> conductor, e.g. a rotating dynamo.

From (4.1), Ampère's law and Faraday's law, we have

$$\frac{\partial \underset{\sim}{H}}{\partial t} = - c \nabla \wedge \underset{\sim}{E} = \nabla \wedge (\underset{\sim}{v} \wedge \underset{\sim}{H}) - c \nabla \wedge \left(\frac{\underset{\sim}{j}}{\sigma}\right) = \nabla \wedge (\underset{\sim}{v} \wedge \underset{\sim}{H}) - \frac{c^2}{4\pi} \nabla \wedge \left(\frac{\nabla \wedge \underset{\sim}{H}}{\sigma}\right)$$

$$= \nabla \wedge (\underset{\sim}{v} \wedge \underset{\sim}{H}) + \frac{c^2}{4\pi\sigma} \nabla^2 \underset{\sim}{H} \ , \tag{4.3}$$

where for simplicity σ has been taken constant, and the condition $\nabla \cdot \underset{\sim}{H} = 0$ has been used. The operator ∇^2 in (4.3) must be applied only to Cartesian components.

We now consider special cases. If the second term on the right of (4.3) is negligible, then

$$\frac{\partial \underset{\sim}{H}}{\partial t} = \nabla \wedge (\underset{\sim}{v} \wedge \underset{\sim}{H}) \quad , \tag{4.4}$$

the perfect conductor approximation. With D a typical scale of variation of the field, we have that this is a good approximation if

$$\frac{4\pi\sigma}{c^2} vD \gg 1 \quad . \tag{4.5}$$

Equation (4.3) is similar to the equation to the vorticity $\underset{\sim}{\omega}$ in an incompressible fluid, with $\underset{\sim}{H}$ replacing $\underset{\sim}{\omega}$ and $1/(4\pi\sigma/c^2)$ the kinematic viscosity ν, a resemblance exploited by Batchelor. For this reason $(4\pi\sigma/c^2)(vD)$ is often called the Magnetic Reynolds Number.

Equation (4.3) is purely kinematic -- the velocity is prescribed, and its effect on the magnetic field studied. As is well known, the approximation (4.4) implies the "freezing of the field" into the moving fluid: as is proved in texts on classical electromagnetism, the rate of change of the flux of $\underset{\sim}{H}$ across a moving circuit is

$$\int_S \left[\frac{\partial \underset{\sim}{H}}{\partial t} - \nabla \wedge (\underset{\sim}{v} \wedge \underset{\sim}{H}) \right] . \underset{\sim}{n} dS \quad . \tag{4.6}$$

where the surface element $\underset{\sim}{n}dS$ has velocity $\underset{\sim}{v}$. More simply, we may appeal to Faraday's law relating the rate of change of the flux across a circuit to the electromotive force round the circuit, valid whether the circuit is at rest or moving. In an infinitely conducting medium the e.m.f. must vanish, whence the conservation of the flux. In fact, the expression

$$\underset{\sim}{E}' = \underset{\sim}{E} + \frac{\underset{\sim}{v} \wedge \underset{\sim}{H}}{c}$$

for the electric field in a moving frame is just the condition that keeps Faraday's law invariant under a Galilean transformation.

Qualitatively, we may say that while fluid flow parallel to the field has no direct electrodynamic effect, flow across the field drags the field with it.

But this does not imply that in a steady state $\underset{\sim}{v}$ must be parallel to $\underset{\sim}{H}$ in all cases. All that is required is that the flux across a moving circuit should be invariant, or that the field $\underset{\sim}{v} \wedge \underset{\sim}{H}/c$ can be balanced by a space-charge electric field $-\nabla\phi$. Thus steady flow across a unifrom field is not forbidden — in fact it can be reached from the case $\underset{\sim}{v}$ parallel to $\underset{\sim}{H}$ by a Galilean transformation. Equally, uniform rotation of a field about an axis of symmetry does not change the field (discussed in detail in Chapter 6).

Now suppose that we have

$$\frac{4\pi\sigma}{c^2} v\ell \ll 1 \;, \tag{4.7}$$

so that (4.3) reduces to

$$\nabla^2 \underset{\sim}{H} = \frac{4\pi\sigma}{c^2} \frac{\partial \underset{\sim}{H}}{\partial t} \;. \tag{4.8}$$

This is just an ordinary diffusion equation, implying that nonuniformities in the field drift so as to cancel each other. A characteristic time for the decay of a field of scale D is

$$\tau = \frac{4\pi\sigma}{c^2} D^2 \;. \tag{4.9}$$

If we divide the magnetic energy density $H^2/8\pi$ by the rate of Ohmic dissipation per cm^3, $j^2/\sigma = [(c/4\pi)(H/D)]^2/\sigma$, we arrive at a time-scale of the same order.

We may interpret the condition (4.5) for the approximate freezing of the field by writing it as

$$\tau \gg \frac{D}{v} = \text{time of flow across region } D \;.$$

Alternatively, we can write it as $v \gg D/\tau$, where D/τ can be thought of as a velocity of diffusion of the field through the fluid. In fact, if $\underset{\sim}{j}$ is perpendicular to $\underset{\sim}{H}$, we may write (4.1) as

$$\underset{\sim}{E} + \frac{\underset{\sim}{v}' \wedge \underset{\sim}{H}}{c} = 0$$

where

$$\underset{\sim}{v}' - \underset{\sim}{v} = \frac{c}{\sigma} \frac{\underset{\sim}{j} \wedge \underset{\sim}{H}}{H^2} \; ; \qquad (4.10)$$

the field then moves with a speed $\sim D/\tau$ through the fluid, in the direction of the magnetic force.

Let us now put in some numbers. For a large-scale field in a star, we take $\sigma \sim 10^{17}$ esu (corresponding to $T \sim 10^{7}$ °K), and $D \sim 10^{10}$ cm -- (one seventh of the solar radius); τ is then $\sim 5 \times 10^{9}$ years -- about the age of the solar system. A more accurate treatment by Cowling yields 10^{10} years for the e-folding time of the slowestdecaying mode. It is thus not impossible that stellar general magnetic fields are "fossils" -- slowly decaying relics of the primeval fields present when the stars were formed. [Such an explanation is not possible, however, for the earth's field -- the factor D^2 in (4.9) reduces the e-folding time much below the age of the earth, and a regeneration process is required.]

Again consider a fairly small sunspot, with $D \sim 10^{8}$ cm, and $\sigma \sim 10^{15}$, corresponding to the lower temperature near the surface. (We assume that the currents maintaining the sunspot field flow deep enough for the matter to be almost fully ionized, so that the present theory is applicable.) Then we have $\tau \simeq 50$ years -- much longer than the time in which a spot disappears (a few weeks or months at the most). Cowling concluded that the observed disappearance of a spot must be due to hydromagnetic dragging of the field below the surface, and not through destruction of the field by Ohmic dissipation.

These enormous time-scales are due to the factor D^2. A decaying field induces enormous electric fields that tend to keep the currents flowing; or, alternatively, by Ampère's law the currents required to maintain a large-scale field are weak, and so the dissipation j^2/σ is small. The difficulty of simulating astrophysical phenomena in the laboratory is due mainly to the requirement that D be large.

However, one must beware of assuming automatically in all cases that D is a "large" length -- e.g., comparable with the radius of a star. In some problems hydromagnetic distortion of the magnetic field, with either magnetic or nonmagnetic foeces driving the fluid, may force the field to change its direction sharply over a narrow region, implying a larger Ohmic dissipation than normal. Also, some simple fields have singular regions in which the Ohmic field must dominate over the motional induction field -- D becomes arbitrarily small. Thus if we put $D \sim 10^{10}$ cm and $\sigma \sim 10^{17}$ esu, condition (4.5) requires $v > 10^{-7}$ cm/sec ~ 3 cm/year, a trivially slow motion; but an arbitrarily small $\mathcal{D}$ requires an arbitrarily large v. This will be further discussed in connection with the dynamo problem.

Let us now see the effect of the Hall and battery terms. Assuming infinite conductivity, we have as a replacement for (4.3)

$$\frac{\partial \underset{\sim}{H}}{\partial t} = \nabla \wedge \left[(\underset{\sim}{v} + \underset{\sim}{V}) \wedge \underset{\sim}{H} \right] + \nabla \wedge \left(\frac{c \nabla p_e}{n_e e} \right) , \qquad (4.10)$$

where we have written the Hall term in its original form $\underset{\sim}{V} \wedge H/c$, V being the drift of the electrons relative to the ions. Without the battery term this equation implies that the flux is invariant through a circuit moving with the electron velocity $(\underset{\sim}{v} + \underset{\sim}{V})$: as electrons are conserved (in a non-recombining plasma), so is magnetic flux. As in most cosmical applications

$V \ll v$, the change is of little importance. However, the thermal term can generate magnetic flux -- whence our calling it the "battery" term. The rate of change of the flux across a surface S spanning a circuit C moving with the electrons is

$$\int_S \nabla \wedge \left(\frac{c\nabla p_e}{n_e e}\right) . d\underset{\sim}{S} = \oint_C \left(\frac{c\nabla p_e}{n_e e}\right) . d\underset{\sim}{s} = \text{"emf" of battery acting round } C \quad .$$

The generation of magnetic flux by the thermal field is important, because the "motional induction" or hydromagnetic term $\underset{\sim}{v} \wedge \underset{\sim}{H}/c$ and the Hall term do not create new flux -- they merely drag existing field lines about (though as we shall see in connection with the dynamo problem they can cause field lines to be used more "efficiently"). For dragging of the field to occur at all -- e.g., for energy to be pumped into the magnetic field from the kinetic field -- an initial field is required.

It will be noted that once Ohm's law is coupled with Faraday's law, attention is focussed on the freezing of the field into the matter, and deviations therefrom. Thus although with σ infinite, the component of $\underset{\sim}{j}$ across $\underset{\sim}{H}$ must be perpendicular to the corresponding component of the "equivalent electric field" $[\underset{\sim}{E} + (\underset{\sim}{v} \wedge \underset{\sim}{H}/c) + (\nabla p_e/n_e e)]$, this result is of small significance compared with the excellent (cosmical) approximation $\underset{\sim}{E} + \underset{\sim}{v} \wedge \underset{\sim}{H}/c \simeq 0$, (the battery term will often be small compared with $\underset{\sim}{v} \wedge \underset{\sim}{H}/c$). However, it can happen that $\underset{\sim}{v} \wedge \underset{\sim}{H}/c$ has a zero component in a particular direction, so that the "small" terms become significant. This will be discussed further in Chapter 7.

CHAPTER 5

DYNAMICAL COUPLING

So far the interaction between the magnetic and kinetic fields has been one-way -- $\underset{\sim}{v}$ has been prescribed, and the consequences for $\underset{\sim}{H}$ and $\underset{\sim}{j}$ examined. We now discuss the effect of magnetic forces on the flow.

A. FLOW UNDER A FIXED MAGNETIC FIELD

We begin by supposing $\underset{\sim}{H}$ to be prescribed, and ignore any changes in $\underset{\sim}{H}$ due to generation of new currents. This $\underset{\sim}{H}$ must be a curl-free field, maintained by currents external to the region of interest. Then if there is a flow across $\underset{\sim}{H}$, Ohm's law (ignoring Hall and thermal terms) must be satisfied, and the current field $\underset{\sim}{j}$ is given by

$$\underset{\sim}{j} = \sigma\left(\underset{\sim}{E} + \frac{\underset{\sim}{v} \wedge \underset{\sim}{H}}{c}\right)$$

where $\underset{\sim}{E}$ is by hypothesis purely "electrostatic" -- we are ignoring $\partial H/\partial t$, and so $\nabla \wedge \underset{\sim}{E} = 0$. Thus at any instant we may write (by Helmholtz's theorem),

$$\frac{\underset{\sim}{v} \wedge \underset{\sim}{H}}{c} = \nabla\phi + \nabla \wedge \underset{\sim}{a} \quad .$$

Then if $\nabla\phi \neq 0$, there is an effectively instantaneous adjustment of the charge density to maintain an electrostatic or polarization field $\underset{\sim}{E} = -\nabla\phi$, while the part $\nabla \wedge \underset{\sim}{a}$ is left to drive currents. If we have

$$\nabla \cdot (\underset{\sim}{v} \wedge \underset{\sim}{H}) = \underset{\sim}{H} \cdot (\nabla \wedge \underset{\sim}{v}) - \underset{\sim}{v} \cdot (\nabla \wedge \underset{\sim}{H}) = 0$$

--e.g., if $\underset{\sim}{H}$ is curl-free, and $\underset{\sim}{v}$ vortex-free -- then $\nabla^2\phi = 0$ everywhere, so that $\nabla\phi = 0$ (with suitable boundary conditions at infinity): then all of $\underset{\sim}{v} \wedge \underset{\sim}{H}/c$ is available to drive currents, provided the hypothesis that self-induction may be ignored is valid. Suppose then that

$$\underset{\sim}{j} = \sigma\left(\frac{\underset{\sim}{v} \wedge \underset{\sim}{H}}{c}\right). \qquad (5.1)$$

The magnetic force due to the flow of $\underset{\sim}{j}$ across $\underset{\sim}{H}$ is then

$$\left(\frac{\sigma \dfrac{(\underset{\sim}{v} \wedge \underset{\sim}{H})}{c} \wedge \underset{\sim}{H}}{c} \right) = - \frac{\sigma}{c^2} H^2 \left\{ \underset{\sim}{v} - \frac{(\underset{\sim}{v} \, . \, \underset{\sim}{H}) \, \underset{\sim}{H}}{H^2} \right\} = - \frac{\sigma}{c^2} H^2 \underset{\sim}{v}_{\perp} \tag{5.2}$$

where $\underset{\sim}{v}_{\perp}$ = velocity across $\underset{\sim}{H}$. The equation to the motion of the matter across $\underset{\sim}{H}$ is

$$\rho \frac{d\underset{\sim}{v}_{\perp}}{dt} = \underset{\sim}{P}_{\perp} - \frac{\sigma}{c^2} H^2 v_{\perp} \tag{5.3}$$

where $P_{\perp}$ is the component across $\underset{\sim}{H}$ of the density of the other forces acting on the fluid. Then (5.3) implies an exponential decay of $v_{\perp}$, with a time constant $\rho/(\sigma/c^2)H^2$, to the value $P_{\perp}/(\sigma/c^2)H^2$. The current density $\underset{\sim}{j}$ approaches

$$\frac{\frac{\sigma}{c} \underset{\sim}{P} \wedge \underset{\sim}{H}}{\frac{\sigma}{c^2} H^2} = c \frac{\underset{\sim}{P} \wedge \underset{\sim}{H}}{H^2} .$$

This is independent of the conductivity; in fact it is the dynamical condition

$$\frac{\underset{\sim}{j} \wedge \underset{\sim}{H}}{c} + P_{\perp} = 0 \tag{5.4}$$

that specifies $\underset{\sim}{j}$. Ohm's law (5.1) merely fixes the velocity of drift across the field, allowed by the finite conductivity or the work done on the fluid by $\underset{\sim}{P}_{\perp}$ required to offset the Ohmic dissipation of energy. Instead of Ohm's law determining the current, and the equation of motion the velocity, the roles are reversed.

It should be emplasised that this conclusion depends on the conductivity being high, so that $v_{\perp} \propto 1/\sigma$ is small, and the nonlinear inertia term in (5.3) is negligible. For example, if $H \sim 1$ gauss, and $\underset{\sim}{P} \sim \rho g$, where g is the surface gravity on the sum, and $\sigma/c^2 \sim 10^{-8}$ (corresponding to $T \sim 10^4$, then $v_\perp \sim 2 \times 10^{12} \rho$. With $\rho < 10^{-7}$ (the photospheric value) this is much less than the thermal speed; while the time constant $\rho/(\sigma/c^2)H^2$ is short compared with the time of gravitational free-fall over a scale height. If $\underset{\sim}{H}$ becomes very small -- e.g., near neutral points -- it may not be correct to neglect the inertia of cross-field flow. Magnetohydrostatic equilibrium will be discussed further in Chapters 7 and 10.

However, the approach to equilibrium studied above depends on self-induction being negligible. The flow of currents $\sigma(\underset{\sim}{v} \wedge \underset{\sim}{H}/c)$ in a region of scale D generates a field $\underset{\sim}{H}'$, satisfying

$$\frac{H'}{D} \sim \frac{4\pi}{c} \left| \sigma \frac{\underset{\sim}{v} \wedge \underset{\sim}{H}}{c} \right|$$

by Ampère's law. A characteristic time for the build-up of this field is D/v , so that by Faraday's law, the induced electric field is given by

$$\frac{E}{D} \sim - \frac{1}{c} \frac{\frac{4\pi}{c} D\sigma \frac{vH}{c}}{\frac{D}{v}} \sim - \frac{4\pi\sigma}{c^2} v^2 H \quad .$$

Thus $|\underset{\sim}{E}|$ becomes comparable with $|\underset{\sim}{v} \wedge \underset{\sim}{H}/c|$ if

$$\frac{4\pi\sigma}{c^2} vD \sim 1$$

-- just the magnetic Reynolds number condition again. When $(4\pi\sigma/c^2)vD >> 1$, we must use, as a zero-order approximation, not (5.1) but

$$\underset{\sim}{E} + \frac{\underset{\sim}{v} \wedge \underset{\sim}{H}}{c} = 0 \tag{5.5}$$

so that at any time

$$\frac{\partial \underset{\sim}{H}}{\partial t} = \nabla \wedge (\underset{\sim}{v} \wedge \underset{\sim}{H}) \quad . \tag{5.6}$$

The current density is then given by Ampère's equation, and so the magnetic force density $(\nabla \wedge \underset{\sim}{H}) \wedge \underset{\sim}{H}/4\pi$ is fixed by (5.6). The dragging of the field by the moving fluid -- implied by (5.6) -- generates a magnetic force which does not automatically vanish when the fluid is at rest, as is the case with (5.2). The magnetic force may be represented as the divergence of the Maxwell stress tensor, consisting of a uniform pressure $H^2/8\pi$, and a tension $H^2/4\pi$ along the direction of the field. With matter moving with the field, we may regard each field line of infinitesimal area A as an elastic string under a tensile force $(H^2/4\pi)A$, and with a line density (ρA), where ρ is the volume density of the fluid. Thus a disturbance to a uniform field will not be damped out, but will generate a set of Alfvén or magnetohydrodynamic waves, traveling with speed $(H^2/4\pi\rho)^{1/2}$. There is a continuous interchange of energy between the kinetic and magnetic fields.

B. ALFVÉN WAVES

Consider a perfect conductor, incompressible and of uniform density ρ. There are no nonmagnetic body forces, and there is a uniform field $\underset{\sim}{H}_0$ pervading the medium, its direction parallel to the z-axis. All quantities are assumed to depend only on z and t.

Then from (5.6) we obtain

$$\frac{\partial \underset{\sim}{H}}{\partial t} = \left(H_z \frac{\partial}{\partial z}\right) \underset{\sim}{v} - \left(v_z \frac{\partial}{\partial z}\right) \underset{\sim}{H} , \qquad (5.7)$$

where use has been made of the incompressibility condition

$$\nabla \cdot \underset{\sim}{v} = 0 . \qquad (5.8)$$

This last implies that $\partial v_z/\partial z = 0$, so that there exists a frame with $v_z = 0$: the fluid motion is transverse. From $\nabla \cdot \underset{\sim}{H} = 0$, we have that H_z is a function of t at the most; from (5.7), $\partial H_z/\partial t = 0$, so that we may write

$$\underset{\sim}{H} = \underset{\sim}{H}_0 + \underset{\sim}{h} , \qquad (5.9)$$

where $\underset{\sim}{h}$ is transverse ($h_z = 0$). Thus

$$\left.\begin{aligned} \frac{\partial h_x}{\partial t} &= H_0 \frac{\partial v_x}{\partial z} \\ \text{and} \qquad & \\ \frac{\partial h_y}{\partial t} &= H_0 \frac{\partial v_y}{\partial z} \end{aligned}\right\} \tag{5.10}$$

The magnetic force density $(\nabla \wedge \underset{\sim}{H}) \wedge \underset{\sim}{H}/4\pi$ is

$$\left[\frac{H_0}{4\pi} \frac{\partial h_x}{\partial z} , \frac{H_0}{4\pi} \frac{\partial h_y}{\partial z} , - \frac{1}{8\pi} \frac{\partial}{\partial z} (h_y{}^2 + h_x{}^2) \right] . \tag{5.11}$$

The equation of motion

$$\rho \left(\frac{\partial \underset{\sim}{v}}{\partial t} + (\underset{\sim}{v} \cdot \nabla) \underset{\sim}{v} \right) = - \nabla p + \frac{(\nabla \wedge \underset{\sim}{H}) \wedge \underset{\sim}{H}}{4\pi} \tag{5.12}$$

yields from the z-component

$$p + \frac{h_y{}^2 + h_x{}^2}{8\pi} = \text{const} = p_0 \tag{5.13}$$

where p_0 is the fluid pressure far from the disturbance. The other components yield

$$\left.\begin{aligned} \rho \frac{\partial v_x}{\partial t} &= \frac{H_0}{4\pi} \frac{\partial h_x}{\partial z} \\ \rho \frac{\partial v_y}{\partial t} &= \frac{H_0}{4\pi} \frac{\partial h_y}{\partial z} \end{aligned}\right\} \tag{5.14}$$

so that v_x, v_y, h_x, h_y all satisfy the one-dimensional wave equation; e.g.,

$$\frac{\partial^2 h_x}{\partial t^2} = V_A{}^2 \frac{\partial^2 h_x}{\partial z^2} \tag{5.15}$$

where $V_A{}^2 = H_0{}^2/4\pi\rho$. Plane-polarized, transverse waves are propagated along the field, without change of shape. No approximations have been made after the initial assumptions of σ infinite and ρ uniform. The analogy with the stretched string is imperfect, in that only small oscillations satisfy (5.15) in the string case; in the Alfvén wave problem, both the magnetic pressure and the tension in the disturbed state contribute to yield the accurate expressions (5.11) for the restoring force.

From (5.15), we obtain

$$\underset{\sim}{v} = \underset{\sim}{v}\,(z \pm V_A t) \quad \text{and} \quad \underset{\sim}{h} = \underset{\sim}{h}\,(z \pm V_A t) \quad . \tag{5.16}$$

It follows from (5.10), (5.14) and (5.16) that

$$\frac{\partial}{\partial t}\,(V_A \underset{\sim}{h} \pm H_0 \underset{\sim}{v}) = \frac{\partial}{\partial z}\,(V_A \underset{\sim}{h} \pm H_0 \underset{\sim}{v}) = 0$$

so that

$$V_A \underset{\sim}{h} \pm H_0 \underset{\sim}{v} = 0 \quad \text{or} \quad \underset{\sim}{v} = \pm \frac{\underset{\sim}{h}}{4\pi\rho} \tag{5.17}$$

as $\underset{\sim}{h}$ and $\underset{\sim}{v}$ vanish together. Thus we have

$$\frac{1}{2}\,\rho v^2 = \frac{(H_0 + h)^2 - H_0^{\,2}}{8\pi} = \frac{h^2}{8\pi} \tag{5.18}$$

as $\underset{\sim}{H_0} \cdot \underset{\sim}{h} = 0$; in an Alfvén wave there is equipartition of energy between the kinetic field and the disturbance in the magnetic field.

Waten generalized Alfvén's work. Still assuming ρ constant and H_0 uniform, he showed that there exist exact solutions for which (5.17) holds, but without requiring that $\underset{\sim}{h}$ depend on the space variables in any particular way. With (5.17) holding, the terms $(\nabla \wedge \underset{\sim}{v}) \wedge \underset{\sim}{v}$ and $(\nabla \wedge \underset{\sim}{h}) \wedge \underset{\sim}{h}$ cancel each other in the equation of motion, and again only a linear magnetic force term is left. If there exist body forces $\underset{\sim}{g} = -\nabla \phi$, ϕ being a scalar potential, then

$$p - \rho\phi + \frac{1}{8\pi}\,(\underset{\sim}{H_0} + \underset{\sim}{h})^2 = \text{const} \tag{5.19}$$

is the analog of (5.13).

Chandrasekhar showed that there exist solutions of the steady state hydromagnetic equations (for a perfect inviscid conductor) such that

$$\underset{\sim}{v} = \pm \frac{\underset{\sim}{H}}{\sqrt{4\pi\rho}} \qquad (5.20)$$

provided ρ is constant along the field-stream lines, and

$$p + \frac{H^2}{8\pi} = \rho \nabla\phi . \qquad (5.21)$$

If ρ is uniform everywhere, and we write $\underset{\sim}{H} = \underset{\sim}{H}_0 + \underset{\sim}{h}$, then it follows that a nonsteady solution of the equation is given by a Galilean transformation to a frame moving with velocity $H_0/\sqrt{4\pi\rho}$ relative to Chandrasekhar's frame; in this frame

$$v' = \underset{\sim}{v} - \frac{H_0}{\sqrt{4\pi\rho}} = \frac{h}{\sqrt{4\pi\rho}} \quad ; \qquad (5.22)$$

i.e., we recover Waten's solution.

By including dissipative terms -- the Ohmic field in Ohm's law, the viscous force in the equation of motion -- one can demonstrate the progressive damping of Alfven waves. Neglect of dissipation, but inclusion of the Hall field makes a further small change. With σ infinite, we have seen that the magnetic field moves with the electrons rather than the ions. Consequently, the modes behaving like $\exp[i(\omega t - kz)]$ are circularly polarized, and have phase velocities which differ slightly for right-handed and left-handed polarizations.

The Alfven speed is less than c provided that

$$H_0^2/4\pi < \rho c^2$$

-- i.e., the magnetic energy density is less than half the material energy density. As the two energies approach, the neglected "inertia of the ether," as represented by the displacement current, becomes important as well as the

the inertia of the matter tied to the lines of force. If the previous calculation is repeated with

$$\underset{\sim}{j} = \frac{c}{4\pi}\left[\nabla \wedge \underset{\sim}{H} - \frac{1}{c}\frac{\partial \underset{\sim}{E}}{\partial t}\right] = \frac{c}{4\pi}\left[\nabla \wedge \underset{\sim}{H} + \frac{1}{c^2}\frac{\partial}{\partial t}(\underset{\sim}{v} \wedge \underset{\sim}{H})\right] \quad ,$$

(substituting from (5.5)), we find that transverse waves are propagated with speed c', where

$$(c')^2 = \frac{c^2 V_A^{\,2}}{c^2 + V_A^{\,2}} \simeq c^2 \quad \text{if} \quad V_A^{\,2} > c^2$$

$$\simeq V_A^{\,2} \quad \text{if} \quad c^2 > V_A^{\,2}$$

As an example, a wave traveling along a field of strength $\sim$ 100 gauss, in a medium of density 1 gram/cm^3, has a speed of $\sim$ 28 cm/sec. This is far less than c, and also less than the sound speed inside a star, thus justifying the neglect of compressibility. If the magnetic energy is comparable with the thermal energy, then coupling occurs between sound and waves. It can be shown that in the presence of a uniform field H_0, there are three waves of <u>small</u> amplitude in any direction in a uniform medium:

(i) the "Alfven" wave, of speed c_A, where $c_A^{\,2} = H_z^{\,2}/4\pi\rho$, H_z being the component of H_0 in the direction of propagation z, and ρ the undisturbed density. In this mode there are no motions along the field and no density variations -- the wave consists of the propagation of a twist along the field.

(ii) slow and fast mixed sound and hydromagnetic waves of speeds c_s and c_f, such that

$$c_s < c_A < c_f \quad ,$$

with values depending on the direction of propagation.

CHAPTER 6

MAGNETISM AND STELLAR ROTATION

A. ISOROTATION

Consider a star with a rotation field $\underset{\sim}{\Omega}$ (not necessarily uniform), but with no "poloidal" motion -- i.e., no circulation of matter in the meridian planes defined by the rotation axis. Let the star possess a magnetic field $\underset{\sim}{H}$, symmetric about the rotation axis; we write

$$\underset{\sim}{H} = \underset{\sim}{H}_p + \underset{\sim}{H}_t \tag{6.1}$$

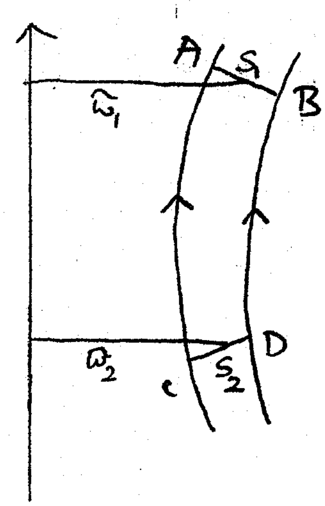
where the suffices p and t refer respectively to "poloidal" and "toroidal" -- the toroidal direction being that of the rotation velocity. Then if we assume the field to be frozen into the gas, it follows that in a steady-state Ω cannot vary along a field line; for nonuniformities in Ω would continually "stretch" the field lines in the toroidal direction. We thus have, as a necessary condition for a steady state, Ferraro's law of isorotation -- a simple and indeed obvious deduction from flux freezing; though when Ferraro deduced it in 1937, before the ideas of magnetohydrodynamics had become commonplace, it was anything but "obvious".

As a dynamical condition for a steady state, we require that there be no magnetic torque on the matter,

$$\left(\frac{\underset{\sim}{j} \wedge \underset{\sim}{H}}{c}\right)_t = \left(\frac{\underset{\sim}{j}_p \wedge \underset{\sim}{H}_p}{c}\right) = 0 \quad ; \tag{6.2}$$

as there is no poloidal circulation, there is no Coriolis force, and we ignore the small viscous force on nonuniform rotation. Condition (6.2) is satisfied if $\underset{\sim}{j}_p = 0$, so that the field is purely poloidal; more generally, we require that poloidal currents $\underset{\sim}{j}_p$ flow parallel to the poloidal field $\underset{\sim}{H}_p$, so that the total field is torque-free.

A formal proof of Ferraro's law is as follows: consider two neighboring poloidal field lines distance S_1 and S_2 from each other at AB and CD, with the segments AB and CD distance $\bar{\omega}_1$ and $\bar{\omega}_2$ from the rotation axis, and with angular velocities Ω_1 and Ω_2 respectively. The condition $\nabla . \underset{\sim}{H} = 0$ implies



$$\underline{2\pi\bar{\omega}_1 S_1 H_1 = 2\pi\bar{\omega}_2 S_2 H_2} \tag{6.3}$$

(in an axially symmetric state, any toroidal component does not contribute to the divergence condition). Also, in a steady state we find

$$\frac{\underset{\sim}{v} \wedge \underset{\sim}{H}}{c} = \nabla\phi, \tag{6.4}$$

where ϕ is the scalar potential of the space-charge field. Thus we have

$$\oint_{ABCD} (\underset{\sim}{v}_t \wedge \underset{\sim}{H}_p) \cdot \underset{\sim}{ds} = 0 \quad , \tag{6.5}$$

there being no poloidal motion; whence

$$\underline{\Omega_1 \bar{\omega}_1 H_1 S_1 = \Omega_2 \bar{\omega}_2 H_2 S_2} \quad . \tag{6.6}$$

Dividing (6.6) by (6.3) yields Ferraro's law

$$\underline{\Omega_1 = \Omega_2} \quad .$$

A less primitive treatment, e.g., using spherical polar coordinates, will yield

$$(\underset{\sim}{H} . \nabla) \Omega = 0 \quad .$$

Thus a uniform rotation of a complete field line does not affect the magnetic field, but merely generates an irrotational electric field

$$\underset{\sim}{E} = - \frac{\underset{\sim}{v}_t \wedge \underset{\sim}{H}_p}{c} \tag{6.7}$$

maintained by a charge density $\rho_e = -\nabla \cdot (\underset{\sim}{v}_t \wedge \underset{\sim}{H}_p)/4\pi c$. If Ferraro's law does not hold, the nonuniform rotation will generate a toroidal magnetic component at the expense of the rotational energy, and a complicated hydromagnetic oscillation will be set up. Equally, if Ferraro's law holds, but the magnetic field is not torque-free, the magnetic force will set up a nonuniform Ω-field, which in turn will react on the magnetic field, again generating waves along the lines of force.

If $\underset{\sim}{H}$ is purely poloidal, the Ohmic field $\underset{\sim}{j}/\sigma$ -- neglected so far -- is toroidal; hence Ferraro's law still holds, as the motional induction field $\underset{\sim}{v}_t \wedge \underset{\sim}{H}_p/c$ is a poloidal vector. But if $\underset{\sim}{H}$ is partly toroidal -- necessarily torque-free in a steady state -- then Ferraro's law must be modified because of the term $\underset{\sim}{j}_p/\sigma$. Other corrections to Ferraro's law are discussed in Chapter 7.

B. THE EFFECT OF MERIDIAN CIRCULATION

Consider for simplicity a star without meridian circulation, but with a purely poloidal field, symmetric about the axis of rotation. Initially isorotation holds, so that conditions are steady. Now imagine that a meridian circulation of matter is started up. Such circulation inside stars is known to be due to the effect on the thermal field of nonspherical perturbing forces -- centrifugal and magnetic. The completely self-consistent problem is difficult; for the moment we postulate the circulation, and examine its consequences for the magnetic and rotation fields. The convection of angular momentum by the circulation immediately causes a variation of angular velocity along field lines, which in turn generates a toroidal field. The resulting magnetic torque changes the angular momentum of circulating elements. Further, the change in the toroidal field at any point depends not only on the non-uniform rotation but also on the convection of the toroidal field by the circulation. Finally, the maintained meridian flow will convect and distort the poloidal field. The complete time-dependent problem is difficult; we suppose that a steady state has been reached, after some dissipation of energy, and look for the distribution of Ω and $\underset{\sim}{H}_t$.

Analogous to (6.1), we write

$$\underset{\sim}{v} = \underset{\sim}{v}_p + \underset{\sim}{v}_t = \underset{\sim}{v}_p + \varpi \, \Omega \underset{\sim}{t} \quad , \tag{6.8}$$

where $\underset{\sim}{t}$ is the unit toroidal vector and ϖ is again the axial distance. We again use the simple perfect conductor approximation, so that

$$\nabla \wedge [(\underset{\sim}{v}_p + \underset{\sim}{v}_t) \wedge (\underset{\sim}{H}_p + \underset{\sim}{H}_t)] = 0 \quad . \tag{6.9}$$

Poloidal and toroidal components of this are

$$\nabla \wedge (\underset{\sim}{v}_p \wedge \underset{\sim}{H}_p) = 0 \quad , \tag{6.10}$$

and

$$\nabla \wedge (\underset{\sim}{v}_t \wedge \underset{\sim}{H}_p + \underset{\sim}{v}_p \wedge \underset{\sim}{H}_t) = 0 \quad . \tag{6.11}$$

Thus from (6.10) we obtain

$$v_p \wedge \underset{\sim}{H}_p = - (\nabla \Phi)_t \quad . \tag{6.12}$$

But in an axially symmetric system, the gradient of a single-valued scalar potential cannot have a toroidal component, so that (6.12) implies

$$\underset{\sim}{v}_p = k \underset{\sim}{H}_p \tag{6.13}$$

where k is a scalar. Introducing cylindrical polar coordinates (ϖ , Φ, z), with origin at the center of the star, and the rotation axis as the z-axis, we have from (6.8), (6.11) and (6.13)

$$\frac{\partial}{\partial \varpi} (\varpi \Omega H_\varpi - k H_\varpi H_\phi) - \frac{\partial}{\partial z} (k H_z H_\phi - \varpi \Omega H_z) = \left(\Omega - \frac{k H_\phi}{\varpi} \right) \left[\frac{\partial}{\partial \varpi} (\varpi H_\varpi) + \frac{\partial}{\partial z} (\varpi H_z) \right]$$

$$+ \varpi \left(H_\varpi \frac{\partial}{\partial \varpi} + H_z \frac{\partial}{\partial z} \right) \left(\Omega - \frac{k H_\phi}{\varpi} \right) = 0 \quad . \tag{6.14}$$

As

$$\nabla \cdot \underset{\sim}{H} = \frac{1}{\varpi}\left[\frac{\partial}{\partial\varpi}(\varpi H_{\varpi}) + \frac{\partial}{\partial z}(\varpi H_z)\right] = 0 \quad , \tag{6.15}$$

(6.14) reduces to

$$\underset{\sim}{H} \cdot \nabla\left[\Omega - \frac{kH_{\phi}}{\varpi}\right] = 0 \quad ;$$

or

$$\Omega - \frac{kH_{\phi}}{\varpi} = \alpha \quad , \tag{6.16}$$

where α is constant on a particular field line. When $k = 0$ (no circulation), (6.16) reduces to Ferraro's law $\Omega = \alpha$. This generalization of Ferraro's law is simply interpreted as follows. One special solution of the steady-state hydromagnetic equation

$$\nabla \wedge (\underset{\sim}{v} \wedge \underset{\sim}{H}) = 0$$

is

$$\underset{\sim}{v} = k\underset{\sim}{H} \quad : \tag{6.17}$$

-- the <u>total</u> velocity parallel to the <u>total</u> magnetic field. This corresponds to putting $\alpha = 0$ in (6.16). Now give a poloidal field line an arbitrary but uniform rotation; then as shown by Ferraro, the motional induction field $\underset{\sim}{v}_t \wedge \underset{\sim}{H}_p/c$ merely polarized the medium and (6.17) continues to hold.

[A consequence of (6.13) is that a steady state is possible only along field lines that be within the star, since the stream-lines of the circulation are all within the star. Thus if the initial poloidal field is of a dipole type, with some field lines leaving and entering the stellar surface, an inexorable, maintained circulation within the star would continuously distort the field until the internal and external parts are separated by Ohmic diffusion (which necessarily becomes locally large if the field lines are progressively distorted). The external part of the field could then be blown away by a stellar "wind," analogous to the much-discussed solar wind, and one could not deduce the nonexistence of a large-scale ordered field within the star, from its absence outside. Perhaps it is significant that the polar plumes in the sun -- indicators of a general dipole-type field -- are confined to the poles, while in the latitudes that show sunspots and the solar cycle, there is no evidence of a general magnetic field above the surface.]

The condition of mass conservation yields

$$\nabla \cdot (\rho \underset{\sim}{v}) = \nabla \cdot (\rho k \underset{\sim}{H}) = \underset{\sim}{H} \cdot \nabla (\rho k) = 0 \tag{6.18}$$

or

$$\rho k = \eta \ , \tag{6.19}$$

η being a constant on a field line. Finally, we have the toroidal equation of motion: the magnetic torque is balanced by the transport of angular momentum by the circulation, or alternatively, the toroidal magnetic force balanced the Codiolis force:

$$\varpi \left[\frac{(\nabla \wedge \underset{\sim}{H}) \wedge \underset{\sim}{H}}{4\pi} \right]_t = \rho \underset{\sim}{v} \cdot \nabla (\Omega \varpi^2) \ . \tag{6.20}$$

Thus H_ϕ must be nonzero, otherwise $\Omega\varpi^2$ would need to be constant on a field line, and this would be inconsistent with (6.16). With

$$(\nabla \wedge \underset{\sim}{H})_{\varpi} = -\frac{1}{\varpi}\frac{\partial}{\partial z}(\varpi H_\phi)$$

$$(\nabla \wedge \underset{\sim}{H})_z = \frac{1}{\varpi}\frac{\partial}{\partial \varpi}(\varpi H_\phi) \ , \tag{6.21}$$

then (6.20) becomes, with the help of (6.17),

$$\frac{1}{4\pi}\left[H_\varpi \frac{\partial}{\partial \varpi} + H_z \frac{\partial}{\partial z} \right] (\varpi H_\phi) - \rho k \underset{\sim}{H} \cdot \nabla (\Omega \varpi^2) = \underset{\sim}{H} \cdot \nabla \left[\frac{\omega H_\phi}{4\pi} - \rho k \Omega \varpi^2 \right] = 0 \ ,$$

or

$$-\frac{\varpi H_\phi}{4\pi} + \rho k \Omega \varpi^2 = -\frac{\beta}{4\pi} , \qquad (6.22)$$

where β is constant on a field line.

Equation (6.22) is easily interpreted. The flow of matter parallel to the poloidal field transports angular momentum

$$\rho v_p \Omega \varpi^2 A = (\rho k \Omega \varpi^2)(H_p A)$$

per second along a poloidal flux tube of infinitesimal area A. This quantity is not constant along the flux tube, but a steady distribution of angular momentum is maintained through the transport by the magnetic stresses of angular momentum

$$-\left(\frac{\varpi H_\phi}{4\pi}\right)(H_p A)$$

-- a particular case of a more general result due to Lüst and Schlüter. Since $(H_p A)$ is constant along a flux-tube, we arrive at (6.22)

We now combine (6.16), (6.19) and (6.22) to yield

$$\Omega \left(1 - \frac{4\pi\eta^2}{\rho}\right) = \left(\alpha + \frac{\eta}{\rho\varpi^2}\beta\right) , \qquad (6.23)$$

$$\varpi H_\phi \left(1 - \frac{4\pi\eta^2}{\rho}\right) = \left(\beta + 4\pi\eta\alpha\varpi^2\right) . \qquad (6.24)$$

We note first that for a steady state to be possible, the quantity

$$\frac{4\pi\eta^2}{\rho} = \frac{v_p^2}{\frac{H_p^2}{4\pi\rho}} = \frac{v_p^2}{(V_{Alfvén})^2} \qquad (6.25)$$

must be either less than unity or greater than unity all the way round a poloidal loop; for if $4\pi\eta^2/\rho = 1$ at a particular density, a nonsingular solution is possible only if

$$\beta + 4\pi\,\alpha\,\bar{\omega}^2 = 0 \qquad (6.26)$$

simultaneously. But in a nearly spherical star, $4\pi\eta^2/\rho = 1$ at a particular radial distance while (6.26) holds at a particular axial distance. Unless the star is violently distorted from the approximately spherical -- e.g., by the centrifugal and magnetic forces becoming comparable with the gravitational -- these two conditions cannot be satisfied simultaneously at two points on a poloidal loop. (It is true that an infinite rotational shear is inconsistent with the neglect of viscous forces in (6.20); but stellar viscosity is so small that a steady state that included viscous forces would still result in local centrifugal and poloidal magnetic forces far larger than the equilibrium of the star could tolerate.)

We thus require that in a steady state the circulation be either "sub-Alfvén" or "super-Alfvén" at all points of a poloidal loop. The parameter (v_p/v_A) is the natural one to turn up; for v_p is the rate at which the circulation is destroying isorotation, and v_A the rate at which hydromagnetic waves tend to restore isorotation.

With η prescribed by the maintained circulation, there are two degrees of freedom left. One parameter is removed by requiring that the angular momentum in a flux tube be fixed: varying η or the other parameter merely redistributes the angular momentum along the flux tube, but in the absence of viscous shear there can be no transverse transport. The remaining indeterminacy in the solution is the analog of the one degree of freedom left in the solution with $\eta = 0$ (no circulation):

$$\Omega = \alpha$$

and

$$\bar{\omega} H_\phi = \beta$$

the last equation being just the torque-free condition: any amount of torque-free field may be imposed on the star (provided the poloidal magnetic force does not become too large for the thermal and gravitational fields). However, we have postulated that in the absence of circulation the magnetic field is purely poloidal, with field lines necessarily closed loops. If we assume the freezing of the field to hold strictly, then the toroidal hydromagnetic distortion that generates H_ϕ will not alter the topology of the field. The field lines are given by

$$\frac{d\varpi}{H_\varpi} = \frac{\varpi d\phi}{H_\phi} = \frac{dz}{H_z} , \tag{6.27}$$

so that the condition of closed loops yields

$$0 = \oint d\phi = \oint \frac{dz}{\varpi^2 H_z} \frac{(\beta + 4\pi\eta\alpha\varpi^2)}{\left(1 - \frac{4\pi\eta^2}{\rho}\right)} = 0 , \tag{6.28}$$

on substituting from (6.24) and (6.27). This fixes β/α in terms of η, ρ and H_p. If $\eta = 0$, $\beta = 0$ -- no motion, no toroidal field (in a steady state).

As $dz/H_z > 0$, we may write (6.28), in either the sub-Alfven or super-Alfven cases

$$\beta + 4\pi\eta\alpha\varpi_c^2 = 0 , \tag{6.29}$$

where ϖ_c is the axial distance of at least two points on the loop (there may be more than two if the equation $z = z(\varpi)$ to the loop is more than doubly-valued). By (6.24), H_ϕ changes sign at ϖ_c -- as H_ϕ has arisen by toroidal distortion of H_p, clearly H_ϕ cannot have the same sign all the way round a loop. Because of the factor ϖ^2 in the denominator of (6.28), the points ϖ_c will be near the axis if the loop considered comes close the the axis.

Substituting (6.29) into (6.23), we have

$$\frac{\Omega}{\alpha} = \frac{\left[1 - \frac{4\pi\eta^2}{\rho}\left(\frac{\bar{\omega}_c}{\bar{\omega}}\right)^2\right]}{\left[1 - \frac{4\pi\eta^2}{\rho}\right]} . \qquad (6.30)$$

Suppose now $4\pi\eta^2/\rho < 1$, and consider a field line with a segment nearly parallel to the stellar surface, so that ρ is constant along this segment. Then as long as $\left(4\pi\eta^2/\rho\right)(\bar{\omega}_c/\bar{\omega})^2 < 1$, (6.30) predicts that Ω/α is positive, and that it increases with $\bar{\omega}$. As $4\pi\eta^2/\rho < 1$, Ω/α is certainly positive for $\bar{\omega} > \bar{\omega}_c$; and if $4\pi\eta^2/\rho << 1$, we may expect Ω/α positive along the whole of the horizontal segment, as $\bar{\omega}_c$ is likely to be a point near the axis.

If $4\pi\eta^2/\rho > 1$ along a whole loop, then (6.30) is best written

$$\frac{\Omega}{\alpha} = \frac{\left[\frac{4\pi\eta^2}{\rho}\left(\frac{\bar{\omega}_c}{\bar{\omega}}\right)^2 - 1\right]}{\left[\frac{4\pi\eta^2}{\rho} - 1\right]} , \qquad (6.31)$$

predicting a decrease in Ω/α along a horizontal segment. If $4\pi\eta^2/\rho > (\bar{\omega}_{max}/\bar{\omega}_c)^2$, $|\Omega|$ decreases along the whole length; otherwise there is a point of zero rotation on the segment, with Ω increasing on either side. When $4\pi\eta^2/\rho >> 1$, then (6.31) yields $\Omega\bar{\omega}^2$ = constant, as expected; for if the circulation is fast, the magnetic forces are negligible.

The surprising prediction, for the sub-Alfvén case, of equatorial acceleration is of special interest in view of the observed 15% acceleration in the sun. If we put for ρ a sub-photospheric value (10^{-7} gram/cm^3), and for v_p the velocity of migration of the sunspot zones ($\simeq 2 \times 10^2$ cm/sec), the relative acceleration $\Delta\Omega/\Omega$ along a horizontal segment, which is $\simeq 4\pi\eta^2/\rho$, is numerically $\simeq 5 \times 10^{-2}/H_p{}^2$, which is 0.15 if $H_p \sim 0.6$ gauss. This is of the same order as the general field observed at the poles, and so presumably of the subphotospheric general field kept beneath the surface by the circulation, as suggested above.

However, no magnetic theory of solar rotation can be considered satisfactory until the solar cycle is included. It may be possible to derive an approximately steady equatorial acceleration from a variable magnetic state -- see Chapter 8 on the dynamo problem. For the moment we note that the assumption of closed field lines is not necessary for equatorial acceleration. Most fields will be ergodic, with lines covering the toroidal surfaces defined by the poloidal loops. But if β/α is negative, we may define a distance ϖ_c by (6.29), so that Ω/α is given by (6.30). Provided ϖ_c is not too large, Ω/α will be positive along the whole of a horizontal segment, and will increase with ϖ .

The existence of a range of $4\pi\eta^2/\rho$ for which there are no steady state solutions is essential to the understanding of the phenomenon. For $4\pi\eta^2/\rho \ll 1$, we expect Ω uniform -- the circulation is unable to upset the magnetic stiffening. For $4\pi\eta^2/ \gg 1$, $\Omega\varpi^2 \simeq$ const -- in a steady state each element has the same angular momentum. If there were a continuum of solutions, one would expect for intermediate values of $4\pi\eta^2/\rho$ that Ω would decrease with ϖ , though less strongly than ϖ^{-2} . The gap allows equatorial acceleration on one side and deceleration on the other. It is tempting to link up magnetic variability with the necessarily nonsteady parameter range. Again, such an identification will not be satisfactory until the velocity field has been deduced from the theory, as in principle it can be, instead of being postulated. [It is known theoretically that in turbulent convective regions, such as that beneath the solar surface, there is a large-scale laminar motion superimposed on the small-scale turbulence.]

CHAPTER 7

THE THERMAL GENERATION OF STELLAR MAGNETIC FIELDS

The work of the last chapter assumed perfect conductivity and an imposed poloidal magnetic field. Such a field could be a relic from the time of formation of the star, in which case there is no obvious reason why the star's primeval field should not have a toroidal component as well, torque-free in the absence of circulation: the effect of circulation would be to generate an extra toroidal component, exerting a torque. The finite conductivity of the star implies that the field will decay unless energy is supplied to offset the Ohmic losses. The discussion naturally bifurcates at this point. In this section we consider primarily the thermal generation of magnetic fields, but include dynamical and hydromagnetic interactions with any primeval fields present. The self-exciting dynamo problem--the supply of magnetic energy from a dynamical field--is discussed in the next chapter.

The introduction of both energy-supplying and energy-dissipating mechanisms is of particular interest, in that we left the problem of Chapter 6 in an unsatisfactory, arbitrary state. It will be of interest to know whether the magnetic state of a star should approach a steady or quasi-steady asymptotic state, depending on its structure and angular momentum, or whether a degree of arbitrariness remains.

A. THERMAL GENERATION IN AN INITIALLY NON-MAGNETIC STAR

Consider a non-magnetic, non-rotating, fully ionized star, necessarily spherically symmetric. Its equations of dynamical equilibrium are

$$\nabla p = \rho \nabla V \qquad (7.1)$$

and

$$\nabla^2 V = 4\pi G\rho , \qquad (7.2)$$

with the equation of state

$$p = \frac{R}{\mu} \rho T \quad . \qquad (7.3)$$

In thermal equilibrium the energy leak down the temperature gradient is balanced by nuclear energy generation; otherwise, the star contracts, the gravitational energy released supplying the energy loss and also heating up the star.

Equation (7.1) treats the stellar material as one gas. In fact, there are separate equations for the ions and electrons, that for the electrons being just Ohm's law (3.8), with redundant terms dropped. As local thermodynamic equilibrium holds very closely, the partial pressures of the ions and the electrons will be of the same order, but the gravitational force on the electrons is much smaller; in fact, it is neglected in (3.8). In equilibrium, then, there must be an electric force acting on the electrons, equal and opposite to their partial pressure gradient: from (3.8),

$$\underset{\sim}{E} + \frac{\nabla p_e}{n_e e} = 0 \quad . \tag{7.4}$$

The electron partial pressure causes a small charge separation until the required space-charge, "electrostatic" field is built up, equilibrium as usual being reached in a time of order σ/ω_p^2 (c.f. Chapter 4). We write, assuming very close charge neutrality,

$$\begin{aligned} p_e &= (Z/Z + 1)\, p \,, \\ n_e &= (Z/Am_H)\rho \,, \end{aligned} \tag{7.5}$$

where m_H is the mass of the hydrogen atom, A the mean ionic weight, and Z the mean ionic charge; then the potential of the space-charge field is, from (7.1), (7.4) and (7.5),

$$\phi = \left(Am_H/(Z + 1)e\right) V \,. \tag{7.6}$$

The net density of charge is

$$\rho_e = -\frac{\nabla^2 \phi}{4\pi} = -\frac{Am_H}{(Z + 1)e}\frac{\nabla^2 V}{4\pi} = \frac{G(Am_H)^2\, n_i}{(Z + 1)e} \,, \tag{7.7}$$

so that

$$\frac{Z n_i - n_e}{n_e} \simeq \frac{G(m_H)^2}{e^2}\frac{A^2}{Z(Z + 1)} \sim 10^{-36} \,. \tag{7.8}$$

This is short by a factor $m_H/m_e \simeq 1840$ of Eddington's magic number. Equation (7.8) is roughly the ratio of the gravitational to the Coulomb forces between two protons, instead of between a proton and an electron.

This space-charge field--the Pannekoek-Rosseland field--is of slight importance even on a cosmological scale; the charge excess required by Bondi and Lyttleton in their recent cosmology is one part in 10^{18}. Its smallness justifies our neglecting electrostatic forces in (7.1); effectively the same force density, with sign reversed, acts on the ions as on the electrons, so that the partial electron pressure is transferred to the ions. However, it was pointed out by Biermann that in non-spherical stars, such equilibrium may not be possible-- $\nabla p_e/n_e e$ need not have a vanishing curl. In a rotating star, (7.1) is replaced by

$$\nabla p = \rho(\nabla V + \Omega^2 \underset{\sim}{\varpi}) , \qquad (7.9)$$

so that

$$\underset{\sim}{E}^i \equiv \frac{\nabla p_e}{n_e e} = \frac{Am_H}{(Z+1)e} (\nabla V + \Omega^2 \underset{\sim}{\varpi}) . \qquad (7.10)$$

The vector $\Omega^2 \underset{\sim}{\varpi}$ may be divided into an irrotational part, and a solenoidal part written $\overline{\Omega^2 \underset{\sim}{\varpi}}$, which vanishes when Ω is a function of ϖ only, and not of displacement parallel to the axis. In general, we may expect $|\overline{\Omega^2 \underset{\sim}{\varpi}}|$ to be of the same order as $|\Omega^2 \underset{\sim}{\varpi}|$. The total irrotational part of $\underset{\sim}{E}_i$ again merely generates a small space-charge field, in a time σ/ω_p^2; but $\overline{\underset{\sim}{E}_i}$ acts like a technological battery, in which non-electric forces continually drive electrons relative to the ions, so generating poloidal currents with an associated toroidal magnetic field. The energy supply comes ultimately from the nuclear sources that maintain the star's thermal and gravitational energies against radiative loss.

At any time, the current density $\underset{\sim}{j}$ is given by Ohm's law,

$$\frac{\underset{\sim}{j}}{\sigma} = (\overline{\underset{\sim}{E}} + \overline{\underset{\sim}{E}^i}) , \qquad (7.11)$$

with

$$\nabla \wedge \underset{\sim}{H} = 4\pi/c \; \underset{\sim}{j} . \qquad (7.12)$$

The Hall field is temporarily dropped from (7.11); meridian circulation is ignored. The induced electric field $\overline{\underset{\sim}{E}}$ satisfies

$$\nabla \wedge \overline{\underset{\sim}{E}} = -\frac{1}{c} \frac{\partial H}{\partial t} . \qquad (7.13)$$

As long as $|j/\sigma| \ll |\overline{\underset{\sim}{E}^i}|$, we have that $\overline{\underset{\sim}{E}} \simeq -\overline{\underset{\sim}{E}^i}$, and the magnetic field grows linearly with time; in order of magnitude, if D is a typical scale of variation of $\overline{\underset{\sim}{E}^{i}}$,

$$H \sim \frac{c\,\overline{E^{i}}}{D}\, t \quad . \tag{7.14}$$

When the maintaining currents

$$j \sim \frac{c}{4\pi}\,\frac{H}{D} \sim \frac{c^2}{4\pi\, D^2}\,\overline{E^{i}}\, t \tag{7.15}$$

are large enough for the Ohmic field j/σ to be comparable with $\overline{E^{i}}$, i.e., when

$$t \sim \frac{4\pi\sigma}{c^2}\, D^2 \ , \tag{7.16}$$

the asymptotic steady state is reached: the dissipation of magnetic energy by Ohmic resistance balances the supply of energy from the battery, and a field

$$H \sim \frac{4\pi\sigma}{c}\,\overline{E^{i}}\, D \tag{7.17}$$

is reached in the time (7.16). More accurately, the approach to the asymptotic field is exponential. If $\overline{\underset{\sim}{E}^{i}} = 0$, an initial toroidal field would decay, with the associated electric field satisfying

$$-\nabla_{\wedge}\,(\nabla_{\wedge}\,\underset{\sim}{E}) = \frac{4\pi\sigma}{c^2}\,\frac{\partial E}{\partial t} \quad . \tag{7.18}$$

This equation has unnormalized eigen-solutions $\overline{\underset{\sim}{E}}^{\tau}\, e^{-t/\tau}$, with $\overline{\underset{\sim}{E}}^{\tau}$ and τ fixed by the conditions of finiteness at the center of the star and at infinity. If we write

$$\overline{\underset{\sim}{E}^{i}} = \sum_{\tau} \overline{\underset{\sim}{E}}^{\tau} \ , \tag{7.19}$$

it is easy to see that the current field behaves like

$$\underset{\sim}{j}/\sigma = \sum_{\tau} (1 - e^{-t/\tau})\, \overline{\underset{\sim}{E}}^{\tau} \quad . \tag{7.20}$$

Each part of the field is built up in the same time that it decays if the "battery" is switched off--analogous to simple results in circuit theory.

As an example, consider the sun, with $\Omega \sim 3 \times 10^{-6}$, $\sigma \sim 10^{17}$ esu, and $D \sim 3 \times 10^{10}$ say. Taking $\overline{E^i} \sim E^i$, and $A = Z = 1$ (pure hydrogen), (7.17) yields an asymptotic field of about 500 gauss, built up in 10^{10} years. This field is comparable with sunspot fields, and is far larger than the sun's polar field.

Equation (7.17) predicts still larger fields for more rapidly rotating stars; when $\Omega \sim 10^{-3}$, so that centrifugal force at the surface equator is comparable with gravity (about as large as it can be without causing surface instability), then $H \sim 5 \times 10^{7}$ gauss, and the magnetic forces are comparable with the gravitational. The magnetic force then affects $\nabla p_e / n_e e$, and the Hall term must be included in Ohm's law, which now becomes

$$\underset{\sim}{j}_p / \sigma = \frac{A m_H}{(Z+1) e} \left[\overline{\Omega^2 \underset{\sim}{\varpi}} - \overline{\frac{\underset{\sim}{j}_p \wedge \underset{\sim}{H}_t}{c \rho Z}} \right] . \qquad (7.21)$$

The time of approach to the asymptotic state is also modified somewhat. However, it is found that the nonlinear term in (7.21) becomes comparable with the Ohmic only when the star is rotating with the maximum angular velocity; otherwise the previous theory is adequate.

The theory should be completed by inclusion of meridian circulation, which will convect the field lines. Such a flow is determined by the rotational and magnetic distortion to the thermal field; in the absence of a poloidal field, no torque acts on the matter, and in a steady state the angular momentum would need to be constant on each stream line. However, this difficult problem need not be pursued, as we shall now show that the conditions for Biermann's mechanism to work are unlikely to be fulfilled.

B. THE EFFECT OF AN INITIAL POLOIDAL FIELD

The magnetic forces exerted by Biermann's field are negligible except when the star's rotation is very high, and in any case they act in meridian planes, and are absorbed by the thermal field. But if the star has a primeval field, with a poloidal component that is not too weak, then the toroidal force due to the flow of Biermann's currents across $\underset{\sim}{H}_p$ reacts on the rotation field, and in the time-scale defined by the travel of hydromagnetic waves along the poloidal field. If H_p is greater than 10^{-4} gauss, the time-scale of adjustment is shorter than the induction time-scale in which Biermann's currents grow. Thus, in the absence of meridian circulation, we must impose as a dynamical constraint on the problem the condition that $\underset{\sim}{j}_p$ is parallel to $\underset{\sim}{H}_p$ --the field allowed must be torque-free. This is achieved by a slight departure from the isorotational state: the sum of the non-irrotational part of $\underset{\sim}{v}_t \wedge \underset{\sim}{H}_p/c$, the Hall terms and the component of $\nabla p_e/n_e e$ across $\underset{\sim}{H}_p$ is at all times a curl-free vector, so that Ohm's law is satisfied without a component of current across $\underset{\sim}{H}_p$.

The strength of the poloidal current field that is built up, flowing parallel to $\underset{\sim}{H}_p$, is given by integrating Ohm's law round a poloidal loop (ignoring the decay of the poloidal field--see Chapter 8). There results

$$\oint \frac{\underset{\sim}{j}_p \cdot \underset{\sim}{ds}}{\sigma} = \frac{Am_H}{(Z+1)e} \oint \Omega^2 \underset{\sim}{\omega} \cdot \underset{\sim}{ds} \quad , \tag{7.22}$$

with the left-hand side essentially positive, as $\underset{\sim}{j}_p$ is parallel to $\underset{\sim}{ds}$. If there were no departure from isorotation, $\oint \Omega^2 \underset{\sim}{\omega} \cdot \underset{\sim}{ds}$ would vanish identically, and there would be no toroidal field built up at all. As it is, the very slight departures from isorotation brought about by the extra terms in Ohm's law--neglected when Ferraro's law was deduced in Chapter 6 --which effectively cancel out the component of $\underset{\sim}{j}_p$ across $\underset{\sim}{H}_p$, do lead to a non-vanishing integral of Biermann's "battery" around the loops of $\underset{\sim}{H}_p$. However, the toroidal field so built up is much smaller than Biermann's estimate--it is reduced by order of the ratio of the macroscopic angular velocity Ω to the gyration frequency of an ion in the field $\underset{\sim}{H}_p$. Instead of about 500 gauss, the field in the sun would be $\sim 10^{-7}/H_p$, with $\underset{\sim}{H}_p$ as low as 10^{-4} gauss, it is still 6 orders of magnitude smaller.

The mechanism here is essentially hydromagnetic. In a <u>solid</u> conductor, the flow of poloidal current across a poloidal field again generates a torque, and may effect a transfer of angular momentum between, e.g., the liquid core and the solid shell of the earth. But the condition of solid body rotation is a zero-order dynamical constraint that must be put into the problem; the magnetic torque generates a stress in the solid, and consequently small elastic distortions, which have small hydromagnetic consequences that can be considered in a higher order approximation. In a fluid, however, the torque cannot be balanced, and the correct zero-order equilibrium state is with a zero torque, which can be and is achieved through the variation of Ω; this is impossible in a rigid body, but also is not required for dynamical equilibrium.

C. THE EFFECT OF MERIDIAN CIRCULATION

The problem is altered somewhat if, as is most likely, the star has a poloidal circulation field, as well as a primeval magnetic field. As discussed in the last chapter, the circulation necessarily generates a toroidal field from an initial poloidal field; if the star's field is purely poloidal in the absence of circulation, and perfect conductivity is assumed, then the steady state in the presence of a prescribed circulation field is described by (6.28). However, there is no good reason for assuming no primeval toroidal component, even in the absence of circulation. Further, although hydromagnetic effects nearly always dominate in non-steady situations, once a steady state (satisfying the simple hydromagnetic condition (6.17)) has been achieved, the slower acting Ohmic, thermal and Hall effects can play a role.

Let us suppose that a hydromagnetic steady state, defined by (6.13), (6.23), and (6.24) has been achieved, with again one linear relation between α and β holding, prescribing the total angular momentum in a flux-tube. We now consider the effect of the terms neglected in deriving (6.13), (6.23), and (6.24). First, there is the inevitable decay of an axially symmetric poloidal field; we shall see in the next chapter that meridian circulation does not prevent this. As, however, we are interested in the toroidal field, we shall temporarily ignore this embarrassment, and regard $\underset{\sim}{H}_p$ as fixed. In discussing the effect of the poloidal components of the neglected terms, we again distinguish between components across $\underset{\sim}{H}_p$ and those along it. Both $\underset{\sim}{v}_t \wedge \underset{\sim}{H}_p/c$ and the zero-order component of $\underset{\sim}{v}_p \wedge \underset{\sim}{H}_t/c$ are perpendicular to $\underset{\sim}{H}_p$; hence the components across $\underset{\sim}{H}_p$ of the neglected terms simply lead to very slight departures of Ω and $\underset{\sim}{H}_t$ from the simple laws (6.23), (6.24)--just as, in the absence of meridian circulation, their effect is to alter slightly the law of isorotation. But parallel to $\underset{\sim}{H}_p$ the "large" motional induction terms vanish to zero-order, and so the neglected terms are not "negligible" when one considers current flow round $\underset{\sim}{H}_p$. If there is a net e.m.f. round a loop of $\underset{\sim}{H}_p$, the current flowing round $\underset{\sim}{H}_p$ will slowly change, and along with it the torque-free part of $\underset{\sim}{H}_t$. Simultaneously, the Ω and $\underset{\sim}{H}_t$ fields adjust themselves to satisfy (6.23) and (6.24) with slightly differing parameters α and β; for the dynamical condition and the very nearly accurate hydromagnetic condition must hold at all times--the time-scale for adjustment being very much shorter than the induction time-scale in which the neglected terms show their effect. Still assuming a steady $\underset{\sim}{H}_p$ field, we conclude that the asymptotic relation between α and β is given by the vanishing of the integral round $\underset{\sim}{H}_p$ of Ohm's law. This constraint on α and β is

$$\oint \frac{\underset{\sim}{j} \cdot \underset{\sim}{H}}{\sigma H_p} \, ds = \oint \frac{\nabla p_e}{n_e e} \cdot d\underset{\sim}{s} = \frac{A m_H}{(Z+1)e} \left[\oint \Omega^2 \underset{\sim}{\varpi} \cdot d\underset{\sim}{s} + \oint \frac{\underset{\sim}{j}_p \wedge \underset{\sim}{H}_t}{c \rho} \cdot d\underset{\sim}{s} \right] \qquad (7.23)$$

--the Hall term being cancelled by the non-zero part of $\mathbf{v}_p \wedge \mathbf{H}_t/c$. The term in $\mathbf{j}_p \wedge \mathbf{H}_t/c$ does not vanish identically because the field exerts a torque, so that $\mathbf{j}_p$ is not parallel to $\mathbf{H}_p$.

On substitution of Ω and $\mathbf{H}_t$ from (6.23) and (6.24), it is found that the $\oint \nabla p_e/n_e e \cdot d\mathbf{s} \equiv 0$ --the thermal field makes no contribution to the constraint, which reduces to

$$\oint \frac{\mathbf{j} \cdot \mathbf{H}}{\sigma H_p} \, ds = 0 \ . \qquad (7.24)$$

Only if we use the corrections to (6.23) and (6.24), due to the Hall, thermal and Ohmic terms, will the right-hand side of (7.24) not vanish; and except for stars with abnormally slow circulation fields we can be sure that (7.24) is an adequate approximation to the asymptotic relation between α and β .

This work is of value in showing that the thermal field is of no importance whether or not there is meridian circulation, as long as the star has even a very weak poloidal field. Thus if, for example, we start with the field of the last chapter, with closed field lines, generated by hydromagnetic distortion, then there is no tendency of the thermal field to pump more energy in, for $\oint \nabla p_e/n_e e \cdot d\mathbf{s} \equiv 0$. However, the time-scale of approach to the asymptotic state (7.24) is of the same order as the e-folding time of decay of the poloidal field; thus, even if we use for $\mathbf{H}_p$ a time-dependent field, given by the decay equation for $\mathbf{H}_p$, we have, in (7.24), at best only a moderately good approximation to the instantaneous β-α relation. Further, even if we ignored the decay of $\mathbf{H}_p$, and improved the theory by computing the dependence of the circulation on Ω and $\mathbf{H}$, there would still be--speaking rather loosely--two degrees of freedom in any steady state; both the rotation field and the poloidal magnetic field would be available for choice. This is due essentially to the absence of feed-back, in an axially symmetric system, from the toroidal to the poloidal field. This will be discussed further in connection with the dynamo problem.

(A detailed treatment of the work of this chapter appears in Microwave Laboratory Report No. 837, August 1961, by L. Mestel and I. W. Roxburgh. This will appear in the ~~Monthly Notices of the Royal Astronomical Society~~ Astrophysical Journal during 1962.)

CHAPTER 8

THE DYNAMO PROBLEM

A magnetic field in a medium of finite conductivity decays through Ohmic dissipation, unless energy is pumped into it from an external source. We have seen that the energy equation to a moving conductor has the term $(j_\wedge \underset{\sim}{H}/c) \cdot \underset{\sim}{v}$, which represents supply of energy to the field if the fluid velocities are driven by non-magnetic forces against the magnetic force. The self-exciting dynamo problem is to find conditions for a supply of kinetic energy to maintain the field against Ohmic losses, without help from a "battery." We thus look for steady solutions of the equation

$$\frac{\partial H}{\partial t} = \nabla_\wedge(\underset{\sim}{v}_\wedge \underset{\sim}{H}) + \frac{1}{\frac{4\pi\sigma}{c^2}} \nabla^2 \underset{\sim}{H} , \tag{8.1}$$

where again σ is taken as constant for simplicity: the flow of matter across the field $\underset{\sim}{H}$ generates currents which maintain the field $\underset{\sim}{H}$, by satisfying the Ampère equation.

At first sight the problem seems trivially simple. We have our Magnetic Reynolds Number condition

$$\frac{4\pi\sigma}{c^2} v D \simeq 1 \tag{8.2}$$

for the induction term in (8.1) to balance the Ohmic term; we found that inside stars, very small circulation speeds are sufficient to satisfy (8.2), if we take for D a length that is even a small fraction of the stellar radius. However, the length D is the "scale of variation" of the field--the distance in which the field substantially changes in direction or strength, and it is not obvious that, for an arbitrary field structure, D is finite at all points of the field.

Thus consider an axially symmetric, poloidal field, with components in cylindrical polars (ϖ, φ, z)

$$H_z = \frac{1}{\varpi}\frac{\partial P}{\partial \varpi} \qquad H_\varpi = \frac{-1}{\varpi}\frac{\partial P}{\partial z} . \tag{8.3}$$

By use of the "stream-function" P, $\nabla \cdot \underset{\sim}{H} = 0$ is automatically satisfied, and $(\underset{\sim}{H} \cdot \nabla) P = 0 \rightarrow P$ is constant on field-lines, which therefore have equations

$$P(z, \varpi) = \text{constant.} \tag{8.4}$$

The total flux of $\underset{\sim}{H}$ across a circle in the (ϖ, ϕ) plane, and centered on the axis, is

$$\int_0^{\varpi} \frac{1}{\varpi} \frac{\partial P}{\partial \varpi} (2\pi \varpi) \, d\varpi = 2\pi P(\varpi, z) \; , \tag{8.5}$$

if we choose $P = 0$ on the axis. If the field is purely local--i.e., not part of a field maintained by currents at infinity--then the limit of the total flux as $\varpi \rightarrow \infty$ must vanish if the field line $P = 0$ consists of the axis and the semi-circle at infinity. All the other field lines are then closed loops within $P = 0$, which must therefore surround at least one O-type neutral point O. (The field may have some field lines of the form of figures of eight, surrounding two or more O-type neutral points.) In the neighborhood of O the field changes direction in a distance which can be made arbitrarily small; with σ finite, no finite velocity field can make the condition (8.2) hold, so that the dissipation term must dominate near O. It is impossible to restrict the zone of decay to a small region surrounding O; for this would lead to an arbitrarily large gradient of the field strength, and so extend the region of arbitrarily small D.

In fact, we know that if $\underset{\sim}{j}$ is perpendicular to $\underset{\sim}{H}$, Ohm's law can be written

$$\underset{\sim}{E} + \frac{\underset{\sim}{v}' \wedge \underset{\sim}{H}}{c} = 0 \; , \tag{8.6}$$

with

$$\underset{\sim}{v}' = \underset{\sim}{v} + \frac{c}{\sigma} \frac{\underset{\sim}{j} \wedge \underset{\sim}{H}}{H^2} \; , \tag{8.7}$$

implying that the field drifts through the fluid. With $\underset{\sim}{v} = 0$, the field decays by loops disappearing into O; with $\underset{\sim}{j}$ finite near O, the drift is infinite at O, and only a <u>source of fluid</u> at O could offset the decay, by creating magnetic flux at the rate that the Ohmic field destroys it.

This theorem--Cowling's anti-dynamo theorem--was proved originally as follows. If the neutral point O is a simple extremum

$$\frac{\partial P}{\partial z} = \frac{\partial P}{\partial \varpi} = 0 \text{ , with } \nabla^2 P \neq 0 \; . \tag{8.8}$$

Ohm's law has a toroidal component

$$\underset{\sim}{j} = \sigma \left(\frac{\underset{\sim}{v} \wedge \underset{\sim}{H}}{c} \right) \tag{8.9}$$

(in an axially symmetric state, there can be no toroidal electrostatic field, while the Hall and thermal fields are poloidal). Substituting for $\underset{\sim}{j}$ from Ampère's equation, (8.9) becomes

$$- \frac{1}{\varpi} \nabla^2 P + \frac{1}{\varpi^2} \frac{\partial P}{\partial \varpi} = - \frac{4\pi\sigma}{c^2 \varpi} \left(v_z \frac{\partial P}{\partial z} + \varpi \frac{\partial P}{\partial \varpi} \right) , \qquad (8.10)$$

which is inconsistent with (8.8), if σ and $\underset{\sim}{v}$ are finite. Essentially, the small poloidal loops near O require by Ampère's law a finite toroidal current, which cannot be driven by motional induction, since $\underset{\sim}{v} \wedge \underset{\sim}{H}/c$ vanishes with $\underset{\sim}{H}$.

The proof is inapplicable if $\nabla^2 P$ vanishes, implying that $\underset{\sim}{j}$ vanishes at O ; however, Chandrasekhar and Backus have extended the theorem to fields with higher order extrema.

In general, the simple form of Ohm's law applicable to a steady-state dynamo is

$$\underset{\sim}{j}/\sigma = \left(-\nabla \phi + \frac{\underset{\sim}{v} \wedge \underset{\sim}{H}}{c} \right) , \qquad (8.11)$$

where ϕ is the potential of the space-charge field. Cowling's theorem may then be simply extended to fields which have a limiting closed curve C , such that the neighboring field lines spiral round it in the same sense all the way round C . In general, C is a field line; in the degenerate case when the local field lines are closed loops, it is a null line. Then the integral of (8.11) round this curve yields

$$\oint_C \frac{\underset{\sim}{j}}{\sigma} \cdot d\underset{\sim}{s} = \oint_C \left(\frac{\underset{\sim}{v} \wedge \underset{\sim}{H}}{c} \right) \cdot d\underset{\sim}{s} = 0 , \qquad (8.12)$$

either because $\underset{\sim}{H} = 0$, or because $(\underset{\sim}{v} \wedge \underset{\sim}{H})$ is perpendicular to $\underset{\sim}{H}$. (For similar reasons, the Hall field, dropped in (8.11), would make no contribution to (8.12).) But by Ampère's law, the currents maintaining the field near C , if they are non-zero, must flow in a fixed sense round C , so that the left-hand side of (8.12) is non-zero, whence a contradiction. Thus, one can show that a toroidal field (necessarily axially-symmetric) cannot be maintained by dynamo action, as a curve C can easily be found, meeting the axis either at $\pm \infty$, or at consecutive points where the neighboring loops reverse their sense. Again, the decay of the fields may be pictured as the disappearance of closed toroidal loops into the axis, which a finite velocity field is unable to prevent.

If Biermann's "battery" is operating, it will determine the amount of toroidal magnetic flux in the field: the creation of flux along the axis will asymptotically balance the destruction by Ohmic diffusion, but the circulation of matter merely convects loops of field, without creating them. No toroidal "battery" is known, able to build up any but an extremely weak poloidal field; and as we have seen, axially symmetric motion will merely drag the field lines about, but does not generate new field lines.

The conclusion is that we must consider fields that are more complicated in structure than those discussed. Analytically speaking, we must avoid fields which have singular closed curves of type C above. Topologically, we must look for motions that convect the field lines in such a way as to regenerate the dipole component of the poloidal field: dynamo action does not create new field lines, so much as use the same field lines more "efficiently." There is a "topological asymmetry" (in Elsasser's words) between axially symmetric poloidal and toroidal fields. From a poloidal component non-uniform rotation generates a toroidal component, but no axially symmetric circulation can generate a poloidal component from a toroidal. If we relax the condition of axial symmetry, it is possible to construct velocity fields that do appear to be able to regenerate a poloidal field. Bullard, following earlier work by Elsasser, has developed a technique for numerical treatment of the problem on an electronic computer. The fluid motion adopted--a simple laminar thermal convection (driven by radioactive heating) and a resulting non-uniform rotation--are physically plausible, but attention is concentrated on the kinematical problem: part of the justification for the particular laminar motion adopted is that it is able to maintain a field with a large dipole component. Dynamical support for this type of dynamo comes from work of Parker, who showed that Coriolis force acting on convection currents would tend to twist the toroidal component--generated by non-uniform rotation--so generating a poloidal component that acts in the correct sense for reinforcing the original poloidal field.

The Elsasser-Bullard technique is unsuited for proving an existence theorem. Essentially, the magnetic field is expanded as a series of eigen-solutions of the decay equation (7.18); the resulting infinite set of algebraic equations contains as a scale factor the strength of the velocity field, which is fixed, for a steady dynamo, by the vanishing of the infinite determinant of the system. Unfortunately, the matrix of the determinant is non-symmetric, so it is not obvious that the eigenvalues are real: in simple symmetries, for which dynamo maintenance is impossible, the eigenvalues are in fact imaginary. Bullard's iterative process yields an eigenvalue that seems to converge rapidly to a real value, but he does not claim to have proved a rigorous existence theorem. Such a theorem has been proved by Herzenberg, for a much simpler model. Two conducting spheres, A and B, rotate with angular velocities ω_A and ω_B about inclined axes. Both are placed eccentrically within a large conducting sphere of radius M, and center D, surrounded by insulating material. The sphere A rotates in the magnetic field arising from motional induction in B, and vice versa. The existence proof depends on taking the radii of the spheres small compared with their distances from O, and these distances small compared with M, so that convergence can be demonstrated; for then only the axially-symmetric parts of the field at each sphere, and of these only the parts which decrease most slowly with distance, need be considered when treating induction in the other sphere.

Herzenberg finds that for about half the possible relative orientations of the spheres one can find a value for $(\omega_A \times \omega_B)$ that maintains a magnetic field. He likens his spheres to hydrodynamic eddies--it is possible that his work could be applied to a model of hydromagnetic turbulence.

Once we are convinced that a steady dynamo is possible--that a given supply of kinetic energy can offset the natural decay of a magnetic field--it is necessary to look at the dynamical problem. Increase of the velocity field above the strength for dynamo maintenance will pump more energy in than is being dissipated, so that the field grows, until the forces it exerts on the motion cannot be ignored. Bullard estimated the strength of the field in the earth's fluid core by equating the mean toroidal component of magnetic force to the mean Coriolis force, obtaining $H_p \sim 4$ gauss, and $H_t \sim 400$ gauss. He is now planning to program the whole problem, kinematical and dynamical, for electronic computation.

The mathematics of non-axially symmetric systems is so complicated that it is tempting to stick to the axially symmetric theory of Chapters 6 and 7, regarding it as a smoothed-out approximation to the actual state of affairs in a convective region. For this to be an adequate simulation, we need not only a $(\theta\text{-}\Omega)$ relation--for which (7.24) may suffice--but also a feedback relation that fixes $\mathbf{H}_p$ in terms of $\mathbf{H}_t$. In a complete dynamo theory, $\mathbf{H}_p$ will be fixed as well as $\boldsymbol{\Omega}_p$ and $\mathbf{H}_t$; with $\mathbf{v}_p$ determined by the magnetic and centrifugal perturbation to the thermal field, the cycle (i) circulation $\rightarrow$ (ii) non-uniform rotation $\rightarrow$ (iii) toroidal field generated from poloidal $\rightarrow$ (iv) feed-back from the toroidal to the poloidal field should determine everything in terms of the angular momentum of the star, provided both the toroidal equation of motion and the hydromagnetic equation (with finite σ) are satisfied.

Any approximate treatment that avoids detailed treatment of the actual non-axially symmetric problem must involve a certain amount of guesswork, or at the best, a phenomenological feeding-in of empirical data; this, however, may be worthwhile, if insight is gained into other problems. For example, Babcock has recently observed that the polar magnetic field of the sun reverses itself with the solar cycle. The most plausible explanation of this is in terms of a non-steady dynamo--the toroidal magnetic loops, that presumably give rise to sunspots, feed the poloidal field and reverse it. This in turn implies that the magnetic torque $\underset{\sim}{j}_p \wedge \underset{\sim}{H}_p / c$ acting on the gas does <u>not</u> change sign with the solar cycle, so that one probably need not fear that the equatorial acceleration found in Chapter 6 would become a deceleration over part of the cycle. It is tempting to try to find a partly physical, partly phenomenological theory of the solar cycle, in the form of solution of the non-steady hydromagnetic equations, supplemented by a feed-back hypothesis.

CHAPTER 9

MAGNETISM AND TURBULENCE

In the dynamo models discussed in the last chapter, the fluid motions are laminar, with the velocity uniquely prescribed as a function of space and time. However, in stellar convective zones the transport of energy outwards is mainly by small-scale turbulent motion which is superimposed in the rotation field, and also on a large-scale laminar circulation discussed recently by Biermann and Kippenhahn. Again, besides the galactic rotation and the radial gas streaming discovered by the radio astronomers (which probably has a hydromagnetic explanation), the interstellar gas clouds have random velocities, driven probably by the heating of gas by radiation from hot stars. We are therefore obliged to grapple with hydromagnetic turbulence.

A. THE EFFECTS OF A PRESCRIBED TURBULENT FIELD ON MAGNETIC DECAY

Consider first a simple, perfectly conducting fluid, in which

$$\frac{\partial \underset{\sim}{H}}{\partial t} = \nabla_\wedge(\underset{\sim}{v}_\wedge \underset{\sim}{H}) = (\underset{\sim}{H}\cdot\nabla)\underset{\sim}{v} - (\underset{\sim}{v}\cdot\nabla)\underset{\sim}{H} - \underset{\sim}{H}(\nabla\cdot\underset{\sim}{v}) , \qquad (9.1)$$

(the term $\underset{\sim}{v}(\nabla\cdot\underset{\sim}{H})$ vanishing identically). The "convective" or Lagrangian time derivative is

$$\frac{d\underset{\sim}{H}}{dt} = \frac{\partial \underset{\sim}{H}}{\partial t} + (\underset{\sim}{v}\cdot\nabla)\underset{\sim}{H} = (\underset{\sim}{H}\cdot\nabla)\underset{\sim}{v} + \frac{\underset{\sim}{H}}{\rho}\frac{d\rho}{dt} , \qquad (9.2)$$

since

$$\frac{d\rho}{dt} = \frac{\partial\rho}{\partial t} + (\underset{\sim}{v}\cdot\nabla)\rho = -\rho\nabla\cdot\underset{\sim}{v} . \qquad (9.3)$$

Thus, by (9.2), the field strength is increased if either the fluid is compressed, or if the field lines are "lengthened" by the motion.

More simply, if P_1 and P_2 are neighboring points on a field line, separated by a distance ℓ, and if A is the area of cross-section of the flux tube through P_1, P_2, then we find

(i) HA = constant (flux conservation)

(ii) $\rho A \ell$ = constant (conservation of matter within the flux-tube—the "freezing of the field").

Thus we have

$$H \propto \rho\ell \qquad (9.4)$$

--essentially equivalent to (9.2).

Now suppose that the velocity field $\underset{\sim}{v}$ is turbulent. By definition, it is overwhelmingly probable that two fluid particles, initially neighboring, should become more and more separated. Thus if there is present a weak magnetic field, its lines will be continually lengthened by the turbulence; if the fluid is incompressible, or if the volume occupied by the gas is constant, we can be certain that an inexorable turbulent flow will begin by increasing the energy of the field.

Simultaneously, the field structure changes radically--the field lines become tangled. This has led some authors (Elsasser, Sweet) to argue that the turbulence will hasten the natural decay of the field. The tangling-up of the field is interpreted to mean that the length-scale of the largest magnetic loops present systematically decreases with time. Thus if two segments of the same field line, pointing in opposite directions, are brought together by the turbulence, rapid Ohmic diffusion takes place, and a loop breaks off. Thus the field becomes a set of small loops, which decay far more rapidly than the original field, because of the factor L^2 in (4.9).

This argument implicitly ignores the possibility of dynamo regeneration. However, if one restricts the "turbulence" to be axially symmetric--small-scale rolling motions about the azimuthal direction--then the tangling of the field within a convective zone does accelerate the decay of the internal field. But, as pointed out by Spitzer, it does not follow that the decay-time of the external dipole field will be decreased substantially. For example, within a fully convective star, those field lines with external parts are forced by the convection to be close to the axis, or just below the surface. In these regions the convective motions are slow, and the work they do just compensates for the locally increased Ohmic dissipation. The decay time of the external field is not altered in its order of magnitude.

B. TURBULENT DYNAMOS

So far we have seen that a turbulent, highly-conducting fluid should initially increase the strength of a "seed" magnetic field. (All dynamo theories require an initial magnetic field, due to a "battery," or a relic of a previous field; for dynamo action convects magnetic flux, but does not create it.) The recent successes of laminar dynamo theory suggest that "turbulent dynamos" may sometimes exist; i.e., a continuous supply of turbulent kinetic energy can maintain against Ohmic loss a statistically steady "turbulent" magnetic field. Even if the turbulence is statistically homogeneous or axially symmetric, the trajectories of individual elements are sure to be complex enough for the peculiar difficulties of laminar dynamo theory not to be relevant. And in fact nearly all treatments of the problem assume without question that there are no difficulties analogous to those summed up in Cowling's theorem. Instead, attention is focused on the dynamical problem--the amount of energy that the magnetic field finally acquires. If it is asserted that a dynamo is impossible, the explanation is in terms of the dissipative properties of the medium rather than the detailed structure of the magnetic or velocity fields.

We limit the discussion to incompressible, homogeneous, isotropic turbulence, acting on an initial weak, large-scale field. Energy is fed into the turbulence at a fixed rate at the large wavelength end of the velocity spectrum, and steadily cascades to smaller and smaller wavelengths until it is destroyed by viscosity. All workers agree that the field is initially strengthened and tangled up, but there are differences of opinion as to whether there will always be ultimate amplification of the field. It is agreed that the Fourier components of the magnetic field cannot have greater energy than the corresponding components of the kinetic field, for then the motion would be suppressed, and with it the cascading of energy (which occurs only through the velocity field); but one school asserts that there will be ultimate equipartition of energy between the kinetic and magnetic fields, while their opponents assert that equipartition is at the best limited to small wavelengths.

Biermann and Schlüter argue in favor of equipartition by denying that the tangling-up of the field implies that energy is fed into smaller and smaller wavelengths. They argue that all wavelengths increase in energy, the small loops more rapidly than the larger ones, so that equipartition of energy is reached first at small wavelengths; but once the twisting of the field is slowed up by the reaction of the magnetic stresses on the motion, the larger wavelengths catch up, until equipartition is reached at all scales. Further supplies of energy cascade down to be destroyed by viscosity, and to maintain the field against Ohmic loss.

A different type of argument for equipartition was used by Elsasser. He showed that the hydromagnetic equation and the equation of motion of an incompressible fluid can be combined into a pair of strikingly similar equations in the variables

$$\underset{\sim}{P}, \underset{\sim}{Q} = \underset{\sim}{v} \pm \frac{\underset{\sim}{H}}{\sqrt{4\pi\rho}} ;$$

$$\rho\left(\frac{\partial \underset{\sim}{P}}{\partial t} + \underset{\sim}{Q}\cdot\nabla\underset{\sim}{P}\right) = -\nabla\left(p + \frac{\underset{\sim}{H}^2}{8\pi}\right) + \underset{\sim}{F} + K_1\nabla^2\underset{\sim}{P} + K_2\nabla^2\underset{\sim}{Q} \tag{9.5}$$

$$\rho\left(\frac{\partial \underset{\sim}{Q}}{\partial t} + \underset{\sim}{P}\cdot\nabla\underset{\sim}{Q}\right) = -\nabla\left(p + \frac{\underset{\sim}{H}^2}{8\pi}\right) + \underset{\sim}{F} + K_1\nabla^2\underset{\sim}{Q} + K_2\nabla^2\underset{\sim}{P} \;, \tag{9.6}$$

where ν is the kinematic viscosity,

$$\begin{aligned} K_1 &= \frac{\rho}{2}\left(\nu + \frac{1}{\frac{4\pi\sigma}{c^2}}\right) \\ K_2 &= \frac{\rho}{2}\left(\nu - \frac{1}{\frac{4\pi\sigma}{c^2}}\right) \;, \end{aligned} \tag{9.7}$$

and $\underset{\sim}{F}$ is the density of nonmagnetic body forces. Equations (9.6) are formally similar to equations of motion, with $\underset{\sim}{Q}$ and $\underset{\sim}{P}$ taking the part of $\underset{\sim}{v}$. This suggests an analogy between $H/\sqrt{4\pi\rho}$ and $\underset{\sim}{v}$; if these vectors behave similarly, then in equilibrium we should have

$$\rho\, \underset{\sim}{v}^2/2 \sim \underset{\sim}{H}^2/8\pi \tag{9.8}$$

or equipartition of energy. However, no unambiguous conclusions can be drawn as to the behavior of $\underset{\sim}{H}$ and $\underset{\sim}{v}$ separately from that of their combinations $\underset{\sim}{P}$ and $\underset{\sim}{Q}$; for example, if $\underset{\sim}{P} \equiv \underset{\sim}{Q}$, we have $\underset{\sim}{H} = 0$. Without detailed solutions of the equations, such arguments from their structure are unconvincing.

A different argument from analogy is due to Batchelor. The hydromagnetic equation of Chapter 4 ,

$$\frac{\partial \underset{\sim}{H}}{\partial t} = \nabla_\wedge (\underset{\sim}{v}_\wedge \underset{\sim}{H}) + \lambda\nabla^2\underset{\sim}{H} \;, \tag{9.9}$$

where

$$\lambda = 1/(4\pi\sigma/c^2) \;, \tag{9.10}$$

is analogous to the equation to the vorticity in an incompressible viscous fluid, unaffected by magnetic forces:

$$\frac{\partial \underset{\sim}{\omega}}{\partial t} = \nabla_\wedge (\underset{\sim}{v}_\wedge \underset{\sim}{\omega}) + \nu\nabla^2\underset{\sim}{\omega} \;, \tag{9.11}$$

where

$$\underset{\sim}{\omega} = \nabla_\wedge \underset{\sim}{v} \;,$$

and ν is the kinematic viscosity. Further, both $\underset{\sim}{H}$ and $\underset{\sim}{\omega}$ are divergence-free vectors. Batchelor therefore expects $\underset{\sim}{H}$ to behave like $\underset{\sim}{\omega}$ rather than $\underset{\sim}{v}$. Certainly analogous processes are occurring in the two cases: the magnetic field lines and the vortex lines are convected with the fluid, except for a degree of relative diffusion determined by the parameters λ and ν.

The first question asked is: when will an arbitrary seed field be amplified by homogeneous turbulence? Because of the similarity of (9.9) and (9.11), Batchelor argues that the initial field $\underset{\sim}{H}$ is rapidly transformed to a structure statistically similar to the vorticity field--i.e., $\underset{\sim}{\omega}$ and $\underset{\sim}{H}$ become essentially the same vector. Granted this, then the motion increases the two fields at the same rate. Now in steady turbulence, we find $\partial \underset{\sim}{\omega}/\partial t = 0$ --the convection term is balanced by the diffusion term; hence, if the motional induction term in (9.9) is to dominate, so that the field grows, we must have

$$\lambda < \nu$$

or

$$4\pi\sigma\nu/c^2 > 1 \quad . \tag{9.13}$$

This Batchelor number is the ratio of the magnetic to the ordinary Reynolds number. When it is greater than unity, then the Ohmic dissipation reduces the energy of a magnetic loop at a <u>slower</u> rate than viscous dissipation reduces the energy of an eddy of comparable scale. Inside stars, the short mean free path ensures that $4\pi\sigma\nu/c^2 < 1$; but in <u>fully ionized</u> interstellar gas, the inequality (9.13) holds. [In <u>lightly</u> ionized gas, the magnetic dissipation is anisotropic--currents flowing across the field are destroyed far more rapidly than by Ohmic dissipation (see Chapter 11). A more complicated equation replaces (9.9). One can represent the change as requiring a resistivity for currents across the field that increases strongly with the field strength. If (9.13) holds both for the Ohmic (parallel) resistivity, and also for the much greater cross-resistivity, perhaps Batchelor's arguments may be applied to the lightly ionized case; but if (9.13) holds only for the parallel resistivity, the analogy with $\underset{\sim}{\omega}$ seems too tenuous for any plausible conclusions to be drawn.]

Accepting for the moment Batchelor's criterion (9.13) for spontaneous growth of the field, we may ask what asymptotic state is reached. Pursuing the $\underset{\sim}{\omega}$-$\underset{\sim}{H}$ analogy, Batchelor argues that the magnetic spectrum is similar to that of vorticity (<u>not</u> velocity) in ordinary nonmagnetic turbulence. As most of the vorticity is in the small eddies, whereas the large eddies contain most of the energy, it follows that equipartition between the kinetic and magnetic fields exists at small wavelengths, but the total magnetic energy is much less than the kinetic.

Batchelor's arguments may be criticised on a number of grounds. His criterion (9.13) for spontaneous growth does depend on his assumption that the initial $\underset{\sim}{H}$ rapidly acquires the same statistical properties as $\underset{\sim}{\omega}$ --a difficult postulate to picture or to justify. The analogy between $\underset{\sim}{\omega}$ and $\underset{\sim}{H}$ is imperfect, because $\underset{\sim}{\omega} = \nabla \wedge \underset{\sim}{v}$, whereas $\underset{\sim}{H}$ is independent of $\underset{\sim}{v}$. H. K. Moffatt has argued that this does not affect the application to turbulence, as the distortion of one Fourier component of the vorticity is performed by a Fourier component of the velocity of smaller wavelength. This, however, leads to the strongest criticism of all. In ordinary turbulence, both energy and vorticity are fed in at large wavelengths, from which they cascade down to the small eddies, there to be destroyed by viscous action. If, then, one accepts the $\underset{\sim}{\omega}$-$\underset{\sim}{H}$ analogy, logically one requires a supply of magnetic energy as well as kinetic energy to be fed in at large wavelengths--i.e., there should be a permanent battery in the system, continually at work pumping in magnetic energy. Such a system would not be a true dynamo: if the battery were switched off, the field would decay, at a rate accelerated by the turbulence (see the earlier discussion).

I do <u>not</u> wish to claim that a true turbulent dynamo is impossible. On the contrary, I feel that the recent successes with the laminar dynamo problem encourage one to believe that in a highly conducting turbulent fluid, in which motions of all symmetries are present, dynamo maintenance should sometimes be possible. My point is that the $\underset{\sim}{\omega}$-$\underset{\sim}{H}$ analogy is actually misleading at this point: if a strict turbulent dynamo is possible, it is more in spite of rather than because of the analogy with vorticity.

However, Batchelor's arguments may be put in a form more easily followed by astrophysicists. Let us imagine a large-scale seed field, and suppose that the motion consists initially of a large-scale component, which subsequently breaks up into smaller and smaller eddies. The field is then simultaneously strengthened and tangled up. Kinetic energy is continually fed in at the largest wavelength; equipartition is first reached at the smallest wavelength, which we assume is far above the level at which either Ohmic or viscous dissipation is important. The problem is then, how does the system adjust itself to a continuing supply of kinetic energy? As already mentioned, Biermann and Schlüter expect a subsequent extension of equipartition up to all wavelengths. If, however, one feels intuitively that "tangling of the field" implies a transfer of energy from large loops to small loops, then the cascading down of more kinetic energy will have the effect of generating smaller loops--i.e., the wavelength at which the magnetic energy is of the order of the kinetic energy continually decreases. At this point the Batchelor criterion (9.13) becomes relevant. If $\lambda > \nu$, then as the size of the loops continually decreases, a scale will be reached at which the destruction of magnetic energy by Ohmic dissipation becomes comparable with the supply of magnetic energy from the kinetic field; whereas the destruction of kinetic energy by viscous dissipation is still much smaller, because $\nu < \lambda$. As cascading continues, the field loops are twisted more, Ohmic dissipation dominates, and the seed field is destroyed. (If the "battery" that produced the original seed field is still operating, there will of course remain a weak field, but its energy will stay negligible.)

On the other hand, if $\nu > \lambda$, then cascading of energy and twisting of the field continue, until viscous dissipation halts the cascading. This occurs at a scale D given by

$$\frac{vD}{\nu} \sim 1 \qquad (9.14)$$

--the ordinary Reynolds number is unity. Equation (9.14) defines the scale of the smallest eddies in ordinary turbulence. The magnetic Reynolds number at scale D is $\left[4\pi\sigma/c^2\right]vD = vD/\lambda > 1$ as $\lambda < \nu$. Thus we should arrive at "magnetic eddies," of a scale given by (9.14), with magnetic and kinetic energies comparable. But the magnetic energy is far less than the kinetic energy in the fluid as a whole.

An attempt to give a detailed description of these magnetic eddies has recently been made by H. K. Moffatt. He points out that the time D^2/ν for which the smallest turbulent eddies persist is again shorter than the decay time of an axially-symmetric magnetic eddy of the same dimension $\left\{(4\pi\sigma/c^2)D^2 > D^2/\nu \text{ by } (9.13)\right\}$. Thus it is consistent to talk of magnetic eddies, with a nearly permanent magnetic field, lasting for the natural lifetime D^2/ν of the smallest turbulent eddy. The real difficulty is to convince oneself that after a magnetic eddy has been destroyed by the shearing effect of the larger eddies, the new eddies that are formed have a regenerated magnetic field. Moffatt's eddies are axially symmetric, and would be subject to Cowling's theorem, if they persisted. As it is, some magnetic flux is destroyed during the lifetime of an individual eddy; a complete theory would show how during the reformation, the existing field lines are used more "efficiently"--as in the laminar dynamo problem--so as to cancel out the previous loss of flux. But although this has not yet been done, Moffatt's work is pioneering, in that he seems to be the first to attempt to produce a detailed theory of the structure of the asymptotic field. It is only when one makes this attempt that the difficulties associated with Cowling's theorem turn up: if one is content with merely semi-quantitative arguments, one can postulate that the field has no inconvenient limiting lines, and concentrate on the dynamical aspects of the problem.

To sum up: it does not seem that the proponents either of general or small-scale equipartition have proved their case. A big step forward would be a clarification of the concept of tangling-up of the initial field. For a homogeneous isotropic medium, does it imply an increase in small-scale energy at the expense of the larger loops, or are all wavelengths increased in energy? If the former, then we can understand the Batchelor criterion, and how a dissipative mechanism can fix the scale and strength of the field; if the latter, then one would expect general equipartition to be approached. One would in any case like more detailed work on the structure of the asymptotic field.

After all this inconclusive discussion of incompressible, homogeneous, isotropic, hydromagnetic turbulence, one is hard put to find a really convincing application. Batchelor's criterion would imply no spontaneous generation inside stars; but the stellar dynamo theory depends strongly on the departures from homogeneity and isotropy. The gravitational field defines a preferred direction, and the large-scale nonuniform rotation plays an essential role. In fact, Parker's discussion of the feed-back process is in terms of the statistical regeneration of small poloidal loops which tend to be aligned by the gravitational field, and coalesce to form a dipole-type field.

The random motions of gas clouds in the galaxy are highly supersonic: the very existence of large density variations implies that incompressibility is a bad approximation, and one is very dubious about applying any theory of incompressible turbulence, or conclusions drawn from terrestrial experiments. With this strong reservation, one notes that the figure of 10^{-6} gauss sometimes quoted for the galactic magnetic field yields an energy comparable with the turbulent energy, though somewhat less. Believers in equipartition can claim this in favor of the turbulent dynamo theory for the origin of the galactic magnetic field. Their opponents would assert that the field must have been built up by large-scale laminar motions in order to acquire both its strength and its large scale. The figure of 10^{-5} gauss quoted by others implies an energy much greater than the turbulent energy: the field would have to have a scale larger than that of the clouds and would strongly resist distortion by cloud motion. It is difficult to see how the turbulent dynamo mechanism could be responsible for such a strong field; it is more likely that the energy has come from the galactic rotation field.

C. THE EFFECT OF A STRONG MAGNETIC FIELD ON TURBULENCE

Suppose now that, instead of the turbulence beginning by acting on a weak magnetic field, a strong field is forced into a turbulent medium. Because of the application to sunspots, we think here of turbulent convection, e.g., in the sun's sub-photospheric convection zone. A super-adiabatic temperature gradient would be required to transmit radiatively through this zone all the energy coming from the solar core; the consequent instability leads to convection currents which reduce somewhat the super-adiabacy.

We adopt the following elementary description of convection. An element of matter rises a distance z, expanding adiabatically so as to stay in pressure equilibrium with the ambient medium. Its temperature, initially T_0, is now

$$T_0 + (dT/dr)_{ad.} z \quad , \tag{9.15}$$

whereas the ambient temperature is (by definition of $(dT/dr)_{amb}$),

$$T_0 + (dT/dr)_{amb.} z \quad . \tag{9.16}$$

In a convectively unstable region, the element is hotter than its surroundings by an amount of approximately

$$\left[(dT/dr)_{ad.} - (dT/dr)_{amb.}\right] z = (\Delta T') z \quad . \tag{9.17}$$

It is therefore less dense than its surroundings, and experiences a buoyancy force density,

$$\frac{\rho g}{T} (\nabla T') z \quad , \tag{9.18}$$

where g is the local value of gravity. According to Prandtl's picture of convection, the element travels a distance ℓ, the mixing length, before mingling with the surroundings: ℓ is a sort of macroscopic mean free path. The turbulent elements therefore have a mean kinetic energy

$$\frac{1}{2} \ell^2 \frac{\rho g}{T} (\Delta T') \tag{9.19}$$

per unit volume. If the turbulence is to be unaffected by the magnetic field, this must exceed the magnetic energy density. Taking for ℓ the scale height, it is found in the sun that

$$H > 6 \times 10^3 .$$

A field much less than this will be forced aside by the convection; otherwise the field will interfere with the convection.

Unfortunately, it is difficult to compute the precise reduction in the efficiency of convection. It is unlikely that the energy transport will be forced to become completely radiative. Although eddying motion is suppressed, elements of matter can rise and fall, and exchange heat laterally. This problem will come up in the next chapter in connection with the structure of sunspots.

CHAPTER 10

MAGNETOHYDROSTATIC EQUILIBRIUM

A. THE EQUILIBRIUM OF MAGNETIC STARS

In Chapters 6-8, we considered various aspects of stellar magnetism. In particular, the equation of poloidal equilibrium was written down including magnetic and centrifugal force:

$$\nabla p = \rho(\nabla V + \Omega^2 \underset{\sim}{\varpi}) + \left(\frac{\underset{\sim}{j} \wedge \underset{\sim}{H}}{c}\right)_p \tag{10.1}$$

where the suffix p again means "poloidal component." We now wish to remark that the pressure field in a star is able to absorb the perturbing, non-spherical forces, because of the extra degree of freedom in the temperature field. There exists a systematic expansion procedure for finding the spherical and nonspherical parts of the perturbations to ρ , T and p , assuming the centrifugal and magnetic forces are not too strong. To do this, one needs Poisson's equation relating ρ and V , and the equation of state $p \propto \rho T$. The equation of radiative equilibrium in general cannot be satisfied except when averaged over a sphere: to preserve thermal equilibrium one requires a system of circulating material currents which carry the required energy. Analogous results follow for convective zones: there is a large-scale laminar flow superimposed on the small-scale turbulence. [To be fully self-consistent, the velocity field used in Chapters 6 and 7 should be computed in terms of the perturbing $\underset{\sim}{\Omega}$ and $\underset{\sim}{H}$ fields.]

However, the time-scale of radiative adjustment of the temperature field is very long -- of the order of the Kelvin-Helmholtz time-scale. In the course of the normal evolution of a star, or during the induction time-scale, we may expect the temperature field to adjust itself steadily to changing conditions. But if, for example, the star is undergoing torsional oscillations, say with the period of the solar sunspot cycle, the temperature changes of the star will be nearly adiabatic -- there will be no time for radiative transfer to take place to balance the fluctuating perturbing forces. This means that there will be induced poloidal oscillations with their own natural frequency, superimposed on the slower oscillation that keeps pace with the toroidal oscillations.

By contrast, suppose the star were liquid, with ρ constant. Then the curl of (10.1) yields

$$\nabla \wedge \left[\Omega^2 \underset{\approx}{\varpi} + \left(\frac{\underset{\approx}{j} \wedge \underset{\approx}{H}}{c\rho} \right)_p \right] = 0 \tag{10.2}$$

-- i.e., the conditions of equilibrium (10.1) imply a constraint on the $\underset{\approx}{\Omega}$ and $\underset{\approx}{H}$ fields. A similar conclusion holds if $\rho = \rho(p)$. We thus see that the incompressible and barytropic simplifications are bad when considering equilibrium, as they remove one essential degree of freedom.

B. THE EQUILIBRIUM OF SUNSPOTS

We have seen in Chapter 9 that an externally maintained, strong magnetic field, forced into a convective zone, can interfere with the normal convective transport of energy. As a tentative model of a sunspot pair, we imagine that part of a toroidal magnetic girdle is dragged up to the photosphere, the ascending and descending field lines defining a pair of "spots" of opposite polarity. We now consider just one such "spot." Following Biermann and Hoyle, we accept that the magnetic field is the basic cause of cooling, through its interference with convection. The temperature outside the spot is that defined by the normal convective transport; thus the spot region is compressed, with an increase in H because of flux-freezing. This cuts down further the convective transport, causes more cooling, with more compression, until quasi-equilibrium is reached.

As a rough first approximation, we write

$$\frac{H^2}{8\pi} = p_e - p_i \qquad (10.3)$$

where H is the axial magnetic field, and suffices refer to external and internal pressures respectively. The quasi-equilibrium of a spot is due to the interference with convection, which allows (10.3) to be satisfied, because $p_e > p_i$.

In attempting a detailed theory, one is up against our lack of an adequate theory of turbulent convection, and even more, a theory of its inhibition by a magnetic field. One can experiment by assuming first complete inhibition (only radiative equilibrium in the spot), and then a convective transport reduced by a variable efficiency factor. Starting with the observed spot surface temperature and radiative flux, one can then integrate downwards, comparing all the time with the normal T-ρ field well outside the spot. All workers agree that the assumption of complete suppression leads to implausible results. The radiative temperature gradient is so high that p_i stays small and H diverges with depth. ~~approaches p_e far too rapidly~~. It is simpler, but less convincing, to compute p_i from (10.3); the "base" of the spot will be where $H^2/8\pi$ is of the order of the turbulent energy density.

Schlüter and Temesvary have produced a model which allows the field lines to splay outwards with decreasing depth. Their field has a monopolic structure above the surface. By means of a simple similarity assumption, they reduce the horizontal equation of equilibrium to an ordinary differential equation involving Δp, the horizontal thermal pressure difference between infinity and the axis. Again they are forced to compute Δp in terms of the field, rather than the reverse: a little way below the surface they find that $\Delta p/p$ becomes small.

The Schlüter-Temesvary model does not allow for any sharp changeover from umbra to penumbra. A model that attempts to include such a feature is being studied by S. K. Chitre. The umbra is thought of as a region of reduced convective efficiency, as before. The limiting field line between umbra and penumbra is defined as that line along which material, rising from the level of the sunspot base, has enough energy to flow continuously all the way up until the field line becomes horizontal: the observed Evershed velocities lend tentative support to this picture. As a boundary condition at the base, we assume this "Evershed" flow starts with a speed equal to the ordinary turbulent speed; again the field energy at the base is taken to be the same as the turbulent energy. Along the limiting field line the condition of continuity, and Bernoulli's equation for approximately adiabatic flow, together fix the (p , T) field in terms of the shape of the field line: the discontinuity in p is balanced by the discontinuity in $(H^2/8\pi)$, whence we deduce the structure of the penumbral field in terms of the umbral.

Again, if the umbral (p , T) field is properly estimated, the magnetic field structure should be given by the condition that the (p , T) field at infinity be the normal field, unaffected by the local magnetic disturbance. It is again likely that we shall be forced to work backwards. We can perhaps expect to show that a reasonable model, yielding Evershed velocities of the observed order, and defining umbral and penumbral zones according to our suggested criterion, is consistent with lateral equilibrium provided again $\Delta p/p \ll 1$ below the surface. How the umbral turbulent field is able to adjust itself so as to keep the magnetic interference very low is still a difficult question to answer quantitatively. There is in fact strong observational support for convective motions inside the umbra.

A further problem is the ultimate fate of the energy not convected into the umbra. The evidence for a bright-ring surrounding the penumbra is considered by many observers quite spurious. A more promising recent suggestion is that the energy defect is carried up by Alfvén waves into the chromosphere and lower corona, probably to be released as heat when the waves are damped.

C. MAGNETOHYDROSTATIC EQUILIBRIUM IN GENERAL

If $\underset{\sim}{F}$ is the total nonmagnetic force per unit mass, including inertial force, the equilibrium condition is

$$\frac{\underset{\sim}{j} \wedge \underset{\sim}{H}}{c} + \rho \underset{\sim}{F} = 0 \quad : \qquad (10.4)$$

enough current flows across $\underset{\sim}{H}$ to yield the correct magnetic force density. In Section A we considered the case when the magnetic force is a perturbation compared with the individual large terms on the right. In the sunspot problem, the emphasis was on the generation by the external thermal pressure of the correct magnetic force able to balance it. A similar process is pictured in the Chandrasekhar-Fermi cylindrical model of a galactic spiral arm: a longitudinal magnetic field is compressed by gravitational contraction towards the axis until the radial magnetic force balances the self-gravitation of the cylinder.

In more general geometries, with $\underset{\sim}{H}$ strong, condition (10.4) has both an active and a passive aspect. For example, consider the theory of solar filaments put forward by Schlüter and Kippenhahn. A segment of a field line has its ends anchored in the dense subphotospheric regions. If the density of matter is negligible in the chromosphere (10.4) is satisfied by the solution of $\nabla \wedge \underset{\sim}{H} = 0$ which has the correct normal component at the photosphere.

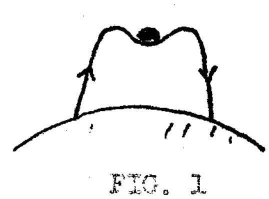


FIG. 1

If matter is added to the horizontal parts of the field line, a sag will develope, so that the vertical magnetic force balances gravity. At the same time there results a horizontal pinching force which increases ρ , giving rise to the filamentary structure. The magnetic force must balance the non-magnetic, but balance is achieved by adjustment of both the magnetic and material fields.

A similar structure arises when we consider the field outside a dense gravitating globule , supposed to have condensed from a region with a uniform magnetic field. If the density is low outside, one is tempted to write (10.4) as

$$\nabla \wedge \underset{\sim}{H} = 0 \qquad (10.5)$$

so that the field has the structure as in Fig. 2, with a neutral point X . But if the density is not zero, the gravitational force inwards at X must exceed the magnetic force, which vanishes. As there is a continuous supply of matter from infinity, the field structure is distorted, until X is pulled into the star. The sharp bends in the field lines correspond to a strong equatorial discharge: the radial magnetic force balances the inward gravitational pull, while the lateral magnetic force builds up a pressure able to withstand it.

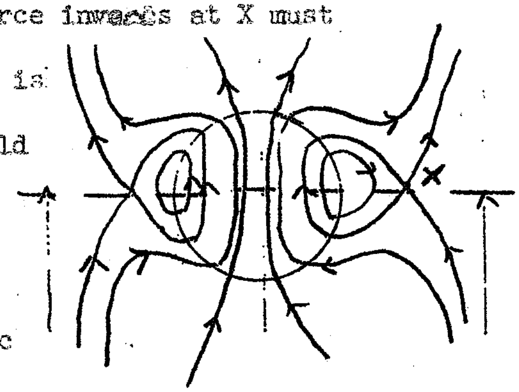


FIG. 2

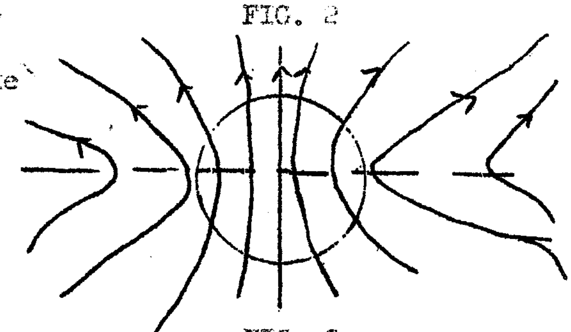


FIG. 3

The hydromagnetic stability of such pinching fields has not yet been adequately discussed. The presence of a field component across the pinched zone probably has a stabilizing effect compared with a simple pinch.

The gravitational distortion of the field of Fig. 2 is an example of the limitation of the force-free field. It has been argued that with H large, equilibrium is achieved only if the currents maintaining the field flow parallel to the field, so that $\underset{\sim}{j} \wedge \underset{\sim}{H}/c = 0$. The field of Fig. 2 is the special case $\underset{\sim}{j} = 0$. The argument implicitly assumes that balance cannot be achieved by the field's building up a sufficiently high material density by compression, or that such a state is unstable. But in any case, if the force-free or curl-free assumption results in a field that, so far from being strong everywhere, is actually zero at critical points, the basic assumption is self-contradictory.

Completely force-free fields -- defined by

$$\underset{\sim}{j} = k\underset{\sim}{H} \tag{10.6}$$

with

$$\underset{\sim}{H} \cdot \nabla k = 0 \tag{10.7}$$

have been studied by Lüst and Schlüter and Chandrasekar, axial symmetry being assumed. They are characterised by linked flux-tubes: the tendency of one tube to contract is resisted by the pressure of the enclosed tube. They also possess neutral points, and so would suffer distortion by a local gravitational field.

It is, however, difficult to see why the whole of a magnetic field should acquire so much more energy than its force-density suggests. One's expectation is rather that the energy would be fed into a kinetic field, even if the material density is low. However, if part of the field is anchored into regions where the nonmagnetic forces are strong, then it is quite plausible that in low density regions outside, the field should adjust itself to be force-free when the rest of the field is distorted. For example, consider an axially-symmetric system, with field lines linking a central mass (1) rotating about the axis, and a ring (2) with a slower rotation about the same axis. The twisting of the field generates a toroidal component which exerts poloidal force. In the

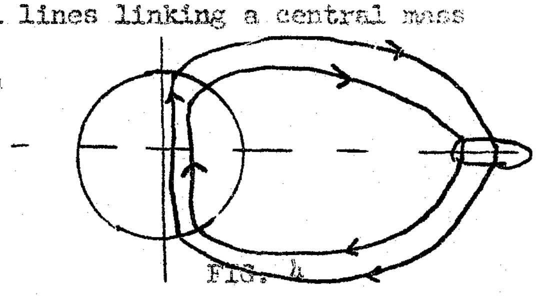


FIG. 4

low density region, this must be balanced by a poloidal force due to the poloidal field -- i.e. the poloidal field adjusts itself so that the whole field in this region is force-free. [In so doing, some of the poloidal loops may snap, breaking magnetic contact between (1) and (2).] Further, the field in (3) must be torque-free -- in the notation of Chapter 6, $\bar{\omega} H_\phi / 4\pi$ = constant on each poloidal loop. Within (1) and (2) the field exerts poloidal forces, which will be absorbed by the pressure field. Also, the field exerts torques, tending to slow up the faster rotating part and accelerate the slower. This is consistent with our interpretation of $(-\bar{\omega} H_\phi / 4\pi)$ in Chapter 6 as a transport of angular momentum along the field lines linking (1) and (2).

CHAPTER 11

STAR FORMATION AND THE GALACTIC MAGNETIC FIELD

In this chapter we consider the effect that a fairly strong, large-scale galactic magnetic field has on the problem of the condensation of stars from interstellar gas. There is strong evidence, from different branches of astrophysical theory, that star formation has occurred at all epochs since the formation of the galaxy itself (2×10^{10} years ago?). Conditions were undoubtedly different when the galaxy was young; e.g., there was probably more random kinetic energy, perhaps a complete absence of dust grains and molecular radiators. However, there seems no good reason for assuming the magnetic field to be a feature only of the galaxy as observed today, and not of earlier epochs. We therefore consider the problem in as general a way as possible.

The difficulties introduced by the field are first cussed, followed by an outline of how they may be got over if the freezing of the field into the matter remains a good approximation. We then discuss in detail the magnetohydrodynamics of a lightly ionized gas, with special reference to the star formation problem, but in sufficiently general terms for application to other problems. In particular, we derive a rate of dissipation of magnetic energy, when there is a component of current across the field, much greater than the ordinary Ohmic dissipation.

A. THE MECHANICAL EFFECT OF A MAGNETIC FIELD

Consider a cosmical gas cloud. If it is not to disperse into the general background, but instead is to collapse to stellar densities, its self-gravitation must dominate over the disruptive forces. A necessary integral condition is provided by the virial theorem, proved in generality by Chandrasekhar and Fermi:

$$-\Omega > \mathfrak{M} + 2T \quad , \tag{11.1}$$

where

Ω = the (negative) total gravitational energy of the system,

$\mathfrak{M}$ = the total magnetic energy,

T = the total kinetic energy (thermal, rotatory, turbulent, etc.).

Strictly, there ould be included in (11.1) a surface term, the effect of the external therma pressure and Maxwell stresses, w ich assists self-gravitation. If the magnetic field were force-free, the magnetic surface term would cancel $\mathfrak{M}$ in (11.1). However, under the necessarily n n-uniform ract on, an initial force-free field would not remain so; the parts of the field well within the cloud would be amplified more rapidly than the outer parts. Condition (11.1) is thus a satisfactory order-of-magnitude estimate, and shows that magnetic energy is disruptive: the isotropic magnetic pressure more than compensates for the one-dimensional tension that acts in the local direction of the field. The magnetic force is, of course, anisotropic: a large-scale field as in the diagram tends to straighten itself, and so exerts a net force in the equatorial direction, but not along the axis of the field.

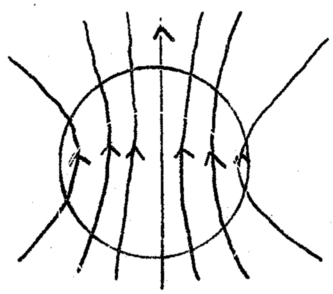


Fig. 1

Suppose that the cloud is roughly spherical. Then we find

$$\Omega \simeq - GM^2/R \quad , \qquad (11.2)$$

where G is the gravitational constant, M the mass, and R the radius. Introducing a mean density ρ , satisfying

$$M = 4\pi/3 \ \rho \ R^3 \quad , \qquad (11.3)$$

we have

$$-\Omega \propto M^{5/3} \rho^{1/3} \quad . \qquad (11.4)$$

Further, we have

$$\mathfrak{M} \simeq (H^2/8\pi)\ (4\pi/3 \ R^3) = \frac{1}{6} H^2 R^3 \propto H^2 M/\rho \quad , \qquad (11.5)$$

and the thermal component of kinetic energy is

$$\mathcal{U} = \frac{3}{2} R M \Theta \propto M \quad , \qquad (11.6)$$

where R is the gas constant, and Θ the absolute temperature (μ is taken as unity).

For simplicity, we ignore rotatory and turbulent energy. Then for a cool HI cloud-- $\Theta \sim 10^{2}$ °K --and with $H \simeq 10^{-6}$ gauss, associated with a density $\rho \sim 10\, m_H$, (11.1) yields a minimum mass for gravitational binding

$$M_c \simeq 10^3 \odot \quad (\odot = \text{solar mass}) \quad . \tag{11.7}$$

With the assumed parameter values, the thermal energy is somewhat larger than the magnetic; a higher Θ, associated with the same ρ and H, requires a larger M_c, by (11.4), (11.5), and (11.6).

The high value found for M_c is not due essentially to the magnetic field; it would be not much lowered if H were put equal to zero. We need to focus attention on how such a massive, gravitationally collapsing cloud can subsequently break up into smaller masses, i.e., we study the formation of a star cluster. There is in fact evidence that the general galactic stellar field is supplied by the slow disintegration of star clusters by "evaporation" through the tail of the Maxwell distribution; it is entirely plausible that in general it is the cluster rather than the individual star that is formed from diffuse gas clouds.

Consider first a non-magnetic cloud, collapsing under its own gravitation. Then if the temperature Θ remains constant, $\mathcal{U}$ stays constant, while $-\Omega$ increases. Thus the minimum mass M_c decreases with increasing density at constant temperature. This suggests that the cloud may "fragment," in Hoyle's terminology: smaller and smaller masses separate out systematically as the cloud as a whole contracts. However, this will certainly not happen in a strictly uniform cloud, as there is then no reason for the density to increase more rapidly locally than in the cloud as a whole. But if one recognizes the existence of a field of "longitudinal turbulence," i.e., a system of pressure waves traveling through the cloud, then one can frame the question: under what circumstances will a local isothermal density increase not "right itself," but grow instead under its self-gravitation more rapidly than the background density? The answer will certainly involve both the length-scale of the perturbation and its strength: for in a local fluctuation the thermal pressure gradient will have the effect of reducing the local self-gravitation, so that only a density increase greater than a minimum will be able to grow more rapidly than the mean background density.

Recent work by C. Hunter on the stability of a uniform contracting, self-gravitating cloud has made the fragmentation picture very plausible; but there remain many problems. In particular it is not easy to estimate how much of the gravitational energy released is thermally dissipated, and how much remains as macroscopic kinetic energy of the condensed globules: i.e., is the ultimate star cluster dense, like the globulars, or open, like the young galactic clusters? Probably the answer depends on whether there is an external source of radiative energy maintaining isothermality or whether every compression followed by rarefaction necessarily dissipates energy.

As pointed out by Hoyle, the fragmentation process halts when the isothermal condition breaks down--i.e., when the globules become too opaque for the compressional heat generated to be radiated away in the time of gravitational collapse of the globule. If such a globule continued to collapse at this rate, its temperature would increase adiabatically-- $\Theta \propto \rho^{2/3} \propto 1/R^2$ --so that by (11.2) and (11.6) U would increase more rapidly than $-\Omega$, and the collapse would be halted. The globule is now a proto-star; its thermal field is in close mechanical equilibrium with its gravitational field and the contraction rate is fixed by the energy leak down the temperature gradient from the center to the surface. However, it is difficult to estimate accurately the final masses, or to determine their statistical distribution.

To sum up: a non-magnetic, non-rotating cloud, contracting isothermally, will ultimately break up into a star cluster, but theory so far gives no adequate prediction of the detailed properties of the cluster.

Now let us suppose the cloud has a frozen-in large-scale field, e.g., as in Fig. 1. Initially, the field energy in the cloud considered is somewhat less than the thermal. Under approximately isotropic contraction, the density is again given by (11.3); by conservation of flux,

$$HR^2 = \text{constant} , \tag{11.8}$$

so that

$$H \propto \rho^{2/3} . \tag{11.9}$$

The magnetic energy is again given by (11.5), and so increases like $1/R$, <u>at the same rate as</u> $\underline{-\Omega}$. Under isothermal, isotropic contraction, the thermal energy becomes negligible, so that the minimum mass is reduced--for our parameter values, to about 500 $\odot$; but no smaller mass can separate out. The mixed sound and hydromagnetic waves traveling across the field, which in the absence of the field would (if sufficiently intense) lead to small local condensations separating out, are kept stable by the magnetic field.

(The difficulty does depend on the field being large-scale and not tangled, so that (11.8) holds. A small-scale field could straighten itself out under contraction, so that the field lines cross the equator a smaller number of times; flux is then conserved without H increasing at the rate (11.8).)

There are two possible ways out. First, we may challenge the assumption that the contraction is nearly isotropic. The virial theorem shows that isotropic contraction--which leads to (11.9)--is not inconsistent with flux conservation: the magnetic forces do not increase too rapidly for the gravitational. However, suppose that initially the lateral magnetic force just balances the gravitational and that the gas is at rest; the unopposed gravitational force acting down the magnetic field lines immediately causes flow down the field, so increasing ρ without simultaneously increasing H. But as soon as ρ changes substantially, the gravitational force density across the field far outweighs the unchanged magnetic force density, and lateral contraction starts. It is, however, possible that a cloud with an initially oblate structure may, by its non-isotropic gravitational field, allow enough preferential flow for ρ to increase more rapidly than $H^{3/2}$, so getting over the difficulty. To demonstrate this conclusively is difficult; it certainly requires finer tools than the virial theorem.

If the cloud is rotating rapidly about the same axis as that of the field, then non-isotropic flow will result if the centrifugal field becomes comparable with the gravitational. The magnetic energy problem is then resolved, but one is left with the problem of the formation of stars from a cloud with high angular momentum. However, to treat this properly, one must consider whether in fact all the cloud will conserve its angular momentum, or whether those magnetic field lines linked with the galaxy are able to transport angular momentum efficiently enough to prevent the centrifugal force becoming high. This in turn demands study of the structure of the magnetic field; in particular, the times at which individual cloud field lines detach themselves from the galactic field must be estimated. An attempt at this problem is in preparation.

It should be noted that if the problem is resolved by preferential flow, we may expect the small condensations that arise to have fairly strong internal magnetic fields. This is in the spirit of the fragmentation picture, discussed above for a non-magnetic cloud. Density fluctuations that initially cannot grow, because of the strong magnetic field, are able to grow once enough preferential flow has occurred; further, even if the cloud as a whole is contracting--i.e., there is no centrifugal field holding it up--we may again expect that a sufficiently strong density fluctuation, of mass rather greater than the minimum set by the virial theorem, will condense more rapidly than the general background. A moderately high internal magnetic energy could have a noticeable effect on stellar structure and evolution.

The other way of resolving the magnetic energy difficulty is to challenge the assumption that the field is frozen into the matter. We have in fact applied the freezing-in concept--derived as an excellent approximation for a plasma--to a lightly-ionized, HI cloud, where in fact the bulk of the matter is neutral hydrogen, and the plasma component consists of a small proportion--1 part in 10^4--of metals with low ionization potentials. The star formation problem provides our first opportunity to generalize the work of Chapters 3, 4, and 5 to a lightly ionized gas. (See Mestel and Spitzer, M. N., 1956)

B. THE MAGNETOHYDRODYNAMICS OF A LIGHTLY IONIZED GAS

We define

n_e ≡ number density of electrons,
n_i ≡ number density of ions, $= n_e/Z$ (quasi-neutrality of plasma),
n_H ≡ number density of neutral hydrogen,
$\underset{\sim}{v}_e$ ≡ mean drift of electrons
$\underset{\sim}{v}_i$ ≡ mean drift of ions
$\underset{\sim}{v}_H$ ≡ mean drift of neutral hydrogen,
$\underset{\sim}{F}_{ei}$ ≡ mean force on an electron due to collisions with ions,
$\underset{\sim}{F}_{eH}$ ≡ mean force on an electron due to collisions with neutral hydrogen,
$\underset{\sim}{F}_{iH}$ ≡ mean force on an ion due to collisions with neutral hydrogen,
p_e ≡ partial pressure of electrons,
p_i ≡ partial pressure of ions,
p_H ≡ partial pressure of neutral hydrogen.

The condition of momentum balance for the electron gas is

$$- e\, n_e \left(\underset{\sim}{E} + \frac{\underset{\sim}{v}_e \wedge \underset{\sim}{H}}{c}\right) + n_e \underset{\sim}{F}_{ei} + n_e \underset{\sim}{F}_{eH} - \nabla p_e = 0 \; , \qquad (11.10)$$

where as in Chapter 3 we have dropped gravitation and inertia. For the ionic gas, we have similarly

$$Z\, e\, n_i \left(\underset{\sim}{E} + \frac{\underset{\sim}{v}_i \wedge \underset{\sim}{H}}{c}\right) + n_i \underset{\sim}{F}_{iH} - n_e \underset{\sim}{F}_{ei} - \nabla p_i + n_i m_i \nabla V = n_i m_i \frac{d\underset{\sim}{v}_i}{dt} \; , \qquad (11.11)$$

where V is the gravitational potential; the law of action and reaction is used when we write $-(n_e \underset{\sim}{F}_{ei})$ for the force density due to ion-electron encounters.

Adding (11.10) and (11.11), we have

$$\frac{\underset{\sim}{j} \wedge \underset{\sim}{H}}{c} + (n_i \underset{\sim}{F}_{iH} + n_e \underset{\sim}{F}_{eH}) - \nabla(p_i + p_e) + n_i m_i \nabla V = n_i m_i \frac{d\underset{\sim}{v}_i}{dt} . \tag{11.12}$$

For the motion of the neutral gas, we have

$$- (n_i \underset{\sim}{F}_{iH} + n_e \underset{\sim}{F}_{eH}) - \nabla p_H + n_H m_H \nabla V = n_H m_H \frac{d\underset{\sim}{v}_H}{dt} . \tag{11.13}$$

The sum of (11.12) and (11.13) yields the equation of motion of the gas as a whole, with all the mutual friction terms cancelled. The magnetic force acts directly on the plasma--hence it appears only in (11.12); its effect on the neutral gas is through the friction terms, which appear with opposite signs in both equations.

For application to our problem, we simplify a little; a full treatment is given by Cowling ("Magnetohydrodynamics"). As by hypothesis we are concerned with a lightly ionized cloud in which the magnetic force is comparable with the total gravitational force and perhaps with the total pressure, we may drop $\nabla(p_i + p_e)$ and $n_i m_i \nabla V$ from (11.12). For $\underset{\sim}{F}_{iH}$ and $\underset{\sim}{F}_{eH}$, we again use the good approximation that after a collision the motion of a light particle relative to a heavy one is purely random: thus

$$\underset{\sim}{F}_{eH} = n_H \sigma_{eH} (v_e)_T m_e (\underset{\sim}{v}_H - \underset{\sim}{v}_e) \tag{11.14}$$

and

$$\underset{\sim}{F}_{iH} = n_H \sigma_{iH} (v_H)_T m_H (\underset{\sim}{v}_H - \underset{\sim}{v}_i) = \gamma(\underset{\sim}{v}_H - \underset{\sim}{v}_i) , \tag{11.15}$$

say. Here σ_{eH} and σ_{iH} are collision cross-sections, and $(v_e)_T$ and $(v_H)_T$ thermal speeds. As in general $F_{eH} \ll F_{iH}$, (the electrons and ions as usual being strongly coupled by electrostatic forces) we shall drop $\underset{\sim}{F}_{eH}$ as well from (11.13). We may further approximate by dropping the inertia-term from (11.13), as the frictional and magnetic forces rapidly approach balance. If the ion velocity $\underset{\sim}{v}_i$ is not equal to $\overline{\underset{\sim}{v}_i}$, where

$$\left(\frac{\underset{\sim}{j} \wedge \underset{\sim}{H}}{c}\right) + n_i \gamma(\underset{\sim}{v}_H - \overline{\underset{\sim}{v}_i}) = 0 , \tag{11.16}$$

then $\underset{\sim}{v}_i$ alters according to the equation

$$\gamma(\overline{\underset{\sim}{v}_i} - \underset{\sim}{v}_i) = m_i \frac{d\underset{\sim}{v}_i}{dt} . \tag{11.17}$$

This implies that $\underset{\sim}{v}_i \to \bar{\underset{\sim}{v}}_i$ is a time of order $m_i/\gamma = \left(\frac{m_i}{m_H}\right)/n_H\sigma_{iH}(v_H)$ — the $(i - H)$ collision time with an extra factor (m_i/m_H). With $\sigma_{iH} \simeq 10^{[illegible]}$ secs (or possibly greater, according to recent calculations) one can check that this equilibrium time is at most $\sim 10^6$ secs, as compared with a time of 10^{15} secs or more for gravitational free-fall--the characteristic time-scale of our problem. We may therefore write

$$\frac{\underset{\sim}{j} \wedge \underset{\sim}{H}}{c} + n_i \underset{\sim}{F}_{iH} = \frac{\underset{\sim}{j} \wedge \underset{\sim}{H}}{c} + n_i\,\gamma(\underset{\sim}{v}_H - \underset{\sim}{v}_i) = 0 \qquad (11.18)$$

as the equation to the motion of the plasma. From this we can estimate the velocity of slip of plasma relative to the neutral gas.

The electric field is given by (11.11), where we may again drop the pressure, gravitation and inertia terms. The term in $\underset{\sim}{F}_{ei}$ is just the Ohmic field, and as usual is small. We may substitute for $n_i \underset{\sim}{F}_{iH}$ from (11.18), yielding

$$\underset{\sim}{E} + \frac{\underset{\sim}{v}_i \wedge \underset{\sim}{H}}{c} - \frac{\underset{\sim}{j} \wedge \underset{\sim}{H}}{c n_e e} = 0 \qquad (11.19)$$

--just Ohm's law for a plasma, without the Ohmic and thermal terms. Again, when combined with Faraday's law, (11.19) implies that the magnetic field moves with the electrons--a trifling change from the usual statement that it moves with the bulk velocity $\underset{\sim}{v}_i$ of the plasma.

Thus in our problem, the magnetic force causes a drift of plasma through the neutrals, such that the $(i - H)$ friction just balances the magnetic force. By self-induction, the magnetic field is tied to the plasma. The motion of the neutral gas is essentially gravitational free-fall, resisted by the magnetic force, via the frictional coupling. If the relative drift of plasma is comparable with or larger than the speed of free-fall, then the neutral density goes up without a large increase in H; otherwise, the field is effectively frozen into the gas as a whole, and the problem of star formation in a strongly magnetic cloud is unaltered.

It is in fact found that normal plasma densities yield a frictional coupling constant $(n_i\gamma)$ in (11.18), too large by a factor 10^2 or more for $|\underset{\sim}{v}_H - \underset{\sim}{v}_i|$ to be comparable with v_H, as determined by (11.13). With an increased estimate for σ_{iH}, this factor will be still further increased. However, a rapid reduction in n_i much below the normal value will allow this frictional uncoupling to occur. Thus if a cloud has a high concentration of dust, the obscuration of galactic star-light may be so increased that the natural tendency of the plasma to decay rapidly--

e.g., by attachment to dust grains--will far outweigh the production of plasma by galactic radiation. If this is so, then the neutral gas--including some neutralized plasma--will contract across the field, as required.

It is essential for the cloud to be kept cool, so that no collisional ionization occurs. Radiation by dust grains and molecules insures this. But in a cloud of atomic hydrogen, without any mechanism of cooling other than collisional ionization followed by radiative recombination, it is clear that the process will not work; for the heat of compression generated by the contracting cloud will automatically maintain a fairly high plasma density. The resolution of the difficulty must then proceed on alternate lines, as discussed earlier.

It is instructive to write down the energy equation for a lightly ionized gas in our approximation. By (11.19),

$$\underset{\sim}{j} \cdot \underset{\sim}{E} = \left(\frac{\underset{\sim}{j} \wedge \underset{\sim}{H}}{c}\right) \cdot \underset{\sim}{v}_i = \left(\frac{\underset{\sim}{j} \wedge \underset{\sim}{H}}{c}\right) \cdot (\underset{\sim}{v}_i - \underset{\sim}{v}_H) + \left(\frac{\underset{\sim}{j} \wedge \underset{\sim}{H}}{c}\right) \cdot \underset{\sim}{v}_H . \quad (11.20)$$

From (11.18), the first term on the right may be written alternatively as

$$\frac{\left(\frac{\underset{\sim}{j} \wedge \underset{\sim}{H}}{c}\right)^2}{n_i \gamma} = \gamma n_i \, (\underset{\sim}{v}_H - \underset{\sim}{v}_i)^2 \qquad (11.21)$$

As $\gamma n_i \, (\underset{\sim}{v}_H - \underset{\sim}{v}_i)$ is the frictional force density acting on the ions, we have for the dissipation of energy/sec, cm^3

$$\left[\gamma n_i (\underset{\sim}{v}_H - \underset{\sim}{v}_i)\right] \cdot \left[-(\underset{\sim}{v}_i - \underset{\sim}{v}_H)\right] = \gamma n_i (\underset{\sim}{v}_H - \underset{\sim}{v}_i)^2 = \frac{\left(\frac{\underset{\sim}{j} \wedge \underset{\sim}{H}}{c}\right)^2}{n_i \gamma}$$

--just (11.21). Thus given a magnetic field $\underset{\sim}{H}$, maintained by currents $\underset{\sim}{j}$, we have (11.21) as an extra rate of dissipation. Currents flowing parallel to $\underset{\sim}{H}$ are not affected; but for $j_{\perp}$, there is an effective "resistivity" which genuinely depends on $\underset{\sim}{H}$, being $H^2/c^2 n_i \gamma$. As long as friction on the ions and magnetic force are in equilibrium, so that we may substitute from (11.18), then (11.21) predicts an <u>increased</u> dissipation for a <u>reduced</u> coupling constant $n_i \gamma$. This peculiarity--pointed out by Cowling--is due to the dissipation being proportional to the square of $|\underset{\sim}{v}_H - \underset{\sim}{v}_i|$, which is itself inversely proportional to $(n_i \gamma)$: the smaller the coupling, the greater the relative motion of plasma and neutral gas, the greater the neutral hydrogen momentum randomized at each collision, and so the more random kinetic energy produced per second, even though the collision rate has decreased. However, should the macroscopic frequency be much greater than the collision frequency, then the plasma equilibrium equation (11.18) no longer holds. The plasma-plus-field is almost completely unaffected by the neutrals, and the plasma motion is essentially a hydromagnetic wave. However, for reasonable plasma densities, one can expect (11.18) to hold.

The ratio of the Ohmic dissipation of transverse currents to this new form of dissipation is

$$\frac{j_1^{\,2}}{\sigma} \Big/ \frac{\dfrac{j_1^{\,2}}{c^2 n_i \gamma}}{H^2} \qquad (11.22)$$

On substituting $\sigma = \dfrac{n_e e^2 \tau_{ei}}{m_e}$, (11.22) reduces to

$$(AZ)\,(\omega_e \tau_{ei})\,(\omega_i \tau_{iH}) \;, \qquad (11.23)$$

where τ_{iH} is the $(i - H)$ collision time. With $\sigma_{iH} \ll \sigma_{ie}$, we can be sure that in an interstellar cloud with even a moderate field H, the ratio (11.22) is much greater than unity, so that this new form of dissipation is much greater than the Ohmic.

The second term on the right of (11.20) can be written $(-n_i \underset{\sim}{F}_{iH}) \cdot \underset{\sim}{v}_H$; it is the rate of working of the $(i - H)$ friction on the neutral gas, and represents the indirect effect of the magnetic force in reducing the kinetic energy of the collapsing neutral gas.

Let us consider again our contracting cloud, and suppose that $|\underset{\sim}{v}_i - \underset{\sim}{v}_H| \ll |\underset{\sim}{v}_H|$ --the plasma, and hence the field, moves closely with the neutral gas. Then by (11.21), the frictional dissipation, though much larger than the Ohmic dissipation, is small compared with the (negative) work done on the neutral gas by the magnetic body force: $\underset{\sim}{j} \cdot \underset{\sim}{E}$ is negative, and some of the gravitational energy released becomes magnetic energy, the rest becoming mainly kinetic energy of mass motion. When $\underset{\sim}{v}_i = 0$, $\underset{\sim}{j} \cdot \underset{\sim}{E} = 0$, and the field energy is steady; the field lines are stationary, while the neutral gas drifts across. The gravitational energy that would have become magnetic energy in the frozen-in case is all dissipated. Finally, if $|\underset{\sim}{v}_H - \underset{\sim}{v}_i| \gg |\underset{\sim}{v}_H|$ and the plasma-plus-field moves back through the cloud, the magnetic energy decreases, part is used up in decelerating the cloud, but most of it goes into heat.