The area, centroid and volume of revolution of the Koch curve

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Abstract

An elementary method to calculate the area, centroid and volume of rotation of the Koch curve is presented. Classroom extensions are provided to allow students to investigate the method used.

The Koch curve is a standard example of a fractal whose construction is illustrated in Figure 1. Starting with an initiating straight line of unit length we remove the middle third and replace it with two lines to form the upper part of an equilateral triangle. This forms the generating motif for the fractal, or the k=1 prefractal. To produce the k=2 prefractal we replace every segment of the k=1 prefractal with a scaled down (by 1/3) version of the motif. To produce the k=3 prefractal every line segment of the k=2 prefractal is replaced with a suitably scaled (by 1/9) version of the motif, and repeating this process of iteration an infinite number of times produces the Koch curve. The key feature of the Koch curve is that each of its four components is an exact scaled copy of the entire fractal.

If we define the length of the kth prefractal be L_k then it is clear that at each iteration the length increases by a factor of 4/3 and since the initiating straight line is of unit length we have that

$$L_k = \left(\frac{4}{3}\right)^k \tag{1}$$

and hence the length of the curve diverges.

The area bounded by the Koch curve and the initiating straight line is typically calculated as follows[1 2]: We define the area bounded by the kth prefractal and the initiator as A_k. Given the area bounded by the initiator and generator is the area of the equilateral triangle of side 1/3

$$A_1 = \frac{\sqrt{3}}{36}$$

Then the area bounded by the initiator and the k=2 prefractal is this area plus the area of four triangles reduced in length scale by a factor of 1/3 (and hence in area by a factor of $(1/3)^2$) ie

$$A_2 = \frac{\sqrt{3}}{36}\left(1+\frac{4}{9}\right).$$

At the next iteration we add 16 triangles each reduced in length scale by a factor of 1/9 (and hence in area by a factor of $(1/9)^2$ giving a total area of,

$$A_2 = \frac{\sqrt{3}}{36}\left(1+\frac{4}{9}+\frac{4^2}{9^2}\right).$$

Repetition of this process leads to

$$A_k = \frac{\sqrt{3}}{36}\sum_{i=0}^{k}\left(\frac{4}{9}\right)^i \qquad (2)$$

And hence summing the geometric series to infinity gives the area bounded by the Koch curve and its initiator as

$$A_\infty = \frac{\sqrt{3}}{20} \ . \qquad (3)$$

An alternative way to calculate this area is as follows. From Figure 2

$$A_\infty = A_T + 4A_S \qquad (4)$$

By definition each of the areas A_S is a scaled version of the entire bounded area. Since length is scaled by a factor of 1/3 – the areas are scaled by a factor of $\left(1/3\right)^2$. Thus, given

$$A_S = \frac{1}{9}A_\infty \qquad (5)$$

and the area of the equilateral triangle is

$$A_T = \frac{\sqrt{3}}{36} \qquad (6)$$

the result (3) immediately follows.

This method not only explicitly utilises the self-similarity of the curve, but it also naturally suggests how to also calculate its centre of area. If the centre of area of the Koch curve lies a distance *h* above the initiator, then following Figure 3, we can think of the centre of area being made up of the centre of area of the initiating triangle plus the four scaled versions of the Koch curve. Each of these scaled versions has area A_S and its centre of area is a perpendicular distance $h/3$ from its corresponding initiating line. Given the centre of area of the central equilateral triangle lies 1/3 of its perpendicular height from the base, ie a distance $\frac{\sqrt{3}}{18}$ from the initiating line, we have that

$$hA_\infty = \frac{\sqrt{3}}{18}A_T + 2\left(\frac{h}{3}A_S\right) + 2\left(\left(\frac{\sqrt{3}}{12} + \frac{h}{6}\right)A_S\right). \qquad (7)$$

Using (3),(5) and (6) gives,

$$h = \frac{\sqrt{3}}{18} \tag{8}$$

Given the second theorem of Pappas, which states that the volume of revolution of lamina about an axis is the area of the lamina multiplied by the distance travelled by the lamina’s centroid we can immediately write that that volume of revolution of the Koch curve about its initiator is

$$2\pi h A_{\infty} = \frac{\pi}{60}. \tag{9}$$

The distance of the centroid of the right half of the Koch curve from the axis of vertical symmetry can be shown to be

$$x = \frac{7}{54}. \tag{10}$$

Thus the volume of revolution about the vertical axis of symmetry is

$$2\pi x \frac{A_{\infty}}{2} = \frac{7\sqrt{3}}{1080}\pi. \tag{11}$$

Classroom Exercises

1. The Koch snowflake, or Koch island is formed by joining three Koch curves together around an equilateral triangle of unit side. Calculate the snowflake’s area and show that its volume of revolution about the axis of symmetry of the unit equilateral triangle is $\frac{11\sqrt{3}}{135}\pi$. What is the volume of revolution about the snowflakes other axis of

symmetry

2. A fractal *S* is formed from an initiating line of unit line. The generator is formed by dividing the line into five segments of equal length, removing the third segment and replacing it with the upper three edges of a square. To create the k=2 prefractal each of the seven line segments is replaced by a version of the generator as shown in Figure 4. Repeating this process an infinite number of times produces the fractal. Show that the length of the fractal is divergent. What is the area bounded by *S* and its initiator? Calculate the centre of area and the volume of rotation of *S* about the initiator.

Figure Captions

Figure 1 The initiator, generator, k=2 and k=3 prefractals of the Koch curve. The Koch curve is the result of an infinite number of iterations.

Figure 2. The Koch curve divided into component self-similar areas.

Figure 3 The centre of area of the Koch curve lies a distance *h* above the initiating line along the axis of symmetry.

Figure 4. The construction of the fractal *S* described in question 2 of the Classroom Exercises

Figure 1

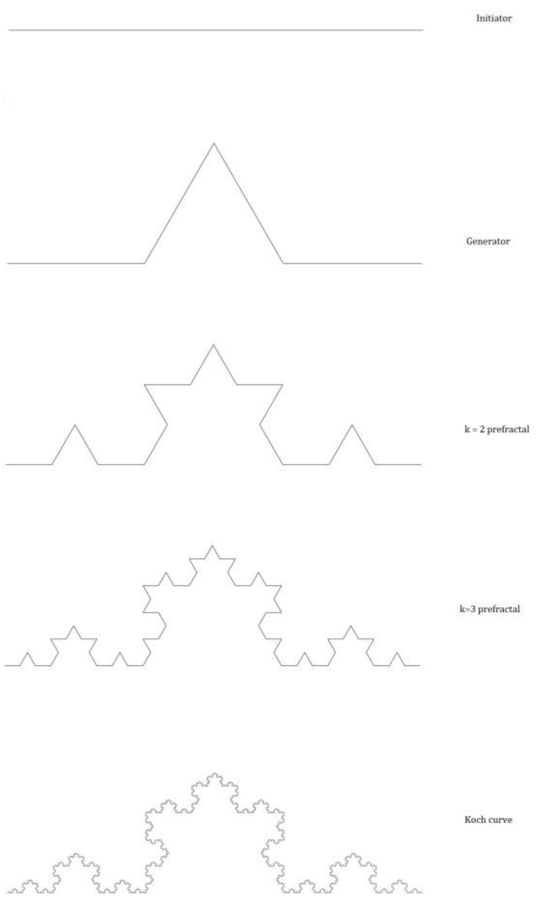


Figure 2

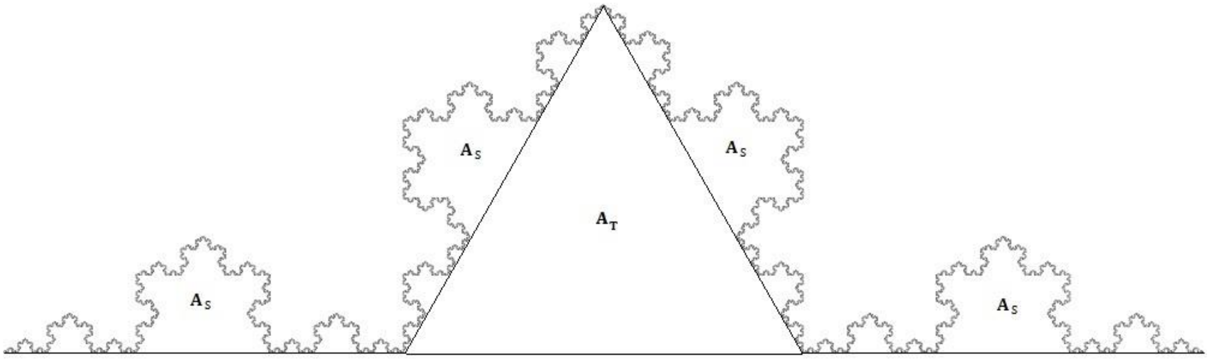


Figure 3

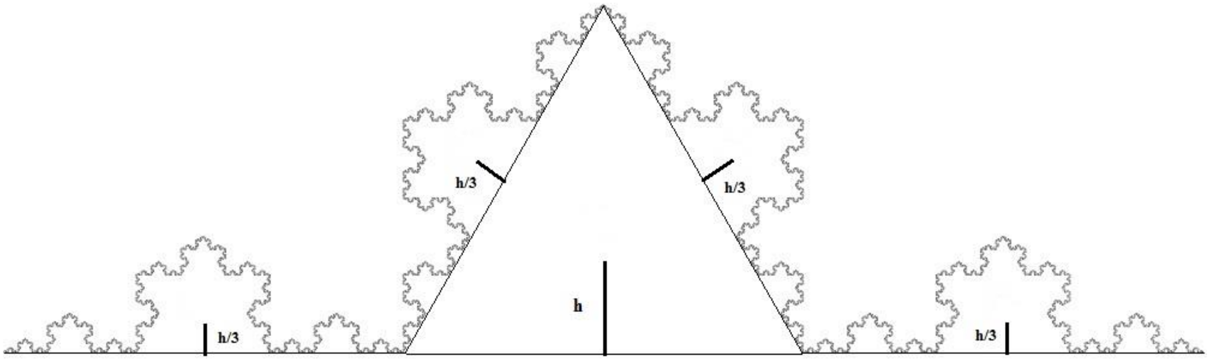
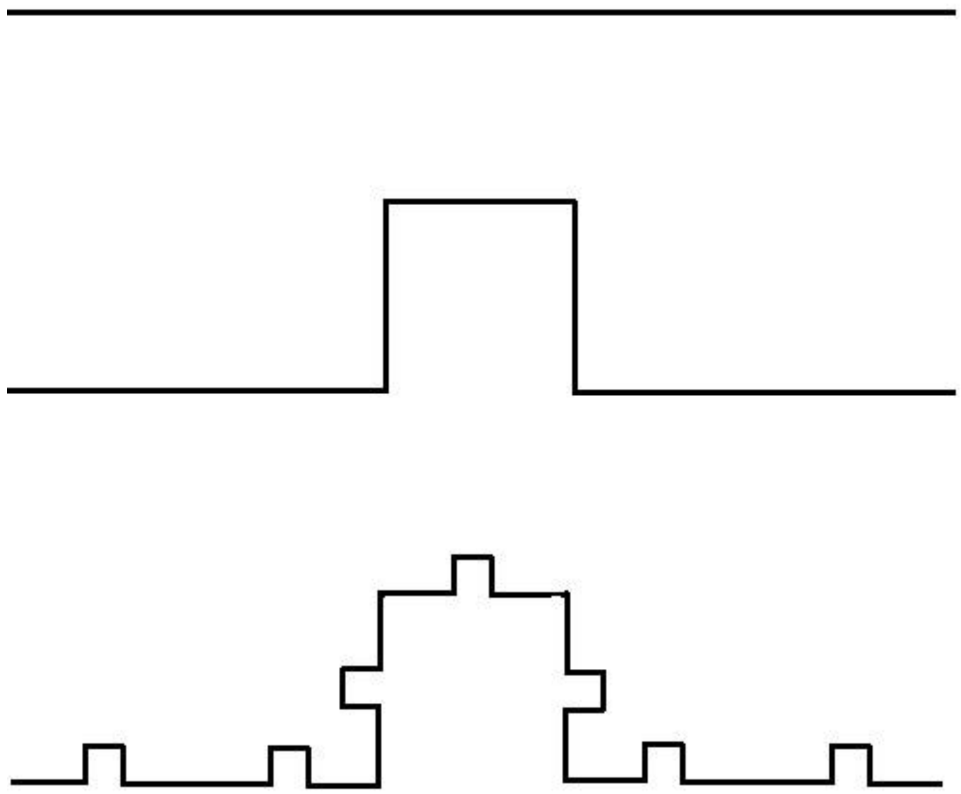


Figure 4



References

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[2] P.S. Addison, Fractals and Chaos: An Illustrated Course, Bristol: IOP, 1997, p.16-21.