Violations of locality and free choice are equivalent resources in Bell experiments

Pawel Blasiak[a,b,1], **Emmanuel M. Pothos**[b], **James M. Yearsley**[b], **Christoph Gallus**[c], **and Ewa Borsuk**[a]

[a]Institute of Nuclear Physics Polish Academy of Sciences, PL-31342 Krakow, Poland; [b]City, University of London, London EC1V 0HB, United Kingdom; [c]Technische Hochschule Mittelhessen, D-35390 Gießen, Germany

This manuscript was compiled on March 16, 2021

Bell inequalities rest on three fundamental assumptions: *realism*, *locality* and *free choice*, which lead to nontrivial constraints on correlations in very simple experiments. If we retain realism, then violation of the inequalities implies that at least one of the remaining two assumptions must fail, which can have a profound consequences for the causal explanation of the experiment. In this paper, we investigate the extent to which a given assumption needs to be relaxed for the other to hold at all costs. Our discussion is based on the observation that an assumption violation need not occur on every experimental trial, even when describing some correlations violating Bell inequalities. How often this needs to be the case determines the degree of, respectively, *locality* or *free choice* in the observed experimental behaviour. Despite their disparate character and interpretation, we show that both assumptions are *equally costly*. Namely, the resources required to explain the experimental statistics (measured by the frequency of causal interventions of either sort) are exactly the same. Furthermore, we compute such defined measures of locality and free choice for any *non-signalling* statistics in a Bell experiment with binary settings, showing that it is directly related to the amount of violation of the so called Clauser-Horne-Shimony-Holt inequalities. This result is theory-independent as it refers directly to the experimental statistics (with quantum predictions being just one example). Additionally, we show how the local fraction results for quantum-mechanical frameworks with infinite number of settings translate into analogous statements for the measure of free choice introduced in the present paper. Such a parallel treatment of both assumptions demonstrates that, as far as the statistics is concerned, causal explanations resorting either to violation of locality or free choice are *fully* interchangeable.

locality | free choice | causal models | non-local correlations | Bell inequalities | measure of locality | measure of free choice

"I would rather discover one true cause
than gain the kingdom of Persia."

– Democritus (c. 460-370 BC)

The study of experimental correlations provides a window into the underlying causal mechanisms, even when their exact nature remains obscured. In his seminal works (1–5), John Bell showed that seemingly innocuous assumptions about the structure of causal relationships leave a mark on the observed statistics. The first assumption, called *realism* (or counterfactual definiteness), presents the worldview in which physical objects and their properties exist, whether they are observed or not. Note that realism allows a standard notion of *causality* (6, 7), which in turn provides us with the language to express the remaining two assumptions. The *locality* assumption is a statement that physical (or causal) influences propagate in accord with the spatio-temporal structure of events (i.e., neither backward in time nor instantaneous causation). The *free choice* assumption asserts that the choice of measurement settings can be made independently from anything in the (causal) past. These three assumptions are enough to derive testable constraints on correlations called Bell inequalities.

Surprisingly, nature violates Bell inequalities (8–15) which means that if the standard causal (or *realist*) picture is to be maintained at least one of the remaining two assumptions, that is *locality* or *free choice*, has to fail. It turns out that rejecting just one of those two assumptions is always enough to explain the observed correlations, while maintaining consistency with the causal structure imposed by the other. Either option poses a challenge to deep-rooted intuitions about reality, with a full range of viable positions open to serious philosophical dispute (16–18). Notably, quantum theory in its operational formulation does not provide any clue regarding the causal structure at work, leaving such questions to the domain of interpretation. It is therefore interesting to ask about the extent to which a given assumption needs to be relaxed, if we insist on upholding the other one (while always maintaining *realism*). In this paper, we seek to compare the cost of *locality* and *free choice* on an equal footing, without any preconceived conceptual biases. As a basis for comparison we choose to measure the weight of a given assumption in terms of the following question:

> *How often a given assumption, i.e. locality or free choice, can be retained, while safeguarding the other assumption, in order to fully reproduce some given experimental statistics within a standard causal (or realist) approach?* (42)

This question presumes that a Bell experiment is performed trial-by-trial and the observed statistics can be explained in

Significance Statement

Faced with a violation of Bell inequalities, a committed realist might pursue an explanation of the observed correlations on the basis of violations of the *locality* or *free choice* (sometimes called *measurement independence*) assumptions. The question of whether it is better to abandon (partially or completely) locality or free choice has been strongly debated since the inception of Bell inequalities, with committed supporters on either side. For the first time, we offer a comprehensive treatment that allows a comparison of both assumptions on an equal footing. This both advances the foundational debate and provides quantitative answers regarding the weight of each assumption for causal (or realist) explanation of observed correlations.

PB developed the mathematical ideas and initial versions of the manuscript. EMP, JMY, CG, and EB contributed equally in refining and polishing this work.

The authors declare no competing interests.

[1]To whom correspondence should be addressed. E-mail: pawel.blasiak@ifj.edu.pl

the standard causal model (or hidden variable) framework (1–7, 19–21), which subsumes *realism*. It means that the remaining two assumptions of *locality* and *free choice* translate into conditional independence between certain variables in the model, whose causal structure is determined by their spatio-temporal relations (6, 22) (for some alternative approach endorsing indefinite causal structures see e.g. (23, 24), or (25) for discussion of retrocausality). Modelling of the experiment implies that in each run of the experiment all variables (including unobserved or hidden ones) always take definite values and the statistics accumulates over many trials. This leaves open the possibility that the violation of the assumptions do not have to occur on each run of the experiment to explain the given statistics. We can thus put flesh on the bones of the above question and seek the maximal proportion of trials in which a given assumption can be retained, while safeguarding the other assumption, so as to fully reproduce some given statistics. In the following, we shall denote so defined *measure of locality* (safeguarding freedom of choice) as μ_L and *measure of free choice* (safeguarding locality) as μ_F. Also, without stating this in every instance, we note that in all subsequent discussion *realism* is assumed.*

There has been some previous research on this theme. A measure of locality analogous to μ_L was first proposed by Elitzur, Popescu and Rohrlich (27) to quantify non-locality in a singlet state. Note, it seems that the original idea of a bound for such a locality measure was expressed earlier, in (28), but a bound was not worked out. In any case, Elitzur, Popescu, and Rohrlich's measure was dubbed *local fraction* (or *content*) and shown (with improvements in (29–31)) to vanish in the limit of an infinite number of measurement settings. A substantial step was made in (32) where the local fraction is explicitly calculated for any pure two-qubit state for an arbitrary choice of settings. We note that those results concern measure μ_L only for the specific case of quantum-mechanical predictions. In this paper we go beyond this framework and consider the case of general experimental statistics (see (33) for extension to the idea of contextuality). To avoid confusion, the term *local fraction* for measure μ_L will be only used in relation to the quantum case. Furthermore, we propose a similar treatment of the free choice assumption quantified by measure μ_F. Natural as it may seem, this approach has not been pursued in the literature, with some other measures proposed to this effect (34–42) (all retaining locality as a principle, but departing from the original notion of free choice introduced by Bell (6, 22)).

We aim to comprehensively consider the extent to which a given assumption, i.e. *locality* or *free choice*, can be preserved through partial violation of the other assumption. To accomplish this, we provide similar definitions and discuss on an equal footing both measures of locality μ_L and free choice μ_F. Then, we derive the following results. First, we prove a general structural theorem about causal models explaining any given experimental statistics in a Bell experiment (for any number of settings) showing that such defined measures are necessarily equal, $\mu_L = \mu_F$. This result consolidates those two disparate concepts demonstrating their deep interchangeability. Second, we explicitly compute both measures for any *non-signalling* statistics in a two-setting and two-outcome Bell scenario. This enables a direct interpretation to the amount of violation of the Clauser-Horne-Shimony-Holt (CHSH) inequalities (43). Third, we consider the special case of the quantum statistics with infinite number of settings, utilising existing results for the local fraction μ_L, which thus translate on the newly developed concept of the measure of free choice μ_F. Fig. 1 summarises the results in the paper.

*As noted, realism is subsumed in the standard notion of causality, which is implicit in the definition of locality and free choice (1–6). So, henceforth, referring to the standard causal framework implies the realist approach. We also remark that, although, "realism" goes under different guises in the literature (e.g. "counterfactual definiteness", '"local causality", "hidden causes", etc.), for our purposes those distinctions are irrelevant and the underlying mathematics remains the same, i.e. it boils down to the hidden variable framework (which beyond physics is frequently referred to as the structural causal models (7)). See (3, 26) for some discussion.

		Locality	*Free choice*	
Any statistics	*any no. settings*	$\mu_L = \mu_F$		**Theorem 1**
Any statistics	*non-signalling two settings*	$\mu_L = \frac{1}{2}(4 - S_{max})$	$\mu_F = \frac{1}{2}(4 - S_{max})$	**Theorem 2**
Quantum statistics	*Bell state infinite no. settings*	$\mu_L \xrightarrow[M\to\infty]{} 0$ (*)	$\mu_F \xrightarrow[M\to\infty]{} 0$	**Theorem 3**
Quantum statistics	*two-qubit state any no. settings*	$\mu_L = \cos\theta$ (*)	$\mu_F = \cos\theta$	**Theorem 4**

Fig. 1. Summary of the results. The main **Theorem 1** is the backbone of the paper, consolidating both measures of locality μ_L and free choice μ_F. **Theorem 2** is a theory-independent result about both measures μ_L and μ_F. It offers a concrete interpretation for the amount of violation of the CHSH inequalities. **Theorems 3** and **4** are specific to the quantum-mechanical statistics stated here for measure μ_F. They are translations of some remarkable local fraction results μ_L in the literature (marked with an $^{(*)}$, cf. (29–32)).

Results

Bell experiment and Fine's theorem. Let us consider the usual Bell-type scenario with two parties, called Alice and Bob, playing the roles of agents conducting experiments on two separated systems (whose nature is irrelevant for the argument). We assume that on each side there are two possible outcomes labelled respectively $a, b = \pm 1$ and M possible measurement settings labelled respectively $x, y \in \mathfrak{M}$ where $\mathfrak{M} \equiv \{1, 2, \ldots, M\}$. A Bell experiment consists of a series of trials in which Alice and Bob each choose a setting and make a measurement registering the outcome. After many repetitions, they compare their results described by the set of $M \times M$ distributions $\{P_{ab|xy}\}_{xy}$, where $P_{ab|xy}$ denotes the probability of obtaining outcomes a, b, given measurements x, y were made on Alice and Bob's side respectively. For conciseness, following the terminology in (5), we will call $\{P_{ab|xy}\}_{xy}$ a *"behaviour"*. Note that without assuming anything about the causal structure underlying the experiment any behaviour is admissible (as long as the distributions are normalised, i.e. $\sum_{a,b} P_{ab|xy} = 1$ for each $x, y \in \mathfrak{M}$). In particular, quantum theory gives a prescription for calculating the experimental statistics $P_{ab|xy}$ for each choice of settings $x, y \in \mathfrak{M}$ based on the formalism of Hilbert spaces.

It is instructive to recall the special case of two measurement settings on each side $x, y \in \mathfrak{M} = \{0, 1\}$ for which Bell derived his seminal result. Briefly, this can be expressed by saying that any *local* hidden variable model with *free choice* has to satisfy the following four CHSH inequalities (43)

$$|S_i| \leqslant 2 \qquad \text{for } i = 1, \ldots, 4, \qquad [1]$$

where

$$S_1 = \langle ab\rangle_{00} + \langle ab\rangle_{01} + \langle ab\rangle_{10} - \langle ab\rangle_{11}\,, \qquad [2]$$

$$S_2 = \langle ab\rangle_{00} + \langle ab\rangle_{01} - \langle ab\rangle_{10} + \langle ab\rangle_{11}\,, \qquad [3]$$

$$S_3 = \langle ab\rangle_{00} - \langle ab\rangle_{01} + \langle ab\rangle_{10} + \langle ab\rangle_{11}\,, \qquad [4]$$

$$S_4 = -\langle ab\rangle_{00} + \langle ab\rangle_{01} + \langle ab\rangle_{10} + \langle ab\rangle_{11}\,, \qquad [5]$$

with $\langle ab\rangle_{xy} = \sum_{a,b} ab\, P_{ab|xy}$ being correlation coefficients for a given choice of settings x, y. Interestingly, by virtue of Fine's theorem (44, 45), this is also a sufficient condition for a *non-signalling* behaviour $\{P_{ab|xy}\}_{xy}$ to be explained by a *local* hidden variable model with *freedom of choice* (for non-signalling see Eqs. (16) and (17)).

It is crucial to observe that, although locality and freedom of choice are two disparate concepts with different ramifications for our understanding of the experiment, they are in a certain sense interchangeable. If locality is dropped with Alice and Bob freely choosing their settings, then the boxes, by influencing one another, can produce any behaviour $\{P_{ab|xy}\}_{xy}$. Similarly, a violation of the free choice assumption can be used to reproduce any behaviour $\{P_{ab|xy}\}_{xy}$, without giving up locality. It is straightforward to see how this might work if one of the two assumptions fails on *every* experimental trial.[†]

However, such a complete renouncement of assumptions so central to our view of nature may seem excessive, especially when the CHSH inequalities are violated only by a little amount (less than the maximal algebraic bound of $|S_i| \leqslant 4$), leaving room for a possible explanation of the experimental statistics by rejecting one of the assumptions *sometimes* only. Here we assess the cost of such a partial violation by asking how often a given assumption can be retained in order to account for a behaviour $\{P_{ab|xy}\}_{xy}$. We will investigate both cases in parallel: (♠) full freedom of choice with *occasional* non-locality (communication), and (♣) the possibility of retaining full locality at a price of compromising freedom of choice (by controlling or rigging measurement settings) on *some* of the trials. We shall use the least frequency of violation, required to model some statistics with a hypothetical simulation, as a natural figure of merit, guided by the principle that the less the violation the better. Notably, such simulations should not restrict possible distributions of measurement settings P_{xy}. In other words, we define a *measure of locality* μ_L as

> *the <u>maximal</u> fraction of trials in which Alice and Bob do not need to communicate trying to simulate a given behaviour $\{P_{ab|xy}\}_{xy}$, optimised over <u>all</u> conceivable strategies with freely chosen settings.* [♠]

Similarly, we define a *measure of free choice* μ_F as

> *the <u>maximal</u> fraction of trials in which Alice and Bob can grant free choice of settings in trying to simulate a given behaviour $\{P_{ab|xy}\}_{xy}$, optimised over <u>all</u> conceivable local strategies.* [♣]

In the quantum-mechanical context the measure μ_L is called a *local fraction* (27–32). By analogy, when considering the quantum-mechanical statistics the measure μ_F might be called a *free fraction*. This provides an equal basis for comparing the two assumptions within the standard causal (or realist) approach, which we formalise in the following section.

Causal models, locality and free choice. The appropriate framework for the discussion of locality and free choice is provided by hidden variable models (1–5). First, a hidden variables model allows a formal statement of the *realism* assumption, understood to mean that properties of a physical system exist irrespective of an act of measurement (counterfactual definiteness). Second, hidden variable models provide the causal language in which the locality and free choice assumptions are expressed (6, 7). The *locality* assumption conveys the requirement that the propagation of physical (or causal) influences have to follow the spatio-temporal structure of events (i.e., preserve the arrow of time and respect that actions at a distance require time). The *free choice* assumption concerns the choice of measurement settings which are deemed cusally unaffected by anything in the past (and thus it is sometimes called *measurement independence*).[‡] Both assumptions take the form of conditional independencies between certain variables in a hidden variables model.

To make this idea more concrete, let us consider a given set of probability distributions (behaviour) $\{P_{ab|xy}\}_{xy}$ which describes the statistics in a Bell experiment. Without loss of generality, by conditioning on λ in some *a priori* unknown hidden variable space Λ, one can always write (4, 5, 7)

$$P_{ab|xy} = \sum_{\lambda\in\Lambda} P_{ab|xy\lambda} \cdot P_{\lambda|xy}\,, \qquad [6]$$

where $P_{\lambda|xy}$ and $P_{ab|xy\lambda}$ are valid (i.e. normalised) conditional probability distributions. The role of the hidden variable (cause in the past) $\lambda \in \Lambda$, distributed according to some P_λ, is to provide an explanation of the observed experimental statistics. This means that at each run of the experiment the outcomes are described by the distribution $P_{ab|xy\lambda}$ with $\lambda \in \Lambda$ fixed in a given trial, so that the accumulated experimental statistics $P_{ab|xy}$ obtains by sampling from some distribution $P_{\lambda|xy}$ over the whole hidden variable space Λ. It is customary to say that

> *the choice of space Λ and probability distribution P_λ along with conditional distributions $P_{ab|xy\lambda}$ and $P_{\lambda|xy}$ satisfying Eq. (6) specify a <u>hidden variable</u> (HV) model of a given behaviour $\{P_{ab|xy}\}_{xy}$.* [⋆]

Note that such a model implicitly describes the distribution of settings chosen by Alice and Bob through the standard formula

$$P_{xy} = \sum_{\lambda\in\Lambda} P_{xy|\lambda} \cdot P_\lambda\,. \qquad [7]$$

So far the framework is general enough to accommodate *any* causal explanation of the statistics observed in the experiment. The assumptions of locality and free choice take the form of constraints on conditional distributions in (⋆). For a *local hidden variable* (LHV) model, we require the following factorisation[§]

$$P_{ab|xy\lambda} = P_{a|x\lambda} \cdot P_{b|y\lambda}\,, \qquad [8]$$

[†] For the simulation of a given behaviour $\{P_{ab|xy}\}_{xy}$ in a Bell experiment one may proceed as follows. Upon rejection of locality, in *each* trial the system on Alice's side, one may not only use input x but also y to generate outcomes (and similarly for the box on Bob's side) that comply with the appropriate distribution. On the other hand, when freedom of choice is abandoned, both settings x, y may be specified in advance on *each* trial and the boxes can be instructed to provide the outcomes needed to simulate the appropriate distribution. It is however unclear how this might work with *occasional* violation of the respective assumptions.

[‡] As noted, the *free choice* assumption is sometimes called *measurement independence*. Instead of on the agent, measurement independence is focussed on the measurement devices and possible correlations between their settings, which can affect the observed statistics. Regardless of interpretation, the mathematics remains the same, with the source of correlations traced to some common factor (in the causal past).

[§] Locality can be seen as a conjunction of two conditions: *parameter independence* $P_{a|xy\lambda} = P_{a|x\lambda}$ & $P_{b|xy\lambda} = P_{b|y\lambda}$, and *outcome independence* $P_{a|bxy\lambda} = P_{a|xy\lambda}$ & $P_{b|axy\lambda} = P_{b|xy\lambda}$. One can show that such defined locality entails the factorisation condition $P_{ab|xy\lambda} = P_{a|x\lambda} \cdot P_{b|y\lambda}$ (46).

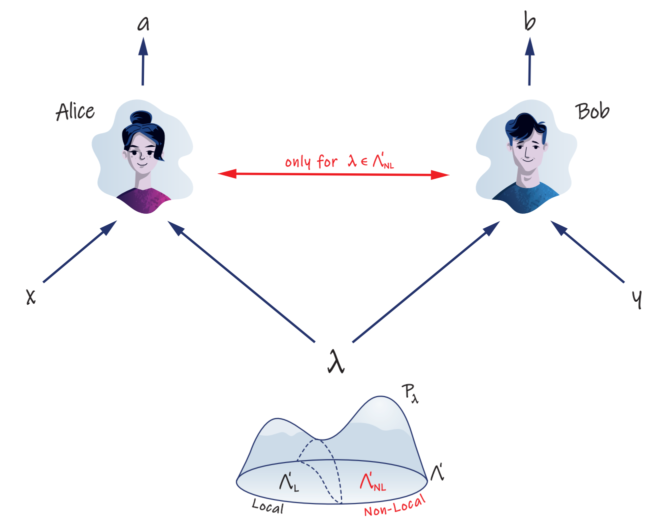

Fig. 2. Causal model with some non-locality (communication). In a Bell scenario, with free choice of settings, correlations between Alice and Bob's outcomes have two possible explanations: common cause in the past or causal influence between the parties. In any causal model the space of hidden variables (representing common causes) splits into two disjoint parts $\Lambda' = \Lambda'_L \cup \Lambda'_{NL}$ distinguished by whether, for a given $\lambda \in \Lambda'$, causal influence occurs or not, Eq. (10). Then, *locality* is measured by the proportion of events when locality is maintained, which is equal to the probability accumulated over subset Λ'_L, i.e. $Prob\,(\lambda \in \Lambda'_L) \equiv \sum_{\lambda\in\Lambda'_L} P_\lambda$.

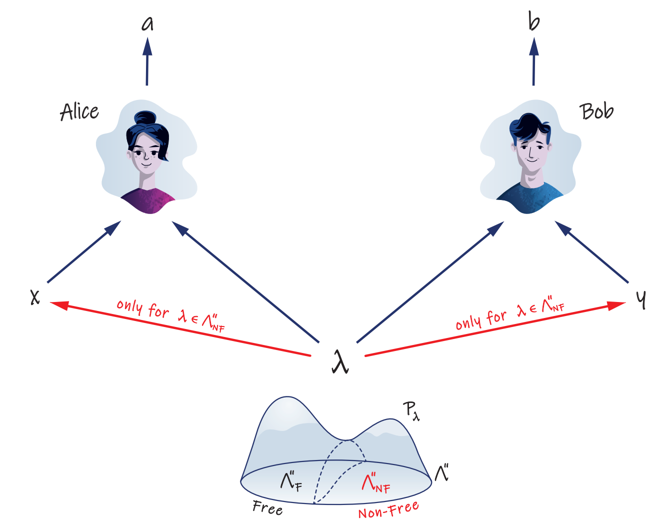

Fig. 3. Causal model with some freedom of choice (rigging). In a Bell scenario, with locality assumption, correlations between the outcomes on Alice and Bob's side can be explained by a common cause affecting choice or not (the latter implies freedom of choice). In any causal model the space of hidden variables (representing common causes) splits into two disjoint parts $\Lambda'' = \Lambda''_F \cup \Lambda''_{NF}$ distinguished by whether, for a given $\lambda \in \Lambda''$, the choice is free or not, Eqs. (11). Then, the parties enjoy *freedom of choice* only on the trials when $\lambda \in \Lambda_F$, which happens with a frequency equal to the probability accumulated over subset Λ''_F, i.e. $Prob\,(\lambda \in \Lambda''_F) \equiv \sum_{\lambda\in\Lambda''_F} P_\lambda$.

for each $x, y \in \mathfrak{M}$ and all $\lambda \in \Lambda$. The *freedom of choice* assumption consists of requiring that λ does not contain any information about variables x, y representing Alice and Bob's choice of measurement settings. This boils down to the independence condition (6, 22)

$$P_{\lambda|xy} = P_\lambda \qquad (\text{or equivalently } P_{xy|\lambda} = P_{xy})\,, \qquad [9]$$

holding for $x, y \in \mathfrak{M}$ and all $\lambda \in \Lambda$. In the following, we will abbreviate a *hidden variable* model with *freedom of choice* as FHV model.

The crucial point is the distinction between *local* vs *non-local* as well as *free* vs *non-free* situations in the individual runs of the experiment modelled by Eq. (6). This means that each condition Eq. (8) and Eq. (9) should be considered separately for each $\lambda \in \Lambda$, i.e. whenever the respective condition does not hold for a given λ the assumption fails on the corresponding experimental trials. Such a distinction leads to a natural splitting of the underlying HV space into two *unique* partitions $\Lambda = \Lambda_L \cup \Lambda_{NL}$ and $\Lambda = \Lambda_F \cup \Lambda_{NF}$. The first one divides Λ by the locality property

$$\begin{aligned} \lambda \in \Lambda_L \quad &\Leftrightarrow \quad \text{Eq. (8) holds for } all\ x, y \in \mathfrak{M}\,, \\ \lambda \in \Lambda_{NL} \quad &\Leftrightarrow \quad \text{Eq. (8) fails for } some\ x, y \in \mathfrak{M}\,, \end{aligned} \qquad [10]$$

while the second one divides Λ with by the free choice property

$$\begin{aligned} \lambda \in \Lambda_F \quad &\Leftrightarrow \quad \text{Eq. (9) holds for } all\ x, y \in \mathfrak{M}\,, \\ \lambda \in \Lambda_{NF} \quad &\Leftrightarrow \quad \text{Eq. (9) fails for } some\ x, y \in \mathfrak{M}\,. \end{aligned} \qquad [11]$$

Figs. 2 and 3 illustrate the causal structures for two extreme cases: FHV and LHV models (in general built on different HV spaces Λ' and Λ''). The first one grants full freedom of choice ($\Lambda' = \Lambda'_F$) while allowing for partial violation of locality ($\Lambda' \supset \Lambda'_L$). The second one retains full locality ($\Lambda'' = \Lambda''_L$) while admitting some violation of free choice ($\Lambda'' \supset \Lambda''_F$).

Thus, for a given experimental trial (with $\lambda \in \Lambda$ fixed) the constraints in Eqs. (10) and (11) indicate, respectively, whether some non-local influence between the parties takes place ($\lambda \in \Lambda_{NL}$) and whether some influence from the past on the measurement settings occurs ($\lambda \in \Lambda_{NF}$). In other words, in a hypothetical simulation scenario these possibilities correspond to, respectively, communication or rigging measurement settings. How often this has to happen depends on the distribution P_λ. This picture lends itself to quantifying the degree of locality and freedom choice in a given HV model.

Remark 1. *For a given HV model ($\star$) locality is measured by* $Prob\,(\lambda \in \Lambda_L) \equiv \sum_{\lambda\in\Lambda_L} P_\lambda$, *and similarly freedom of choice is measured by* $Prob\,(\lambda \in \Lambda_F) \equiv \sum_{\lambda\in\Lambda_F} P_\lambda$.

This remark captures the intuition of measuring locality and freedom of choice by considering the proportion of trials when the respective property is maintained across the whole experimental ensemble. We note that this quantity is model-dependent, since it is a property of a particular HV model adopted to explain some given experimental statistics $\{P_{ab|xy}\}_{xy}$ (including the distribution of measurement settings P_{xy}, cf. Eq. (7)).

The concepts just introduced allow a precise expression for the informal definitions (♠) and (♣) given above.

Definition 1. *For a given behaviour $\{P_{ab|xy}\}_{xy}$ the measure of locality μ_L and freedom of choice μ_F are defined as*

$$\mu_L := \min_{P_{xy}} \max_{FHV} \sum_{\lambda\in\Lambda_L} P_\lambda\,, \qquad [12]$$

$$\mu_F := \min_{P_{xy}} \max_{LHV} \sum_{\lambda\in\Lambda_F} P_\lambda\,, \qquad [13]$$

where the maxima are taken respectively over __all__ hidden variable models with freedom of choice (FHV) or __all__ local hidden variable models (LHV) simulating given behaviour $\{P_{ab|xy}\}_{xy}$, with a fixed distribution of settings P_{xy}, minimized over __any__ choice of the latter.

This definition follows the intuition of, respectively, locality or free choice as properties that can be relaxed only to the extent that is required to maintain the other assumption in every experimental situation (i.e., for any distribution of measurement settings P_{xy}). Formally, the measures μ_L and μ_F count the maximal frequency of, respectively, local or free choice events optimised over all protocols simulating $\{P_{ab|xy}\}_{xy}$ without violating of the other assumption, cf. **Remark 1**. The minimum over all P_{xy} amounts to the worst case scenario, which takes into account the possibility that P_{xy} is *a priori* unspecified (i.e., this amount of freedom is enough to simulate an experiment with any arbitrary choice of distribution P_{xy} in compliance with Eq. (7)).

At first glance, even if conceptually appropriate, such a definition might seem too general to provide a manageable notion, due to the range of experimental scenarios that need to be taken into account (i.e. arbitrariness of P_{xy}). However, the situation considerably simplifies because of the following lemma (see **Methods** section for further discussion and proof). This lemma also provides additional support for **Definition 1**.

Lemma 1. *In both Eqs. (12) and (13) in **Definition 1** the first minimum can be omitted, i.e. we have*

$$\mu_L = \max_{FHV} \sum_{\lambda \in \Lambda_L} P_\lambda \,, \qquad [14]$$

$$\mu_F = \max_{LHV} \sum_{\lambda \in \Lambda_F} P_\lambda \,, \qquad [15]$$

where the respective maxima are taken for some fixed nontrivial distribution P_{xy} (i.e., the expression is insensitive to this choice provided all settings are probed, $P_{xy} \neq 0$ for all x, y).

It is in this way that the present measure of locality μ_L extends the notion of *local fraction* (27–32) to arbitrary experimental behaviour $\{P_{ab|xy}\}_{xy}$. Remarkably, the twin concept, which is the measure of free choice μ_F has not been considered at all. Perhaps the reason for this omission is the issue of arbitrariness of the distribution P_{xy}, for which there are non-trivial constraints when freedom of choice is violated (note that for the measure μ_L this problem does not occur). Those concerns can be dismissed only after the proper treatment in **Lemma 1**. This allows a so defined measure of freedom μ_F on a par with the more familiar measure of locality μ_L.

So far the concepts of violation of locality and freedom of choice, and the corresponding measures μ_L and μ_F, have been kept separate. This is expected given their disparate character. First, each concept plays a different role in the description of an experiment and hence offers a different explanation for any observed correlations, this is, direct influence (communication during the experiment) vs measurement dependence (employing common past for rigging measurement settings). Second, on the level of causal modelling those assumptions are expressed differently, Eq. (8) vs Eq. (9). Third, violating free choice gives rise to subtle issues regarding constraints on the distribution of settings P_{xy} (as noted, these concerns are addressed in **Lemma 1**).

Having brought all those issues to the spotlight, it is surprising that the assumption of locality and free choice are intrinsically connected. We now present the key result in this paper showing the exchangeability of both concepts, while maintaining the same degree of locality and freedom of choice so defined. It holds for any number of settings $x, y \in \mathfrak{M} = \{0, 1, \ldots, M\}$ (see **Methods** for the proof).

Theorem 1. *For a given behaviour $\{P_{ab|xy}\}_{xy}$ the degree of locality and freedom of choice are the same, i.e. both measures in **Definition 1** coincide $\mu_L = \mu_F$.*

This is a general structural theorem about causal modelling of a given behaviour $\{P_{ab|xy}\}_{xy}$. It means that the resources measured by the frequency of causal interventions of either sort, required to explain an experimental statistics, are equally costly. Thus, as far as the statistics is concerned, causal explanations resorting either to violation of locality or free choice (or measurement dependence) should be kept on an equal footing. Preference should be guided by a better understanding of a particular situation (design of the experiment as well as ontological commitments in its description).

Let us emphasise two features of **Theorem 1**. First, this is a *theory-independent* result in the sense that it applies directly to experimental statistics irrespective of the design or theoretical framework behind the experiment (with the quantum predictions being just one example). Second, the connection between those two seemingly disparate quantities μ_L and μ_F has a practical advantage: knowledge of one suffices to compute the other. Both features are illustrated by the following results.

Non-signalling behaviour with binary settings. Consider the case of Bell's experiment with only two measurement settings on each side $x, y \in \mathfrak{M} = \{0, 1\}$. Let us recall that *non-signalling* of some given behaviour $\{P_{ab|xy}\}_{xy}$ means that Alice *cannot* infer Bob's measurement setting (whether it is $y = 0$ or 1) from the statistics on her side alone, i.e.

$$P_{a|x0} = \sum_b P_{ab|x0} = \sum_b P_{ab|x1} = P_{a|x1} \quad \text{for all } a, x\,, \qquad [16]$$

and similarly on Bob's side (whether Alice chooses $x = 0$ or 1), i.e.

$$P_{b|0y} = \sum_a P_{ab|0y} = \sum_a P_{ab|1y} = P_{b|1y} \quad \text{for all } b, y\,. \qquad [17]$$

Now we can state another result which explicitly computes both measures μ_L and μ_F in a surprisingly simple form (see **Methods** for the proof).

Theorem 2. *For a given non-signalling behaviour $\{P_{ab|xy}\}_{xy}$ with binary settings $x, y \in \mathfrak{M} = \{0, 1\}$ both measures of locality μ_L and free choice μ_F from **Definition 1** are equal to*

$$\mu_L = \mu_F = \begin{cases} \frac{1}{2}(4 - S_{max})\,, & \text{if } S_{max} > 2\,, \\ 1\,, & \text{otherwise}\,, \end{cases} \qquad [18]$$

where $S_{max} = \max\{|S_i| : i = 1, \ldots, 4\}$ is the maximum absolute value of the four CHSH expressions in Eqs. (2)-(5).

We thus obtain a systematic method for assessing the degree of locality and free choice directly from the observed statistics $\{P_{ab|xy}\}_{xy}$ without reference to the specifics of the experiment (the only requirement is non-signalling of the observed distributions). In this sense, this is a general *theory-independent* statement.

Overall, **Theorem 2** allows an interpretation of the amount of violation of the CHSH inequalities in Bell-type experiments as a fraction of trials violating locality (granted freedom of choice) or equivalently trials without freedom of choice (given locality).

The quantum case: Binary settings and beyond. Let us restrict our attention to the special case of the quantum statistics. Notably, various aspects of non-locality have been extensively researched in relation to the quantum-mechanical predictions, see (4, 5) for a review. This includes the notion of *local fraction* (27–32), which is the same as measure μ_L here defined for a general behaviour $\{P_{ab|xy}\}_{xy}$. As noted, it may be thus surprising that the equally natural measure of freedom μ_F has not been explored. **Theorem 1** bridges the gap between those two seemingly disparate notions: there is no actual need for separate study. We next review some crucial results for the *local fraction* in the quantum-mechanical framework, which allows us to make similar statements for the measure of freedom μ_F.

We first observe that **Theorem 2** can be readily applied to the quantum-mechanical statistics (where non-signalling holds). In a Bell experiment, quantum probabilities obtain through the standard formula $P_{ab|xy} = Tr\,[\,\rho\,\mathbb{P}_x^a \otimes \mathbb{P}_y^b\,]$ where ρ is a (bipartite) mixed state with two PVMs $\{\mathbb{P}_x^{a=\pm1}\}$ and $\{\mathbb{P}_y^{b=\pm1}\}$ representing Alice and Bob's choice of measurement settings $x, y \in \mathfrak{M} = \{0,1\}$. Calculating the CHSH expressions Eqs. (2)-(3) in each particular case is straightforward, which gives explicitly the expression for both measures μ_L and μ_F via Eq. (18). The result of special significance concerns the famous Tsirelson bound $S_{max}^{QM} = 2\sqrt{2}$ for the maximal violation of the CHSH inequalities in quantum mechanics (47). By virtue of **Theorem 2**, this means that in order to locally recover the quantum predictions in a Bell experiment with two settings, Alice and Bob can enjoy freedom of choice in the worst case, at most, with a fraction $\mu_F = 2 - \sqrt{2} \approx 0.59$ of all trials (corresponding to the choice of measurements on a maximally entangled state that saturate the Tsirelson bound). Clearly, the same applies to local fraction μ_L in a two-setting scenario.

Interestingly, relaxing the constraint on the number of settings for Alice and Bob's measurements $x, y \in \mathfrak{M} = \{1, 2, 3, \ldots, M\}$ the quantum statistics forces us to further constrain, respectively, locality or free choice. The case of local fraction μ_L with arbitrary number of settings $M \to \infty$ has been thoroughly investigated for statistics generated by quantum states. Let us refer to two interesting results in the literature on local fraction μ_L which readily translate via **Theorem 1** to the measure of freedom μ_F. The first one concerns the statistics of a maximally entangled state, cf. (27, 29) (see **SI Appendix** for a direct proof).

Theorem 3. *For every local hidden variable (LHV) model that explains the statistics of a Bell experiment for a maximally entangled state the amount of free choice tends to zero with increasing number of measurement settings M, i.e.* $\mu_F \xrightarrow[M\to\infty]{} 0$.

Apparently, for less entangled states the amount of freedom increases, reaching the maximal value $\mu_F = 1$ for separable states. This is a consequence of the result in (32), which explicitly computes the local fraction μ_L for all pure two-qubit states. Stated for measure μ_F this takes the following form.

Theorem 4. *For a pure two-qubit state, which by appropriate choice of the basis can always be written in the form* $|\psi\rangle = \cos\frac{\theta}{2}\,|00\rangle + \sin\frac{\theta}{2}\,|11\rangle$ *with* $\theta \in [0, \frac{\pi}{2}]$, *the amount of freedom is equal* $\mu_F = \cos\theta$, *whatever the choice and number of settings on Alice and Bob's side.*

Note that both **Theorem 3** and **Theorem 4** assume a specific form of behaviour $\{P_{ab|xy}\}_{xy}$ as obtained by the rules of quantum theory. The theorems should be contrasted with **Theorem 2** which is a *theory-independent* statement, not limited to a particular theoretical framework.

Discussion

The ingenuity of Bell's theorem lies in the fundamental nature of the premises from which the result is derived. Within the standard causal (or *realist*) approach, it is hard to assume less about two agents than having *free choice* and their systems being *localised* in space. Yet in some experiments nature refutes the possibility that both assumptions are concurrently true (8–15). It is not easy to reject either one of them without carefully rethinking the role of observers and how cause-and-effect manifests in the world.[¶] Our objective in this paper is this: *instead of pondering the question of how this could be possible, we ask about the extent to which a given assumption has to be relaxed in order to maintain the other*. Expressed more colloquially, it is natural for a realist to ask what is the cost of trading one concept for the other: *Is it possible to save free choice by giving up on some locality? Or, maybe is it better to forego a modicum of free choice in exchange for locality?* These questions can be compared on equal footing by computing a proportion of trials across the whole experimental ensemble in which a given assumption must fail, when the other holds at all times. Surprisingly, the answer can be obtained by looking at the observed statistics alone (avoiding the specifics of the experimental setup). The first question was formulated in the quantum-mechanical context by Elitzur, Popescu and Rohrlich (27) who introduced the notion of *local fraction* further elaborated in (29–32) (see (28) for an early indication of these ideas). Here, we generalise this notion to arbitrary experimental statistics (see also (33)). Furthermore, we answer the second question by adopting a similar approach to measuring the amount of free choice (which by analogy may be called *free fraction*). The first main result, **Theorem 1**, compares such defined measures in the general case (arbitrary statistics with any number of settings), showing that both assumptions are *equally costly*. This demonstrates a deeper symmetry between locality and free choice, which may come as a surprise, given our intuition of a profound difference in the role these concepts play in the description of an experiment.

In this paper, the notions of locality and free choice are understood in the usual sense required to derive Bell's theorem (6, 22). They are expressed in the standard causal model framework (which subsumes realism) as unambiguous yes-no criteria for each experimental trial (i.e. when all past variables are fixed), determining whether there is a causal link between certain variables in a model (without pondering its exact nature). The measures μ_L and μ_F count the fraction of trials when such a connection needs to be established, breaking locality or free choice respectively, in order to explain the observed statistics. This problem is prior to a discussion of how this actually occurs, which is particularly relevant when the exact nature of the phenomenon under study is obscured. **Theorem 1** shows no intrinsic reason for a realist to favour one assumption vs the other. The minimal frequency of the required causal influences of either sort, measured by μ_L and μ_F, is exactly the

[¶] We note that the conventional understanding of causality and the language of counterfactuals has recently gained a solid mathematical basis; see e.g. the work of J. Pearl (7). However, in view of the apparent difficulties with embedding quantum mechanics in that framework, the standard approach to causality based on Reichenbach's principle or claims regarding spatio-temporal structure of events might need reassessment; see e.g. indefinite causal structures (23, 24) or retrocausality (25).

same. Notably, this is a general result which holds for *any* behaviour $\{P_{ab|xy}\}_{xy}$. What remains is explicit calculation of those measures for a given experimental statistics.

The second main result, **Theorem 2**, evaluates both measures μ_L and μ_F for any *non-signalling* behaviour in a Bell experiment with two outcomes and two settings. It provides a direct interpretation to the amount of violation of the CHSH inequalities (43). The key motivation behind this result is that the degree by which the inequalities are violated has not been given tangible interpretation so far, beyond its use as a binary test of whether the inequalities are obeyed or not in study of Bell non-locality. Furthermore, **Theorem 2** has the advantage of being *theory-independent* in the sense of being applicable to the experimental statistics regardless of its theoretical origin (i.e., beyond the quantum-mechanical framework). This makes it suitable for quantitative assessment of the degree of locality and free choice across different experimental situations, with prospective applications beyond physics, e.g. in neuroscience, cognitive psychology, social sciences or finance (48–52).

We also state two results, **Theorem 3** and **Theorem 4**, for the measure of free choice μ_F in the case of the quantum statistics generated by the pure two-qubit states. Both are direct translation, via **Theorem 1**, of the corresponding results for the local fraction μ_L (27–32).

It is worth noting a related idea of quantifying non-locality through the amount of information transmitted between the parties that is required to reproduce quantum correlations (under free choice assumption). Together with the development of the specific models (53–57), this has led to various results regarding communication complexity in the quantum realm (58). However, in this paper we take a different perspective on measuring non-locality by changing the question from *"how much"* to *"how often"* communication needs to be established between the parties to simulate given correlations. **Theorem 2** gives the exact bound in the case of non-signalling statistics in the two-setting and two-outcome Bell experiment. In the quantum case, such a simulation requires communication in at least 41 % of trials (because of Tsirelson's bound (47)) and for maximally entangled states increases to 100 % of trials when the number of settings is arbitrary (cf. **Theorems 3** and **4**).

Natural as it may seem, the idea of measuring freedom of choice by measure μ_F has not been developed in the literature. The reason for this omission can be traced to the conceptual and technical issues with handling arbitrariness of the distribution of settings P_{xy}. Those concerns are properly addressed in the present paper with **Lemma 1**, which considerably simplifies and supports **Definition 1**. We note that various measures have been developed as a means of quantifying freedom of choice (or *measurement independence*, as it is sometimes called). They include maximal distance between distributions (35, 37), mutual information (38, 42) or measurement dependent locality (39–41). Furthermore, some explicit models simulating correlations in a singlet state with various degrees of measurement dependence have been proposed (34, 36) and analysed (e.g. see (42) for comparison of causal vs retrocausal models). However, these attempts depart from the original understanding of the free choice as introduced by Bell (6, 22) (strict independence of choice from anything in the past) in favour of more sophisticated information-theoretic accounts. Notably, the proposed measure of free choice builds on the Bell's original framework assessing the maximal frequency with which such a freedom *can* be retained in a model strictly consistent with locality. It thus benefits from a direct interpretation within the established causal framework of Bell inequalities and has a clear-cut operational meaning.

Regarding **Theorem 3**, which rules out *any* freedom of choice *so defined*, it is interesting to take an adversarial perspective on the problem of free choice in relation to quantum cryptography and device independent certification (59, 60). In this narrative an eavesdropper controls the devices trying to simulate the quantum statistics of a Bell test, which is impossible as long as the parties enjoy freedom of choice. However, any breach of the latter, i.e. control of measurement settings, shifts the balance in favour of the eavesdropper in her malicious task. Taking the view that any causal influence comes with a cost or danger of being uncovered there are two diverging strategies that reduce the cost/risk to be considered: *(a)* resort to the use of control of choice as seldom as possible during the experiment, or *(b)* minimise the intensity of each act of control. **Theorem 3** completely rules out the first possibility when simulating quantum statistics, i.e., the eavesdropper needs to manipulate both settings on each trial in order to simulate the quantum statistics. The question about the intensity of the control is left open in our discussion, but amenable to information-theoretic methods (35–42). This gives additional security criteria for quantum cryptography and device independent certification by forcing the eavesdropper to a more challenging sort of attack (not only can she not miss a trial, but the control has to be subtle enough).

We remark that the main **Theorem 1** readily extends to the case of larger number of parties and outcomes $\{P_{abc...|xyz...}\}_{xyz...}$. This should be also possible for **Theorem 2** when characterisation of the local polytope is known, cf. (61–67). Yet another valuable avenue for research in that case consists of completing the analysis to include signalling scenarios (68, 69). As for the quantum case, we considered the simplest Bell-type scenario with two parties involved in the experiment, but extensions may prove even more surprising (see (5) for a technical review of the vast field of Bell non-locality). In particular, in three-party scenarios the methods discussed presently can be used to eliminate freedom of choice already for two settings per party sharing the GHZ state (cf. Mermin inequalities which saturate in that case (70)). We should also mention an intriguing result (71) for a triangle quantum network in which non-locality can be proved with all measurements fixed. Remarkably, there is nothing to choose in that setup, but there is another assumption of preparation independence which plays a crucial role in the argument.

In this paper we are trying to remain impartial as to which assumption — *locality* or *free choice* — is more important on the fundamental level. This is certainly a strongly debated subject in general, both among physicists and philosophers, with strong supporters on each side (16–18). As just one example depreciating the role of freedom of choice let us quote Albert Einstein[||]: *"Human beings, in their thinking, feeling and acting are not free agents but are as causally bound as the stars in their motion."* As a counterbalance, it is hard to resist the objection that was eloquently stated by Nicolas Gisin (72): *"But for me, the situation is very clear: not only does free will exist, but it is a prerequisite for science, philosophy, and our very ability to think rationally in a meaningful way."* Without entering into this

[||] Statement to the Spinoza Society of America. September 22, 1932. AEA 33-291.

debate, we remark that both assumptions are interchangeable on a deeper level. Namely, for a given experimental statistics $\{P_{ab|xy}\}_{xy}$ in a Bell-type experiment the measure of locality μ_L and measure of free choice μ_F are exactly the same. This makes an even stronger case regarding the inherent impossibility of inferring causal structure from experimental statistics alone.

Materials and Methods

In order to facilitate the following discussion we begin with two technical lemmas. See **SI Appendix** for the proofs.

The first one holds for a Bell experiment with arbitrary number of settings $x, y \in \mathfrak{M} = \{1, 2, 3, \dots, M\}$.

Lemma 2. *For any behaviour $\{P_{ab|xy}\}_{xy}$ and distribution of settings P_{xy} there exists a local hidden variable model (LHV) which fully violates the freedom of choice assumption. [i.e. if $\tilde{\Lambda}$ is the relevant HV space, then we have $\tilde{\Lambda} = \tilde{\Lambda}_L = \tilde{\Lambda}_{NF}$, cf. Eqs. (10) and (11))].*

The second one concerns a Bell scenario with binary settings $x, y \in \mathfrak{M} = \{0, 1\}$.

Lemma 3. *Each non-signalling behaviour $\{P_{ab|xy}\}_{xy}$ with binary settings $x, y \in \mathfrak{M} = \{0, 1\}$ can be decomposed as a convex mixture of a local behaviour $\{\bar{P}_{ab|xy}\}_{xy}$ and a PR-box $\{\tilde{P}_{ab|xy}\}_{xy}$ in the form*

$$P_{ab|xy} \;=\; p \cdot \bar{P}_{ab|xy} + (1-p) \cdot \tilde{P}_{ab|xy}\,, \qquad [19]$$

with $p = \frac{1}{2}(4 - S_{max})$ for all $x, y \in \{0, 1\}$.

Recall that a PR-box (73) is a non-signalling behaviour for which one of the CHSH expressions in Eqs. (2)-(5) reaches the maximal algebraic bound of $|S_i| = 4$. Here, local behaviour means existence of a LHV+FHV model of $\{\bar{P}_{ab|xy}\}_{xy}$ and $S_{max} = \max\{|S_i| : i = 1, \dots, 4\}$.

We are now ready to proceed with the proofs.

Proof of Lemma 1. Suppose we have a HV model ($\star$) of some behaviour $\{P_{ab|xy}\}_{xy}$ for some nontrivial distribution of settings P_{xy}. The latter obtains via Eq. (7) from the conditional probabilities $P_{xy|\lambda}$ which are related to probabilities specified by the model, $P_{\lambda|xy}$ and P_λ, by the usual Bayes' rule. The point at issue is whether a given HV model can simulate any other distribution of settings $\tilde{P}_{xy}$ via Eq. (7) by changing $P_{xy|\lambda} \rightsquigarrow \tilde{P}_{xy|\lambda}$, while keeping the remaining components of the HV model ($\star$) intact. This requires consistency with Bayes' rule, i.e.

$$\tilde{P}_{xy|\lambda} \;=\; \frac{P_{\lambda|xy} \cdot \tilde{P}_{xy}}{P_\lambda}\,, \qquad [20]$$

which should be a well-defined probability distribution for each λ. Since distributions $P_{\lambda|xy}$ and P_λ are fixed by the HV model ($\star$), then the distribution of settings $\tilde{P}_{xy}$ is arbitrary as long as the expression in Eq. (20) is less then 1 for each $\lambda \in \Lambda$ (normalisation is trivially fulfilled). Now, whenever freedom of choice from Eq. (9) holds, this condition is always satisfied, and hence such a HV model can be trivially adjusted for any distribution $\tilde{P}_{xy}$ (by redefining $\tilde{P}_{xy|\lambda} := \tilde{P}_{xy}$ in compliance with Eq. (20), and keeping all the remaining components of the HV model ($\star$) unchanged). Of course, for FHV models in the definition of μ_L in Eq. (12) this is the case, which thus entails the simpler expression for μ_L in Eq. (14).

Clearly, such a simple argument falls apart for models without freedom of choice, like those in the definition of μ_F in Eq. (13), when $P_{\lambda|xy}$ and P_λ *do not* cancel out and the probability in Eq. (20) may be ill-defined. In that case, some deeper intervention into the model is required as shown below.

Let us take some LHV model ($\star$) simulating a given behaviour $\{P_{ab|xy}\}_{xy}$ with nontrivial distribution of settings P_{xy}. Then the related HV space decomposes as $\Lambda = \Lambda_F \uplus \Lambda_{NF}$ and the degree of freedom is measured by $p_F := \sum_{\lambda \in \Lambda_F} P_\lambda$, cf. **Remark 1**. Now, consider a restriction of the model to the respective subspaces Λ_F and Λ_{NF} which amounts to the following rescaling

$$P^F_\lambda \;:=\; \tfrac{1}{p_F}\, P_\lambda\,, \quad P^F_{\lambda|xy} \;:=\; \tfrac{1}{p_F}\, P_{\lambda|xy}\,, \quad P^F_{ab|xy\lambda} \;:=\; P_{ab|xy\lambda}\,, \qquad [21]$$

for $\lambda \in \Lambda_F$, and similarly

$$P^{NF}_\lambda \;:=\; \tfrac{1}{1-p_F}\, P_\lambda\,, \quad P^{NF}_{\lambda|xy} \;:=\; \tfrac{1}{1-p_F}\, P_{\lambda|xy}\,, \quad P^{NF}_{ab|xy\lambda} \;:=\; P_{ab|xy\lambda}\,, \qquad [22]$$

for $\lambda \in \Lambda_{NF}$. Both are LHV models with marginals

$$P^F_{ab|xy} \;=\; \sum_{\lambda \in \Lambda_F} P^F_{ab|xy\lambda} \cdot P^F_{\lambda|xy}\,, \qquad [23]$$

$$P^{NF}_{ab|xy} \;=\; \sum_{\lambda \in \Lambda_{NF}} P^{NF}_{ab|xy\lambda} \cdot P^{NF}_{\lambda|xy}\,, \qquad [24]$$

which provide a convex decomposition of the original behaviour $\{P_{ab|xy}\}_{xy}$, i.e.

$$P_{ab|xy} \;=\; p_F \cdot P^F_{ab|xy} + (1-p_F) \cdot P^{NF}_{ab|xy}\,. \qquad [25]$$

The crucial point is a careful adjustment of these two models to recover some arbitrary distribution of settings $\tilde{P}_{xy}$, while maintaining the respective marginals Eqs. (23) and (24). For the first one (restriction to Λ_F) the situation is trivial as explained above: since it is a FHV model, then it suffice to redefine $\tilde{P}^F_{xy|\lambda} := \tilde{P}_{xy}$ (in compliance with Eq. (20)) and leave all rest intact. As for the second one (restriction to Λ_{NF}), we can use **Lemma 2** for constructing another HV space $\tilde{\Lambda}_{NF}$ with a LHV model without any free choice, that simulates behaviour $\{P^{NF}_{ab|xy}\}_{xy}$ with the required distribution of settings $\tilde{P}_{xy}$. Then, such modified models can be stitched back together on the compound HV space $\tilde{\Lambda} := \Lambda_F \uplus \tilde{\Lambda}_{NF}$ with respective weights p_F and $1 - p_F$. This guarantees reconstruction of the original behaviour $\{P_{ab|xy}\}_{xy}$ (see Eq. (25)) with the new distribution of settings $\tilde{P}_{xy}$. The model is local and has the same degree of freedom equal to p_F (the first component has full freedom of choice, while in the second one it is entirely missing).

The above construction shows that for every LHV model of some behaviour $\{P_{ab|xy}\}_{xy}$ there is always another one adjusted for any other distribution of settings $\tilde{P}_{xy}$ with the same degree of freedom. This justifies the simpler expression for μ_F in Eq. (15) and hence concludes the proof of **Lemma 1**.

Proof of Theorem 1. Note that **Lemma 1** Eqs. (14) and (15) can be taken as a definition of measures μ_L and μ_F. This is very convenient, since it allows a discussion free from any concerns about the distribution of settings P_{xy} (this is particularly relevant in the case of μ_F as explained above).

It is instructive to observe that the calculation of both measures μ_L and μ_F can be succinctly formulated as a convex optimisation problem. Suppose, we can decompose some given behaviour $\{P_{ab|xy}\}_{xy}$ as a mixture

$$P_{ab|xy} \;=\; p_L \cdot P^L_{ab|xy} + (1-p_L) \cdot P^{NL}_{ab|xy}\,, \qquad [26]$$

where $\{P^L_{ab|xy}\}_{xy}$ is a local behaviour with full freedom of choice (i.e., has a LHV+FHV model), and $\{P^{NL}_{ab|xy}\}_{xy}$ is a free behaviour (i.e., has a FHV model). And similarly, suppose that

$$P_{ab|xy} \;=\; p_F \cdot P^F_{ab|xy} + (1-p_F) \cdot P^{NF}_{ab|xy} \qquad [27]$$

where $\{P^F_{ab|xy}\}_{xy}$ is a local behaviour with full freedom of choice (i.e., has a LHV+FHV model), and $\{P^{NF}_{ab|xy}\}_{xy}$ is a local behaviour (i.e., has a LHV model). In both cases we assume that $0 \leqslant p_L, p_F \leqslant 1$, and both Eq. (26) and Eq. (27) have to hold for all $a, b = \pm 1$ and $x, y \in \mathfrak{M}$. Then, we have

Remark 2. *Measures μ_L and μ_F evaluate the maxima over all possible decompositions in Eqs. (26) and (27) of behaviour $\{P_{ab|xy}\}_{xy}$, i.e.*

$$\mu_L \;=\; \max_{\text{decomp. (26)}} \; p_L\,, \qquad [28]$$

$$\mu_F \;=\; \max_{\text{decomp. (27)}} \; p_F\,. \qquad [29]$$

Proof. We will justify only Eq. (28), since the argument for Eq. (29) is analogous.

Let us observe that every HV model ($\star$) of behaviour $\{P_{ab|xy}\}_{xy}$ as described by Eq. (6) splits into two components (cf. Eq. (10))

$$P_{ab|xy} \;=\; \underbrace{\sum_{\lambda \in \Lambda_L} P_{ab|xy\lambda} \cdot P_\lambda}_{p_L \cdot P^L_{ab|xy}} + \underbrace{\sum_{\lambda \in \Lambda_{NL}} P_{ab|xy\lambda} \cdot P_{\lambda|xy}}_{(1-p_L) \cdot P^{NL}_{ab|xy}}\,, \qquad [30]$$

which defines decomposition of the type in Eq. (26) with $p_L := \sum_{\lambda \in \Lambda_L} P_\lambda$. Therefore, by Eq. (14), we get $\mu_L \leqslant \max_{\text{decomp. (26)}} p_L$.

To see the reverse, we note that every decomposition of the type in Eq. (26) implies existence of a LHV+FHV model of behaviour $\{P^{L}_{ab|xy}\}_{xy}$ on some HV space $\tilde{\Lambda}_L$ and a FHV model of behaviour $\{P^{NL}_{ab|xy}\}_{xy}$ on some HV space $\tilde{\Lambda}_{NL}$. Those two models, when combined on a compound HV space $\Lambda := \tilde{\Lambda}_L \uplus \tilde{\Lambda}_{NL}$ with the respective weights p_L and $1 - p_L$, provide a HV model of behaviour $\{P_{ab|xy}\}_{xy}$. Since the local domain of such a model contains $\tilde{\Lambda}_L$, then from Eq. (14) we have $\mu_L \geqslant p_L$, which entails $\mu_L \geqslant \max_{\text{decomp. (26)}} p_L$. This concludes the proof of Eq. (28). □

Now, in order to prove **Theorem 1** it is enough to show that for every decomposition of the type in Eq. (26) there exists a decomposition of the type in Eq. (27) with the same weight $p_L = p_F$, and vice versa. A closer look at both expressions reveals that behaviours $\{P^{L}_{ab|xy}\}_{xy}$ and $\{P^{F}_{ab|xy}\}_{xy}$ are both local with full freedom of choice (i.e., share the same LHV+FHV model). Thus, the problem can be reduced to justifying that: *(a)* behaviour $\{P^{NL}_{ab|xy}\}_{xy}$ also has a LHV model (possibly a non-FHV model), and *(b)* behaviour $\{P^{NF}_{ab|xy}\}_{xy}$ also has a FHV model (possibly a non-LHV model).

Ad. (a) This readily follows from **Lemma 2**.

Ad. (b) Here, a trivial model will suffice. Let us take $\Lambda := \{\lambda_o\}$ (a single-element set) with $P_{\lambda_o} \equiv P_{\lambda_o|xy} := 1$ and conditional distribution defined as $P_{ab|xy\lambda_o} := P^{NF}_{ab|xy}$. Clearly, it is a FVH model of behaviour $\{P^{NF}_{ab|xy}\}_{xy}$.

Thus, we have shown equivalence of both decompositions Eqs. (26) and (27), which, by virtue of **Remark 2**, proves **Theorem 1**.

Proof of Theorem 2. By virtue of **Theorem 1** it suffices to prove the result for one of the measures. Let it be measure μ_L evaluated by means of Eq. (28) in **Remark 2**.

Consider some arbitrary decomposition Eq. (26) of behaviour $\{P_{ab|xy}\}_{xy}$. Then, by linearity, the four CHSH expressions Eqs. (2)-(5) decompose as well, i.e. we get

$$S_i \;=\; p_L \cdot S_i^L + (1 - p_L) \cdot S_i^{NL}\,, \qquad [31]$$

where S_i^L and S_i^{NL} are calculated for the respective behaviours $\{P^{L}_{ab|xy}\}_{xy}$ and $\{P^{NL}_{ab|xy}\}_{xy}$. Since the first one is a local behaviour with full freedom of choice (i.e. having a LHV+FHV model), then from the CHSH inequalities Eq. (1) we have $|S_i^L| \leqslant 2$. For the second one there is nothing interesting to be said other than noting the maximal algebraic bound $|S_i^{NL}| \leqslant 4$. As a consequence, the following inequality obtains

$$|S_i| \;\leqslant\; p_L \cdot 2 \,+\, (1 - p_L) \cdot 4 \;=\; 4 - 2\,p_L\,, \qquad [32]$$

and we get $p_L \leqslant \frac{1}{2}(4 - |S_i|)$. Thus, by assumed arbitrariness of decomposition, Eq. (26) gives the upper bound on expression in Eq. (28)

$$\mu_L \;\leqslant\; \tfrac{1}{2}(4 - |S_i|)\,, \qquad [33]$$

where $S_{max} = \max\{|S_i| : i = 1, \ldots, 4\}$. By **Lemma 3** we conclude that the bound is tight, which ends the proof of **Theorem 2**.

ACKNOWLEDGMENTS. We thank R. Colbeck, J. Duda, N. Gisin, M. Hall, L. Hardy, D. Kaiser, M. Markiewicz, S. Pironio, D. Rohrlich and V. Scarani for helpful comments. PB acknowledges support from the Polish National Agency for Academic Exchange in the Bekker Scholarship Programme. PB and EMP were supported by Office of Naval Research Global grant N62909-19-1-2000.

1. JS Bell, *Speakable and unspeakable in quantum mechanics.* (Cambridge University Press), (1987).
2. ND Mermin, Hidden variables and the two theorems of John Bell. *Rev. Mod. Phys.* **65**, 803–815 (1993).
3. HM Wiseman, The two Bell's theorems of John Bell. *J. Phys. A: Math. Theor.* **47**, 424001 (2014).
4. N Brunner, D Cavalcanti, S Pironio, V Scarani, S Wehner, Bell nonlocality. *Rev. Mod. Phys.* **86**, 419 (2014).
5. V Scarani, *Bell Nonlocality.* (Oxford University Press), (2019).
6. JS Bell, Free variables and local causality in *Speakable and unspeakable in quantum mechanics.* (Cambridge University Press), (1987).
7. J Pearl, *Causality: Models, Reasoning, and Inference.* (Cambridge University Press), 2nd edition, (2009).
8. A Aspect, J Dalibard, G Roger, Experimental Test of Bell's Inequalities Using Time-Varying Analyzers. *Phys. Rev. Lett.* **49**, 1804 (1982).
9. M Giustina, et al., Significant-Loophole-Free Test of Bell's Theorem with Entangled Photons. *Phys. Rev. Lett.* **115**, 250401 (2015).
10. LK Shalm, et al., Strong Loophole-Free Test of Local Realism. *Phys. Rev. Lett.* **115**, 250402 (2015).
11. B Hensen, et al., Loophole-free Bell inequality violation using electron spins separated by 1.3 kilometres. *Nature* **526**, 682 (2015).
12. A Aspect, Closing the Door on Einstein and Bohr's Quantum Debate. *Physics* **8** (2015).
13. J Gallicchio, AS Friedman, DI Kaiser, Testing Bell's Inequality with Cosmic Photons: Closing the Setting-Independence Loophole. *Phys. Rev. Lett.* **112**, 110405 (2014).
14. D Rauch, et al., Cosmic Bell Test Using Random Measurement Settings from High-Redshift Quasars. *Phys. Rev. Lett.* **121**, 080403 (2018).
15. C Abellán, et al., Challenging local realism with human choices. *Nature* **557**, 212 (2018).
16. T Maudlin, *Philosophy of Physics: Quantum Theory.* (Princeton University Press), (2019).
17. F Laloë, *Do we really understand quantum mechanics?* (Cambridge University Press), 2nd edition, (2019).
18. T Norsen, *Foundations of Quantum Mechanics: An Exploration of the Physical Meaning of Quantum Theory*, Undergraduate Lecture Notes in Physics. (Springer), (2017).
19. CJ Wood, RW Spekkens, The lesson of causal discovery algorithms for quantum correlations: causal explanations of bell-inequality violations require fine-tuning. *New J. Phys.* **17**, 033002 (2015).
20. R Chaves, R Kueng, JB Brask, D Gross, Unifying Framework for Relaxations of the Causal Assumptions in Bell's Theorem. *Phys. Rev. Lett.* **114**, 140403 (2015).
21. EG Cavalcanti, Classical Causal Models for Bell and Kochen-Specker inequality Violations Require Fine-Tuning. *Phys. Rev. X* **8**, 021018 (2018).
22. R Colbeck, R Renner, A short note on the concept of free choice. *arXiv:1302.4446* (2013).
23. C Brukner, Quantum causality. *Nat. Phys.* **10**, 259 (2014).
24. JMA Allen, J Barrett, D Horsman, CM Lee, RW Spekkens, Quantum Common Causes and Quantum Causal Models. *Phys. Rev. X* **7**, 031021 (2017).
25. KB Wharton, N Argaman, Colloquium: Bell's theorem and locally mediated reformulations of quantum mechanics. *Rev. Mod. Phys.* **92**, 021002 (2020).
26. T Norsen, Against 'Realism'. *Found. Phys.* **37**, 311 (2007).
27. AC Elitzur, S Popescu, D Rohrlich, Quantum nonlocality for each pair in an ensemble. *Phys. Lett. A* **162**, 25 (1992).
28. L Hardy, A new way to obtain Bell inequalities. *Phys. Lett. A* **161**, 21 (1991).
29. J Barrett, A Kent, S Pironio, Maximally Nonlocal and Monogamous Quantum Correlations. *Phys. Rev. Lett.* **97**, 170409 (2006).
30. R Colbeck, R Renner, Hidden Variable Models for Quantum Theory Cannot Have Any Local Part. *Phys. Rev. Lett.* **101**, 050403 (2008).
31. R Colbeck, R Renner, The Completeness of Quantum Theory for Predicting Measurement Outcomes in *Quantum Theory: Informational Foundations and Foils*, eds. G Chiribella, RW Spekkens. (Springer), pp. 497–528 (2016).
32. S Portmann, C Branciard, N Gisin, Local content of all pure two-qubit states. *Phys. Rev. A* **86**, 012104 (2012).
33. S Abramsky, RS Barbosa, S Mansfield, Contextual Fraction as a Measure of Contextuality. *Phys. Rev. Lett.* **119**, 050504 (2017).
34. CH Brans, Bell's theorem does not eliminate fully causal hidden variables. *Int. J. Theor. Phys.* **27**, 219 (1988).
35. MJW Hall, Local Deterministic Model of Singlet State Correlations Based on Relaxing Measurement Independence. *Phys. Rev. Lett.* **105**, 250404 (2010).
36. MJW Hall, The Significance of Measurement Independence for Bell Inequalities and Locality in *At the Frontier of Spacetime*, ed. T Asselmeyer-Maluga. (Springer), pp. 189–204 (2016).
37. MJW Hall, Relaxed Bell inequalities and Kochen-Specker theorems. *Phys. Rev. A* **84**, 022102 (2011).
38. J Barrett, N Gisin, How Much Measurement Independence Is Needed to Demonstrate Nonlocality? *Phys. Rev. Lett.* **106**, 100406 (2011).
39. G Pütz, D Rosset, TJ Barnea, YC Liang, N Gisin, Arbitrarily Small Amount of Measurement Independence Is Sufficient to Manifest Quantum Nonlocality. *Phys. Rev. Lett.* **113**, 190402 (2014).
40. D Aktas, et al., Demonstration of Quantum Nonlocality in the Presence of Measurement Dependence. *Phys. Rev. Lett.* **114**, 220404 (2015).
41. G Pütz, N Gisin, Measurement dependent locality. *New J. Phys.*, 055006 (2016).
42. MJW Hall, C Branciard, Measurement-dependence cost for Bell nonlocality: Causal versus retrocausal models. *Phys. Rev. A* **102**, 052228 (2020).
43. JF Clauser, MA Horne, A Shimony, RA Holt, Proposed Experiment to Test Local Hidden-Variable Theories. *Phys. Rev. Lett.* **23**, 880–884 (1969).
44. A Fine, Hidden Variables, Joint Probability, and the Bell Inequalities. *Phys. Rev. Lett.* **48**, 291–295 (1982).
45. JJ Halliwell, Two proofs of Fine's theorem. *Phys. Lett. A* **378**, 2945–2950 (2014).
46. JP Jarrett, On the Physical Significance of the Locality Conditions in the Bell Arguments. *Nous* **18**, 569 (1984).
47. BS Tsirelson, Quantum generalizations of Bell's inequality. *Lett. Math. Phys.* **4**, 93–100 (1980).
48. JR Busemeyer, PD Bruza, *Quantum Models of Cognition and Decision.* (Cambridge University Press), (2012).
49. E Haven, A Khrennikov, *Quantum Social Science.* (Cambridge University Press), (2013).
50. EM Pothos, JR Busemeyer, Can quantum probability provide a new direction for cognitive modeling? *Behav. Brain Sci.* **36**, 255–274 (2013).
51. L Hardy, Bell inequalities with retarded settings in *Quantum [Un]speakables II: Half a Century of Bell's Theorem*, eds. R Bertlmann, A Zeilinger. pp. 261–272 (2017) arXiv:1508.06900.
52. L Hardy, Proposal to use humans to switch settings in a bell experiment. *arXiv: 1705.04620* (2017).
53. T Maudlin, Bell's Inequality, Information Transmission, and Prism Models in *PSA: Proceedings of the Biennial Meeting of the Philosophy of Science Association 1992.* (The University of Chicago Press) Vol. 1, pp. 404–417 (1992).
54. G Brassard, R Cleve, A Tapp, Cost of Exactly Simulating Quantum Entanglement with Classical Communication. *Phys. Rev. Lett.* **83**, 1874 (1999).

55. M Steiner, Towards quantifying non-local information transfer: finite-bit non-locality. *Phys. Lett. A* **270**, 239 (2000).
56. BF Toner, D Bacon, Communication Cost of Simulating Bell Correlations. *Phys. Rev. Lett.* **91**, 187904 (2003).
57. RD Gill, The Triangle Wave Versus the Cosine: How Classical Systems Can Optimally Approximate EPR-B Correlations. *Entropy* **22**, 287 (2020).
58. H Buhrman, R Cleve, S Massar, R de Wolf, Nonlocality and communication complexity. *Rev. Mod. Phys.* **82**, 665–698 (2010).
59. J Kofler, T Paterek, C Brukner, Experimenter's freedom in Bell's theorem and quantum cryptography. *Phys. Rev. A* **73**, 022104 (2006).
60. DE Koh, et al., Effects of Reduced Measurement Independence on Bell-Based Randomness Expansion. *Phys. Rev. Lett.* **109**, 160404 (2012).
61. P Suppes, M Zanotti, When are probabilistic explanations possible? *Synthese* **48**, 191–199 (1981).
62. A Garg, ND Mermin, Farkas's Lemma and the nature of reality: Statistical implications of quantum correlations. *Found. Phys.* **14**, 1–39 (1984).
63. M Zukowski, C Brukner, Bell's Theorem for General N-Qubit States. *Phys. Rev. Lett.* **88**, 210401 (2002).
64. AA Klyachko, MA Can, S Binicioğlu, AS Shumovsky, Simple Test for Hidden Variables in Spin-1 systems. *Phys. Rev. Lett.* **101**, 020403 (2008).
65. J Barrett, et al., Nonlocal correlations as an information-theoretic resource. *Phys. Rev. A* **71**, 022101 (2005).
66. J Barrett, S Pironio, Popescu-rohrlich correlations as a unit of nonlocality. *Phys. Rev. Lett.* **95**, 140401 (2005).
67. NS Jones, L Masanes, Interconversion of nonlocal correlations. *Phys. Rev. A* **72**, 052312 (2005).
68. JV Kujala, EN Dzhafarov, JA Larsson, Necessary and Sufficient Conditions for an Extended Noncontextuality in a Broad Class of Quantum Mechanical Systems. *Phys. Rev. Lett.* **115**, 150401 (2015).
69. EN Dzhafarov, JV Kujala, JA Larsson, Contextuality in Three Types of Quantum-Mechanical Systems. *Found. Phys.* **45**, 762–782 (2015).
70. ND Mermin, Quantum mysteries revisited. *Am. J. Phys.* **58**, 731–734 (1990).
71. MO Renou, et al., Genuine Quantum Nonlocality in the Triangle Network. *Phys. Rev. Lett.* **123**, 140401 (2019).
72. N Gisin, *Quantum Chance: Nonlocality, Teleportation and Other Quantum Marvels.* (Springer), (2014) [Ch. 9, p. 90].
73. S Popescu, D Rohrlich, Quantum Nonlocality as an Axiom. *Found. Phys.* **24**, 379 (1994).