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THE ELECTRICAL FIELD INDUCED BY OCEAN CURRENTS AND WAVES, WITH APPLICATIONS TO THE METHOD OF TOWED ELECTRODES

BY

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PREFACE

This paper originated in an informal conference between the authors at Woods Hole in September, 1951. A preliminary draft, entitled "A manual of examples of the electrical effects of different types of water motion in the ocean," was circulated in May, 1952. Since then the paper has been revised and extended to its present form.

One of us (M. S. L. H.) is indebted to the Commonwealth Fund, New York, for a Fellowship which enabled him to study at the Woods Hole Oceanographic Institution and the Scripps Institution of Oceanography during 1951 and 1952. The work of the other two (M. E. S. and H. S.) was supported from funds of the Office of Naval Research, Contract No. N6onr-27701 (NR-083-004).

CONTENTS

INTRODUCTION

THE purpose of this paper is to discuss the nature of the electrical field induced in the ocean by particular types of velocity distribution. It is believed that these examples will be helpful in the interpretation of measurements by towed electrodes in the sea.

The electrical field induced by waves and tidal streams, originally predicted by Faraday (1832), was first measured experimentally by Young, Gerrard and Jevons (1920),* who used both moored and towed electrodes in their observations. Recently, the technique of towed electrodes has been developed by von Arx (1950, 1951) and others into a useful means of detecting water movements in the deep ocean. While the method has been increasingly used, the problem of interpreting the measurements in terms of water movements has become of great importance. Two of the present authors have made theoretical studies (Longuet-Higgins 1949, Stommel 1948) dealing with certain cases of velocity fields, and Malkus and Stern (1952) have proved some important integral theorems. There seems, however, to be a need for a more extended discussion of the principles underlying the method, and for the computation of additional illustrative examples. This is all the more desirable since some of the theoretical discussions published previously have been misleading.

A fundamental difficulty is that the electrical field at any point depends not only on the local water velocity, but also on the electrical current-density, which is determined to a greater or less extent by the whole velocity field. Thus it is generally impossible to deduce the velocity at the surface from measurements of the electrical field taken only at the surface. However, such measurements may provide very useful information when taken in conjunction with other observations; and, in some cases when the form of the streams is known (e.g., if it is known that they are mainly confined to a shallow surface layer), a more or less accurate estimate of the surface velocity can be made. In this paper we attempt to evaluate the degree of uncertainty involved in such estimates, by computing the electrical potential in specific cases and comparing it with the actual surface velocity.

We begin, in Chapter 1, by giving an account of the principle of induction, with simple circuit analogies to illustrate the relation between observations by stationary and moving electrodes, in moving water. The theory of towed electrodes is set out in detail. Next, in Chapter 2, we give a qualitative discussion of an important practical case — that of an infinitely extended stream of finite width in water of uniform depth. The influence of the conductivity of the bottom and of variations in water velocity with depth are roughly evaluated. In Chapter 3 we give the field equations necessary for an exact analysis, and discuss two-dimensional motion in particular. In Chapters 4, 5 and 6 we work out some particular solutions for the cases of a sinusoidal stream, and streams of rectangular and elliptical cross-section, respectively. The solution for the rectangular section is of special interest, since it can be used to build up the solution for a stream of arbitrary cross-section, to any degree of approximation. Chapter 7 deals with the field induced by sea waves, and of the effect of the magnetic field on them. Fluctuations of the velocity or of the magnetic field with time, and variations in the electrical conductivity of the oceans are considered in Chapter 8. The conclusions are summarised in Chapter 9.

Parts of the paper (for example Chapters 1 and 2) are mainly expository, and may serve as an introduction to the subject. For those not wishing to go into the mathematical details of the solutions, a general picture will be provided by Chapters 1, 2, 8 and 9.

* Tidally induced potentials had previously been noticed in submarine telegraph cables; for references see Longuet-Higgins (1949).

1. GENERAL PRINCIPLES

1.1 Faraday's Law of Induction

Suppose that a simple closed circuit C moves in the presence of a magnetic field $\mathbf{H}$ (see Figure 1). By Faraday's law of induction an electromotive force (e.m.f.) is induced in C equal to minus the

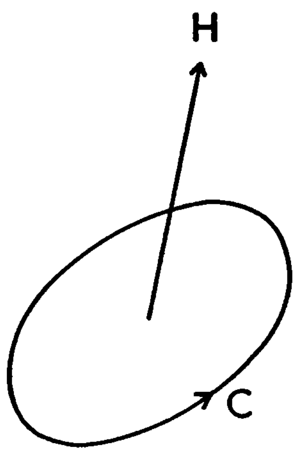

Figure 1

rate of change of magnetic flux through C. This change of flux may be due to either of two causes: a variation of the magnetic field with time, or the movement of the circuit. It will be supposed in the following that $\mathbf{H}$ is constant (whether this is permissible in the case of the Earth's magnetic field will be considered later). Then the rate of change of flux through C is due solely to the motion of C in the field, and is equal to the rate at which

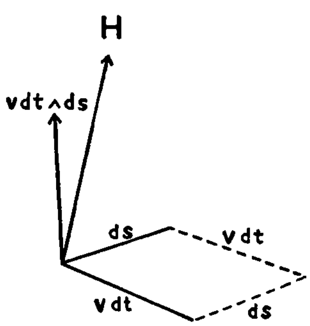

Figure 2

elements of the circuit are, on the whole, cutting lines of magnetic flux. Now an element ds of the circuit, moving with velocity $\mathbf{v}$, in a short time dt sweeps out an area whose projection in the direction of $\mathbf{H}$ is $(\mathbf{v}dt \wedge d\mathbf{s}) \cdot \mathbf{H}$ (see Figure 2).

This equals $-\mathbf{v} \wedge \mathbf{H} \cdot d\mathbf{s}\, dt$, so that the rate of cutting of lines of magnetic flux by the circuit element ds is $-\mathbf{v} \wedge \mathbf{H} \cdot d\mathbf{s}$, and the total e.m.f. E developed in the circuit C is the integral of $\mathbf{v} \wedge \mathbf{H} \cdot d\mathbf{s}$ taken round C:

$$E = \int_C \mathbf{v} \wedge \mathbf{H} \cdot d\mathbf{s} \qquad 1.1.1$$

Thus an alternative expression of Faraday's law is to say that at each point in space the motion induces an e.m.f. per unit distance represented by the vector $\mathbf{v} \wedge \mathbf{H}$. This vector is perpendicular to $\mathbf{v}$ and $\mathbf{H}$, and is proportional to both $\mathbf{v}$, $\mathbf{H}$ and the sine of the angle between them.

In the case of a simple wire circuit C the e.m.f. induced in the wire would cause a current I to flow such that

$$E = RI, \qquad 1.1.2$$

where R is the resistance of the wire. In the case of continuous media we suppose that there is an electrical current-density $\mathbf{i}$ such that the circulation of $\rho\mathbf{i}$ round any circuit C just equals the circulation of the e.m.f. $\mathbf{v} \wedge \mathbf{H}$ (ρ being the electrical resistivity). In other words the circulation of $(\mathbf{v} \wedge \mathbf{H} - \rho\mathbf{i})$ round C vanishes.

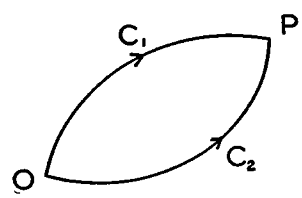

Figure 3

It follows that we may define an electrical potential ϕ, at each point P in space, as the integral of $(\mathbf{v} \wedge \mathbf{H} - \rho\mathbf{i})$ along any path from a fixed point O to the point P:

$$\phi\,(P) = \int_O^P (\mathbf{v} \wedge \mathbf{H} - \rho\mathbf{i}) \cdot d\mathbf{s}. \qquad 1.1.3$$

The path of integration is arbitrary; for if C_1 and C_2 are any two paths from O to P (see Figure 3) the integral round the closed circuit consisting of C_1 and $-C_2$ is zero; the integral along C_2 therefore equals the integral along C_1. From 1.1.3 we have also

$$\nabla\phi = \mathbf{v} \wedge \mathbf{H} - \rho\mathbf{i} \cdot \qquad 1.1.4$$

The potential ϕ is the same as would be measured by a stationary potentiometer connected between O and P. For suppose this to consist of an external wire circuit C' (see Figure 4) which contains a cell of voltage E' so adjusted as to reduce the current in C' to zero. If C is any circuit from O

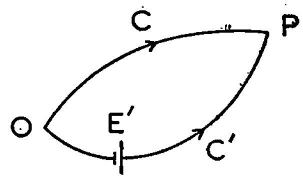

FIGURE 4

to P through the fluid, the total e.m.f. developed in the circuit C and $-C'$ is

$$\int_{C-C'} \mathbf{v} \wedge \mathbf{H} \cdot d\mathbf{s} - E', \qquad 1.1.5$$

and this must equal

$$\int_{C-C'} \rho \mathbf{i} \cdot d\mathbf{s}. \qquad 1.1.6$$

But on C' both $\mathbf{v}$ and $\mathbf{i}$ vanish. Hence on equating 1.1.5 and 1.1.6 we have

$$E' = \int_C (\mathbf{v} \wedge \mathbf{H} - \rho \mathbf{i}) \cdot d\mathbf{s} = \phi(P). \qquad 1.1.7$$

It is essential to this argument that the potentiometer be stationary. If on the other hand the potentiometer circuit is in motion with velocity $\mathbf{v}'$, then the contribution to 1.1.5 from C' does not vanish and we find

$$E' = \phi(P) - \int_{C'} \mathbf{v}' \wedge \mathbf{H} \cdot d\mathbf{s}. \qquad 1.1.8$$

By taking P sufficiently close to O we might obtain a measure of the potential gradient at O. For then we have for the stationary electrodes

$$E' = \phi(P) - \phi(O) = \nabla \phi \cdot \overline{OP} \qquad 1.1.9$$

(to first order in $\overline{OP}$). For the moving electrodes we should obtain

$$E' = (\nabla \phi - \mathbf{v}' \wedge \mathbf{H}) \cdot \overline{OP}, \qquad 1.1.10$$

so that the apparent gradient measured would be

$$\nabla \phi - \mathbf{v}' \wedge \mathbf{H}. \qquad 1.1.11$$

It should be noticed, however, that $\mathbf{v}' \wedge \mathbf{H}$ is not necessarily the gradient of a potential function as was incorrectly assumed by von Arx in Section 3 of his 1950 paper.

1.2 EXAMPLES

(a) Imagine a conducting rod AB of length L moving horizontally at right angles to itself, with velocity V, say, and let there be a uniform magnetic field of strength directed vertically upwards (see Figure 5). Then an e.m.f. will be induced in AB, of magnitude VH per unit distance and directed from left to right as one faces in the direction of motion. Suppose the ends of the rod are connected to a stationary potentiometer P by a rectangular circuit $BCDA$, as in Figure 5. The sides BC and DA of the rectangle are not cutting any lines of magnetic flux, and CD is stationary. Since no electric current flows, the voltage measured by P is simple VHL.

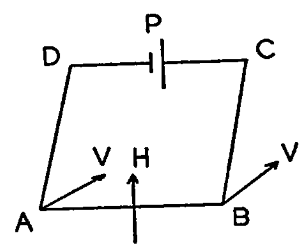

FIGURE 5

(b) Suppose now that the potentiometer, instead of being stationary, moves horizontally with the same velocity as AB. The e.m.f. induced in CD will just balance that induced in AB, and P will record zero voltage.

(c) Now let the potentiometer be stationary as in example (a), but suppose that the ends of the rod are made to move along two parallel conducting rails in electrical contact with one another as in Figure 6. If the resistance R_1 of the external path is low compared with R, the system will be

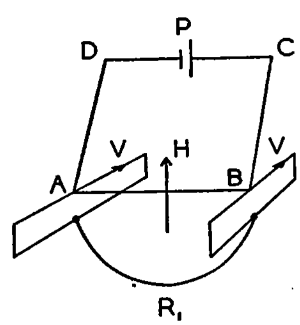

FIGURE 6

short-circuited: a current I will flow in AB (returning through R_1) which will reduce the voltage between A and B to zero. The additional voltage produced by the current must therefore be $-VHL$. If the ratio R_1/R is at first large and then decreases to zero the voltage recorded by P

will gradually fall from its maximum value VHL to zero.

(d) Assume the same arrangement as in (c) but with the potentiometer in motion as in (b). The rate at which the circuit $ABCD$ is cutting lines of magnetic force is, on the whole, zero. But a current I still flows in AB. Thus when R_1/R is small the potentiometer records a voltage $-VHL$, the negative of the voltage measured in example (a). Again, if R_1/R gradually decreases to zero, the voltage will gradually increase (in absolute magnitude) from O to $-VHL$.

(e) In examples (a) to (d) the moving rod may be replaced by a uniform stream of water moving in a long channel of rectangular cross-section and width L; the external path may be replaced by the walls of the channel (see Figure 7). If the walls of

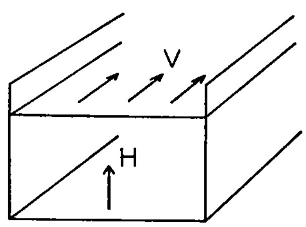

FIGURE 7

the channel are nonconducting, the potential difference between the sides of the channel will be VHL, if measured by a stationary potentiometer, or zero if measured by a potentiometer moving with the fluid. If on the other hand the walls of the channel are perfectly conducting the stationary potentiometer will measure zero voltage, and the moving potentiometer $-VHL$.

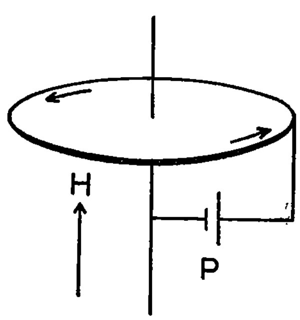

FIGURE 8

(f) *The Faraday disc.* Suppose a conducting circular disc spins about a vertical axis, in the presence of a vertical magnetic field (see Figure 8). The vector $\mathbf{v} \wedge \mathbf{H}$ is now directed radially outwards from the centre of the disc. However, because of the circular symmetry, no electrical current will normally be able to flow. Let a stationary

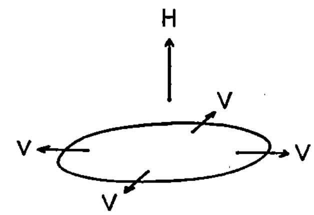

FIGURE 9

potentiometer be connected between the centre of the disc and its circumference (where there must be a brush or sliding contact). If a is the radius and ω the angular velocity of the disc, a voltage E', given by

$$E' = \int_0^a \omega r \, H \, dr = \tfrac{1}{2} \omega a^2 H, \qquad 1.2.1$$

will be recorded. This voltage could be entirely short-circuited by electrical contact between the rotating disc and a parallel, stationary disc of high conductivity. If the potentiometer were made to rotate with the moving disc, but without any connection to the stationary disc, the voltage measured would again be zero; but if the external connections were made, the measured voltage would be $-\frac{1}{2} \omega a^2 H$.

(g) In the previous example the disc may be replaced by a whirlpool or vortex, with circular symmetry about a vertical axis. If the vortex extends uniformly in the vertical direction there will be a potential difference (with regard to stationary electrodes) between points at different distances from the axis; but if the current is only very shallow in relation to its radius, and if the water below is stationary, much of this voltage may be short-circuited; this would be expected, for example, in the case of a shallow Ekman current.

(h) Imagine a circular wire circuit expanding radially in all directions in the presence of a magnetic field parallel to its axis of symmetry (Figure 9). The induced e.m.f. will be everywhere along the circuit. There will be no hindrance to the flow of electric current, which will therefore reduce the measured potential to zero.

(i) The comparable case for a continuous medium would be that of flow outwards from a line source, or of a symmetrical upwelling (see Figure 10). The electrical current-density would circulate freely round the axis, and the measured potential gradient would be zero.

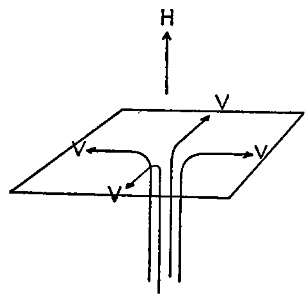

FIGURE 10

1.3 THE METHOD OF TOWED ELECTRODES

Suppose two electrodes *A* and *B* are towed in line behind a ship (Figure 11). Let the velocity of the water be $\mathbf{v}$, assumed to be the same for both the ship and the electrodes. Then the direction of the vector $\overline{AB}$ joining the electrodes may be expected to be the same as that of the velocity $\mathbf{v}_1$ of the ship relative to the velocity of the water; this is given by

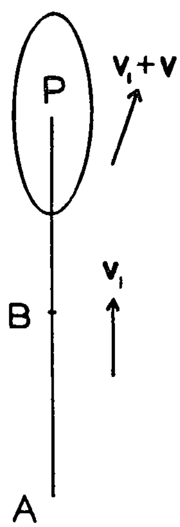

FIGURE 11

$$\mathbf{v}_1 = \mathbf{v}_p + \mathbf{v}_w \qquad 1.3.1$$

where $\mathbf{v}_p$ is the velocity due to the ship's propellors (which may be estimated) and $\mathbf{v}_w$ the additional velocity due to the wind (windage). The absolute velocity of the ship is

$$\mathbf{v} + \mathbf{v}_1 \qquad 1.3.2$$

which, if a steady state has been reached, will also be the absolute velocity of the electrodes. If the electrodes were stationary the voltage between them would be simply that due to the potential gradient in the direction of the electrodes, i.e.,

$$\nabla \phi \cdot \overline{AB}. \qquad 1.3.3$$

However, since the system is in motion with velocity $\mathbf{v} + \mathbf{v}_1$, the actual voltage recorded is, by 1.1.10,

$$[\nabla \phi - (\mathbf{v} + \mathbf{v}_1)] \cdot \overline{AB}. \qquad 1.3.4$$

But since $\overline{AB}$ is parallel to $\mathbf{v}_1$ we have

$$\mathbf{v}_1 \wedge \mathbf{H} \cdot \overline{AB} = 0, \qquad 1.3.5$$

that is to say, the electrode-line itself is cutting no lines of magnetic flux by virtue of being towed through the water. Thus the voltage measured is

$$[\nabla \phi - \mathbf{v} \wedge \mathbf{H}] \cdot \overline{AB}, \qquad 1.3.6$$

which is independent of the towing-velocity $\mathbf{v}_1$ (except in so far as this determines the direction of $\overline{AB}$). Since

$$\nabla \phi = \mathbf{v} \wedge \mathbf{H} - \rho \mathbf{i} \qquad 1.3.7$$

the above may also be written

$$-\rho \mathbf{i} \cdot \overline{AB}. \qquad 1.3.8$$

The electrodes therefore measure, in effect, the component of the electrical current-density $-\rho\mathbf{i}$ *in the direction of the electrode-line* $\overline{AB}$.

Now in many cases the potential gradient is completely short-circuited, as for example in Section 1.2 (b), (d), (e), (h) and (i). We then have

$$\nabla \phi = 0 \qquad 1.3.9$$

and so

$$\rho \mathbf{i} = \mathbf{v} \wedge \mathbf{H}. \qquad 1.3.10$$

The voltage recorded is therefore

$$-\mathbf{v} \wedge \mathbf{H} \cdot \overline{AB}, \qquad 1.3.11$$

and is a direct measure of the component of velocity at right-angles to the electrode-line.

Although this case may be regarded as normal for currents in the deep ocean, it is essential to consider under what conditions $\nabla \phi$ may be expected to vanish, and what errors will be introduced when these conditions are not exactly satisfied. This will be done in later Sections.

It will be convenient to consider here one or two errors that may be introduced when the conditions are different from those assumed.

Variation of water velocity with depth. Suppose the water velocity varies with depth in such a way that the effective water velocity for the ship is different from the velocity $\mathbf{v}$ for the electrodes. Suppose the water velocity for the ship is $\mathbf{v} + \mathbf{v}_2$, so that the absolute velocity of the ship and of the electrodes is $\mathbf{v} + \mathbf{v}_1 + \mathbf{v}_2$. The vector $\overline{AB}$ may be expected to be parallel to $\mathbf{v}_1 + \mathbf{v}_2$ which is the velocity of the ship relative to the water velocity for the electrodes. Hence all the previous equations are valid if only $\mathbf{v}_1$ is replaced by $\mathbf{v}_1 + \mathbf{v}_2$. The direction of $\overline{AB}$ will be slightly altered; but the measured voltage will still depend only on the velocity $\mathbf{v}$ and the current-density $\mathbf{i}$ in the neighbourhood of the electrodes.

Deflection of the electrode-line. Suppose that the electrode-line $\overline{AB}$, instead of being parallel to $\mathbf{v}_1 + \mathbf{v}_2$, makes an angle with this direction (in a horizontal plane). Such an effect might be produced, for example, by a wind $\mathbf{W}$ acting at right-angles to the electrode-line (see Figure 12). Then

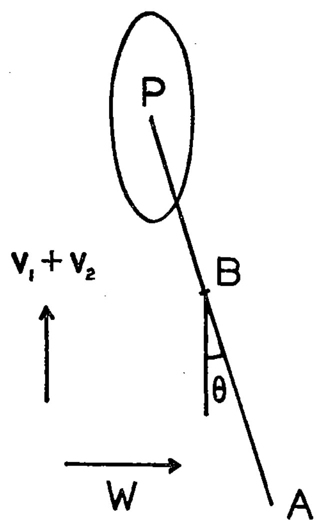

FIGURE 12

the direction of the measured component of potential gradient will again be slightly different; and there will also be a contribution from the term

$$-(\mathbf{v}_2 + \mathbf{v}_1) \wedge \mathbf{H} \cdot \overline{AB}. \qquad 1.3.12$$

The magnitude of this term is

$$V_0 H_z \cdot \overline{AB} \sin\theta, \qquad 1.3.13$$

where V_0 is the absolute value of $\mathbf{v}_1 + \mathbf{v}_2$ and H_z is the vertical component of $\mathbf{H}$. The voltage acts in the same sense as that which would be produced by a water velocity

$$V_0 \tan\theta \qquad 1.3.14$$

flowing across the electrodes in the same sense as the deflecting wind.

Similarly if the cable droops, so that A is lower than B by an amount $AB \sin\theta'$, a signal will be recorded from the horizontal component of field H_x, and will be of magnitude $V_0 H_x \sin\theta'$.

General Case. We shall now discuss the general case where the water velocity varies horizontally and is not necessarily steady. The following analysis might apply, for example, to small eddies or to waves where length was not large compared with the electrode-separation AB.

If, in Figure 11, the ship and electrodes were stationary, the measured e.m.f. between them would be

$$\phi(B) - \phi(A) = \int_A^B (\mathbf{v} \wedge \mathbf{H} - \rho\mathbf{i}) \cdot d\mathbf{s}. \qquad 1.3.15$$

The motion of the cables induces an additional e.m.f.

$$-\int_{AP+PB} \mathbf{v}' \wedge \mathbf{H} \cdot d\mathbf{s}, \qquad 1.3.16$$

the above integral being taken along the electrode cables. But between B and P the two cables follow the same paths though they are opposite in direction. Therefore 1.3.16. reduces to

$$-\int_{AB} \mathbf{v}' \wedge \mathbf{H} \cdot d\mathbf{s} \qquad 1.3.17$$

and the measured e.m.f. E' is

$$E' = \int_A^B (\mathbf{v} \wedge \mathbf{H} - \rho\mathbf{i}) \cdot d\mathbf{s} - \int_A^B \mathbf{v}' \wedge \mathbf{H} \cdot d\mathbf{s}. \qquad 1.3.18$$

Now in general $\mathbf{v}'$ is different from $\mathbf{v}$, but if, as the cable is trailed through the water, it has no transverse motion relative to the water we shall have

$$\mathbf{v}' = \mathbf{v} + \mathbf{v}'' \qquad 1.3.19$$

where $\mathbf{v}''$ is parallel to $d\mathbf{s}$ and so

$$\mathbf{v}' \wedge \mathbf{H} \cdot d\mathbf{s} = \mathbf{v} \wedge \mathbf{H} \cdot d\mathbf{s}, \qquad 1.3.20$$

since $\mathbf{v}'' \wedge \mathbf{H} \cdot d\mathbf{s}$ vanishes, or in other words the component of $\mathbf{v}'' \wedge \mathbf{H}$ along the cable contributes nothing to the rate at which it is cutting lines of magnetic flux. From 1.3.4. and 1.3.6. we have

$$E' = -\int_A^B \rho\mathbf{i} \cdot d\mathbf{s} \qquad 1.3.21$$

the path of integration being along the cable. In the case when all the induced e.m.f. is short-circuited we have

$$\rho\mathbf{i} = \mathbf{v} \wedge \mathbf{H} \qquad 1.3.22$$

and so

$$E' = -\int_A^B \mathbf{v} \wedge \mathbf{H} \cdot \mathrm{ds} \qquad 1.3.23$$

If the cable is deflected by wind, as in Figure 12, or if the path of the cable is curved, as in Figure 13, the cable may have a small transverse component relative to the water. In place of equation 1.3.19. we have, in general

$$\mathbf{v}' = \mathbf{v} + \mathbf{v}'' + \mathbf{v}''', \qquad 1.3.24$$

where $\mathbf{v}'''$ is perpendicular to ds. The additional e.m.f. due to $\mathbf{v}'''$ is given by

$$-\int_A^B \mathbf{v}''' \wedge \mathbf{H} \cdot \mathrm{ds}. \qquad 1.3.25$$

The extent of this "side-slipping" might be measured by towing the electrodes in a full circle, first clockwise and then anti-clockwise.

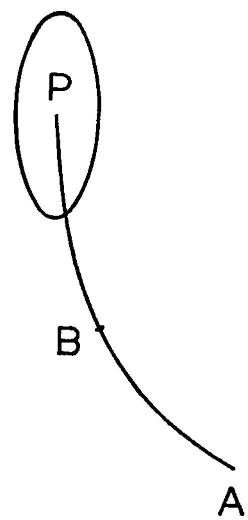

FIGURE 13

2. A LONG, STRAIGHT STREAM: QUALITATIVE TREATMENT

WE shall now consider some simple ideal cases of ocean currents, and investigate by rough physical reasoning the effect of the conductivity of the sea bed, and other factors, on the electrical potential. The methods used will mostly be approximate; some exact solutions are given later, in Chapters 4–6.

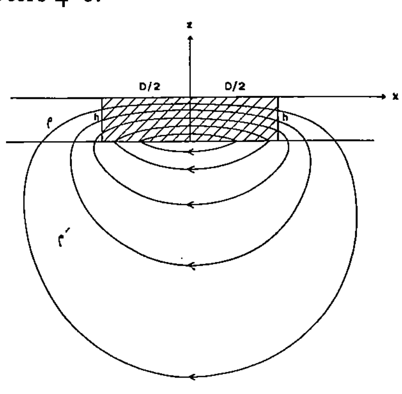

FIGURE 14. The electrical current-density due to H_z (sketch only).

2.1 THE EFFECT OF A CONDUCTING SEA BED

Suppose that the water velocity is everywhere horizontal and parallel to the y-axis, and that its strength is independent of y. Let the depth of water be h, and let the resistivity of the water and of the "sea bed" be ρ and ρ', respectively. It will be assumed for the purpose of the discussion that both ρ and ρ' are uniform, though this may be far from true in the case of the actual ocean floor. However, it may certainly be assumed that $\rho << \rho'$.

Consider first the case when a rectangular block of water of width D ($D >> h$) moves parallel to the y-axis with uniform velocity V, the remaining water being at rest (see Figure 14).

Assuming the magnetic field $\mathbf{H}$ to be uniform, we may consider separately the potentials induced by the three components H_x, H_y and H_z. The magnetic component H_y, being parallel to the water velocity, induces no electrical field. In this and the following section we shall consider the field due to the vertical component H_z; the field due to H_x will be considered in Section 2.3.

H_z will induce a horizontal e.m.f. acting from left to right (see Figure 14), which will tend to cause an electric current to flow from left to right through the water, with a return current back through the sea bed. The system is analogous to the wire circuit shown in Figure 15, which consists of a cell of voltage E and internal resistance R_i connected in series to an external resistor R_e. The voltage E corresponds to the total induced e.m.f. across the circuit, i.e. VH_zD; the internal resistance R_i is of the order of $\rho D/h$; and, assuming that the electric current spreads downwards into the "sea bed" to a depth comparable with the width of the current, the external resistance is of the order of $\rho' D/D$, or ρ'. Thus

$$E \sim VH_zD; \quad R_i \sim \rho D/h; \quad R_e \sim \rho'. \qquad 2.1.1$$

The total current I flowing round the circuit of Figure 15 is

$$I = \frac{E}{R_i + R_e} \qquad 2.1.2$$

and the potential difference between A and B is

$$\phi(A) - \phi(B) = I R_e = \frac{E}{1 + R_i/R_e} \qquad 2.1.3$$

Hence the potential difference between A and B in Figure 14 is given (to an order of magnitude) by

$$\phi(A) - \phi(B) \sim \frac{VH_zD}{1 + \rho D/\rho' h} \qquad 2.1.4$$

and the horizontal component of potential gradient is given by

$$\frac{\phi(A) - \phi(B)}{D} \sim \frac{VH_z}{1 + \rho D/\rho' h}. \qquad 2.1.5$$

Thus the effect of the conductivity of the sea bed depends upon the ratio

$$R_i/R_e \sim \rho D/\rho' h. \qquad 2.1.6$$

When this ratio is small, the conductivity of the sea bed has little effect and the sea bed can be regarded as an insulator; when it is large, the potential gradient is effectively short-circuited by the sea bed; and when it is of order unity the potential gradient depends critically on the sea-bed conductivity.

Let us consider some practical examples:

(a) *The English Channel* (see Longuet-Higgins, 1949). The resistivity of the water is nearly uniform, on account of the high degree of mixing, and has a mean value of about 25 ohm-cm. On the other hand the resistivity of the rocks beneath the sea bed is highly variable. In the uppermost sediments the resistivity can be expected to be nearly the same as that of the water; but at lower levels, the resistivity, if equal to that of crystalline rocks, may be as much as 10^6 ohm-cm. Thus the ratio ρ/ρ' may vary, so far as can be estimated, between 1 and $2{\cdot}5 \times 10^{-5}$. Now for most sections of the English Channel the ratio of the width D to the mean depth h is of the order of 10^3. Hence the ratio $\rho D/\rho' h$ may lie anywhere between 10^3 and 0·025. Thus the potential gradient due to tidal or other streams cannot be accurately predicted; it becomes more useful to find a value of ρ' from the observed potential gradients.

Measurements by Cherry and Stovold (1946) on submarine telephone cables have shown that

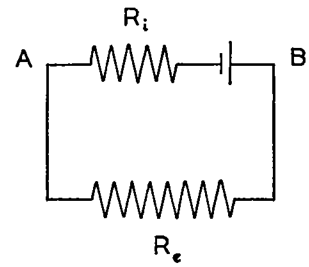

FIGURE 15

there is a considerable potential gradient across the English Channel which can lie between 17 and 71 per cent of the maximum mean potential gradient. Values for ρ', deduced from the above observations, lie between $1{\cdot}3 \times 10^4$ and $2{\cdot}7 \times 10^4$ ohm-cm (see Longuet-Higgins, 1949).

(b) *Currents in the deep ocean.* The conductivity of the different layers in the bed of the deep ocean is also little known. Let us assume that the effective conductivity over the regions concerned is of the same order as for the English Channel, so that the ratio of the conductivities of the water and of the sea bed is of the order of 10^{-3}. If the depth of water is, say, 3 km, it appears that the conductivity of the sea bed will be of importance only if the width of the current considered is of the order of 3,000 km. For a narrow current such as the Gulf Stream, for example, the effect of the sea bed conductivity is probably negligible (in contrast to the English Channel, where it is critically important). However for very broad currents, if they exist, the conductivity may be important.

In the English Channel the tidal stream velocity is not, of course, uniform with depth (in the Straits of Dover the mean velocity is about 0·83 times the surface velocity; see Longuet-Higgins, 1949). In the deep oceans the difference between mean velocity and surface velocity may be much greater. We shall now consider the effect of variation of water velocity with depth.

2.2 The Effect of a Vertical Variation of Velocity

Suppose now that the velocity V of the stream, instead of being uniform, is a function of the vertical co-ordinate z. Consider the field due to H_z. If the width D of the stream is large compared to the depth h, the electrical current-density **i** will be mainly horizontal, in the direction of the induced e.m.f. and hence the equipotential lines in the ocean will be vertical. Hence, in the water, $\nabla\phi$ is nearly horizontal and independent of the depth. The horizontal component i_x of the current-density is found from equation 1.1.4:

$$\frac{\partial\phi}{\partial x} = V(z)\, H_z - \rho i_x, \qquad 2.2.1$$

and hence the total current I is given by

$$\rho I = \int_{-h}^{0} \rho i_x\, dz = \int_{-h}^{0} V(z)\, H_z\, dz - h\,\frac{\partial\phi}{\partial x}, \qquad 2.2.2$$

that is

$$\rho I = h\left(\overline{V}H_z - \frac{\partial\phi}{\partial x}\right), \qquad 2.2.3$$

where

$$\overline{V} = \frac{1}{h}\int_{-h}^{0} V(z)\, dz. \qquad 2.2.4$$

Thus the total current, and so the potential gradient, are the same as if the water moved with its mean velocity $\overline{V}$.

The circuit analogy of Section 2.1 may be extended to this case also. Imagine the sea to be divided into horizontal layers of infinitesimal thickness dz, in each of which the velocity is constant. The vertical component of field will induce in each layer a horizontal e.m.f. $V(z)H_zD$. The system may be compared to a circuit of the same kind as in Figure 15, except that the cell and internal resistance R_i must now be replaced by a number of cells connected in parallel, each of strength $V(z)\, H_zD$ and of resistance $\rho D/dz$ (see Figure 16). If ϕ_{AB} denotes the potential difference between A and B, the current flowing in each cell is

$$\frac{dz}{\rho D}\,(\phi_{AB} - VH_zD). \qquad 2.2.5$$

The total current I is given by

$$I = \frac{1}{\rho D}\int_{-h}^{0} (\phi_{AB} - VH_zD)\, dz$$

$$= \frac{h}{\rho D}\,(\phi_{AB} - \overline{V}H_zD). \qquad 2.2.6$$

This is the same as for a single cell of voltage

$$\overline{E} = \overline{V}H_zD \qquad 2.2.7$$

and internal resistance

$$\overline{R}_i = \rho D/h \qquad 2.2.8$$

Thus I is given by

$$I = \frac{\overline{E}}{\overline{R}_i + R_e} \qquad 2.2.9$$

and the potential difference between A and B by

$$\overline{E} - I\overline{R}_i = \frac{\overline{E}}{1 + \overline{R}_i/R_e} \qquad 2.2.10$$

Thus the induced potential is the same as if the stream moved with its mean velocity $\overline{V}$.

If the conductivity of the sea bed is relatively high ($\rho D/\rho' h >> 1$) the potential gradient will be small; the current density **i** at the free surface

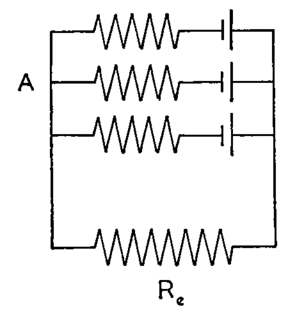

FIGURE 16

(which is what is measured by towed electrodes) will depend only on the surface stream velocity.

Suppose now that the conductivity of the sea bed is low, so that I, the return electric current, is negligible. Then we have from 2.2.6.

$$\phi_{AB} = \overline{V}H_zD. \qquad 2.2.11$$

The potential gradient is given by

$$\frac{\partial\phi}{\partial x} = \overline{V}H_z \qquad 2.2.12$$

and the current-density i_x by

$$\rho i_x = VH_z - \frac{\partial\phi}{\partial x} = (V - \overline{V})\, H_z. \qquad 2.2.13$$

If the stream extends uniformly to the bottom, $\overline{V} = V$ and $\rho i_x = 0$; the towed electrodes would then record zero signal (from the vertical field, at least). On the other hand the largest velocities in ocean currents are usually confined to the surface, so that $\overline{V} << V_o$ where V_o is the surface velocity. In this case

$$\frac{\partial\phi}{\partial x} = \overline{V}H_z << V_o H_z, \qquad 2.2.14$$

i.e. most of the induced e.m.f. is short-circuited, not through the sea bed, but through the lower, relatively slow-moving, parts of the ocean itself. We then have

$$\rho i_x = V_o H_z \qquad 2.2.15$$

very nearly, so that the towed electrodes give the "expected" signal. We may note that the conductivity of the sea bed, so far as it has any effect, tends to diminish the potential gradient and to increase the current-density, making it more nearly equal to the "expected" value.

Suppose that the streams extend uniformly to a depth h' below the upper surface, with strength V_o, and are zero below this depth (see Figure 19). Then we have

$$\overline{V} = V_o h'/h \qquad 2.2.16$$

and

$$\rho i_x = V_o H_z (1 - h'/h) \qquad 2.2.17$$

Thus the "expected" signal is reduced by a relative amount h'/h, which is small when the surface streams are very shallow.

Even if the velocity-distribution is not of exactly the above form, but the velocities are concentrated mainly near the upper surface, it is sometimes convenient to define the *equivalent depth* h_{equ} of the streams by the equation

$$\overline{V} = V_o h_{\text{equ}}/h \qquad 2.2.18$$

so that ρi_x is given in terms of V_o and h_{equ} by

$$\rho i_x = V_o H_z (1 - h_{\text{equ}}/h). \qquad 2.2.19$$

2.3 The Field Due to H_x.

The horizontal component of magnetic field H_x will induce a vertical component of e.m.f. equal to VH_x (acting downwards). In the case considered in Section 2.2., where the streams extend

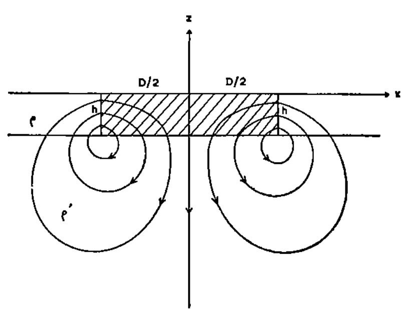

Figure 17. The electrical current-density due to H_x (sketch only).

for a width D, and may vary vertically. this will produce a symmetrical circulation of current as shown in Figure 17. Each half of the system may be compared to a simple circuit similar to that of Figure 16, where now

$$E \sim \overline{V} H_x h, \quad R_i \sim \rho D/h, \quad R_e \sim \rho'. \qquad 2.3.1$$

Since the resistances are of the same order as before, but the e.m.f. E is reduced by a factor (h/D) (H_x/H_z), the electrical current I will also be

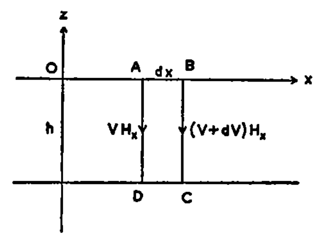

Figure 18

reduced by this factor, in general. When H_x/H_z is of order unity, the factor is small, since $h/D << 1$ by hypothesis.

Thus H_x will contribute little, in general, to the horizontal electrical current-density or to the horizontal potential gradient. It will however produce a vertical potential gradient equal, very nearly, to VH_x. This is approximately the gradient that would be measured by a pair of electrodes suspended, one below the other, in the water.

Suppose, however, that we are in a region where the water velocity varies with x. In any vertical line x = constant, there is a total e.m.f. equal to $\overline{V}hH_x$ acting vertically downwards (see Figure 18). At a distance dx to the right the e.m.f. equals $(\overline{V} + d\overline{V})\, hH_x$. The total circulation of $\mathbf{v} \wedge \mathbf{H}$ round the circuit $ABCD$ in Figure 18 equals $d\overline{V} \cdot hH_x$. This must equal the circulation of $\rho\mathbf{i}$ round the same circuit. The contribution to this circulation from the horizontal component i_x is

$$\rho \left(i_x \right)_{z=-h}^{z=o} dx \qquad 2.3.2$$

and from the vertical component i_z is

$$-\int_{-h}^{o} \left(\frac{\partial i_z}{\partial x}\, dx \right) dz. \qquad 2.3.3$$

Assuming the second term is not large compared with the first, we see that

$$\rho \left(i_x \right)_{z=-h}^{z=o} dx \sim d\overline{V}\, h\, H_x. \qquad 2.3.4$$

Thus the value of $\rho\mathbf{i}$ at the upper surface will differ from that at the lower surface by an amount of the order of $h\, H_x\, d\overline{V}/dx$. If the velocity gradient $d\overline{V}/dx$ is large, a strong horizontal component of current-density will be recorded. At a discontinuity in $\overline{V}$, such as at the edge of the current in Figure 19, i_x is theoretically infinite.

2.4 Conclusions, and Some Integral Theorems

To sum up, we shall consider the form of the signal recorded by towed electrodes when the ship crosses a stream of the form shown in Figure 19, that is a stream of width D with sharply defined edges, and of an equivalent depth h' which is small compared with the total depth h.

In Figure 20 the dotted line shows the actual surface velocity as a function of the horizontal

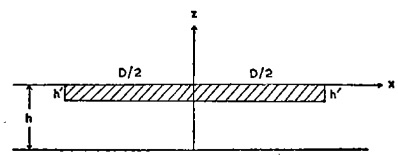

Figure 19. A cross-section of the rectangular stream.

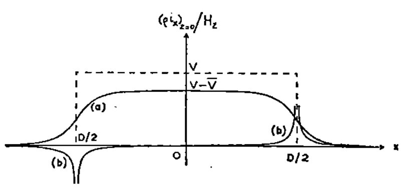

Figure 20. Form of the "apparent velocity" as measured by towed electrodes at the surface, when crossing the stream shown in Figure 19: (a) the signal due to H_z, (b) the signal due to H_x.

coordinate x. Thus the velocity equals V_o when $|x| < D/2$, and equals zero when $|x| > D/2$. The signal due to the vertical component of magnetic field is shown by the full line (a). It is a symmetrical curve which at the mid-point $x = O$ lies just below the line $V = V_o$; the departure from the theoretical value at this point is roughly equal to $\overline{V}$ (assuming that the sea-bed is effectively non-conducting). Outside the area of the stream the curve falls away to zero. The transition across the edge of the stream, however, is not as abrupt as the discontinuity in the actual velocity, since the horizontal component of current-density, which produces the signal, must always be continuous. At the discontinuity, the signal is about half its value at the centre, since if an exactly similar velocity-distribution were placed next to the first, the signal at that point would be doubled and become nearly $(V_o - \overline{V})$, as at the centre of the first stream.

The signal due to H_x is shown in Figure 20 by the curve (b). It is anti-symmetrical about the mid-point $x = O$, since the corresponding electrical current-density is in opposite directions on opposite sides of the stream. There is theoretically an infinity at the edges of the stream, but in fact a slight smoothing of the actual velocity-curve will reduce it to a finite magnitude. In general the order of magnitude of the signal is $V_o\,(h'/h)\,(H_x/H_z)$.

The total signal is the sum of the two curves (a) and (b); this will lie slightly below curve (a) on the left of the stream, and slightly above curve (b) on the right.

One property of the curves in Figure 20 may be mentioned here. From equation 1.1.4 we have

$$\frac{\partial\phi}{\partial x} = VH_z - \rho i_x. \qquad 2.4.1$$

Let this equation be integrated with respect to x right across the current, i.e. from $x = -X$ to $x = X$ where $X - D/2 >> h$. Since $\partial\phi/\partial z = -\rho i_z$, which is small there, we may write $\phi\,(x, z) = \phi_x$ and so

$$\phi_x - \phi_{-x} = \int_{-X}^{X} VH_z\,dx - \int_{-X}^{X} \rho i_x\,dx. \qquad 2.4.2$$

Thus the area beneath the curve representing the surface velocity V_s differs from the area beneath the curve representing the "apparent velocity" $\rho i_x/H_z$ by an amount $(\phi_x - \phi_{-x})/H_z$.

If the sea bed were effectively conducting, e.i. if

$$\rho/\rho' << X/h, \qquad 2.4.3$$

then $\phi_x - \phi_{-x}$ would tend to O as X tended to infinity, and we should have

$$\int_{-\infty}^{\infty} VH_z\,dx = \int_{-\infty}^{\infty} \rho i_x\,dx, \qquad 2.4.4$$

i.e. the total areas beneath the two curves would be equal. If, on the other hand, the sea bed is effectively non-conducting, for which we must have

$$\rho'/\rho >> X/h, \qquad 2.4.5$$

then we may assume

$$\int_{-h}^{o} \rho i_x\,dz = O \qquad 2.4.6$$

Therefore on integrating both sides of 2.4.2 with respect to z from $-h$ to O we have

$$h(\phi_x - \phi_{-x}) = \int_{-X}^{X} \int_{-h}^{o} VH_z \, dx \, dz = TH_z \quad 2.4.7$$

or

$$\frac{\phi_x - \phi_{-x}}{H_z} = \frac{T}{h}, \quad 2.4.8$$

where T denotes the total transport of the stream.

Hence the difference in area between the two curves in Figure 20 equals T/h. Of course, if X is increased to such an extent that the inequality 2.4.5 were reversed, then the total area between the two curves would vanish.

The theorem 2.4.8 was proved by Malkus and Stern (1952). We have seen that its validity depends on the conductivity of the sea bed being negligible.

3. ANALYTICAL METHODS

3.1 Field Equations and Boundary Conditions

The basic field equations are

$$\nabla \phi = \mathbf{v} \wedge \mathbf{H} - \rho \mathbf{i} \quad 3.1.1$$

(see equation 1.1.4) and

$$\nabla \cdot \mathbf{i} = O. \quad 3.1.2$$

Equation 3.1.2 expresses the condition that no electrical charge is created or destroyed in the fluid. At a boundary or a surface of discontinuity we have

$$\Big[\mathbf{i} \cdot \mathbf{n}\Big]_1^2 = O, \quad 3.1.3$$

where $\mathbf{n}$ is the unit normal to the surface and the suffixes 1 and 2 denote the two sides of the surface.

We have also the following equations:

(a) in the absence of magnetic material

$$\nabla \cdot \mathbf{H} = O; \quad 3.1.4$$

(b) if the magnetic field associated with the induced current-density $\mathbf{i}$ is small

$$\nabla \wedge \mathbf{H} = O; \quad 3.1.5$$

(c) if the fluid is incompressible

$$\nabla \cdot \mathbf{v} = O; \quad 3.1.6$$

(d) if the electrical conductivity is locally uniform

$$\nabla \rho = O \quad 3.1.7$$

(We may, however, allow surfaces of discontinuity in $\mathbf{v}$ or ρ.)

On taking the divergence of both sides of 3.1.1 we have

$$\begin{aligned} \nabla^2\phi &= \nabla \cdot (\mathbf{v} \wedge \mathbf{H} - \rho\mathbf{i}) \\ &= (\mathbf{H} \cdot \nabla \wedge \mathbf{v} - \mathbf{v} \cdot \nabla \mathbf{H}) \\ &\quad - (\rho \nabla \cdot \mathbf{i} - \mathbf{i} \cdot \nabla \rho). \end{aligned} \quad 3.1.8$$

Using equations 3.1.2, 3.1.5, and 3.1.7 we have

$$\nabla^2\phi = \mathbf{H} \cdot \nabla \wedge \mathbf{v}. \quad 3.1.9$$

If the water velocity is irrotational, i.e. if

$$\nabla \wedge \mathbf{v} = O, \quad 3.1.10$$

then ϕ satisfies Laplace's equation

$$\nabla^2\phi = O. \quad 3.1.11$$

At a boundary we have from 3.1.2 and 3.1.3

$$\Big[k\mathbf{n} \cdot (\nabla \phi - \mathbf{v} \wedge \mathbf{H})\Big]_1^2 = O, \quad 3.1.12$$

where $k = \rho^{-1}$, is the electrical conductivity. At the surface of an insulator the above expression is to vanish.

One consequence of the field equations may be noted. On taking the curl of both sides of equation 3.1.1 we have

$$\mathbf{O} = \nabla \wedge (\mathbf{v} \wedge \mathbf{H} - \rho\mathbf{i}) \quad 3.1.13$$

and therefore

$$\begin{aligned} \nabla \wedge (\rho\mathbf{i}) &= \nabla \wedge (\mathbf{v} \wedge \mathbf{H}) \\ &= (\mathbf{H} \cdot \nabla \mathbf{v} - \mathbf{v} \cdot \nabla \mathbf{H}) + \\ &\quad (\mathbf{v} \nabla \cdot \mathbf{H} - \mathbf{H} \nabla \mathbf{v}) \end{aligned} \quad 3.1.14$$

(see Weatherburn 1943, p. 9). Using 3.1.4 and 3.1.6 we have

$$\nabla \wedge (\rho\mathbf{i}) = \mathbf{H} \cdot \nabla \mathbf{v} - \mathbf{v} \cdot \nabla \mathbf{H}. \quad 3.1.15$$

Thus, even when $\nabla \mathbf{H}$ vanishes ($\mathbf{H}$ is uniform) $\nabla \wedge (\rho\mathbf{i})$ is not in general zero, so that $\rho\mathbf{i}$ and $\mathbf{v} \wedge \mathbf{H}$ are not in general the gradients of potential functions, although their difference is, by equation 3.1.1.

3.2 Two-dimensional motion

The solution is particularly simple when the motion is two-dimensional. Let (x, y, z) be rectangular coordinates, and suppose that the velocity is independent of y, and has no component in the y-direction. Thus

$$\mathbf{v} = (v_x, O, v_z) = \left(\frac{\partial\psi}{\partial z}, O, -\frac{\partial\psi}{\partial x}\right), \qquad 3.2.1$$

where ψ is the stream-function. It is convenient to divide the magnetic field into two parts, one parallel and one perpendicular to the y-axis:

$$\mathbf{H} = \mathbf{H}_1 + \mathbf{H}_2 \qquad 3.2.2$$

where

$$\mathbf{H}_1 = (O, H_y, O), \quad \mathbf{H}_2 = (H_x, O, H_z). \qquad 3.2.3$$

The parts of ϕ and $\mathbf{i}$ corresponding to $\mathbf{H}_1$ and $\mathbf{H}_2$ will be denoted by suffixes 1 and 2.

Consider first the field due to $\mathbf{H}_1$. The induced e.m.f. may be written

$$\mathbf{v} \wedge \mathbf{H} = \left(\frac{\partial\psi}{\partial x}H_y, O, \frac{\partial\psi}{\partial z}H_y\right) = \nabla\,(H_y\,\psi). \qquad 3.2.4$$

This vector lies in the (x, z)-plane and is everywhere at right-angles to the stream-lines $\psi =$ constant. Further

$$\rho\mathbf{i}_1 = \mathbf{v} \wedge \mathbf{H}_1 - \nabla\,\phi_1 = \nabla\,(H_y\,\psi - \phi_1). \qquad 3.2.5$$

Now the field equation 3.1.9 and the boundary condition 3.1.12, for ϕ_1, follow from the conditions that $\nabla \cdot \mathbf{i}_1$ shall vanish (3.1.2) and that the normal component of $\mathbf{i}_1$ shall be continuous (3.1.3). These conditions, and 3.2.5, are identically satisfied by

$$\mathbf{i}_1 = O, \qquad \phi_1 = H_y\,\psi, \qquad 3.2.6$$

which is therefore the solution. In other words the e.m.f. is such that no electrical current flows, and the stream-lines are all equipotential surfaces. In particular, on each fixed boundary, which must be a stream-line, the potential is constant, whether or not it is the boundary of a perfect conductor.

Consider now the field due to $\mathbf{H}_2$. We have

$$\mathbf{v} \wedge \mathbf{H}_2 = \left(O, -H_x\frac{\partial\psi}{\partial x} - H_z\frac{\partial\psi}{\partial z}, O\right). \qquad 3.2.7$$

This vector is always parallel to the y-axis, and tends to make current flow in that direction. Further, it is independent of y. Therefore, if there are no boundaries hindering the path of the current, we have

$$\rho\mathbf{i}_2 = \left(O, -H_x\frac{\partial\psi}{\partial x} - H_z\frac{\partial\psi}{\partial z}, O\right), \quad \phi_2 = O. \qquad 3.2.8$$

In other words, the current flows freely, and the potential gradient is reduced to zero. However, not only must all the boundaries be two-dimensional, but the current must be free to flow to infinity in the y-direction. Whether this is true will depend on the conditions of the problem. It will be true if the total current demanded from infinity is zero, i.e. if

$$\iint_S \mathbf{v} \wedge \mathbf{H}_2 \cdot dx\,dz = O \qquad 3.2.9$$

where S is the total area of the (x, z) plane in which there is motion, or a typical part of the plane, supposing the motion is periodic in space (as for waves, see Section 7.1). If 3.2.9 is not satisfied there may still be a free flow of current if there is an external path for the current of sufficiently low resistance. For example, if the streams extend a finite distance Y in the y-direction in water of depth h, a sufficient condition is

$$Y/h >> \rho'/\rho \qquad 3.2.10$$

where ρ and ρ' are mean resistivities of the sea and sea bed (compare Section 2.1). But in general, when neither 3.2.9 nor 3.2.10 is satisfied, an additional current-density $\mathbf{i}_2'$ must be superposed on that given by 3.2.8.

3.3 A Generalization of the Previous Solution

The simplicity of the solution for two-dimensional motion suggests a generalization. By the equation of continuity 3.1.6 we may write in general

$$\mathbf{v} = -\nabla \wedge \Psi \qquad 3.3.1$$

where Ψ is a vector such that

$$\nabla \cdot \Psi = O \qquad 3.3.2$$

(see for example Weatherburn, 1943, p. 44). Ψ is the vector stream function of the motion. In the case of two-dimensional flow we have simply

$$\Psi = (O, \psi, O), \qquad 3.3.3$$

where $\psi\,(x, z)$ is the two-dimensional stream function. Ψ can be regarded as determined by the velocity field $\mathbf{v}$. Since

$$\begin{aligned}\nabla \wedge \mathbf{v} &= -\nabla \wedge \nabla \wedge \Psi\\ &= -\nabla\,(\nabla \cdot \Psi) + \nabla^2\,\Psi\\ &= \nabla^2\,\Psi,\end{aligned} \qquad 3.3.4$$

a general expression for Ψ, assuming $\mathbf{v}$ to vanish at infinity, is

$$\Psi\left(x, y, z\right) = -\frac{1}{4\pi} \times \iiint \frac{\nabla \wedge \mathbf{v}\,(x', y', z')\,dx'\,dy'\,dz'}{[(x-x')^2 + (y-y')^2 + (z-z')^2]^{1/2}}, \qquad 3.3.5$$

the integral being taken over all space (see Weatherburn, 1943, p. 45).

From 3.3.4 we see that the differential equation for ϕ is

$$\nabla^2\phi = \mathbf{H} \cdot \nabla^2\Psi. \qquad 3.3.6$$

If $\mathbf{H}$ is uniform, this may be written

$$\nabla^2\chi = 0 \qquad 3.3.7$$

where

$$\chi = \phi - \mathbf{H} \cdot \Psi. \qquad 3.3.8$$

The boundary-condition at an insulator is

$$\mathbf{n} \cdot \nabla\phi = \mathbf{n} \cdot \mathbf{v} \wedge \mathbf{H} = \mathbf{n} \cdot \mathbf{H} \wedge (\nabla \wedge \Psi). \qquad 3.3.9$$

But if $\mathbf{H}$ is a constant vector we have identically

$$\nabla (\mathbf{H} \cdot \Psi) = \mathbf{H} \cdot \nabla \Psi + \mathbf{H} \wedge (\nabla \wedge \Psi). \qquad 3.3.10$$

Equation 3.3.9 may therefore be written

$$\mathbf{n} \cdot \nabla\phi = \mathbf{n} \cdot [\nabla (\mathbf{H} \cdot \Psi) - \mathbf{H} \cdot \nabla \Psi] \qquad 3.3.11$$

or

$$\mathbf{n} \cdot \nabla \chi = -\mathbf{n} \cdot \nabla (\mathbf{H} \cdot \nabla \Psi). \qquad 3.3.12$$

The problem is thus reduced to that of finding a solution χ of Laplace's equation 3.3.7 to satisfy the boundary condition 3.3.12. The right-hand side of 3.3.12 does not in general vanish, but when it does so, as in the two-dimensional case we have simply

$$\chi = 0 \qquad 3.3.13$$

and so

$$\phi = \mathbf{H} \cdot \Psi = H_y\psi. \qquad 3.3.14$$

4. A STREAM WITH SINUSOIDAL PROFILE

4.1 Introduction

In this and the two following Chapters we shall give a precise determination of the electrical field in three cases when the stream-velocity is given by simple analytical expressions. These examples will not only serve to illustrate the general principles stated in Chapter 2, but will also provide quantitative estimates of the different anomalies that are to be expected.

As in Chapter 2 it is supposed that the stream-velocity $\mathbf{v}(x, z)$ is everywhere parallel to the y-axis, (x, y, z) being rectangular coordinates, and z being measured vertically upwards from the "sea surface." ρ and ρ' denote the resistivities of the "sea" and "sea bed" respectively, which for simplicity are assumed to be uniform. From 3.1.9 the field equation for the potential is then

$$\nabla^2\phi = H_z \frac{\partial v_y}{\partial x} - H_x \frac{\partial v_y}{\partial z} \qquad 4.1.1$$

in the "sea" and

$$\nabla^2\phi = 0 \qquad 4.1.2$$

in the "sea bed."

In the present Chapter we shall suppose that the velocity $\mathbf{v}$ is given by

$$v_y = \begin{cases} V\cos\alpha x, & 0 > z > -h', \\ 0, & -h' > z > -h, \end{cases} \qquad 4.1.3$$

where h is the total depth, that is, the velocity is confined to a layer of depth h' in the upper part of the sea, being zero below this depth, and the velocity varies sinusoidally across the stream. The field due to H_z has been previously evaluated and partly discussed by Stommel (1948). We shall also evaluate the field due to H_x and give a more complete discussion.

4.2 The Field Due to H_z

If ϕ_1, ϕ_2 and ϕ_3 denote the values of ϕ in the three regions $0 > z > -h'$, $-h' > z > -h$ and $-h > z > -\infty$ we have as field equations

$$\nabla^2\phi_1 = -VH_z\alpha\sin\alpha x, \quad \nabla^2\phi_2 = 0, \quad \nabla^2\phi_3 = 0. \qquad 4.2.1$$

From 3.1.12 and the condition that ϕ shall be continuous, we have the boundary conditions

$$\left.\begin{array}{ll} \text{when } z = 0, & \dfrac{\partial\phi_1}{\partial z} = 0 \\ \text{when } z = -h', & \begin{cases} \phi_1 - \phi_2 = 0 \\ \dfrac{\partial\phi_1}{\partial z} - \dfrac{\partial\phi_2}{\partial z} = 0 \end{cases} \\ \text{when } z = -h, & \begin{cases} \phi_2 - \phi_3 = 0 \\ \dfrac{1}{\rho}\dfrac{\partial\phi_2}{\partial z} - \dfrac{1}{\rho'}\dfrac{\partial\phi_3}{\partial z} = 0 \end{cases} \\ \text{when } z \to -\infty, & \phi_3 \to 0 \end{array}\right\} \qquad 4.2.2$$

Equations 4.2.1 and 4.2.2 show that

$$(\phi_1 - VH_z\alpha^{-1}\sin\alpha x),$$

ϕ_2 and ϕ_3 must be the sums of functions of the form $e^{\pm\alpha z}\sin\alpha x$. From the last of equations 4.2.2, ϕ_3 is a multiple of $e^{+\alpha z}\sin\alpha x$ only. It can be verified by direct substitution that the full solution is

$$\phi_1 = \frac{VH_z}{\alpha}\left[1 - \frac{\sinh\alpha(h-h') + (\rho/\rho')\cosh\alpha(h-h')}{\sinh\alpha h + (\rho/\rho')\cosh\alpha h} \times \cosh\alpha z\right]\sin\alpha x$$

$$\phi_2 = \frac{VH_z}{\alpha}\sinh\alpha h' \times \frac{\cosh\alpha(z+h) + (\rho/\rho')\sinh\alpha(z+h)}{\sinh\alpha h + (\rho/\rho')\cosh\alpha h}\sin\alpha x$$

$$\phi_3 = \frac{VH_z}{\alpha}\sinh\alpha h' \times \frac{e^{\alpha z}}{\sinh\alpha h + (\rho/\rho')\cosh\alpha h}\sin\alpha x. \qquad 4.2.3$$

Thus the horizontal component of the potential gradient at the free surface is

$$\left(\frac{\partial\phi_1}{\partial x}\right)_{z=0} = VH_z(1-\gamma)\cos\alpha x \qquad 4.2.4$$

and the horizontal component of the electrical current-density is

$$(\rho i_z)_{z=0} = VH_z\,\gamma\cos\alpha x \qquad 4.2.5$$

where

$$\gamma = \frac{\sinh\alpha(h-h') + (\rho/\rho')\cosh\alpha(h-h')}{\sinh\alpha h + (\rho/\rho')\cosh\alpha h}. \qquad 4.2.6$$

The "apparent" velocity is everywhere proportional to the actual velocity but reduced by the constant factor γ.

(a) Suppose the streams are broad compared with the total depth of water. Then αh, $\alpha(h-h')$ and αz are small compared with unity and $\sinh\alpha h$, etc., may be replaced by αh, etc. We have then

$$\gamma = \frac{\alpha(h-h') + (\rho/\rho')}{\alpha h + (\rho/\rho')} \qquad 4.2.7$$

and

$$\left(\frac{\partial\phi_1}{\partial x}\right)_{z=0} = \frac{VH_z\,\alpha h'}{\alpha h + (\rho/\rho')}\cos\alpha x. \qquad 4.2.8$$

This shows that the effect of the conductivity of the bottom on the potential gradient is small provided

$$\rho/\rho' << \alpha h \qquad 4.2.9$$

or

$$\rho a/\rho' h << 1, \qquad 4.2.10$$

where $2a$, $= 2\pi/\alpha$, is the "width" of the stream. Equation 4.2.10 is similar to the condition $\rho D/\rho' h << 1$ found in Section 2.1. Further we may write

$$\left(\frac{\partial\phi_1}{\partial x}\right)_{z=0} = \frac{VH_z(h'/h)}{1+\pi^{-1}(\rho a/\rho' h)}\cos\alpha x, \qquad 4.2.11$$

which illustrates the theorem of Section 2.2, namely that the potential gradient depends only on the mean velocity in each vertical line (in this case $V(h'/h)\cos\alpha x$). When the conductivity of the bottom is negligible we have simply

$$\gamma = 1 - h'/h \qquad 4.2.12$$

and

$$\left(\frac{\partial\phi_1}{\partial x}\right)_{z=0} = VH_z(h'/h)\cos\alpha x. \qquad 4.2.13$$

In particular (1) when the stream extends to the bottom, ($h' = h$), we have $\gamma = 0$ and $(\partial\phi_1/\partial x)_{z=0} = VH_z\cos\alpha x$. This is because there is no stationary water to form a return path for the electric current circulation. The "apparent" velocity, proportional to ρi_z, therefore vanishes. (2) when the stream is very shallow (i.e. $h' << h$ and $\alpha h << 1$) we have $\gamma = 1$ and $(\partial\phi_1/\partial x)_{z=0} = 0$. The potential gradient is now completely short-circuited by the water below and the full "apparent" velocity is recorded.

(b) Suppose the streams are very narrow compared with the total depth of the "ocean." We have then $\alpha h >> 1$, so that $\sinh\alpha h$ and $\cosh\alpha h$ can both be replaced by $\frac{1}{2}e^{\alpha h}$. We assume also that $\alpha(h-h') >> 1$. Thus from 4.2.4 – 4.2.6

$$\gamma = e^{-\alpha h'} \qquad 4.2.14$$

and

$$\left(\frac{\partial\phi_1}{\partial x}\right)_{z=0} = VH_z(1-e^{-\alpha h'})\cos\alpha x. \qquad 4.2.15$$

The effect of the bottom is then negligible in any case (the ocean, in effect, is infinitely deep) and γ depends only on $\alpha h'$, or on the ratio of the depth of the surface streams to their width. When the streams are shallow compared with their width ($\alpha h' << 1$), γ is nearly unity, and the maximum "apparent" velocity is recorded. When the streams are very deep and narrow ($\alpha' h >> 1$), γ tends to zero. This case, however, is unlikely to occur in practice.

4.3 The Field Due to H_x

We have now as field equations

$$\nabla^2 \phi_1 = 0, \quad \nabla^2 \phi_2 = 0, \quad \nabla^2 \phi_3 = 0 \qquad 4.3.1$$

and as boundary conditions

$$\text{when } z = 0, \qquad \frac{\partial \phi_1}{\partial z} = - VH_x \cos \alpha x,$$

$$\text{when } z = -h', \begin{cases} \phi_1 - \phi_2 = 0, \\ \dfrac{\partial \phi_1}{\partial z} - \dfrac{\partial \phi_2}{\partial z} = - VH_x \cos \alpha x, \end{cases}$$

$$\text{when } z = -h, \begin{cases} \phi_2 - \phi' = 0, \\ \dfrac{1}{\rho}\dfrac{\partial \phi_2}{\partial z} - \dfrac{1}{\rho'}\dfrac{\partial \phi_3}{\partial z} = 0, \end{cases}$$

$$\text{when } z \to -\infty \quad \phi_3 \to 0. \qquad 4.3.2$$

It may be verified that the solution of these equations is

$$\phi_1 = \frac{VH_x}{\alpha}\Bigg[\frac{(\cosh \alpha(h - h') - \cosh \alpha h)}{\sinh \alpha h} + \frac{(\rho/\rho')\ (\sinh \alpha(h - h') - \sinh \alpha h)}{\sinh \alpha h} \cosh \alpha z - \sinh \alpha z\Bigg] \cos \alpha x,$$

$$\phi_2 = \frac{VH_x}{\alpha} \frac{\cosh \alpha h' - 1}{\sinh \alpha h} \times \Big[\cosh \alpha(z+h) + (\rho/\rho') \sinh \alpha(z+h)\Big] \cos \alpha x,$$

$$\phi_3 = \frac{VH_x}{\alpha} \frac{\cosh \alpha h' - 1}{\sinh \alpha h} e^{\alpha(z+h)} \cos \alpha x. \qquad 4.3.3$$

Thus

$$\left(\frac{\partial \phi_1}{\partial x}\right)_{z=0} = V H_x \gamma' \sin \alpha x \qquad 4.3.4$$

and

$$(\rho i_z)_{z=0} = - V H_x \gamma' \sin \alpha x \qquad 4.3.5$$

where

$$\gamma' = \frac{[\cosh \alpha h - \cosh \alpha(h - h')]}{\sinh \alpha h} + \frac{(\rho/\rho')\ [\sinh \alpha h - \sinh \alpha(h - h')]}{\sinh \alpha h}. \qquad 4.3.6$$

$(\rho i_z)_{z=0}$ therefore represents a sinusoidal signal which is in quadrature with the stream velocity, that is, it is a maximum or a minimum where the stream velocity is zero, and vice versa.

(a) When the width of the stream is large compared with the total depth ($\alpha h << 1$), then on replacing $\cosh \alpha h$ by $1 - \frac{1}{2}\alpha^2 h^2$ etc., we find

$$\gamma' = [\alpha(2h - h') + (\rho/\rho')]\ (h'/h). \qquad 4.3.7$$

The *relative* effect of the conductivity of the bottom is therefore small under the same conditions as in the previous Section, namely when $\rho/\rho' << \alpha h$; but the absolute value of γ' is in any case only of order αh at most, and may be even smaller for shallow streams, on account of the factor h'/h.

(b) When the streams are very narrow compared with the total depth ($\alpha h >> 1$ and $\alpha(h - h') >> 1$), then we have

$$\gamma' = (1 - e^{-\alpha h'})\ (1 + \rho/\rho'). \qquad 4.3.8$$

ρ/ρ' being small, the second factor can be replaced by unity. The situation is now the reverse of that for the signal due to the vertical component of magnetic field. For now when the streams are shallow ($\alpha h' << 1$) γ' is small, but when they are deep ($\alpha h' >> 1$) γ' is nearly unity. In general γ' is of the same order as $\alpha h'$, the ratio of the depth of the stream to its width.

4.4 The Combined Field

The combined signal $(\rho i_z)_{z=0}$ will be a sine-wave:

$$(\rho i_z)_{z=0} = V H_o \cos(\alpha x + \epsilon) \qquad 4.4.1$$

where

$$H_o = (H_z^2 \gamma^2 + H_x^2 \gamma'^2)^{1/2} \qquad 4.4.2$$

and

$$\tan \epsilon = \frac{H_x \gamma'}{H_z \gamma}, \qquad 4.4.3$$

ϵ being the apparent phase-shift. In the case when the streams are broad compared with the total depth and the conductivity of the bottom is negligible we find from 4.2.7 and 4.3.7.

$$\frac{\gamma'}{\gamma} = \frac{\alpha h'\ (2h - h')}{h - h'} \qquad 4.4.4$$

and when the streams are narrow compared with the total depth

$$\frac{\gamma'}{\gamma} = e^{\alpha h'} - 1. \qquad 4.4.5$$

When $\alpha h'$ and h'/h are both small (the streams are broad surface streams) both these expressions become, approximately

$$\frac{\gamma'}{\gamma} = \alpha h'. \qquad 4.4.6$$

The effect of the horizontal component of field, therefore, is to shift the whole pattern of velocities through a distance

$$h' = h\, H_x/H_z \qquad 4.4.7$$

laterally, that is through a distance comparable with the depth of the surface streams.

5. A STREAM OF RECTANGULAR CROSS-SECTION

5.1 Introduction

In this Chapter we shall consider the more realistic case when the streams are constant over a certain interval of the x-axis, of width $2a$, say, and are zero outside this region, as in Figure 19, where $2a = D$. To be precise, we assume that

$$v_y = \begin{cases} V, & -a < x < a, \quad 0 \geq z > -h', \\ 0, & \text{elsewhere} \end{cases} \qquad 5.1.1$$

The solution to this problem is especially useful, since from it one can build up the solutions to more complicated problems, and approximate to the actual electrical field to any degree of accuracy.

Since the equations governing the present problem are linear in the velocities, it follows that any two or more solutions may be superposed, in the sense that the current-density ρi_x generated by the sum of a number of different velocity-distributions is the sum of the current-densities due to each velocity-distribution separately. Now any velocity-distribution $v_y(x, z)$ which is (a) symmetrical about the line $x = 0$, (b) uniform down to $z = -h'$ and (c) zero below this depth, can be represented as the sum of sinusoidal velocity-distribution of the type discussed in Chapter 4. For

$$v_y(x, z) = \begin{cases} \int_{-\infty}^{\infty} v_y^*(\alpha) \cos \alpha x \, d\alpha, & 0 \geq z > -h', \\ 0, & -h' > z > -h. \end{cases} \qquad 5.1.2$$

where $v_y^*(\alpha)$ is the Fourier cosine transform of $v_y(x, 0)$:

$$v_y^*(\alpha) = \frac{1}{2\pi}\int_{-\infty}^{\infty} v_y(x, 0) \cos \alpha x \, dx = \frac{1}{2\pi}\int_{-\infty}^{\infty} v_y(x, 0)\, e^{i\alpha x}\, dx. \qquad 5.1.3$$

If the current-density ρi_x due to the sinusoidal velocity-distribution of Chapter 4 is of the form $V f(x; \alpha)$, the current-density due to the distribution 5.1.1 is given by

$$\rho i_x = \int_{-\infty}^{\infty} v_y^*(\alpha)\, f(x\,;\,\alpha)\, d\alpha, \qquad 5.1.4$$

which represents, formally, the solution to the problem.

In the present case, when $v_y(x, 0)$ vanishes except when $-a < x < a$ we have from 5.1.3.

$$v_y^*(\alpha) = \frac{V}{\pi}\frac{\sin \alpha x}{\alpha}. \qquad 5.1.5$$

5.2 The Field Due to H_z

We shall suppose that the conductivity of the "sea bed" is negligible. Then the current-density at the surface associated with the sinusoidal velocity-distribution is found from 4.1.5 and 4.1.6 to be

$$(\rho i_x)_{z=0} = VH_z \frac{\sinh \alpha(h - h')}{\sinh \alpha h} \cos \alpha x. \qquad 5.2.1$$

The current-density due to the rectangular stream is therefore

$$(\rho i_x)_{z=0} = \frac{VH_z}{\pi}\int_{-\infty}^{\infty} \frac{\sinh\, \alpha(h - h')}{\sinh \alpha h} \times \cos \alpha x \frac{\sin \alpha a}{\alpha}\, d\alpha. \qquad 5.2.2$$

To evaluate this integral we put $\alpha h = \beta$ and write

$$(\rho i_x)_{z=0} = \frac{VH_z}{2\pi}\int_{-\infty}^{\infty} \frac{\sinh \frac{h - h'}{h}\beta}{\sinh \beta} \left(\sin \frac{a + x}{h}\beta + \sin \frac{a - x}{h}\beta\right)\frac{d\beta}{\beta} \qquad 5.2.3$$

$$= \frac{VH_z}{2\pi}\left[I\left(\frac{h - h'}{h}, \frac{a + x}{h}\right) + I\left(\frac{h - h'}{h}, \frac{a - x}{h}\right)\right] \qquad 5.2.4$$

where

$$I\ (p, q) = \int_{-\infty}^{\infty} \frac{\sinh p\beta}{\sinh \beta} \frac{\sin q\beta}{\beta}\, d\beta \qquad 5.2.5$$

Now

$$\frac{\partial I}{\partial q} = \int_{-\infty}^{\infty} \frac{\sinh p\beta}{\sinh \beta} \cos q\beta\, d\beta = \int_{-\infty}^{\infty} \frac{\sinh p\beta}{\sinh \beta} e^{iq\beta}\, d\beta\,. \qquad 5.2.6$$

This may be evaluated by means of contour integration. Consider the rectangular contour C

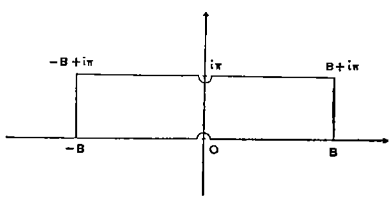

FIGURE 21. Contour of integration in the β-plane.

determined by the points $-$ B, B, B $+ i\pi$, $-$ B $+ i\pi$ and indented at the points o and $i\pi$ (see Figure 21). The function

$$\frac{e^{(p+iq)\beta}}{\sinh \beta} \qquad 5.2.7$$

has no singularities inside C, and, if $|p| < 1$, it tends to zero uniformly on the vertical sides of the rectangle. Hence we have, by Cauchy's residue theorem,

$$\int_{-\infty}^{\infty} \frac{e^{(p+iq)\beta}}{\sinh \beta} d\beta - \int_{-\infty}^{\infty} \frac{e^{(p+iq)(\beta+i\pi)}}{\sinh (\beta + i\pi)} d\beta = i\pi(1 - e^{(p+iq)i\pi}) \qquad 5.2.8$$

or

$$\int_{-\infty}^{\infty} \frac{e^{(p+iq)\beta}}{\sinh \beta} d\beta\, (1 + e^{(p+iq)i\pi}) = i\pi\, (1 - e^{(p+iq)i\pi}) \qquad 5.2.9$$

and so

$$\int_{-\infty}^{\infty} \frac{e^{(p+iq)\beta}}{\sinh \beta} d\beta = \pi \tan\, (\rho + iq)\pi/2. \qquad 5.2.10$$

On subtracting the corresponding formula for $-p$, and dividing by 2, we find

$$\frac{\partial I}{\partial q} = \frac{\pi}{2}[\tan\, (p + iq)\pi/2 + \tan\, (p - iq)\pi/2]. \qquad 5.2.11$$

Now $I\ (p, q)$ vanishes when $q = 0$. Therefore on integrating both sides of 5.2.11 with respect to q we have

$$\begin{aligned} I &= i\, [\log \cos\, (p + iq)\pi/2 - \log \cos\, (p - iq)\pi/2] \\ &= -\,2\, \mathrm{Im} \log \cos\, (p + iq)\pi/2 \\ &= -\,2 \arg \cos\, (p + iq)\pi/2 \\ &= 2 \tan^{-1}\left(\tan \frac{p\pi}{2} \tanh \frac{q\pi}{2}\right). \end{aligned} \qquad 5.2.12$$

Thus the current-density, from 5.2.4, is given by

$$(\rho i_x)_{z=0} = \frac{VH_z}{\pi}\left[\tan^{-1}\left(\tan \frac{h-h'}{h}\frac{\pi}{2} \tanh \frac{x+a}{h}\frac{\pi}{2}\right) - \tan^{-1}\left(\tan \frac{h-h'}{h}\frac{\pi}{2} \tanh \frac{x-a}{h}\frac{\pi}{2}\right)\right], \qquad 5.2.13$$

which is the formal solution of the problem.

Equation 5.2.13 can be written

$$(\rho i_x)_{z=0} = VH_z \left[F\left(\frac{x+a}{h}\right) - F\left(\frac{x-a}{h}\right)\right], \qquad 5.2.14$$

where

$$F\,(\xi) = \frac{1}{\pi}\tan^{-1}\left(\tan \frac{h-h'}{h}\frac{\pi}{2} \tanh \frac{\xi\pi}{2}\right). \qquad 5.2.15$$

Thus $(\rho i_x)_{z=0}$ is the difference of two terms of the same form, centered on $x = -a$ and $x = a$ respectively. We may say that $VH_z F\,(\xi)$ represents the voltage gradient due to a discontinuity in the surface velocity, of magnitude V, at $\xi = 0$. The voltage gradient 5.2.14 is the result of two such discontinuities, one at $x = -a$ and the other at $x = a$.

The form of $F(\xi)$ is shown in Figure 22 for various values of h'/h. As ξ passes from $-\infty$ to ∞, so $F(\xi)$ passes from $-\frac{1}{2}\frac{h-h'}{h}$ to $\frac{1}{2}\frac{h-h'}{h}$, and the transition takes place in a ξ-interval of order unity, that is, in an x-interval of order h, the total depth. The limiting values of $F(\xi)$ are approached exponentially. Thus when the width $2a$ of the stream is greater than the depth h, the two transition curves hardly interfere with one another. At the centre $x = 0$ the voltage gradient is then given by

$$(\rho i_x)_{z=0} = VH_z \frac{h-h'}{h} = (V_s - \overline{V})\, H_z \qquad 5.2.16$$

and at the end-points it is about half this value, as was shown in Section 2.

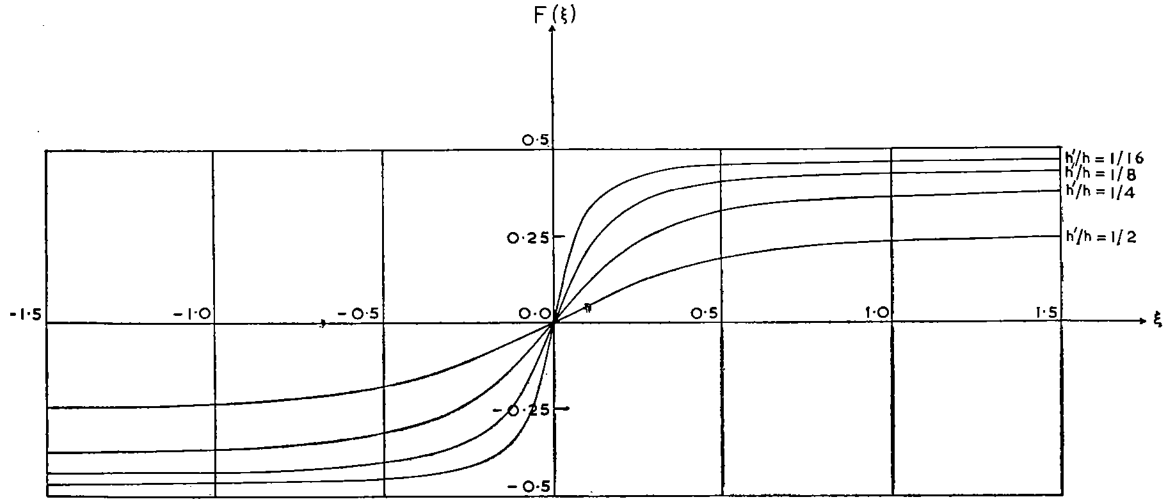

FIGURE 22. Graph of $F(\xi)$ for $h'/h = \frac{1}{2}, \frac{1}{4}, \frac{1}{8}$ and $\frac{1}{16}$, showing the form of the electrical current-density due to H_z near one edge of the stream.

However, when h'/h is small

$$\tan \frac{h - h'}{h}\frac{\pi}{2} = \cot \frac{h'}{h}\frac{\pi}{2} \doteqdot \frac{h}{h'}\frac{2}{\pi}, \qquad 5.2.17$$

so that when $1 > \xi > 2\,h'/h$, say,

$$F(\xi) \doteqdot \frac{1}{\pi}\left(\frac{\pi}{2} - \frac{h'/h}{\xi}\right). \qquad 5.2.18$$

This shows that the greater part of the transition has taken place within a ξ-interval of order $2h'/h$, that is, within a distance of order $2h'$ from the edge of the stream.

An interesting case is when a/h is small, that is, when the stream is very narrow. This is the case of a narrow "pulse" of water velocity at $x = 0$. Its contribution to the measured voltage can be found by differenting $F(\xi)$:

$$(\rho i_z)_{z=0} = 2a\,\frac{d}{dx}\,VH_z F\left(\frac{x}{h}\right)$$

$$= \frac{a}{h} VH_z \frac{\tan \frac{h - h'}{h}\frac{\pi}{2}}{\cosh^2 \frac{\xi\pi}{2} + \tanh^2 \frac{h - h'}{h}\frac{\pi}{2}\,\sinh^2 \frac{\xi\pi}{2}}. \qquad 5.2.19$$

5.3 THE FIELD DUE TO H_x.

From equation 4.2.3, the current density associated with the sinusoidal velocity-distribution is

$$\rho i_x = -\,VH_x \times \left[\frac{\cosh \alpha(z+h) - \cosh \alpha(h-h')\cosh \alpha z}{\sinh \alpha h}\right] \sin \alpha x \qquad 5.3.1$$

(the conductivity of the bottom being neglected). By 5.1.4 the current-density for the stream of rectangular cross-section is then

$$(\rho i_x)_{z=0} = \lim_{z \to 0} H_x \int_{-\infty}^{\infty} v_y^*(\alpha) \times \frac{\cosh \alpha(z+h) - \cosh \alpha(h-h')}{\sinh \alpha h} \sin \alpha x \, d\alpha \qquad 5.3.2$$

(z has been replaced by 0 in $\cosh \alpha z$ but is retained in $\cosh \alpha(z + h)$ in order to make the subsequent integrals determinate). If we now follow the same procedure as for the vertical magnetic field H_z we find that the corresponding integrals are divergent. We therefore proceed as follows: Since $v_y^*(\alpha)$ is an even function of α we have

$$(\rho i_x)_{z=0} = \lim_{z \to 0} iH_x \int_{-\infty}^{\infty} v_y^*(\alpha) \left[\frac{\cosh \alpha(x+h) - \cosh \alpha(h-h')}{\sinh \alpha h}\right] e^{i\alpha x}\, d\alpha. \qquad 5.3.3$$

On substituting from 5.1.3 and changing the order of integration we have

$$(\rho i_x)_{z=0} = \lim_{z \to 0} \int_{-\infty}^{\infty} v_y(x', 0)\, A(x + x')\, dx' \qquad 5.3.4$$

where

$$A(x) = \frac{i}{2\pi} \int_{-\infty}^{\infty} \frac{\cosh \alpha(z+h) - \cosh \alpha(h-h')}{\sinh \alpha h} e^{i\alpha x} d\alpha. \qquad 5.3.5$$

It will be seen from 5.3.4 that $A(x)$ describes the current-density from a pulse distribution of velocity, of infinitesimal width, at the origin. This is now to be multiplied by the velocity and integrated across the stream. In the case of the pulse-function 5.1.1 we have simply

$$(\rho i_x)_{z=0} = \lim_{z \to 0} VH_x \int_{-a}^{a} A(x+x')\,dx'. \qquad 5.3.6$$

Now from 5.2.10

$$\int_{-\infty}^{\infty} \frac{\cosh p\beta}{\sinh \beta} e^{iq\beta} d\beta = \frac{\pi}{2}\left[\tan (p+iq)\frac{\pi}{2} - \tan (p-iq)\frac{\pi}{2}\right] \qquad 5.3.7$$

and so

$$A(x) = \frac{i}{4h}\left[\tan\left(\frac{h+z+ix}{h}\frac{\pi}{2}\right) - \tan\left(\frac{h+z-ix}{h}\frac{\pi}{2}\right) - \tan\left(\frac{h-h'+ix}{h}\frac{\pi}{2}\right) + \tan\left(\frac{h-h'-ix}{h}\frac{\pi}{2}\right)\right]. \qquad 5.3.8$$

Therefore from 5.3.6

$$(\rho i_x)_{z=0} = \lim_{z \to 0} \frac{VH_x}{2\pi} \times \left[\log \frac{\cos\left(\frac{h-h'+ix}{h}\frac{\pi}{2}\right)\cos\left(\frac{h-h'-ix}{h}\frac{\pi}{2}\right)}{\cos\left(\frac{h+z+ix}{h}\frac{\pi}{2}\right)\cos\left(\frac{h+z-ix}{h}\frac{\pi}{2}\right)}\right]_{-a-x}^{a-x}$$

$$= \lim_{z \to 0} \frac{VH_x}{2\pi}\left[\log \frac{\cosh\frac{x\pi}{h} - \cos\frac{h'\pi}{h}}{\cosh\frac{x\pi}{h} - \cos\frac{z\pi}{h}}\right]_{x+a}^{x-a} \qquad 5.3.9$$

We may now write $z = 0$ and obtain finally

$$(\rho i_x)_{z=0} = VH_x\left[G\left(\frac{x+a}{h}\right) - G\left(\frac{x-a}{h}\right)\right] \qquad 5.3.10$$

where

$$G(\xi) = -\frac{1}{2\pi}\log\left(\frac{\cosh \xi\pi - \cos\frac{h'}{h}\pi}{\cosh \xi\pi - 1}\right) = -\frac{1}{2\pi}\log\left(1 + \frac{\sin^2\frac{h'}{h}\frac{\pi}{2}}{\sinh^2\frac{\xi\pi}{2}}\right). \qquad 5.3.11$$

Thus, as in the previous Section, ρi_x is the difference of two terms, each of which can be regarded as arising from a discontinuity in the surface velocity.

In Figure 23 $G(\xi)$ is plotted against ξ for various values of h'/h. It will be seen that $G(\xi)$ is appreciably large only when ξ is of order h'/h, that is, when $x + a$ or $x - a$ is of order h'. Thus the field due to H_x is appreciable only within a distance of order h' from the edges of the stream. At the edge itself ($\xi = 0$) there is a weak infinity; when $\xi << h'/h$

$$G(\xi) \doteqdot \frac{1}{\pi}\log\left|\sinh\frac{\xi\pi}{2}\right| \doteqdot \frac{1}{\pi}\log|\xi|. \qquad 5.3.12$$

When h'/h is small and $1 >> \xi >> h'/h$ we have

$$G(\xi) \doteqdot -\frac{1}{2\pi}\left(\frac{h'/h}{\xi}\right)^2 \qquad 5.3.13$$

and when $\xi >> 1$ (that is when $x \pm a >> h$) we have

$$G(\xi) \doteqdot -\frac{2}{\pi}e^{-\xi\pi}\sin^2\frac{h'}{h}\frac{\pi}{2} \qquad 5.3.14$$

so that

$$(\rho i_x)_{z=0} \doteqdot \frac{4\,VH_x}{\pi}e^{-\xi\pi}\sinh\frac{a\pi}{h}\sin^2\frac{h'}{h}\frac{\pi}{2}. \qquad 5.3.15$$

In other words, $(\rho i_x)_{z=0}$ vanishes exponentially beyond a distance of order h from the edges of the stream.

5.4 Summary

The form of the signal near the left edge of the stream of rectangular section (Figure 19) is shown in Figures 22 and 23. The corresponding analytical expressions are 5.2.15 and 5.3.11. To obtain the total signal, the curves for the right edge must be subtracted from those for the left edge, as indicated in 5.2.14 and 5.3.10.

When $h << a$, the signal in the interior is approximately $VH_z(1 - h'/h)$. Within a distance of order h' from the edges there is an additional signal of order $V(|H_x| + |H_z|)$ which at first falls off with distance from the edge $x = a$, say, like $h'/|x-a|$; but when $|x-a|$ is of order h the signal diminishes exponentially,* like $e^{-|x-a|/h}$.

It should be remembered that the bottom has been treated as an insulator. The conductivity of the bottom will introduce other errors, whose order of magnitude will be evaluated in the next chapter.

* This is a special case of a more general theorem; see Longuet-Higgins (1949).

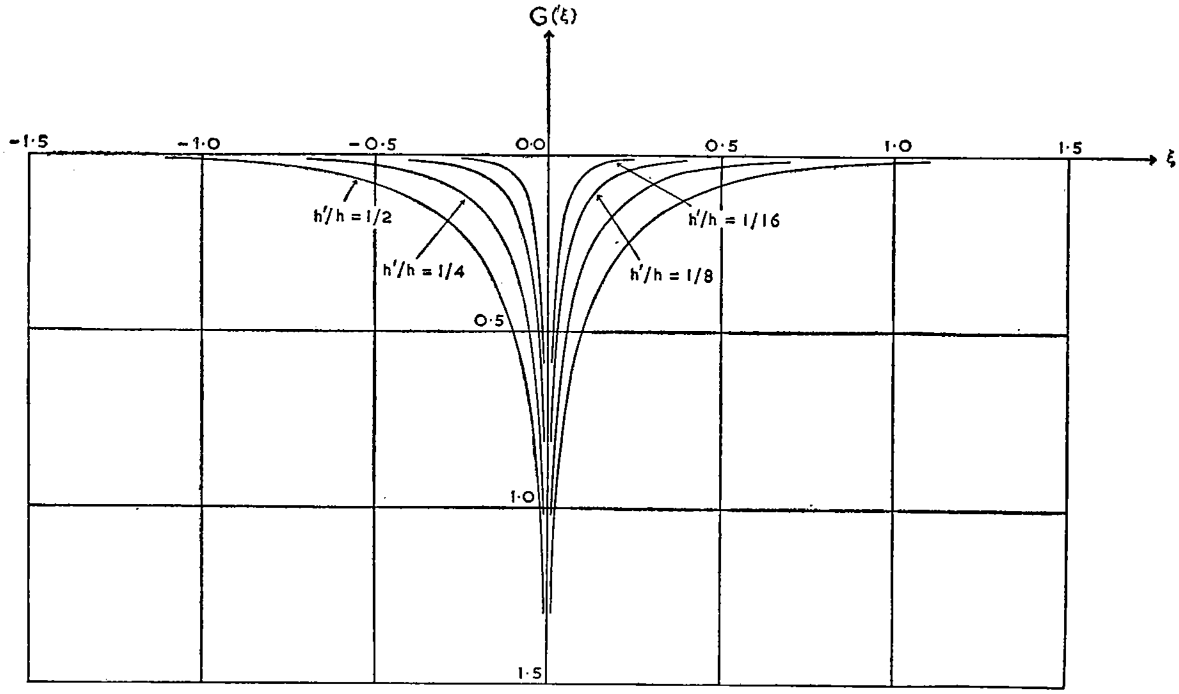

FIGURE 23. Graph of $G(\xi)$ for $h'/h = \frac{1}{2}, \frac{1}{4}, \frac{1}{8}$ and $\frac{1}{16}$, showing the form of the electrical current-density due to H_z near one edge of the stream.

6. A STREAM OF ELLIPTICAL CROSS-SECTION

6.1 INTRODUCTION

The following exact solution will frequently be of use in considering the field due to a stream which is narrow compared with the depth of the ocean, or a stream bounded on either side by land.

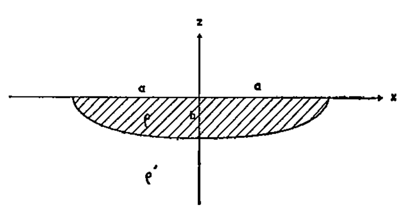

FIGURE 24. A cross-section of the elliptical stream.

The solution has been partly evaluated previously (see Longuet-Higgins 1949).

It is supposed that the cross-section of the stream is half of an ellipse of semi-axes a and b, the major axis, of length $2a$, being horizontal (see Figure 24). The velocity of the stream is uniform and of strength V, and the resistivities of the water and the channel bed are also assumed to be uniform and equal to ρ and ρ' respectively.

Rectangular coordinates are shown as in Figure 24, with the origin at the centre of an elliptical section and the z-axis vertically upwards. We transform to new coordinates (ξ, η) defined by

$$\begin{cases} x = c \cosh \xi \cos \eta \\ -z = c \sinh \xi \sin \eta \end{cases} \qquad 6.1.1$$

where

$$c = \sqrt{a^2 - b^2}. \qquad 6.1.2$$

When z is negative ξ and η take values in the ranges

$$0 < \xi < \infty, \qquad 0 < \eta < \pi. \qquad 6.1.3$$

The channel bed now corresponds to the surface $\xi = \xi_1$, where ξ_1 is defined by

$$a = c \cosh \xi_1, \qquad b = c \sinh \xi_1 \qquad 6.1.4$$

and $x = 0$ is given by $\xi = 0$ between the foci of the ellipse, and by $\eta = 0, \pi$ to the left and to the right respectively.

The potential ϕ obeys Laplace's equation which, since the coordinate-transformation is conformal, becomes

$$\left(\frac{\partial^2}{\partial \xi^2} + \frac{\partial^2}{\partial \eta^2}\right)\phi = 0 \qquad 6.1.5$$

in the new coordinates.

6.2 The field due to H_z

Let ϕ and ϕ' denote the expression for the potential inside and outside the stream. The boundary conditions are:

$$\left.\begin{aligned}
&\text{when } \xi = 0, && \frac{\partial \phi}{\partial \xi} = 0;\\
&\text{when } \eta = 0, \pi, && \frac{\partial \phi}{\partial \eta} = \frac{\partial \phi'}{\partial \eta} = 0;\\
&\text{when } \xi = \xi_1, && \phi = \phi';\\
&\text{when } \xi = \xi_1, && \\
&\frac{1}{\rho}\frac{\partial}{\partial \eta}\left(\phi - VH_z\, c\cosh \xi \cos \eta\right) = \frac{1}{\rho'}\frac{\partial \phi'}{\partial \eta};\\
&\text{when } \xi \to \infty, && \phi' \to 0.
\end{aligned}\right\} \qquad 6.2.1$$

The fourth equation, which expresses the condition 3.1.12, follows from the fact that $\mathbf{v} \wedge \mathbf{H}$ in this case is the gradient of the function $VH_z\, x$ in the interior of the stream. The solution of these equations is

$$\left.\begin{aligned}
\phi &= -VH_z \frac{\rho\, c \cosh \xi \cos \eta}{\rho + \rho' \coth \xi_1}\\
\phi' &= -VH_z \frac{\rho\, c \cosh \xi_1\, e^{\xi_1 - \xi} \cos \eta}{\rho + \rho' \coth \xi_1}.
\end{aligned}\right\} \qquad 6.2.2$$

Since $\tanh \xi_1 = b/a$, these may be written

$$\left.\begin{aligned}
\phi &= -VH_z \frac{x}{1 + \rho' a/\rho b}\\
\phi' &= -VH_z \frac{a\, e^{\xi_1 - \xi} \cos \eta}{1 + \rho' a/\rho b}.
\end{aligned}\right\} \qquad 6.2.3$$

In the interior of the stream the potential gradient is uniform and horizontal, being given by

$$\frac{\partial \phi}{\partial x} = -\frac{VH_z}{1 + \rho' a/\rho b}. \qquad 6.2.4$$

This may be compared with equation 2.1.5. The current-lines in the stream itself are horizontal and uniformly spaced. The potential gradient at the surface, outside the stream, is given by

$$\begin{aligned}
\left(\frac{\partial \phi'}{\partial x}\right)_{\eta = 0} &= \frac{1}{c \sinh \xi}\frac{\partial \phi'}{\partial \xi}\\
&= VH_z \frac{a/c}{1 + \rho' a/\rho b}\frac{e^{\xi_1 - \xi}}{\sinh \xi},
\end{aligned} \qquad 6.2.5$$

that is

$$\left(\frac{\partial \phi'}{\partial x}\right)_{\eta = 0} = VH_z \frac{1}{1 + \rho' a/\rho b}\,\frac{a}{a - b} \times \frac{1}{(x^2/c^2 - 1) + x/c\,(x^2/c^2 - 1)^{1/2}}. \qquad 6.2.6$$

Thus the potential gradient falls off at infinity like c^2/x^2, as for a dipole at the origin.

For a stream of elliptical cross-section in an infinite depth of fluid we may write $\rho = \rho'$. Hence the current-density at the surface is given by

$$(\rho i_x)_{z=0} = \begin{cases} VH_z \dfrac{1}{1 + b/a} & |x| \le a,\\[2ex] VH_z \dfrac{ab/c^2}{(x^2/c^2 - 1) + x/c\,(x^2/c^2 - 1)^{1/2}}, & |x| \ge a. \end{cases} \qquad 6.2.7$$

$(\rho i_x)_{z=0}$ is plotted against x in Figure 25 for b/a 0.5, 0.1 and 0.02, that is, when the maximum depth b is 0.25, 0.05 and 0.01 of the total width $2a$. It will be seen that not only does the "apparent velocity" approach the actual velocity as the ratio b/a is decreased, but also the width of the spurious fringe at the edges of the stream diminishes very rapidly. In fact when b/a is small, $(\rho i_x)_{z=0}$ falls off proportionally to

$$\frac{b/c}{(x^2/c^2 - 1)^{1/2}} = \frac{b/c}{[(x^2 - a^2 - b^2)/c^2]^{1/2}} = \frac{1}{[(x^2 - a^2)/b^2 - 1]^{1/2}}, \qquad 6.2.8$$

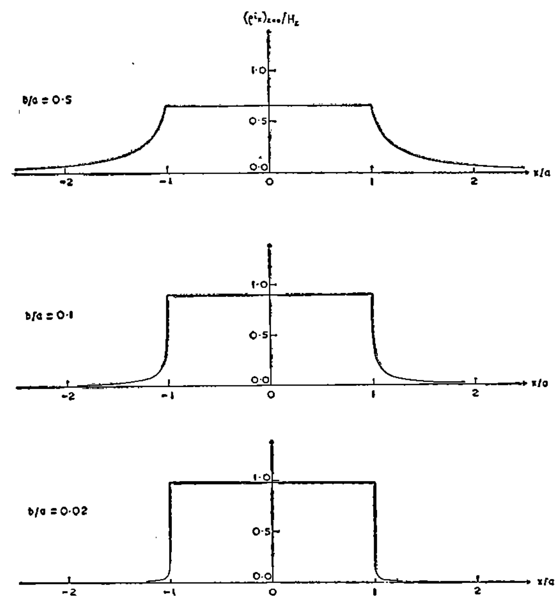

Figure 25. The form of the electrical current-density due to H_z, for b/a = 0.5, 0.1 and 0.02.

so that the width of the fringe is of the order of b^2/a.

6.3 The field due to H_x

The boundary conditions in this case are:

$$\left.\begin{aligned}
&\text{when } \xi = 0, \\
&\quad \frac{\partial}{\partial \xi}(\phi + VH_x c \sinh \xi \sin \eta) = 0; \\
&\text{when } \eta = 0, \pi, \\
&\quad \frac{\partial}{\partial \eta}(\phi + VH_x c \sinh \xi \sin \eta) = \frac{\partial \phi'}{\partial \eta} = 0; \\
&\text{when } \xi = \xi_1, \qquad \phi = \phi'; \\
&\text{when } \xi = \xi_1, \\
&\quad \frac{1}{\rho}\frac{\partial}{\partial \xi}(\phi + VH_x c \sinh \xi \sin \eta) - \frac{1}{\rho'}\frac{\partial \phi'}{\partial \eta} = 0; \\
&\text{when } \xi \to \infty, \qquad \phi' \to \infty.
\end{aligned}\right\} \qquad 6.3.1$$

Since $\sin \eta$ can be expanded in the cosine series:

$$\sin \eta = \frac{2}{\pi} - \frac{4}{\pi}\sum_{m=1}^{\infty} \frac{1}{4m^2 - 1} \cos 2m\eta, \qquad (0 \leq \eta \leq \pi), \qquad 6.3.2$$

we find

$$\begin{aligned}
\phi + VH_x c \sinh \xi \sin \eta &= \frac{4}{\pi} VH_x c \sinh \xi_1 \times \\
&\sum_{m=1}^{\infty} \frac{\rho' \cosh 2m\xi \cos 2m\eta}{(4m^2-1)(\rho' \cosh 2m\xi_1 + \rho \sinh 2m\xi_1)} \\
\phi' = -\frac{4}{\pi} VH_x c \sinh \xi_1 \times \\
&\sum_{m=1}^{\infty} \frac{\rho \sinh 2m\xi_1 \, e^{2m(\xi-\xi_1)} \cos 2m\eta}{(4m^2-1)(\rho' \cosh 2m\xi_1 + \rho \sinh 2m\xi_1)}
\end{aligned} \qquad 6.3.3$$

We shall consider only the case $\rho = \rho'$, corresponding to a surface stream, of limited width, in an infinitely deep ocean. The horizontal component of the current-density is given by

$$(\rho i_x)_{z=0} = -\left(\frac{\partial \phi}{\partial x}\right)_{z=0} = \begin{cases} \dfrac{1}{c \sin \eta}\left(\dfrac{\partial \phi}{\partial \eta}\right)_{\xi=0}, & |x| < c; \\[2ex] \dfrac{-1}{c \sinh \xi}\left(\dfrac{\partial \phi}{\partial \xi}\right)_{\eta=0}, & |x| > c. \end{cases} \qquad 6.3.4$$

that is,

$$(\rho i_x)_{z=0} = \begin{cases} -\dfrac{4 VH_x \sinh \xi_1}{\pi \sin \eta} \times \displaystyle\sum_{m=1}^{\infty} \frac{2m}{4m^2-1} e^{-2m\xi_1} \sin 2m\eta, & |x| < c; \\[2ex] -\dfrac{4 VH_x \sinh \xi_1}{\pi \sinh \xi} \times \displaystyle\sum_{m=1}^{\infty} \frac{2m}{4m^2-1} e^{-2m\xi_1} \sinh 2m\xi, & c < |x| < a; \\[2ex] -\dfrac{4 VH_x \sinh \xi_1}{\pi \sinh \xi} \times \displaystyle\sum_{m=1}^{\infty} \frac{2m}{4m^2-1} e^{-2m\xi} \sinh 2m\xi_1, & |x| > a. \end{cases} \qquad 6.3.5$$

The above series can be summed by writing

$$\frac{2m}{4m^2-1} = \tfrac{1}{2}\left(\frac{1}{2m-1} + \frac{1}{2m+1}\right) \qquad 6.3.6$$

and making use of the fact that, if $R(\alpha) > 0$,

$$\sum_{m=1}^{\infty} \frac{e^{-(2m-1)\alpha}}{2m-1} = \log(1 - e^{-\alpha}) - \tfrac{1}{2}\log(1 - e^{-2\alpha}). \qquad 6.3.7$$

As a result we find, after some simplification,

$$(\rho i_x)_{z=0} = \begin{cases} \dfrac{VH_x \sinh \xi_1}{\pi \sin \eta}\left[2 \cosh \xi_1 \cos \eta \tan^{-1} \dfrac{\sin \eta}{\sinh \xi_1} + \sinh \xi_1 \sin \eta \log \dfrac{\cosh \xi_1 - \cos \eta}{\cosh \xi_1 + \cos \eta}\right], & |x| < c \\[2ex] \dfrac{VH_x \sinh \xi_1}{\pi \sinh \xi}\left[\cosh(\xi + \xi_1) \log \tanh \left|\dfrac{\xi + \xi_1}{2}\right| - \cosh(\xi + \xi_1) \log \tanh \left|\dfrac{\xi - \xi_1}{2}\right|\right], & |x| > c. \end{cases} \qquad 6.3.8$$

$(\rho i_x)_{z=0}$ is plotted against x in Figure 26 for $b/a = 0.5$, 0.1 and 0.02. As in the case of the stream of rectangular cross-section, there is a weak singularity at each edge of the stream; near the right-hand edge, for example, where $\xi \doteqdot \xi_1$,

$$\begin{aligned} (\rho i_x)_{z=0} &\doteqdot -\frac{VH_x}{\pi} \log |\xi - \xi_1| \\ &\doteqdot -\frac{VH_x}{\pi} \log |x - a| \end{aligned} \qquad 6.3.9$$

(c.f. equation 5.3.12). However the effect of the

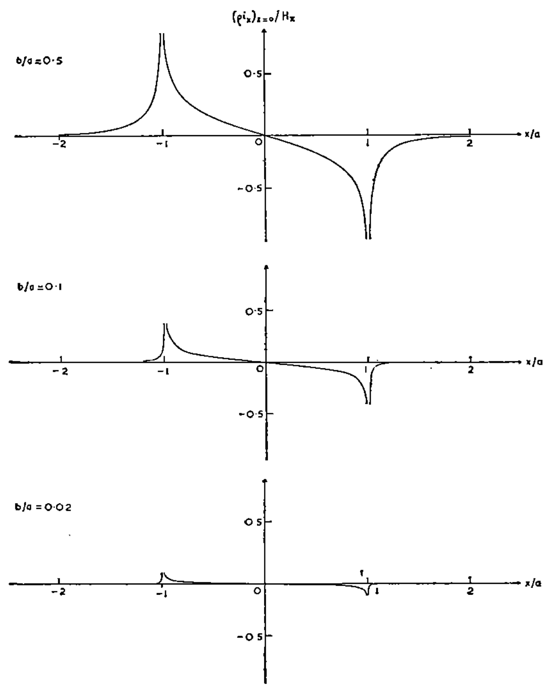

FIGURE 26. The form of the electrical current-density due to H_x, for $b/a = 0.5$, 0.1 and 0.02.

singularity again diminishes as b/a is decreased, being limited to the region where η, or ξ, is of the same order as ξ_1. This implies that $|x - a|$ is of order b^2/a. In the main part of the stream we have (when $\xi_1 << 1$)

$$(\rho i_x)_{z=0} = VH_z\, \xi_1 \cot \eta = VH_z \frac{x\, b/c}{(x^2 - c^2)^{1/2}} \quad 6.3.10$$

and outside the stream ($\xi >> \xi_1$) we find

$$(\rho i_x)_{z=0} = \frac{2\, VH_z}{\pi} \xi_1^2 \left[\log \tanh \frac{\xi}{2} + \frac{\cosh \xi}{\sinh^2 \xi} \right]. \quad 6.3.11$$

Thus outside the stream $(\rho i_x)_{z=0}$ is proportional only to b^2/a^2. For large values of x/a, $(\rho i_x)_{z=0}$ diminishes like $e^{-3\xi}$ or a^3/x^3, in the same way as a quadripole at the origin.

6.4 Summary

The form of the signal generated by the stream of elliptical cross-section (Figure 24 with $\rho = \rho'$) is shown in Figures 25 and 26. The corresponding analytical expressions are 6.2.7 and 6.3.8. In the interior of the stream the signal is constant and equal to $VH_z/(1 + b/a)$, so that the "error" is approximately $VH_z\, b/a$. Near the edges of the stream there is a spurious signal which is appreciable within a distance b^2/a from the edge. (b^2/a is also the order of magnitude of the depth of the stream near the edge.) At some distance from the stream there is a signal of order $VH_z\, b/a$ which falls off proportionally to a^2/x^2.

In Chapter 5 the conductivity of the bottom was entirely neglected. If this, though small, is taken into account, 6.2.3 shows that there will be an additional signal in the interior of order $VH_z\, (\rho' a/{}^{\rm l}\rho b)$ and a similar signal outside the stream which will fall off like a^2/x^2.

7. THE ELECTRICAL FIELD INDUCED IN WATER WAVES

7.1 Long Crested Waves

Let rectangular axes be taken with the origin in the mean free surface, the x-axis in the direction of wave propagation, the y-axis parallel to the wave crests and the z-axis vertically upwards. If

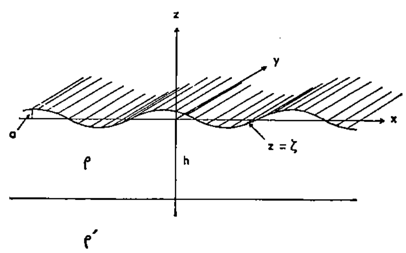

Figure 27

a denotes the wave amplitude and if $2\pi/\sigma$ and $2\pi/k$ denote the wave period and wave-length, the elevation ζ of the free surface is given by

$$\zeta = a \cos (kx - \sigma t). \qquad 7.1.1$$

The corresponding stream function (see Lamb, 1932) is

$$\psi = \frac{a\sigma}{k} \frac{\sinh k(z+h)}{\sinh kh} \cos (kx - \sigma t). \qquad 7.1.2$$

The motion is two-dimensional, as in Section 3.2. Moreover, since the components of velocity vary harmonically with x, the total electrical current demanded from infinity, when integrated over a complete wavelength, is zero. Hence the two parts ϕ_1 and ϕ_2 of the electrical potential are given by

$$\begin{cases} \phi_1 = H_y\psi = \dfrac{a\sigma H_y}{k} \dfrac{\sinh k(z+h)}{\sinh kh} \cos (kx - \sigma t), \\ \phi_2 = 0, \end{cases} \qquad 7.1.3$$

and the corresponding current-densities $\mathbf{i}_1$ and $\mathbf{i}_2$ are given by

$$\rho \mathbf{i}_1 = (0, 0, 0), \qquad \rho \mathbf{i}_2 = (0, \rho i_{y2}, 0) \qquad 7.1.4$$

where

$$\rho i_{y2} = a\sigma H_z \frac{\sinh k(z+h)}{\sinh kh} \sin (kx - \sigma t) - a\sigma H_x \frac{\cosh k(z+h)}{\sinh kh} \cos (kx - \sigma t). \qquad 7.1.5$$

Consider first the potential gradient as measured by fixed electrodes. The bottom $z = -h$ is a streamline, and hence $\phi_{z=-h}$ is a constant. Thus stationary electrodes placed on the bottom would not record the presence of the waves. The free surface, however, is not a streamline. The potential at the free surface $\phi_{z=\zeta}$ can be obtained to the present order of approximation (a^2k^2 being neglected) by writing $z = 0$ in 7.1.3:

$$\phi_{z=\zeta} = \frac{a\sigma H_y}{k} \cos (kx - \sigma t), \qquad 7.1.6$$

that is

$$\phi_{z=\zeta} = \frac{\sigma H_y}{k} \zeta. \qquad 7.1.7$$

Thus the potential is a maximum at the wave crests and a minimum at the wave troughs. The greatest potential difference $2a\sigma H_y/k$ would be obtained by placing the electrodes near the surface half a wavelength apart in the x-direction, or $(n + \frac{1}{2})$ wavelengths apart, where n is an integer. (The electrodes must of course be submerged sufficiently to maintain contact with the water.)

Consider now the voltage measured by towed electrodes. As shown in Section 1.3, the quantity measured by electrodes A and B towed in tandem is

$$\int_A^B \rho \mathbf{i} \cdot d\mathbf{s}, \qquad 7.1.8$$

where the path of integration is along the cable. Now the component of $\mathbf{i}$ in the plane of motion is zero (from 7.1.4). Thus electrodes towed in the direction of propagation of the waves would register zero voltage. The component of $\mathbf{i}$ parallel to the crests, however and in the free surface, is given by

$$(\rho i_y)_{z=\zeta} = a\sigma H_z \sin (kx - \sigma t) - a\sigma H_x \coth kh \cos (kx - \sigma t). \qquad 7.1.9$$

It is clear that the path of the cable is important in this case. For suppose the straight line AB joining the electrodes lies along a wave crest, say, then the voltage between A and B is simply

$$la\sigma H_x \coth kh, \qquad 7.1.10$$

where l is the distance between the electrodes. If, however, the cable describes three sides $ADCB$ of a rectangle, as in Figure 28, where DC lies along a wave trough, the recorded voltage will be just negative of 7.1.10.

Equation 7.1.9 can also be written

$$(\rho i_y)_{z=\zeta} = -v_x H_z + v_z H_x, \qquad 7.1.11$$

which shows that the signal is in two parts: one, proportional to H_z, depends on the horizontal motion v_x at the surface, and is in phase with the

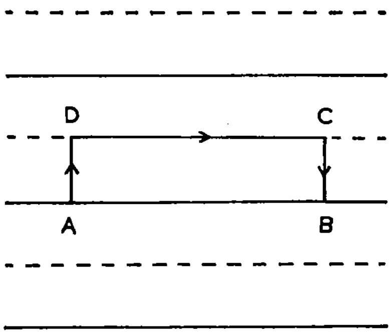

FIGURE 28

surface elevation ζ; the other, proportional to H_x, depends on the vertical motion v_z at the surface and is in quadrature with ζ. If the waves are travelling along a line of magnetic latitude (i.e. $H_x = 0$) only the first term is important. At the magnetic equator ($H_z = 0$) only the second is important. For waves travelling along the magnetic equator ($H_x = H_z = 0$) there will be no signal.

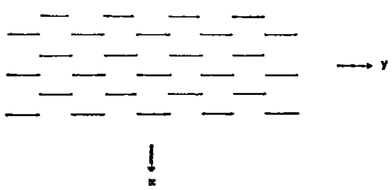

FIGURE 29

7.2 SHORT-CRESTED WAVES

Consider a wave motion in which the surface elevation $\zeta\,(x, y, t)$ is given by

$$\zeta = a \cos (kx - \sigma t) \cos k'y. \qquad 7.2.1$$

This may be split up into two long-crested waves as follows:

$$\zeta = \frac{a}{2}\Big[\cos (kx + k'y - \sigma t) + \cos (kx - k'y - \sigma t)\Big]. \qquad 7.2.2$$

The waves are propagated in the direction of the vectors

$$-\frac{(k, k')}{(k^2 + k'^2)^{1/2}}, \; -\frac{(k, -k')}{(k^2 + k'^2)^{1/2}}. \qquad 7.2.3$$

The components of **H** parallel to the wave-crests are

$$\left(\frac{kH_y - k'H_x}{(k^2 + k'^2)^{1/2}}, \frac{kH_y + k'H_x}{(k^2 + k'^2)^{1/2}}\right) \qquad 7.2.4$$

respectively. Hence the electrical potential ϕ is given by

$$\phi = \tfrac{1}{2}\,\frac{a\,\sigma}{k^2 + k'^2}\,\frac{\sinh (k^2 + k'^2)^{1/2} (z + h)}{\sinh (k^2 + k'^2)^{1/2} h} \times [(k H_y - k'H_x) \cos (kx + k'y - \sigma t) + (k H_y + k'H_x) \cos (kx - k'y - \sigma t)] \qquad 7.2.5$$

or

$$\phi = \frac{a\sigma}{k^2 + k'^2}\,\frac{\sinh (k^2 + k'^2)^{1/2} (z + h)}{\sinh (k^2 + k'^2)^{1/2} h} \times [kH_y \cos (kx - \sigma t) \cos k'y + k'H_x \sin (kx - \sigma t) \sin k'y]. \qquad 7.2.6$$

Thus there is also a component of the potential gradient in the y-direction. However, if $k' << k$, that is, if the crests are long compared with the wavelength, the y-component is small compared with the x-component.

7.3 THE ACTION OF THE MAGNETIC FIELD ON WATER WAVES

It is perhaps worth considering the mechanical reaction of the Earth's magnetic field on surface waves, if only to show that it is negligible.

For simplicity we shall deal only with orders of magnitude. If the magnetic field intensity **H** is measured in gauss and the particle speed **v** in cm/sec., then the induced e.m.f. is of the order $v H \times 10^{-8}$ volt/cm. The electrical current-density **i** is then of order $\rho^{-1} v H \times 10^{-8}$ amp/cm, where ρ is measured in ohm-cm. The force $\mathbf{F} = \mathbf{i} \wedge \mathbf{H}$ per unit volume acting on the fluid due to the interaction of the electrical current and the magnetic field is then of order $\rho^{-1} v H^2 \times 10^{-9}$ dyne/cm^3. If we take

$$H = 0.5 \text{ gauss}, \quad v = 50 \text{ cm/sec}, \quad \rho = 25 \text{ ohm-cm} \qquad 7.3.1$$

we have

$$F = 0.5 \times 10^{-9} \text{ dyne/cm}^3 \qquad 7.3.2$$

This is to be compared with the force of gravity, which chiefly controls the waves, and is of order 10^3 dyne/cm^3. The ratio of the two forces is therefore of order 5×10^{-13}, which is entirely negligible.

8. MISCELLANEOUS EFFECTS

In this chapter we shall determine the effect of variations, in time or space, of some of the quantities hitherto assumed constant. This will enable us to test the validity of some of our previous assumptions.

8.1 Variation of the Velocity with Time

Let us first consider Faraday's law of induction (c.f. Section 1.1) when the magnetic field varies with the time. In general, the rate of change of magnetic flux through a surface S bounded by a circuit C is

$$\frac{d}{dt}\int_S \mathbf{H}\cdot d\mathbf{s} = \int_S \frac{\partial \mathbf{H}}{\partial t}\cdot d\mathbf{s} - \int_C \mathbf{v}\wedge\mathbf{H}\cdot d\mathbf{s} \qquad 8.1.1$$

and therefore

$$\int_C \rho\mathbf{i}\cdot d\mathbf{s} = -\int_S \frac{\partial \mathbf{H}}{\partial t}\cdot d\mathbf{s} + \int_C \mathbf{v}\wedge\mathbf{H}\cdot d\mathbf{s} \qquad 8.1.2$$

or

$$\int_C \mathbf{G}\cdot d\mathbf{s} = \int_S \frac{\partial \mathbf{H}}{\partial t}\cdot d\mathbf{s} \qquad 8.1.3$$

where

$$\mathbf{G} = \rho\mathbf{i} - \mathbf{v}\wedge\mathbf{H}. \qquad 8.1.4$$

Since the left-hand side of 8.1.3 equals

$$\int_S \nabla\wedge\mathbf{G}\,.\,d\mathbf{s} \qquad 8.1.5$$

it follows that

$$\nabla\wedge\mathbf{G} = -\frac{\partial \mathbf{H}}{\partial t}. \qquad 8.1.6$$

We have also Oersted's law for the magnetic field induced by a quasi-stationary current $\mathbf{i}$:

$$\int_C \mathbf{H}\cdot d\mathbf{s} = 4\pi\int_S \mathbf{i}\,.\,d\mathbf{s}, \qquad 8.1.7$$

from which, by a similar transformation, it follows that

$$\nabla\wedge\mathbf{H} = 4\pi\mathbf{i}. \qquad 8.1.8$$

Further, if the electrical charge is conserved,

$$\nabla\cdot\mathbf{i} = 0. \qquad 8.1.9$$

Equations 8.1.6, 8.1.8 and 8.1.9 are special cases of Maxwell's equations when the field quantities vary slowly with the time, and the magnetic permeability is equal to unity.

On eliminating $\mathbf{H}$ from equations 8.1.6 and 8.1.8 we have

$$\nabla\wedge\nabla\wedge\mathbf{G} = -4\pi\frac{\partial \mathbf{i}}{\partial t} \qquad 8.1.10$$

and so from 8.1.4

$$\nabla\wedge\nabla\wedge(\rho\mathbf{i}) - 4\pi\frac{\partial \mathbf{i}}{\partial t} = \nabla\wedge\nabla\wedge(\mathbf{v}\wedge\mathbf{H}). \qquad 8.1.11$$

Expanding the first term and using 8.1.9 we get

$$\rho\nabla^2\mathbf{i} - 4\pi\frac{\partial \mathbf{i}}{\partial t} = \nabla\wedge\nabla\wedge(\mathbf{v}\wedge\mathbf{H}). \qquad 8.1.12$$

Now let us suppose that the velocity $\mathbf{v}$ and the current-density $\mathbf{i}$ fluctuate harmonically with period τ. Further, that the smallest dimensions of the current-system are of the order of λ. Then the orders of magnitude of the terms on the left of 8.1.12 are in general $\rho\mathbf{i}\times(2\pi/\lambda)^2$ and $4\pi\mathbf{i}\times(2\pi/\tau)$ respectively. The ratio of the second term to the first is of order

$$2\lambda^2/\rho\tau. \qquad 8.1.13$$

Thus, provided

$$\tau \gg 2\lambda^2/\rho \qquad 8.1.14$$

the term $\partial\mathbf{i}/\partial t$ is negligible, and the system may be treated as quasi-static.

If practical units are used, there is a factor 10^{-8} on the right-hand side of 8.1.6 and a factor 10^{-1} on the right-hand side of 8.1.8. Hence the right-hand side of 8.1.14 is to be multiplied by 10^{-9}. If we take

$$\lambda = 1 \text{ km} = 10^5 \text{ cm} \qquad 8.1.15$$
$$\rho = 25 \text{ ohm-cm}$$

the equation 8.1.14 gives

$$\tau \gg 0.8 \text{ sec.} \qquad 8.1.16$$

For ocean streams whose least dimensions are 1 km. or more we should therefore expect that the time-variations of velocity can be completely neglected.

The above argument may fail in special instances. For example, in a long cylindrical shell of radius r and thickness δ ($\delta \ll r$), the magnetic field is given by

$$H = 4\pi\delta i \qquad 8.1.17$$

where i is the mean strength of the current-density flowing in the shell in planes perpendicular to the axis of the cylinder. Also by Faraday's law of induction

$$2\pi r\rho i = -\pi r^2\frac{\partial H}{\partial t}, \qquad 8.1.18$$

leading to the equation

$$\mathbf{i} = -\frac{2\pi r\delta}{\rho}\frac{\partial \mathbf{i}}{\partial t}. \qquad 8.1.19$$

Thus the time constant of the system is of order $r\,\delta/\rho$ and not δ^2/ρ as we would deduce from 8.1.14. This is because the largest term in $\rho\,\nabla^2\,\mathbf{i}$ happens to be of order $\rho\,\mathbf{i} \times (4\,\pi^2/r\,\delta)$ and not of order $\rho\,\mathbf{i} \times (4\,\pi^2/\delta^2)$.

One may conclude from the above example that if the sea bed acts as a conductor, so that the current-system extends to a depth comparable with the width D of the stream, then a necessary condition for the time variation of velocity to be negligible is

$$\tau >> 2\,h\,D/\rho \qquad 8.1.20$$

If D is of order 1,000 km., then τ must be much larger than 13 minutes. This would be fairly well satisfied by streams of tidal period.

8.2 Geomagnetic Disturbances

So far it has been assumed that the magnetic field $\mathbf{H}$ is constant in time. As is well known, however (see Chapman and Bartels, Geomagnetism, 1940), there exist small fluctuations in the Earth's magnetic field, whose effect we shall now consider.

The fluctuations are of two kinds: periodic disturbances, for example the solar daily and lunar daily variations, and transient disturbances.

The transient disturbances are closely associated with other solar and terrestrial phenomena, including sun-spot activity, solar flares, aurora polaris, and ionisation of the atmosphere. It is practically certain that the terrestrial effects have their origin in streams of charged particles or high-frequency radiation emitted from the sun. During magnetic "storms" the disturbances may be as much as 100 γ and on rare occasions may be several times greater. On magnetically "quiet days," however, the disturbances are only of the order of 1 γ. ($1\,\gamma = 10^{-5}\,\Gamma$.)

At most places on the Earth's surface there is an appreciable daily variation: the solar-daily variation is of the order of $\pm$ 10 γ, while the lunar-daily variation may be of the order of $\pm$ 5 γ.

Since the total magnetic field is of the order of 1 Γ, it will be seen that the fluctuations mentioned above scarcely affect the total intensity; but the indirect effects of these fluctuations may be important.

Accompanying the fluctuations in the magnetic field are found fluctuations in the electrical field. These are usually interpreted as indicating the existence of natural "earth-currents" flowing in the Earth. Such earth-current gradients form part of the background of measurements with stationary or towed electrodes, and may limit their accuracy. In general character, earth-currents are similar to geomagnetic fluctuations. During earth-current "storms", electrical gradients of 100 mV/km or more may be recorded, though on quiet days the transient disturbances are normally less than 1 mV/km. At most stations there is a solar-daily variation of the order of 5 mV/km and a rather smaller lunar-daily variation. These figures refer to observations at stations well inland. Similar observations in the sea near the coast or on land adjacent tidal waters are very similar except that the lunar-daily variation may be exaggerated, owing to the tidal streams in the neighbourhood (see Longuet-Higgins 1949). Observations in the deep oceans are so far not numerous, but there are reasons, given below, for expecting that the earth-currents may be more active in coastal regions than in the deep oceans.

It is debatable whether the earth-current potentials should be regarded as induced by the fluctuations in the magnetic field, or whether the magnetic perturbations represent the field of the earth-currents. Probably the two effects are not separable, as the following rough calculations show.

For definiteness, let us imagine a circular ocean of radius r and depth h. The e.m.f. E due to a fluctuation of the vertical component of magnetic field H is given by

$$2\,\pi\,r\,E = -\,\pi\,r^2\,\frac{\partial H}{\partial t} \times 10^{-8}\ V/\text{cm.} \qquad 8.2.1$$

so

$$E = \frac{r}{2}\frac{\partial H}{\partial t} \times 10^{-8}\ V/\text{cm.} \qquad 8.2.2$$

For a magnetic storm we may suppose a change in the vertical field of 100 γ in 30 minutes. Then

$$\frac{\partial H}{\partial t} = \frac{100\,\gamma}{30\ \text{min.}} = 5 \times 10^{-7}\,\Gamma/sec. \qquad 8.2.3$$

Assuming r = 1000 km we find

$$E = 25\ \text{m}V/\text{km} \qquad 8.2.4$$

which is of the same order of magnitude as the gradients observed on land. On the other hand if we estimate the total field H at the centre of the ocean induced by an earth-current gradient, say

$$\rho i = E = E_0\,\frac{r}{r_0} \qquad 8.2.5$$

we have

$$H = 2\,\pi \int_0^{r_0} \frac{hi}{r}\, dr \qquad 8.2.6$$

where

$$i = E/\rho. \qquad 8.2.7$$

Thus

$$H = 2\,\pi \int_0^{r_0} \frac{h}{r}\,\frac{E_0 r}{\rho r_0}\, dr = 2\,\pi \frac{hE_0}{\rho}. \qquad 8.2.8$$

Assuming

$$\begin{aligned} E_0 &= 25\ mV/\text{km} = 25 \times 10^{-8}\ V/\text{cm} \\ h &= 2\ \text{km} = 2 \times 10^5\ \text{cm} \\ \rho &= 25\ \text{ohm-cm} \end{aligned} \qquad 8.2.9$$

we find

$$H = 120\ \gamma, \qquad 8.2.10$$

which again is of the same order of magnitude as that observed.

However, for the diurnal variation, if we assume

$$\frac{\partial H}{\partial \gamma} = \frac{10\ \gamma}{6\ \text{hrs}} = 5 \times 10^{-9}\ \Gamma/\text{sec} \qquad 8.2.11$$

we find from (2)

$$E = 0{\cdot}25\ \text{m}V/\text{km}, \qquad 8.2.12$$

which is much smaller than the observed fluctuations, and if we assume

$$E_0 = 2{\cdot}5\ \text{m}V/\text{km} \qquad 8.2.13$$

then (8) gives

$$H = 12\ \gamma, \qquad 8.2.14$$

which is at least of the same order as the daily variation on land. Therefore it seems more likely in the case of the daily variation that the magnetic field can be considered as due to the earth-currents.

Whatever their cause, it is certain that earth-currents can at times have an important effect on measurements with towed electrodes, in coastal regions at least. If 20 mV/km is equivalent to 1 knot of water velocity, electrical gradients equivalent to 5 knots may occur during storms, though on quiet days the transient disturbances may be equivalent to less than 0·1 knot. There may be solar- and lunar-daily variations equivalent to about ¼ knot.

One encouraging feature may be pointed out. In equation 8.2.2 the induced e.m.f. is proportional to r, the radial distance. Hence we may reasonably expect that earth-currents will be less active near the centre than near the edges of an ocean.

8.3 Heterogeneities in ρ

In the analysis of the previous chapters it was assumed that the ocean water is homogeneous so far as resistivity is concerned. Actually the resistivity is a function of salinity and temperature (Thomas, Thompson and Utterback, 1934). In the deep ocean the resistivity is controlled mainly by the temperature. Thus, in the surface slope water to the west of the Gulf Stream, the resistivity might be estimated as 1.2 times that in the hot water in the core of the stream. In estuaries, however, the resistivity is controlled mainly by the salinity, and may vary widely. For example, in the upper reaches of Alberni Inlet we could expect the resistivity of the upper water to be 5 times that of the deeper water. The following Table gives values of resistivity for a small range of salinity and temperature.

ρ, *Resistivity of Sea Water (ohms-cm)*

T°F / S°/∞	40	50	60	70	80
30	34.5	30.3	26.3	23.3	20.8
35	30.3	26.0	23.0	20.4	18.2
40	27.0	23.3	20.4	18.2	16.3

The effect of a vertical stratification of resistivity is really included in the discussion of Section 2.2, if we restrict ourselves to the case of broad streams, where the width is large compared to the depth.

Assuming the equipotential lines in the water to be vertical, we may adopt the circuit analogy of Figure 16, but with the difference that ρ is a function of z. Thus we have

$$\begin{aligned} I &= \frac{1}{D}\int_{-h}^{0} \kappa\,(\phi_{AB} - VH_z D)\, dz \\ &= \bar{\kappa}\, h \left[\frac{\phi_{AB}}{D} - \frac{1}{h}\int_{-h}^{0} \frac{\kappa\, VH_z}{\bar{\kappa}}\, dz\right], \end{aligned} \qquad 8.3.1$$

where $\kappa_1 = \rho^{-1}$, is the conductivity and κ is the mean value of κ. Thus the total current I, and hence the mean potential gradient depends only on the mean conductivity $\bar{\kappa}$ and on a weighted mean of the velocity:

$$\bar{V} = \frac{1}{h}\int_{-h}^{0} \frac{\kappa V}{\bar{\kappa}}\, dz. \qquad 8.3.2$$

To estimate orders of magnitude we may consider a particular example, in which the velocity and resistivity are equal to V_s and ρ_1 in a surface layer of depth h', and below this layer the water is stationary and the resistivity equals ρ_2. The sea bed is assumed to be effectively non-conducting. The total current I is therefore zero, and hence the potential difference between A and B in Figure 16 is

$$\phi_{AB} = \frac{R_2 E}{R_1 + R_2} \qquad 8.3.3$$

where

$$E = V_s H_z D, \quad R_1 = \rho_1 D/h', \quad R_2 = \rho_2 D/(h-h'). \qquad 8.3.4$$

The measured current-density at the surface is given by

$$\rho_1 i = \frac{E}{D} - \frac{\phi_{AB}}{D} = \frac{R_1 E/D}{R_1 + R_2}, \qquad 8.3.5$$

that is

$$\rho_1 i = V_s H_z \, \frac{\rho_1 (h - h')}{\rho_2 h' + \rho_1 (h - h')}, \qquad 8.3.6$$

which can also be written

$$\rho_1 i = V_s H_z \left(1 - \frac{h}{h'}\right) \Big/ \left(1 + \frac{(\rho_2 - \rho_1)\, h'}{\rho_1 h}\right). \qquad 8.3.7$$

When $\rho_1 = \rho_2 = \rho$, this reduces to equation 2.2.17. The relative "error" introduced by ρ_2 being different from ρ_1 is

$$\frac{\rho_2 - \rho_1}{\rho_1} \cdot \frac{h'}{h} \qquad 8.3.8$$

Since $h'/h < 1$, this is less than $(\rho_2 - \rho_1)/\rho_1$, the relative error in the resistivity. It is also less than h'/h, the error due to the finite depth of the stream. Since usually $\rho_2 > \rho_1$, the effect tends to diminish slightly the measured signal, that is, it acts in the same direction as an increase in the depth of the stream.

In the deep ocean, the factor $(\rho_2 - \rho_1)/\rho_1$ varies from perhaps 0.0 in polar regions to 0.3 in the tropics. Thus almost all the error is due to h'/h, which may vary widely.

Suppose that an attempt is made to calculate the equivalent depth h_{equ} from the formula

$$\frac{h_{equ}}{h} = 1 - \frac{\rho i}{V_s H_z}, \qquad 8.3.9$$

which is true for a constant resistivity (see Section 2.2). From 8.3.7 we have

$$\frac{h_{equ}}{h} = 1 - \frac{\rho i_1}{V_s H_z} = \frac{\dfrac{\rho_2}{\rho_1}\dfrac{h'}{h}}{1 + \dfrac{(\rho_2 - \rho_1)\, h'}{\rho_1 h}}. \qquad 8.3.10$$

Hence the relative error in h_{equ} is

$$\frac{h_{equ}}{h'} - 1 = \frac{\rho_2 - \rho_1}{\rho_1} \cdot \frac{\rho_1 i}{V_s H_z} \qquad 8.3.11$$

(using 8.3.7 again). Thus if from observation $\rho_1 i/V_s H_z$ were found to be 0.80 and if ρ were assumed constant, we should compute $h_{equ}/h = 0.20$. But if there were actually a stratification represented by the factor $(\rho_2 - \rho_1)/\rho_1 = 0.25$, our estimate would be greater than the actual depth of current h' by 0.25 x 0.80 or 20%.

9. CONCLUSIONS

The method of towed electrodes does not give a direct measure of the surface velocity; the quantity measured is $-\rho\mathbf{i} \cdot \overline{AB}$, or the component of $\rho\mathbf{i}$ in the direction of the electrodes. The current-density $\mathbf{i}$ is related to the velocity $\mathbf{v}$ by the equation

$$\nabla\phi = \mathbf{v} \wedge \mathbf{H} - \rho\mathbf{i},$$

ϕ being the potential measured by stationary electrodes. Only when there is good reason for expecting $\nabla\phi$ to be small can we assume that the quantity measured equals $-\mathbf{v} \wedge \mathbf{H} \cdot \overline{AB}$.

In general, $\nabla\phi$ will be small when there is a free return path for the flow of electric current. Such a path may be supplied, for example, by the sea bed or by deeper stationary layers of the ocean. For a stream of width D in an ocean of depth h, the condition that the sea bed shall form an effective conducting path is that

$$\rho'/\rho << D/h$$

where ρ' is a mean value for the resistivity of the sea bed down to a depth of order D. The condition that the lower layers of the ocean shall form a good conducting path is that

$$\overline{V} << V_s$$

where V_s is the surface velocity and $\overline{V}$ the mean velocity in a vertical line.

The potential gradient at the surface also vanishes identically in two other instances: a circular upwelling (Figure 10) and a steady, cylindrical rolling motion where the total transport across each vertical plane is zero — e.g. the transverse motion sometimes observed at the edge of a strong surface stream or the upwelling along a coast.

In tidal streams, however, the potential gradient does not vanish identically, unless the sea bed is very highly conducting. On the contrary, if the bottom is non-conducting and if the streams extend uniformly to the bottom, towed electrodes will record zero signal. In general, tidal streams are more easily detected by moored electrodes.

The most important case for oceanographers is perhaps that of surface streams in the deep ocean. Fortunately, towed electrodes should then record an electrical gradient $V_s H_z$ very nearly. However, the following errors, due to the form of the streams, can be expected.

(a) a gradient $-\overline{V}H_z$, due to the finite depth of water (Section 2.2).

(b) a gradient of order $h\, H_x\, \mathrm{d}\overline{V}/\mathrm{d}x$ due to the horizontal velocity-gradient (Section 2.3).

(c) when the edge $x = a$ of the stream is sharply defined, a gradient of order $V_s\,(|H_z| + |H_x|)$. This is largest within a distance of order h' from the edge, where h' is the depth of the stream near $x = a$ (Chapter 5). At first it diminishes like $h'/|x-a|$, but at greater distances it falls off exponentially, like $e^{-|x-a|/h}$.

(d) a gradient of order $VH_z\,(\rho\, a/\rho' h)$ due to the conductivity of the sea bed. This extends to a distance of order a either side of the stream, and falls off with distance like $a^2/(x-a)^2$ (Chapter 6).

If the effect (d) happens to be negligible, the largest error will normally be that due to (a), except in the vicinity of strong horizontal velocity-gradients. Hence, if the true surface velocity is known independently, for example by LORAN fixes, an estimate of the mean velocity $\overline{V}$ can be obtained (as in von Arx 1951). Such an estimate, however, is very uncertain, being found from the difference between two nearly equal quantities $V_s H_z$ and ρi_x, both of which may be subject to error. Estimates of the total transport of a current based on equation 2.4.8 are liable to fewer uncertainties.

Among other possible sources of error, the rate of change of the stream velocity with time is unlikely to have any appreciable effect (Section 8.1). Variation of the conductivity of sea water with depth will not much alter the electrical potential, but may have an important effect on estimates of the mean depth and the total transport (Section 8.3). The most important, and the least predictable, source of error is that due to natural earth-current potentials associated with geomagnetic disturbances (Section 8.2). On magnetically disturbed days the earth-current gradient may be equivalent to 5 knots or more of water velocity; on quiet days, less than 0.1 knot. A daily variation of perhaps ¼ knot can be expected. A thorough experimental investigation of natural earth-currents in the sea appears desirable, and observations should be confined to magnetically quiet days, at least in coastal water.

The signal due to ocean waves is a maximum when electrodes are towed parallel to the wave-crests. It consists of two parts, due to the horizontal and the vertical components of motion at the surface. If the electrodes are towed at right-angles to the wave-crests, supposing the crests are very long and regular, there is theoretically no signal, though in practice there will be a greater or smaller signal corresponding to the irregularity of the waves.

The wave signal is clearly recognisable on a record and, moreover, being periodic, can easily be suppressed by a suitable device (von Arx 1950).

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