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Extending and exploring students' problem solving via problem posing

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EXTENDING AND EXPLORING STUDENTS' PROBLEM SOLVING VIA PROBLEM POSING:

A STUDY OF YEARS 8 AND 9 STUDENTS INVOLVED IN MATHEMATICS CHALLENGE AND ENRICHMENT STAGES OF EULER ENRICHMENT PROGRAM FOR YOUNG AUSTRALIANS

by

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USE OF THESIS

The Use of Thesis statement is not included in this version of the thesis.

ABSTRACT

Despite the fact that problem posing has been recommended as a useful mathematical activity in the curriculum documents of several countries, research about the types of problem-posing situations in which students could be involved, and about the effects of these problem-posing activities on students' mathematical performance is limited. The application of problem posing in school mathematics has been hindered by the absence of a framework which links problem posing, problem solving and mathematics curricula.

In this study problem-posing is viewed as a teacher's as well as a student's activity, and as a means for facilitating students' problem solving.

In particular, this study involved:

- Designing a Program suitable for the participants of the study by adapting and extending the content of the Euler Program - the first stage of a four-level national program for working with mathematically able students;

- Developing a framework of problem-posing situations and a system of a teacher's "hidden" problem-posing questions aimed at assisting students to understand the problem and solution structures. Krutetskii's (1976) system of mathematical problems (which were intended to reveal the structure of students' mathematical abilities) was adapted and extended as a system of structured and semi-structured problem-posing situations;

- Application of this framework as part of an instructional *open problem-solving approach.* This approach aimed to create environments which can support students as they analyse problem and solution structures more deeply and to

encourage students to solve mathematical problems by using different solution strategies;

- Developing schemes for assessing students' problem-posing and problem-solving products. The schemes were pre-defined and then used for evaluating the effects of the experimental treatment on selected aspects of students' problem-solving and problem-posing performances;

- Detecting and examining the major characteristics of problem-posing strategies used by Years 8 and 9 students. The problem-posing strategies identified were classified in four categories (reformulation, reconstruction, imitation and invention);

- Developing two case studies which explored the problem-posing and problem-solving performances of two students;

- Suggesting implications of the findings of this study for learning and teaching mathematics and for further research investigations.

The thesis content consists of four interrelated parts:

Part I comprises Chapters 1 through Chapter 5 and outlines the theoretical frameworks used for the design of the study and the premises which underlie application of the problem-posing situations. In particular, Chapter 1 presents a broad introduction to the thesis content, the research questions and the aim of the study. Chapter 2 summarises the literature on problem posing. The literature review is presented under three subheadings: (a) research studies in which problem posing has been used as a research tool; (b) studies on investigating the impact of problem posing on mathematical instruction, and (c) a summary of the types of problem-posing activities recommended for use in mathematics classrooms. When the research was undertaken it was an open question to what extent students would reflect, via problem-posing actions, on the researcher's verbal and written prompts which were designed on the basis of the initial framework. In Chapter 3 the aims and the methodology are presented. The organisation of the study, the goals of the Euler Program and the Program used for the purpose of the study are outlined. The frameworks developed to describe problem-posing situations, "hidden" problem-posing questions and the open problem-solving approach, are discussed in Chapter 4. Data collection and data analysis procedures employed in the study are outlined in Chapter 5.

Part II consists of Chapters 6 to 9. Here the results of several studies which relate directly to the research questions are presented. Chapter 6 presents a classification of the types of problem-posing situations developed in the project classroom on the basis of the initial framework. Chapter 7 summarises the categories of the problem-posing strategies employed by Years 8 and 9 students. Chapter 8 looks at the effects of the open problem-solving approach on students' problem-solving and problem-posing performances. The results of the participants in the project classroom are compared with those of students who were exposed only to problem-solving activities. Chapter 9 presents two case studies. It throws additional light on the ways in which problem-posing situations can be used as a means to help students to improve their problem solving in a range of classroom contexts.

The discussion and the implications of the study for further research and for the teaching and learning of mathematics are presented in Part III which comprises Chapters 10 and 11.

In order to provide a glimpse of the mathematical content of the Program, samples of teaching materials (individual worksheets, revision papers, additional materials, etc.) developed for the purposes of this study are presented in the Appendices which form the final part of this thesis.

Figure 0.1 presents a guide to the overall structure of the thesis:

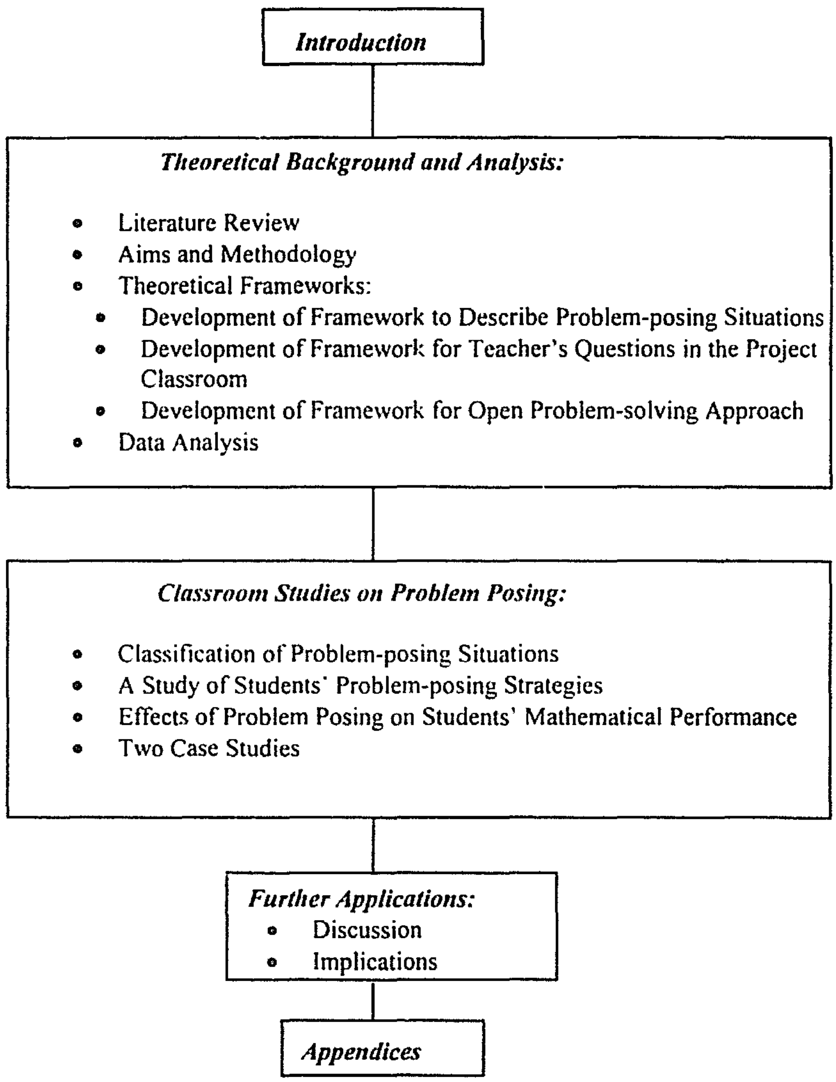

Figure 0.1. Structure of the thesis content.

DECLARATION

I certify that this thesis does not, to the best of my knowledge and belief:

(i) incorporate without acknowledgment any material previously submitted for a degree or diploma in any institution of higher education;

(ii) contain any material previously published or written by another person except where due references is made in the text; or

(iii) contain any defamatory material.

Date: 30. 06. 1997

Signature:

Elena Nikolova Stoyanova

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CHAPTER ONE

INTRODUCTION

The Purpose of this Study

After a decade of studies which have focused on problem solving, researchers have slowly begun to realise that developing a student's ability to *pose* quality problems in mathematics is at least as important, educationally, as developing the student's ability to *solve* them.

A number of studies have looked at the effects of specific types of problem-posing activities on students' mathematical performance. As Silver (1993) has stated,

> . . . despite this interest, however, there is no coherent, comprehensive account of problem posing as a part of mathematics curriculum and instruction, nor has there been systematic research of mathematical problem posing. (p. 66)

The literature review will establish that research into the potential of problem posing as an important means for the development of students' understanding of mathematics has been hindered by the absence of a framework which links problem solving, problem posing and mathematics curricula.

This study represents a first step in the development of such a framework and explores the effects of what will be referred to as an "open problem-solving approach," designed on the basis of this framework, on students' mathematical performances and their problem-posing strategies.

Definitions

To help structure the literature review, broad definitions will be introduced now rather than later in the thesis.

The Problem

According to Mayer (1983, p. 4) a problem has certain characteristics:

Givens — The problem begins in a certain state with certain conditions, objects or pieces of information;

Goals — The desired or terminal state of the problem is the goal state, and thinking is required to transform the problem from the given to the goal state;

Obstacles — The thinker has, at his or her disposal, certain ways to change the given state or the goal state of the problem. The thinker, however, does not already know the correct answer; that is, the correct sequence of behaviours which are needed to solve the problem is not immediately obvious.

Schoenfeld (1989) gave an alternative definition of a mathematical problem:

> For any student, a mathematical problem is a task (a) in which the student is interested and engaged and for which he wishes to obtain a resolution; and (b) for which the student does not have a readily accessible means by which to achieve that resolution. (p. 87)

Thus, according to Schoenfeld, a task is a problem when a student does not know how to resolve the task immediately.

In this study, the definition given by Wickelgren (1974, p. 10) will be used. She described mathematical problems as composed of three types of information: (a) information concerning givens (given expressions); (b) information concerning operations that transform one or more expressions into one or more new expressions; and (c) information concerning goals (goal expressions).

Givens — refer to the set of expressions that we accept as being present in the world of the problem at the onset of work on the problem.

Operations — refer to the actions one is allowed to perform on the givens or on expressions derived from the givens by some previous sequence of actions.

Goals — refer to those parts of a problem which can be described as terminal expressions that one wishes to cause to exist in the world of the problem.

The Solution

In this study the definition given by Wickelgren (1974) will be used. A solution, according to Wickelgren, is

> an ordered succession or sequence of problem states, starting with the given state, such that each successive state is obtained from the preceding state by means of an allowable action (operation applied to one or more expressions in the preceding state). (p. 10)

Thus, a problem solution is a set of successive interrelated problem states, obtained on the basis of a set of allowable actions.

Problem Solving

Krulik and Rudnick (1984) defined problem solving as "a process by which the individual uses previously acquired knowledge to resolve a problem which confronts him or her" (p.123).

For the purposes of this study the *problem-solving process* used by a student will be defined as the process by which the student uses her or his previous mathematical experience and knowledge to solve and write a solution to a problem.

The *problem-solving performance of a student* will be defined as the way in which a student uses his or her previous mathematical experience to: (a) understand the problem statement (the concepts and relationships); (b) identify appropriate problem-solving strategies and methods; (c) solve the problem; (d) write the solution; and (e) evaluate the solution method(s) used and the result (correctness and appropriateness).

Problem Posing

The notion of problem posing has been explored by different researchers from contrasting perspectives. For example, problem posing has been viewed as the generation of a new problem or reformulation of a given problem (Duncer, 1945), as the formulation of a sequence of mathematical problems from a given situation (Leung, 1993), or as the resultant activity when a given problem invites the generation of other problems (Mamona-Downs, 1993). Dillon (1982) conceptualised "problem finding as a process resulting in a problem to solve."

Silver (1995) referred to problem posing as involving the creation of a new problem from a situation or experience, or the reformulation of given problems. Such problem posing, according to Silver, could occur prior to problem solving (when problems are being generated from a given contrived or naturalistic situation), during problem solving (the individual intentionally solving the problem can change some of the problem's goals or conditions), or after solving a particular problem (as would be the case when problems are generated on the basis of the experience gained by solving a particular problem or a set of problems).

The notion of problem posing in this study will be defined through the notion of *problem structure.* Halford (1987) defined a structure as "a set of elements, with a set of relations or functions defined on the elements." Taking Halford's perspective, mathematical problems will be referred to as structures whose elements and relations are mathematical notions. Thus, a specific problem is *well-structured* when the goal can be determined by all given elements and relationships. Problems which are not well-structured will be referred to as *situations.*

In this thesis mathematical problem posing will be defined *as the process by which, on the basis of their mathematical experience, students construct personal interpretations of concrete situations and formulate them as meaningful well-structured mathematical problems.*

Mathematical situations which involve problem posing will be termed *problem-posing situations.*

The Problem-posing Performance

This study explores the effects of a range of problem-posing situations on students' problem-posing performance. The *problem-posing performance of a student* will be defined as the way in which the student uses his or her previous experience to: (a) understand the conceptual and procedural knowledge needed to resolve a particular problem-posing situation; (b) apply a set of appropriate problem-posing actions; and (c) formulate (or write) well-structured mathematical problems which are (somehow) connected with the given problem-posing situation.

Categories of Problem-posing Situations

Problem-posing situations used in the study are classified on the basis of the characteristics and certain structural features of the situations themselves. Central to this study is the thesis that any problem-posing situation can be classified as *free, semi-structured* or *structured.*

A problem-posing situation is described as *free* when students are simply asked to pose a problem from a contrived or naturalistic situation. Thus the structure of the situation is open and students have to select a set of elements, define relationships among them and present this information as a well-structured mathematical problem. Some directions may be given to prompt particular actions. For example, free problem-posing situations can involve asking students to pose a problem which they enjoy solving, or to suggest a problem for a coming mathematics competition.

A problem-posing situation will be referred to as *semi-structured*, when students are given a *situation* in which they are invited to explore and to formulate a problem which would draw on the knowledge, skills, concepts and patterns gained from their previous mathematical experiences. For example, students can be involved in posing problems based on different interpretations of the asterisks in a set of symbols such as: 2 4 6 * 10 * 12 *.

A problem-posing situation will be called *structured* when problem-posing activities are based on a specific problem or solution. The problem-posing task for the student is to develop new problems which are derived from a given problem or problem solution. For example, a specific problem, such as "Calculate

$3 \times 25 + 15 \div 5 - 4$" might be presented to the students and they could be invited to pose other problems based on this calculation.

It should be stated that the boundaries between free, semi-structured and structured problem-posing situations are not always well defined. The definitions have been used to facilitate the design process and to help the researcher to make proper choices with regard to the situations which should be used for particular instructional goals.

Identification and classification of the types of problem-posing strategies which the students use to pose problems was a central theme for this investigation. A *problem-posing strategy* will be taken to refer to the main features of the sequence of steps used by a student to pose a problem. These sets of steps are categorised as problem-posing strategies.

Rationale for this Research

Professional mathematicians and scientists have recognised problem solving and problem posing as two of the essential elements of their intellectual work (Einstein & Infeld, 1938; Polya, 1957). In mathematics education problem solving has gained a significant place in the school classroom and in mathematics education research during the last forty years (Kilpatrick, 1987). Problem solving and having students become competent problem solvers has been accepted by many educators as a primary goal of mathematical instruction (Australian Educational Council, 1991; National Council of Teachers of Mathematics, 1980, 1989).

On the other hand, problem formulation, identified by Einstein and Infeld (1938) as more essential than problem solution, and by Polya (1957) as an inseparable part of problem solving, has received far less attention in the school classroom and in education research (Kilpatrick, 1987). Getzels (1984) observed that

> although there are dozens of theoretical statements, hundreds (^ psychometric instruments, and literally thousands of empirical studies of problem solving, there is virtually no such work on problem finding. (p. 9)

Thus, the lack of systematic work on problem finding forms a strong contrast with the recognition of the importance of problem posing by professional scientists.

In recent years, problem posing by students has begun to receive increased attention, and the potential impact of problem posing on mathematical instruction is being recognised (eg. Brown & Walter, 1983, 1993; Ellerton, 1980, 1986a, 1988; Ellerton & Clarkson, 1996; Ellerton & Clements, 1996; Kilpatrick, 1987; Leung, 1993, 1995, 1997; Moses, Bjork & Goldenberg, 1990; Mousley, 1990; Nohda, 1988, 1991; Pehkonen, 1993, 1995; Shimada, 1977; Shukkwan & Silver, 1997; Silver, 1993, 1995; Silver & Cai, 1996; Silver & Mamona, 1989; Silver, Mamona-Downs, Leung & Kenney, 1996; Silver & Shapiro, 1990; Silver, Kilpatrick & Schlesinger, 1990; Sullivan & Clarke, 1991a, 1991b, 1993; Stacey, 1995; Stoyanova & Ellerton, 1996; Sweller, 1984, 1992, 1993).

Important school curriculum documents such as *Curriculum and Evaluating Standards for School Mathematics* (National Council of Teachers of Mathematics, 1989) and *The National Statement on Mathematics for Australian Schools* (Australian Education Council, 1991) advocated the use of problem posing in mathematics classrooms (see Chapter 2).

However, some researchers (e.g. Ellerton, 1986a; Kilpatrick 1987; Silver, 1993), claim that problem posing is not in fact, an inseparable part of problem-solving environments in most mathematics classrooms and the types of problem-posing situations used in mathematics textbooks is limited. Few appropriate problem-posing activities in mathematics textbooks for students are available (Mousley, 1990). Kilpatrick (1987) generalised that most of the literature on problem posing deals with the re-formulation of ill-formulated problems, or the formulation of sub-problems and related problems. Thus, the lack of a link between problem solving, ways of designing problem-posing situations on the basis of school textbook problems, and modes of applications into mathematics classroom, is clearly evident.

The effects of problem posing on specific goals of mathematical instruction have been explored by education researchers from different perspectives (see Chapter 2). In a few instructional studies researchers reported observing a positive effect on students' mathematical performance when a particular type of problem-posing activity was adopted (Perez, 1985; Winograd, 1990). Silver (1993) claimed that the incorporation of problem posing in mathematics classrooms is associated with its perceived potential to improve student's problem-solving performance. However, until now, no instructional studies have investigated the systematic use of a range of problem-posing activities and their effects on students' mathematical performance.

More research is needed into how problem-posing activities can interact as an inseparable part of problem-solving environments in order to meet the goals of mathematical instruction (Kilpatrick, 1987), and to investigate what modes of interaction between problem posing and problem solving are likely to facilitate students' understanding of mathematics.

Limitations of the Study

One of the limitations of this study is related to the students' wish to take part in the research program, since all participants were volunteers (not randomly selected).

A second limitation of this study, inherent in the research methodology used, is the lack of generalisability. According to Krutetskii (1976), the mathematical problem-solving processes used by students with high mathematical aptitude differ from those with average or low aptitude. Applying Krutetskii's system of mathematical problems as problem-posing situations required a sample which also contained students with above average mathematical abilities. Although a description of how the problem-posing situations used can be generated from school textbooks is provided, a similar study should be conducted in a natural classroom setting.

A further limitation may arise from the method used to determine some of the characteristics of the strategies employed by students in posing problems, because inferences about problem-posing processes suggested by students' verbal and written explanations may differ from the thinking processes the students used when they posed problems (Kantowski, 1977).

CHAPTER TWO

LITERATURE REVIEW

Introduction

The literature review will be summarised under three interrelated headings. First, commentary supporting the recognition of problem-posing by professional scientists and curriculum documents will be presented; second, problem posing as a research tool for investigating students' understanding of mathematics will be discussed; third, studies which deal with the impact of problem-posing activities on mathematical instruction will be described; and fourth, teaching applications of problem-posing situations in mathematics classrooms will be discussed.

Recognition of Problem Posing

By Professional Scientists

Many prominent scientists have recognised the development of skills for *posing* significant questions as an equally important part of their scientific work as the ability to *solve* them. Einstein and Infeld (1938), for example, wrote:

> The formulation of a problem is often more essential than its solution, which may be merely a matter of mathematical or experimental skills. To raise new questions, new possibilities, to

> regard old questions from a new angle, requires creative imagination and marks real advance in science. (p. 92)

Thus, according to Einstein and Infeld, the first step of the discovery is the formulation of a significant question.

The same opinion is shared by researchers from other scientific fields. For example, the biologist Charles Darwin (Immegart & Boyd, 1979), also recognised that for him it was more difficult to identify the problems than to solve them.

Max Wertheimer (1945), a pre-eminent psychologist, acknowledged that

> the function of thinking is not just solving an actual problem, but discovering, envisaging, going into deeper questions. Often in great discoveries the most important thing is that a certain question is found. Envisaging, putting the productive question is more important, often a greater achievement than a solution of a set question. (p. 123)

Thus, according to Wertheimer, "going into deeper questions" and formulating a significant question has to be regarded as an achievement in itself.

Francis Upton (Immegart & Boyd, 1979), a mathematician and physician on Edison's staff who provided ingenious solutions to technical problems in the laboratory, placed far higher value on Edison's ability to pose original questions than on his own ability to provide an answer. He wrote: "I can answer questions very easily after they are asked, but I find great trouble in framing any to answer" (p. 26). Indeed, Thomas Edison's talent for asking original questions has brought humanity many inventions.

By Artists

The ability to pose significant questions is also recognised in the artistic field. Geszels and Csikszentmihalyi (1976) found that better artists are better problem posers. Their art is distinguished by the problems they pose for themselves as well as by the way in which they approach these problems.

By Professional Mathematicians

Problem posing has also received recognition from professional mathematicians. It is well known that, although questions posed by the greatest mathematicians become targets of many mathematicians around the world, it sometimes takes centuries before a solution is found. It was only recently for example, that the Fermat's last theorem, which was posed 350 years ago, was proven (Lemonick, 1993).

The significance of the solution to a specific problem depends, to a very large extent, on the significance of the question asked. In his investigation Zuckerman has found that elite scientists differ from others not so much in the answers as in the questions that these two groups of scientists pose (Zuckerman, 1977).

Recognition of Problem Posing in School Curriculum Documents

In mathematics education, after over a decade of studies which have focused on problem solving, researchers have slowly begun to realise that developing the ability to *pose* quality mathematical problems is at least as important, educationally, as developing the ability to *solve* them. Silver, Kilpatrick and Schlesinger (1991) suggested that incorporation of problem-posing activities into regular classroom situations could be a powerful approach for developing students' mathematical thinking.

The application of problem posing in school mathematics has been advocated in curriculum documents since the late 1980s (Stacey, 1995). *The National Statement on Mathematics for Australian Schools* (Australian Education Council, 1990) support the use of open-ended problems in mathematics classrooms by suggesting that

> students should engage in extended mathematical activities which encourage problem posing, divergent thinking, reflection and persistence. They should be expected to pursue alternative strategies, and to pose and attempt to answer their own mathematical questions. (p. 39)

Thus, the *National Statement* requires teachers to adopt strategies for helping students to pose and solve mathematical questions.

The Education Department of Western Australia (1994) advocated problem-posing activities as an important outcome of mathematics education. Extending mathematical tasks by asking questions like "What would happen if . . ." has been recommended as appropriate for helping to prepare students for future scientific work.

In the United States, *The Curriculum and Evaluation Standards for School Mathematics*, (National Council of Teachers of Mathematics, 1989) acknowledged the importance of presenting to students some basic knowledge about the nature of the work of mathematicians. In this context it was suggested that

> students in grades 9-12 should also have some experience recognising and formulating their own problems, an activity that is at the heart of doing mathematics. (p. 138)

Thus, increasing recognition is being given to the need for teachers to involve students in problem-formulating activities as a part of mathematics classroom work.

In the mathematics curriculum of many countries, students' work on investigative projects is encouraged and supported. In Bulgaria, for example, the winners in the Students' Scientific Conferences have the right to free tertiary admission (Stoyanova, 1994). It is also well known, that on a regular basis, some students' magazines, for example — *Kvant*, *Mathematica v Schkole* (Russia), *Matematika Plus* (Bulgaria), and many others — publish original students' problems.

Problem Posing as a Component of Preservice and Inservice Teacher Education Programs

Researchers have started to consider problem posing not only as part of mathematical instruction, but also as an important component of preservice and inservice teacher education programs (Clements, 1994; Gonzales, 1994; Mousley, 1990; Pehkonen, 1993).

For their every-day work teachers need to find a range of closed, open, interrelated or equivalent problems, which best suit specific instructional goals. Therefore the posing and reformulation of problems has been considered by some educators as an important component in both preservice and inservice teacher education programs, and a valuable skill for teachers to develop (Pehkonen, 1992, 1993).

Suggestions for pedagogical innovation in some recent reforms in mathematics education have been based on the involvement of prospective teachers in problem-posing activities. Gonzales (1994), for example, described a scheme designed to help prospective teachers become more efficient at posing and solving mathematical problems.

In Australia, the first problem-solving and problem-posing unit to be offered as part of a Masters Education Degree Program was introduced to University of

Newcastle by Ken Clements in 1994. The unit was called *Teaching Mathematics Through a Problem-Posing and a Problem-Solving Approach.*

Pehkonen (1993) investigated the nature of teachers' preferences for using, on a regular basis, *open-ended* problems in mathematics classrooms. He found that the criteria given by the teachers could be classified into three main categories: (a) convenience of use; (b) the pupil's motivation; and (c) support for learning objectives. His findings suggest that authors of mathematics textbooks as well as teachers will need specific knowledge and skills for transforming the structures of closed problems into problem-posing situations. Similar opinions were expressed by Anderson and Sullivan (1995). In their work they suggested specific strategies for creating open-ended problems and mathematical investigations.

Problem Posing as Research Tool for Investigating Students' Understanding of Mathematics

Problem-posing has been used as a research tool for investigating students' understanding of mathematics. The frameworks used by the researchers involve *free*, *semi-structured* or *structured* problem-posing situations.

Free Problem-posing Situations Used by Researchers

Many researchers have used free problem posing in their studies as a framework to describe students' mathematical understanding. For example, Ellerton (1980, 1986a, 1986b, 1986c, 1988) introduced creative writing in mathematics by asking students to make up mathematics problems. She asked a large sample of

Australian and New Zealand students to pose a problem which would be difficult for a friend to solve and used this framework as a window for exploring students' perceptions of mathematics. According to Ellerton (1988)

> children's expression of mathematical ideas through the creation of their own mathematics problems demonstrates not only their understanding and level of concept development, but also reflects their perception about the nature of mathematics. (p. 281)

Indeed, Ellerton claims that in a free problem-posing situation the child will respond by creating a problem which is coloured by previous experiences and by the child's perception about the nature of mathematical knowledge.

Richardson and Williamson (1982) used another form of free writing. They asked children to make up mathematical problems for each other. In his study Kennedy (1985) asked his mathematics students to write letters about what were they were studying, to keep logs, and to devise mathematical problems about a particular topic. Problem-posing activities involving much younger children have been described by Skinner (1991), a primary teacher from Australia. She found that free problem-posing activities engaged young students for a prolonged period of time. In a study conducted by Van den Brink (1985), Grade 2 children were asked to make up problems and games for Grade 1 children.

Semi-structured Problem-posing Situations Used by Researchers

Semi-structured problem-posing situations have been used as a research framework by several investigators. For example, Krutetskii (1976), in his study of mathematically talented students, used problems with *unstated questions, surplus information* and *insufficient information* to investigate the structure of students' mathematical abilities.

Semi-structured problem-posing activities were adopted as a research framework by Hashimoto (1987). He described a lesson in which students posed problems on the basis of problems they had solved the previous day. Hashimoto found that asking students to pose a problem *similar* to a solved problem can be a useful teaching technique which reflects students' understanding of mathematical concepts.

Structured Problem-posing Situations Used by Researchers

Another type of problem-posing activity was used by Hart (1981) as a mirror to reflect students' understanding of mathematics. She used *structured* problem posing to examine students' understanding of important mathematical concepts. Hart asked children to make up mathematics problems to *fit* given computations. Her aim was to study how children draw on concrete situations in describing symbolic expressions.

In their "What-if-not?" and "What-if?" instructional approach, Brown and Walter (1983) suggested students could be involved in working through systematic variations of the structure variables in a specific problem.

The Impact of Students' Problem-posing Activities on Mathematical Instruction

Introduction

For more than a decade, problem solving has been regarded as the ultimate goal of mathematical instruction (Schoenfeld, 1995). On the other hand, problem posing — considered, for example by Polya (1957), as an inseparable part of problem solving — has received far less attention (Getzels, 1984).

According to Kilpatrick (1987), problem posing has to be regarded not only as a goal, but also as means of instruction, and as an important companion to problem solving. Silver (1993), for example, viewed problem posing as the ultimate goal of inquiry-oriented instruction. Problem posing was advocated as a way of helping students to construct general rules, theories or principles, and also as a strategy to help them solve problems through the use of self-questioning and self-regulatory techniques and metacognitive skills (Collins, 1988).

The potential of using problem-posing activities in the teaching and learning of mathematics has been explored by mathematics education researchers from a range of contrasting perspectives. These studies, and the impact made by problem-posing activities on the goals of mathematical instruction, will now be summarised under a series of subheadings related to teaching and learning processes.

As a Way of Extending Students' Understanding of Important Mathematical Concepts

The role of problem posing as a way of extending students' understanding of mathematical concepts has been recognised for a long time (Toshikazu, 1993). For example, more than half a century ago, Brueckner (1932) cited the use of problem posing for improving students' ability to solve problems. He used student-generated problems as a means of helping them to develop a sense of number relations and to generalise number concepts. Only a few years later, Connor and Hawkins (1936) argued that having students generate their own problems improved their ability to acquire arithmetic concepts and skills in solving problems.

In recent curriculum documents such as *The Professional Standards for Teaching Mathematics* (National Council of Teachers of Mathematics, 1991) problem posing was advocated as a way of helping students to understand the meaning of mathematical procedures. Problem-posing activities have also been seen as an approach which could improve pupils' awareness of learning mathematics (Shell Centre, 1991). It has been suggested that students can be encouraged to reflect on and discuss mathematical concepts and ideas by incorporating problem-posing activities such as: making up questions from answers, making up questions from data, making up questions from situations, and constructing tests and marking schemes.

As a Means for Improving Students' Skills in Problem Solving

According to Silver (1993), the most frequently cited motivation for curricular and instructional interest in problem posing is its perceived potential for assisting students to become better problem solvers.

Polya (1957) included problem posing as a useful problem-solving strategy. He saw the connection between problem solving and problem posing in the nature of problem solving itself, and considered problem solving as a sequence of successful problem reformulations. Polya recommended four strategies for problem solving: *analogy* (considering an auxiliary element or a problem), *decomposing* and *recombining* (varying the problem), *generalisation* (inventing the general problem), and *specialisation* (concrete interpretations). In fact, Polya recommended the use of some strategies which incorporate problem posing as a means for helping students to become better problem solvers.

Another educator who recommended the use of problem posing was Koenker (1958). He employed problem posing as one of 20 ways which he used to help his students to improve their problem-solving skills.

A number of researchers have explored the effects of specific types of problem-posing activities on students' mathematical problem-solving performance. Jonathan Smilansky, quoted in Getzels (1988, p. 101), for example, investigated the relationship between being able to solve problems and to pose problems in the same domain. Smilansky administered the Ravens Progressive Matrices instrument to 129 Year 10 and Year 11 students in selected high schools. After students completed the test, Smilansky distributed a skeleton test page and invited students to create a new problem which they would consider particularly difficult for a future version of the test.

The Ravens Progressive Matrices Instrument is a compilation of visually presented problems of increasing complexity. It is possible to specify the relative difficulty of a given problem, and to ascertain the difficulty of a newly formulated problem. Thus it was possible to determine the relationship between the scores students obtained in solving the problems on the test and the difficulty level of the problem they were able to formulate. Smilansky (1984) found a low correlation between the performance on the problem-solving task and the problem-posing task in the same domain. He suggested that considerably different thought processes were involved in the two tasks. An analysis of individual performances on the two tasks revealed that not one of the fifty-three students at the lower level in the problem solving task was able to formulate any high level problem, indicating that the ability to solve problems is necessary in order to be able to pose problems. In fact only twelve of the fifty-seven students, who scored at the highest level of the problem-solving tasks posed a problem at the highest level in the problem-posing task. Smilansky concluded that this is an indication that the ability to solve problems does not automatically assure ability to formulate problems.

Palincsar and Brown, quoted in Resnick (1985), investigated the relationship between students' ability in problem formulating and their acquisition of comprehension skills. The researchers divided a cohort of middle-school children with weak reading comprehension skills into small subgroups and engaged each group in an instructional program they called "reciprocal teaching." The children took turns posing questions and summarising a short passage of text they were asked to read. The other members of the group and the teacher commented on the quality of the questions, and tried to help formulate better questions. At the beginning, many of the children had no idea about how to abstract a question from an expository or narrative passage. As the reciprocal teaching sessions progressed the children's ability to pose coherent questions increased, and after several weeks they were able to formulate a core question which addressed the main idea of the passage—a task which had been beyond their ability at the beginning of the program. Moreover, as the children's skills in problem-posing increased, they also improved in their reading comprehension – an improvement which generalised to comprehension of social studies and science texts in their regular classrooms.

These findings have close parallels with those reported by Michael Meyer (1983). He pointed out that writing and reading may be conceived of as complementary question-answer processes. In writing, the author produces an answer to a question, albeit an unstated question. To comprehend what is being written, according to Meyer, the reader must find the question to which the text relates. The author proceeded from a question which was implicit toward an answer which is stated in a figurative or rhetorical way; the reader proceeded in the reverse direction from the answer, i.e. the text to the implicit question in a more explicit literal way. As Meyer (1983b) said: "When we look for the meaning of a text, we try to find the question to which it answers as a text" (p. 157). Thus, according to Meyer, problem posing goes hand in hand with the reading process.

The method of using open-ended problems in the classroom for promoting mathematical discussion — the so called "open-approach" — was developed in Japan in the 1970s. Shimada (1977), Hashimoto and Swada (1984), and Nohda (1986) have described various styles of teaching, termed "open approach teaching," in which problem posing was used to assist students to analyse problems more completely. By promoting classroom discussion about various aspects of the problem, and the range of solutions obtained could be demonstrated, as could a variety of approaches to solving a particular problem. Particular problem-solving strategies used were also discussed. Nohda (1995) claimed that the main aim of instruction using "open-ended problems" is to foster simultaneously both the creative activities of the students and their mathematical thinking during problem solving. He also argued that "open-ended approaches" are effective methods of mathematical problem solving.

Applying problem-posing situations in the classroom setting assumes that both the teacher and the students are important resources for problem-posing situations. Several studies have investigated the student's role as problem poser. Most problem-posing situations reported in the literature are based on having students write word problems. Keil (1964) found that students who wrote and solved word problems on their own perform better on problem solving than students who had the traditional textbook experience. On the basis of this result, she recommended the writing and solving of original problems as one of the approaches for improving students' problem-solving skills.

Perez (1985) found that students who have experience in writing and solving their own story mathematical problems did better on a problem-solving test than students who were not exposed to this experience. His conclusion was that the "process of writing 'word problems' had improved students' abilities to solve word problems" (p. 87).

Lodholz (1980) also investigated the effect on mathematical problem-solving performance of having students pose word problems. His approach was based on asking students to illuminate particular linguistic elements (pronouns, conjunctions, relative clauses) and specific mathematical components (hidden numbers, multiple operations) in the process of writing a mathematical word problem. Analysis of the results did not show any significant difference in student performance in favour of the strategy adopted. Lodholz concluded that having students write their own word problems improved the students' attitude toward mathematics, while at the same time, their achievement on computation or understanding associated concepts was not hampered.

The study designed by Graham (1978) compared three approaches: (a) pupils constructed an open number sentence after understanding the formal structure of the problem; (b) a guided step approach; and (c) practice only. Students from the first group were involved in specific problem-solving procedures and subsequent construction of problems. Graham did not find a significant difference in students' problem-solving performance, even though students involved in the first approach were more successful in writing open-number sentences than the other students.

Silver and Cai (1993) found a strong positive relationship between problem posing based on a brief story which included an unstated question, and the problem-solving performance of middle-school students on open-ended mathematical problems.

Students' engagement with problem solving and conjecture which involved the formulation of problems, in the context of solving "goal-free" problems from the domains of geometry and trigonometry, has been shown to result in improved performance in solving subsequent test problems (Owen & Sweller, 1985; Sweller, 1992, 1993; Sweller, Mawee & Ward, 1983;).

In recent curriculum documents, the perceived potential of problem posing to have a positive effect on students' problem-solving skills has found promising support. According to *Mathematics Student Outcome Statements with Pointers and Work Samples* (Education Department of WA, 1994), one of the important aims of mathematics education is to help students develop their ability to identify those features of a problem which are likely to be relevant to its solution, and to pose and answer their own mathematical questions.

As an Important Component of Students' Assessment

The use of mathematical investigations (referred to by some as open-ended problems) became popular in mathematics teaching in England in the 1970s (William, 1994). In the United States, Australia and other countries the "open problems" have been popular as "projects." In Australia, one of the main areas in which the use of open-ended questions and investigations has been advocated was as a component of students' assessment (Clarke & Sullivan, 1991a). In Victoria, for example, in the late 1980s, two open-ended problem-solving tasks for Year 12 were incorporated as a part of the final overall assessment. From 1990 - 1992 two tasks, each worth 25 percent of the final assessment were used — an investigative project and a challenging problem (Stacey, 1995).

Pegg and Davey (1991) discussed some practical insights and ideas about how problem posing can be used for assessing students' geometrical understanding in the classroom. The recent *Mathematics Student Outcome Statements with Pointers and Work Samples* (Education Department of Western Australia, 1994), which attempt to describe student outcomes in mathematics education, have advocated the inclusion of problem posing in the assessment of students' problem-solving strategies.

In California, the inclusion of open-ended problems is advocated in addition to assessment of tasks involving standard multiple-choice tests (referred to in Pehkonen, 1995). Burjan (1993) has suggested that problem-posing activities should be part of students' mathematics competition activities.

Students' work on original investigative projects has been accepted to have the same importance as their mathematical problem-solving performance. In Bulgaria, for example, the winners in the investigative project section of the Students' National Conferences have the right to free tertiary enrolment (Stoyanova, 1994).

As a Way of Changing the Nature of the Communication in the Mathematics Classroom

Many researchers and educators have suggested that some types of problem-posing activities should be used as a way of bringing about change in the modes of communication in mathematics classrooms (Clarke & Sullivan, 1991b; Del Campo & Clements, 1987; Pehokonen, 1993; Silver, 1993; Stone, 1994; Todd, 1987). The main goal of the "open-ended-approach" developed by Shimada (1987) was to promote classroom discussion. In fact, incorporating semi-structured problem-posing situations into mathematics lessons will involve an increased need for communication between the teacher and the students as well as changes in the character of communication in mathematics classrooms.

As a Window into Students' Difficulties in Mathematics

Problem posing has been used by a number of researchers for investigating contrasting levels of students' mathematical performances. Some researchers have suggested that problem posing is a sensitive reflection of students' learning difficulties. Hosmer (1986) went on to suggest that problem posing can help teachers to improve their work by helping them to become aware of students' difficulties (1986). Indeed, Winograd (1990) observed that children generally composed problems which they themselves had difficulty understanding or solving.

According to Caldwell (1984), the problem format can be one of the reasons affecting problem difficulty. The effect on students' ability to solve problems of requiring them to make format changes themselves has been reported by Cohen and Stover (1981) and by Stover (1982). Grade 6 students were asked to modify one of three structural format variables (adding a diagram, removing extraneous information, and reordering information) in the statement of a problem. The researchers observed a substantial improvement in students' ability to solve word problems of the type they had learned to modify.

Errors made by students provide another mirror which reflects the types of individual learning difficulties. Radatz (1980) suggested that students' errors in solving problems can be a powerful tool for diagnosing learning difficulties and consequently can be of assistance in direct remediation. Borasi (1987) went on to suggest that students' errors can sometimes be interpreted by the teachers as the result of an involuntary change of problem attributes or of making assumptions, and this may provide a natural stimulus and starting point for classroom discussions. In fact, Borasi suggested interpreting the errors as problem-posing situations and engaging students in exploring the new structure.

As a Way of Investigating the Highest Level of Students' Mathematical Performances

Other researchers, Krutetskii and Ellerton for example, used problem posing as a research tool for investigating students' mathematical abilities. Krutetskii (1976) used several forms of problem posing to investigate some of the components in the structure of students' mathematical abilities. He concluded that mathematically able students grasp the problem structure with a greater ease than students with less mathematical ability. Ellerton (1986a) found that asking students to write a mathematical problem can open a window into understanding their mathematical abilities.

There is a strong acceptance among researchers and educators of the notion that students' ability in posing quality problems provides a useful indication of potential mathematical talent. According to Hadamard (1945), the ability to identify key research questions is the hallmark of potential talent in mathematics. He suggested that the ability to choose good questions was a matter of aesthetics. Those students who learned or instinctively identified elegant research questions distinguished themselves from those students who Hadamard described as "second rate." Like Hadamard, Beeridge (1957), the author of *The Art of Scientific Investigation* described students who were talented in science as ones who could find suitable problems. According to Sternberg (1987)

> . . . intelligent people not only answer questions better, but also ask better questions. The time has come to measure and to teach not only how to answer questions, but also how we ask them. (p. 13.)

Thus educating students to grasp the *quality* of the problems solved and to pose *quality* problems is viewed by Sternberg as an imprtant aspect of students' mathematical performance.

As a Way of Preparing Students to be Intelligent Users of Mathematics in Their Every-day Life

Problem posing is regarded as an activity which is central to the discipline of mathematics. Self-directed problem posing is considered as an important characteristic in the work of mathematicians (Polya, 1957) and of scientists (Einstein & Infeld, 1938; Immegart & Boyd, 1979). But most students do not choose mathematics as their profession. The main aim, therefore, of problem solving and problem posing in school mathematics, is to prepare these students to be intelligent users of mathematical knowledge and approaches in every-day life. Out of school, students have to be able to pose and solve real-world problems. It is therefore important that teachers assist students to learn ways of thinking through problems. Problem posing can be an important component of instruction aimed towards achieving this goal (Blum & Niss, 1991). Writz and Kahn (1982) observed that having students make up applications helps them to bridge the gap between concrete situations and mathematical abstractions. Furthermore, such activities help students learn how to generalise and assist in making mathematics more meaningful to them.

Many observers are beginning to recognise that helping students become competent thinkers is a central challenge for all educators (Resnick & Klopfer, 1989). Incorporating problem posing by students into regular classroom situations has begun to be recognised by some researchers as a powerful approach for developing students' mathematical thinking (Silver, Kilpatrick & Schlesinger, 1991).

According to Mason (1991), one of the broad goals of education must be to stimulate students to ask questions, and to learn enough about various disciplined modes of inquiry in order to know where to seek assistance in the future. One of the ways of accomplishing this might be by involving students in a wide range of problem-posing situations.

As a Way of Linking Students' Own Interests With Their Mathematical Education

Several aspects of problem posing are thought to have important roles for linking students' personal interests with the experiences they gain through formal schooling.

Research into metacognition has demonstrated the value of helping students to reflect on, and take control of, their learning (Baird & White, 1984; Garofalo & Lester, 1985). Brown and Walter (1990) suggested that problem posing can help students to develop independent thinking processes. Writing problem stories has the additional advantage of integrating mathematics with other subject areas and of helping to develop creative writing skills (Bush & Fiala, 1986).

There is a growing recognition of the importance of the social context in which teaching and learning occurs. In mathematics education, the importance of social factors and belief systems in the learning of supposedly value-free mathematics topics has been demonstrated (Bauersfeld, 1980; Bishop, 1988; Clarke, 1985; Erwanger, 1975).

Researchers have reported that students appeared to be highly motivated when asked to pose problems that their classmates would find interesting or difficult (Ellerton, 1986b; Winograd, 1991). Mamona-Downs (1993) suggested that

> it is always helpful to imagine that you are addressing questions to a second person in this kind of activity. Whom this second person is will influence the type of questions posed. (p. 47)

Thus addressing questions to another person can help students to reflect in a specific way on the problem-posing task.

In their work, Moses, Bjork and Goldenberg (1990) reported the observation that students' personal interests can be supported by sharing problems with others and that this can help to reduce students' mathematical anxiety (1990).

Mellin-Olsen (1987) developed the Vygotskian notion of Activity Theory to suggest that real-life problem solving in which students work on questions arising from their experience is the best way to attract and involve students in mathematical thinking. According to Mellin-Olsen, this provides an opportunity for students to gain a little power and control over their lives.

Students' preferences for the type of the problem-posing activities also has been explored. Momona-Downs (1993) found that the questions written very rarely broke away from contexts familiar to the students. She found that the students in her study did not use the extra freedom offered by problem posing. It might therefore be expected that students' preferences for a particular type of the problem-posing situation would have an important role in planning for the incorporation of problem posing in the mathematics classroom.

The fact that education should take into account students' interests is being recognised (UNESCO, 1992). For example, in the curriculum documents for comprehensive schools in Hamburg, Germany (quoted in Pehkonen, 1993), in order to encourage mathematics activities, about one-fifth of the teaching time is left content free.

In Bulgaria, account is taken of students' interests by introducing the mathematics subject "facultative mathematical instruction," whose content is chosen by students and teachers, according to their personal preferences (Sendov et al., 1988; Stoyanova, 1994).

In the United States, *The Curriculum and Evaluation Standards* (National Council of Teachers of Mathematics, 1989) also gave a *green* light to the support of students' personal interests through mathematics education. It is suggested that "students should have opportunities to formulate problems and questions that stem from their own interests" (p. 67). Thus, teachers are encouraged to find ways of involving students in discussions about their particular areas of future needs in mathematics education.

Models for the Application of Problem-posing Activities in the Mathematics Classroom

This section of the literature review addresses the different types of problem-posing activities which have been recommended for the mathematics classroom by researchers and educators. These activities will be classified on the basis of the initial source of the problem-posing situation: (a) the problem task; (b) the teacher, and (c) the student.

Problem-task Environment as a Source of Problem Posing

According to Kilpatrick (1987), much of the literature about problem posing which is based on a specific problem deals with students' "formulation of sub-problems and related problems" and with "reformulation of ill-formulated problems" (p. 124). In fact, many educators have recommended the use of various structured problem-posing activities. Polya (1957), for example, mentioned three approaches for constructing a new problem from a proposed problem: "Firstly, keep the unknown and change the rest (data and the condition); or secondly keep the data and change the rest (the unknown and the condition); or thirdly change both the unknown and the data" (p. 78). Brown and Walter (1983, 1993) designed an instructional problem-formulating approach, which could be linked to Polya's ideas. They termed the strategy for posing problems which relate to a specific problem as a "What-if-not" strategy. The main idea underlying the design process of this strategy is that of posing new problems from a problem which has already been solved by varying the conditions or the goals of the original problem.

Another three ways for the application of problem-posing activities to the learning of mathematics were recommended by the authors of the book, *Improving Pupil's Awareness of Learning in Mathematics* (Shell Centre, 1991). The models which were felt to encourage students to reflect in more meaningful ways on what they are learning can be summarised as: (a) questions formulated from data (given a collection of given data, pupils try to formulate questions which may be answered); (b) question from answers (given an answer or a calculation, pupils construct an appropriate contextual question); and (c) questions from situations (given a context, or topic, pupils try to devise a number of questions). In fact, all these problem-posing situations can be referred to as semi-structured.

In order to provide a tool to enable teachers to create equivalent problems, Caldwell (1984) has considered changes in *syntax* (those variables which account for the arrangement of and the relationships among words, phrases and symbols in problem statement), *content* (the key-words which refer to mathematical substance of the task), and *context* (the variables which refer to non mathematical meaning of the problem statement). She illustrated a range of activities in which students might be engaged in the classroom. Some of the situations she has recommended can be referred to as *structured* and *semi-structured* problem-posing situations.

The "open-approach teaching" discussed by Hashimoto and Swada (1984), Nohda (1984, 1986, 1988, 1991) and Shimada (1977) as a form of instruction in which the ways of interaction between the mathematical content and the students are aimed at promoting variety in the problem-solving approaches adopted. In other words, in this approach the problem solution structure is regarded as open and students' activities are aimed at presenting different solution structures.

The term "open-ended problems" is also used in the literature to describe problems with an open structure (Ellerton & Clarkson, 1996; Silver, 1995). Problem-posing tasks of an open problem structure are often termed mathematical investigations (Kissane, 1988) or open-ended problems (Pehkonen, 1995). The concept of "open problems" was described by Pehkonen (1995) in the following way:

> We will begin with its opposite, and say that a problem is not closed if its starting situation and goal situation and/or the goal situation are open. i.e. if they are not closed, we have an open problem. (p. 1)

Thus, Pehkonen relates the openness of a problem to the openness of the *Goal*, the *Given* or to both.

Mathematical investigations, as mentioned earlier in this Chapter, can be regarded as problems with an open structure. Through mathematical investigations, according to Kissane (1988), students are engaged in exploring open situations in a relatively unstructured way. He wrote, that "a critical, defining feature of an investigation is that the student is responsible for devising, refining, and pursuing the questions" (p. 521). In other words, in an investigation the problem structure and students' activity are open. The goal for the student is to create a structure by exploring the possibilities and to provide a solution.

Evans (1987) explained the differences between investigations and the open problems by suggesting that problem solving is a convergent action through which pupils have to find a solution for a certain problem. By way of contrast to this, investigations are usually more divergent, with pupils being encouraged to think of alternative strategies, to consider what would happen if a certain route is followed and to look at whether different approaches will produce different results.

Pehkonen (1992) stated that the boundary between problem solving and mathematical investigations is not at all sharp. He suggested that most problems can become investigations if the task conditions are changed.

Stacey (1995, p. 63) and Silver (1995, p. 68) gave other interpretations of "open problems." Stacey defined a problem as *open* when students do not know immediately how to solve it (they do not know the solution method) and they therefore try to apply their own approach. Silver (1995, p. 68) suggested that the term "open problem" has several different meanings: (a) unsolved for some time; (b) has several methods of solution; or (c) naturally suggests other problems or generalisations. A Discussion Group which met at the 17th International Conference of the Group for the Psychology of Mathematics Education in Japan in 1993 accepted the notion of "open problem" as an umbrella class of problems which contains several categories: investigations, problem posing, real-life situations, projects, problem fields (or problem sequences), problems without questions, and problem variations ("what-if-method"), (Pehkonen, 1995).

A set of problems which are connected in some way is called a *problem field* (Pehkonen, 1989). Problem fields may be designed by varying the conditions of the starting point and the goal state (Pehkonen, 1992). He stated that any problem can generate new problems if its starting (and/or goal) conditions are changed. Tasks in a problem field are mostly closed problems, but the solution of one problem may give helpful ideas for posing a new problem. Problem fields can therefore be considered as "partially structured investigations." In the United States, the term "project" is used to describe such investigations (Trowell, 1990).

In Russia, Dorofeev (1983) used the term "cycles of problems" for a specific type of interconnected problems. Every problem in the cycle represents a sub-goal of a larger problem—the *goal* problem (Georgiev, 1988).

Hoehn (1991) described a series of problems whose design process was based on an application of a specific theorem. Hoehn admitted that, in creating the series, he used some of the same techniques which are used in problem solving — special cases, generalisation, related problems, converses, symmetry, useful notation, accident, previous results, useful figures, looking back, patterns. His insight into the approach he used for posing problems makes a clear link between problem solving and problem posing.

Problem-solution Environment as a Source for Problem Posing

The problem-solution environment is the second source suggested for nurturing appropriate problem-posing environments. According to Kilpatrick (1987), there are two phases in the solution process during which new problems can be created:

> As a mathematical model is being constructed for a problem, the solver can intentionally change some or all of the problem conditions to see what new problem might result. After a problem has been solved, the solver can look back to see how the solution might be affected by various modifications in the problem. (p. 127)

In other words, Kilpatrick recommended drawing students' attention to the changes in problem conditions which affect the mathematical model adopted and which investigate the connection between modifications to the problem and the solution method.

Polya (1957) suggested that students could be asked to make up a problem with the same method or solution as the problems students have solved. Drawing students' attention to the features of the solution idea used is also recommended by Lester (1985). He claimed:

> It is at least as important for the problem solver to identify the key features of a solution effort which may prove to be useful in future problem solving. A step in the direction of making students better able to look back at their efforts might be for teachers to focus more attention on solution attempts and less on correct answers. (p. 46)

Thus for Lester activities based on problem solutions are not less important than those based on problem statements.

Goldman and Zvavitch (1990) have described types of interconnected problems ordered in sequence with an increasing level of difficulty. They suggest that such sequences can be used in mathematics classrooms to enable students to explore specific mathematical topics in depth and to apply different types of reasoning—inductive, deductive or generalisation.

Every-day Life Situations as a Source for Problem Posing

Outside-of-school problems may arise from *free* or *semi-structured* situations, which are ill-structured and contain incomplete or surplus information. The first step involved in solving a real-life problem is to give it an initial formulation. This is in sharp contrast with the activities in which students are engaged in school—solving well-structured problems. Sullivan and Clarke (1991b) observed that

> the problems of the professional mathematician and those of the person in the street do not come well researched and appropriately labelled. (p. 33)

In other words, Sullivan and Clarke drew the attention of researchers to th similarities which might exist in problem-posing processes in science and in real life.

In the same light are the works of Ling (1977) and Lovitt and Clarke (1988) who have presented a range of applications of mathematical knowledge across the curriculum. In fact, the notion that problem solving and problem posing go hand-in-hand when a practical or scientific problem has to be resolved is shared by many other researchers and educators.

Research Studies on Students' and Teachers' Problem-posing Strategies

Goldin (1984) defined an algorithm as a well-defined procedure for solving a class of problems in a given representation. He approached the notion of *strategy* as a generalisation of an algorithm:

> A strategy is any procedure which narrows the set of possible moves, without necessarily singling out a unique move. (p. 148)

Thus a strategy does not lead down a unique path to the solution, but rather it generates a set of possible paths, which may or may not include a solution path.

Although students' problem-solving strategies have been the focus of much research, the literature on students' problem-posing strategies is limited. Several researchers have suggested that the processes involved in problem posing and problem solving might be different. For example, Gage (1982) conducted a study based on interviewing forty community college students. The structure of each interview included asking students to pose problems on the basis of four problem-formulating situation tasks. He found that the processes and strategies used by students who were exposed to formulating and solving problems were not the same as those used by students who were asked to solve ready-made problems. The study also showed that the number of strategies used by less able students when they formulated problems was greater than that used by the same students in solving ready made problems. In fact Gage recognised that students have a capacity to use their own problem-posing strategies. He concluded that

> although problem forming can not guarantee that a poor problem solver will become a good problem solver, problem forming can substantially benefit mathematics students. . . . Problem forming appears to actually decrease the random selection of solutions. (p. 120)

Thus if problem posing can help students learn to minimise the number of the choices for a solution method, this is by itself a significant improvement.

Although the processes involved in problem solving and problem posing might be different, some authors have suggested that there seem to be some similarities in the strategies used for posing and solving problems. In his article, Hoehn (1991) mentioned that for creating the problems he described, he has used *the same techniques* that are used in problem solving (special cases, generalisation, etc.). It is clear that there is a strong resemblance between the strategies he has used and Polya's twelve principle articles described in his "Short Dictionary of Heuristic" (1957, p. 37ff, pp. 129-130).

In their study, Silver et al. (1996) investigated the types of problems posed by middle-school teachers within a reasonably complex task setting. They also analysed the differences between the problems posed prior to solving a problem embedded in that setting, and the kinds of problems posed in the setting during and after solving the problem. The results of the study showed that teachers have some personal capacity for problem posing. For example, they posed problems by generating goal statements while keeping problem constraints fixed, and by manipulating the task's implicit assumptions and initial conditions. The results showed that teachers posed more problems *before* problem solving than during or *after* problem solving. Although the participants in this study were middle-school teachers, one can expect that students might also be able to show personal capacity to pose problems, and may also feel greater freedom when working on activities that are based on manipulating the structure of a specific problem.

A series of problem-posing strategies to assist teachers in developing specific types of problems was suggested by Butts (1980). He classified mathematical problems into five arbitrarily titled subsets: (a) recognition exercises; (b) algorithmic exercises; (c) application problems; (d) open-search problems; and (e) problem situations. Butts demonstrated specific strategies for obtaining problems of the first four types.

Two problem-posing strategies for designing open-ended questions have been proposed by Anderson and Sullivan (1995). These types of problems are referred to by Sweller (1993) as goal-free problems, since they eliminate the final goal from the problem. According to Anderson and Sullivan (1995) these questions provide students with the opportunity of making a start on a problem regardless of previous experiences or mathematical ability. Their statement is in contrast with the observations made by Mamona-Downs (1993) that some students do not use the extra freedom which the problem-solving task has provided. One could argue that open-ended problems can provide a rich environment for structuring and extending students' knowledge from a specific topic rather than an initial inquiry when a student does not possess any knowledge.

Anderson and Sullivan (1995) suggested two practical methods to help teachers to create open-ended problems for the mathematics classroom. The first, called "working backwards," consists of three steps: (a) Think of a problem, (b) Think of the answer to the standard question, and (c) Make up an open-ended question which includes (or addresses) that answer.

The second method is referred by the authors to as "adapting a standard question." Its steps are: (a) Identify the topic; (b) Think of a standard question; and (c) Adapt it to make an open-ended question. According to the definitions given in Chapter 1, the strategies suggested by Anderson and Sullivan are aimed at helping teachers to pose *semi-structured* problem-posing situations from the domain of a specific topic.

Synthesis and Conclusion

The broad spectrum of the literature review presented in this chapter shows four main trends which need further investigation:

First, in recent years problem posing has begun to receive increased attention and the potential impact of problem posing is being recognised by professional scientists, researchers and educators. In a few instructional studies researchers have reported observing a positive effect on students' mathematical performance when a *particular* type of problem-posing activity was adopted (Perez, 1985; Winograd, 1990). However, until now, no instructional studies have investigated the *systematic* use of a range of problem-posing activities and their effects on students' mathematical performance. According to Kilpatrick (1986) most of the literature on problem posing deals with the re-formulation of ill-formulated problems, or the formulation of sub-problems and related problems. Thus, it is clear that research into the potential of problem-posing as an important strategy for the development of students' understanding of mathematics has been hindered by the absence of account of educationally rich classroom problem-posing situations.

Second, in the mathematics classroom, problem posing has a narrow interpretation: working on a specific problem or on a situation (an investigation for example) which has been prepared for the student *in advance* and which eliminates any difficulties the student might face during problem solving. In fact, the lack of a framework for linking problem solving, problem posing and mathematics content has prevented problem posing from making the contribution that it could make to students' understanding of mathematics.

Third, although in mathematics education problem posing tends to be seen not as a goal itself but as an approach facilitating the achievement of other goals of mathematical instruction, the research about how problem-posing activities can interact as an inseparable part of classroom problem-solving environments in order to meet broader goals of mathematical instruction is limited (Kilpatrick, 1987). More research is needed into what modes of interaction between problem posing and problem solving are likely to facilitate students' understanding of mathematics. Before the effects of problem posing and its application for the teaching and learning of mathematics can be adequately researched, however, a teaching approach which incorporates a range of problem-posing situations as a part of problem-solving activities needs to be developed and refined in the light of data gained from its application in the classroom.

Fourth, no research reports have been found about whether students possess natural abilities to pose mathematical problems. We can assume that, after receiving special instruction on problem-posing, the students' "kit" is also presumed to contain problem-posing skills. Including a new tool with the ones which already exist, according to Schoenfeld (1992), may reflect even on the way the tools are used. Hence, if students' problem posing is to be important, it is not because it can potentially make one a better problem solver or problem poser, but because the ability to pose quality well-structured problems might be valuable in its own right.

These are the four areas of research which become the focus of this thesis.

CHAPTER THREE

AIMS AND METHODOLOGY

Introduction

In the last chapter, a number of key issues which relate to the growing recognition of problem posing and the contrasting perspectives from which problem posing has been investigated were summarised and the types of problem-posing situations used in mathematics classroom were described.

The first part of Chapter 3 presents the aims of this study and defines the research questions. In the second section, an overview of the selection and organisation of the sample classes, the participants' background, and the instructional settings are presented. The reasons underlying the choice of the Euler Program are discussed and the program content is outlined. Data collection instruments, interviews and observation procedures are described in the third section.

Aims and Research Questions

The purpose of this study is to design an instructional environment which incorporates problem posing as an inseparable part of problem solving, and to explore the effects of these environments on students' mathematical performance.

In particular, the study aims to:

- Develop a framework of problem-posing situations in which students could be engaged;

- Develop a framework of teacher's "hidden" problem-posing questions aimed at assisting students to understand the problem and solution structures;

- Design a conceptual framework for a teaching approach which incorporates problem posing in order to facilitate students' problem solving;

- Adopt and extend the first level of a national program for students with above average mathematical abilities and implement it by applying two different approaches—a problem-solving approach based on Polya's (1957) recommendations and an *open problem-solving approach* which is defined for the first time in this study;

- Classify the problem-posing categories identified from the project classroom data and describe ways of applying particular problem-posing situations in a variety of classroom contexts;

- Identify from the project classroom data strategies used by the researcher for generating problem-posing situations and for prompting students to react with a specific problem-posing activity;

- Design a scheme for assessing students' problem-posing performance;

- Explore the effects of the instructional environments on students' problem-posing and problem-solving performances;

- Investigate the categories of problem-posing strategies employed by Years 8 and 9 students.

This study seeks answers to the following research questions:

1. How can problem-posing situations be integrated with problem-solving environments to help students solve mathematical problems?

1.1 How can Krutetskii's system of mathematical problems for revealing the structure of students' mathematical abilities be adapted and extended for generating problem-posing situations?

1.2 How can problem-posing situations be classified ?

1.3 Is it possible to identify strategies for the teacher to generate problem-posing situations?

2. What effects do different problem-posing environments have on students' mathematical performance?

2.1 What effects do different problem-posing environments have on students' performance on mathematical skills tasks?

2.2 What effects do different problem-posing environments have on students' problem-solving performance?

2.3 What influence do different problem-posing environments have on students' problem-posing performance?

3. To what extent do students develop their own problem-posing strategies?

3.1 What are the characteristics of the problem-posing strategies developed by students?

Methodology

In order to observe the changes which occur when problem posing is introduced to a mathematics classroom, the study took the form of a teaching experiment and was carried out in two main stages: *The Pilot Study* and *The Main Study*.

The Pilot Study

The Pilot Study was undertaken at the beginning of the 1995 academic school year, from February 8, 1995 until April 4, 1995. It comprised eight one-hour sessions with a group of forty Years 8 and 9 students. The first step focused on the instructional feasibility of the proposed program, on the refinement of the problem-posing situations, and on the development of ways of interactions between problem posing and problem solving. The integration of these as inseparable components of the teaching approach was the second goal of the Pilot Study. The final goal of the Pilot Study was to obtain data which would facilitate the selection of students for the case study.

The Main Study

Thirty-five Years 8 and 9 students took part in the Main Study. The students self-divided themselves into the two groups (Groups A and B), depending on their preference for time for attending sessions. They participated in the Program from April 4, 1995 until November 22, 1995, for one hour per week, giving a total of 32 hours altogether for each group.

Groups A and B were involved in the same program with the researcher acting as a teacher[1] for both groups. The mathematical content of all lessons was the

[1] Throughout this thesis, the researcher will be referred to as "the teacher" because this was the role I chose as an appropriate one with which to apply the problem-posing framework developed in this thesis. First person, will in general, not be used.

same, and the problem-solving tasks were presented to both groups in the same order. The students from Group B were engaged in problem-solving activities. Students from Group A were engaged in problem-solving and a wide range of specially designed problem-posing activities (see Chapter 4).

In addition to Groups A and B, a third group (Group C), comprising 112 Year 8 and Year 9 students was chosen for investigating students' problem-posing strategies.

The Classes

As it was important that the study involved students who had a range of mathematical abilities, and for the students to be observed over a prolonged period of time as they worked on content which was not part of the normal school curriculum, it was decided that the classes should be organised outside of school hours and away from any normal school context. The instructional sessions took place at Edith Cowan University, Mount Lawley Campus, on Thursday afternoons, after school hours, from 4pm – 5pm (Group B) and 5.10pm – 6.10 pm (Group A). The lessons were held consecutively, with a 10 minute break in between. Two independent observers attended all lessons.

The Participants

1. Selection of the participants for the Program. The participants for the Program were from different schools – government and non-government. The selection of students in the classes was carried out according to the procedure which

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The classroom settings in which the research was carried out will be referred to as a "the project classroom" (for group A) and "the program classroom" (for Group B).

has been used for four years at Edith Cowan University. At the end of August each year a letter is sent to all schools in the northern part of the Perth metropolitan area explaining the goal of the Mathematics Enrichment Program and inviting expressions of interest to be submitted by students (see Appendices 1 and 2). In addition to their willingness to take part in these mathematics classes, students interested in the program are required to sit for a qualifying mathematical problem-solving test. Because the main aim for the sample was to include students with a wide range of mathematical backgrounds and abilities, a decision has been made all students who applied to participate in the program to be accepted.

2. The participants' mathematical background. The students involved in the Program had different mathematical backgrounds, and different mathematical abilities and ages. Although three of the students had already attended a 12-hour enrichment program at Edith Cowan University in the previous year, the others had not participated in any such extracurricular activities. The youngest participant was only 7-years-old (and in Year 3), 30 were 12-years-old (in Year 8) and the remaining 4 students were 13-years-old (in Year 9). Until that time, none of the students had achieved any significant result in any national mathematics competition.

All students were free to take part or not to take part in the research.

3. Selection of the students for investigating their problem-posing strategies. The sample for investigating students' problem-posing strategies (Group C) comprised 112 students: (a) the students from Groups A and B; (b) 65 volunteers from three classes (two Year 8 and one Year 9) from a government secondary high school in Perth; and (c) a group of twelve Year 9 students from a private high school in Perth, who were involved in an extracurricular mathematics program.

The main aim was for the sample to include Years 8 and 9 students with a range of mathematical backgrounds and aptitudes in order to investigate a broader range of students' problem-posing strategies.

The Program

1. The Mathematics Challenge for Young Australians. For several years, the Australian Mathematical Olympiad Committee has been running a national four-level program — the *Mathematics Challenge for Young Australians* — for mathematically able students. The topic areas covered by this program are not a part of the school curriculum. Each level is supported by *Students' Notes, Teacher's Reference Book* and a *Challenge Problem Booklet.* The program has two stages. During the first stage, called the *Challenge Stage*, students are given six *Challenge Problems* and are required to solve them within four weeks. The second stage, which is termed the *Enrichment Stage*, comprises 12 one-hour lessons and students have to submit the solutions of another 16 *Challenge Problems.* At the end of the program all participants receive certificates summarising their achievements in the program.

2. The Euler Program. The Euler Program is the first level of the *Mathematics Enrichment Stage of the Challenge Program for Young Australians.* The first level has been designed for Years 8 — 9 students with above average mathematical abilities.

The Euler Program includes the following topics: "Problems I like," "Primes and Composites," "Least Common Multiple," "Highest Common Factor and

Euclidean Algorithm," "Arithmetic Sequences," "Figurate Numbers," "Congruences," "Problems I Like Sharing," "Find that Angle," "Counting Techniques," "The Pigeon-hole Principle," and "Problems I Enjoy Sharing."

There are two reasons underlying the choice of the Euler Program. First, most of the structured problem-posing situations developed in the study were inspired by Krutetskii's work and involved the use of higher-order thinking skills. Second, in order to apply Krutetskii's system to a variety problem-posing contexts, a program which allowed the use of a range of problem-posing situations based on three interconnected areas — Algebra, Geometry and Arithmetic — needed to be adopted.

3. The Program of the study. At the beginning of the study, most participants did not have the mathematical background required for participation in the Euler Program. On the other hand, the Euler Level is designed to be covered in 12 hours (spread between April and October). To meet the goals of the study the Euler Program was extended to 32 hours. The program chosen meant that it was possible to engage students for the whole school year, and that the difficulty of the mathematical content needed to be suitable for most participants. In addition to the content of the Euler level, the following new topics were included: "Indices," "Find the last digit," "Diofantine Equations," "Congruent Triangles," "Tangents," "Pythagoras' Theorem" and "Number Bases." In a natural way, under the headings "Problems I Like Sharing," a range of non-trivial problems, specially designed for the study, was incorporated.

The choice of the additional topics was designed around the possibility of applying new types of problem-posing situations. Students had to have the necessary mathematical background to understand the new mathematical content.

In fact, the Program content in which students participated, was an adaptation and extension of the mathematical content of the Euler Program. For every lesson students in both groups were given written materials which comprised individual worksheets (see Appendix 4), additional materials such as revision and extension papers (see Appendix 5 and Appendix 6) and *Hints for Challenge Problems* (see Appendix 7). All written materials were designed by the researcher specifically for this study.

Instructional Setting

The main differences in the teaching approaches used for the two groups (Group A and Group B) were based on the following principles:

1. Instruction for both groups should be inquiry-oriented (Collins, 1988) and it should be based on the same mathematical content.

2. Students from Group A should be involved in different types of problem-posing and "hidden" problem-posing activities (see Chapter 4), according to the academic content and the possible methods for solving or posing particular mathematics problems.

3. The teacher's questions to students in Group B should be based on Polya's recommendations (1957). These questions should be phrased in general terms so that they are applicable to a range of situations. Teacher's questions in Group A should be aimed at encouraging students to reflect actively via problem-posing activities

applied to situations which involved problem-solving (see Chapter 4 and also Chapter 6).

4. Wherever possible, the teacher's responses to students' questions should be indirect.

5. Most problem-posing and problem-solving questions should encourage students to use higher order thinking skills (Resnick, 1987; Romberg, Zarinnia & Collins, 1990).

6. Students from both groups should be reminded frequently to present arguments to support their ideas (see Chapter 6 and Chapter 9).

In order to incorporate problem-posing activities into the lessons, every session was divided into four sections as follows:

Section 1: This comprised the main part of the session. Students were given the mathematical background necessary for solving specific types of problems. Examples illustrating applications of any method which was new to the children, as well as precisely written solutions, were demonstrated. Comments on submitted solutions f some *Challenge Problems* were made.

Section 2: The teacher introduced the section on group/individual work. Students in both groups were given work sheets which contained the same problem-solving tasks. For Group A, some of the problem-solving tasks were presented as problem-posing situations (see Appendix 4, for sample worksheets).

Section 3: Students worked as individuals (or in groups of two) on problem-posing and problem-solving tasks (in Group A) and on problem-solving tasks (in Group B). They shared their ideas and discussed the problems.

Section 4: The students from Group A presented the problems they had developed or solved to the class, and responded to questions from their peers and the teacher. The students from Group B were asked to present only their solutions and answer questions from their peers. In both groups the features of different solution approaches and their elegance were discussed.

Data Collection

The data collection procedure occurred in five phases throughout the Program.

Phase 1

During Phase 1 background data on the mathematical skills of students participating in the *Challenge Stage* (of the *Mathematics Enrichment Stage of the Challenge Program for Young Australians*) were collected. Two sets of tests were administered: a problem-posing test (Mathematics Questions, Set 1) (see Figure 3.1) and a problem-solving test (Mathematics Questions, Set 2) (see Figure 3.2).

Phase 2

Observational data were collected during the entire study. All lessons were tape-recorded and transcribed. Tape recordings of discussions between students, and discussion between different students and the teacher were also collected and transcribed. The teacher kept a journal before and after lessons at regular intervals throughout the program. Copies of students' work were compiled throughout the program.

Two students from Group A were selected according to Patton's (1990) recommendations. The set of observations which was carried out was according to the following plan:

Observation 1: The first set of observations occurred as students worked on an individual basis, posing or solving mathematical problems. Individual interviews and talk-aloud protocols were used by researcher. In addition, field notes and tape recordings were the techniques adopted for observing the work of individual students. Representative samples of pupils' work both on specific problem-posing activities and other mathematical activities were collected over the year.

Observation 2: The same two selected students were observed as they explained and discussed their problems in pairs. Some of these discussions were tape-recorded. Field notes and selected samples of students' work were taken in addition to the recorded data.

Observation 3: The same two selected students were observed in a similar way as they presented their ideas, problems or problem solutions to the class.

Observation 4: All students in the class were observed by the teacher as they presented and discussed the features of their own and other students' problems and solutions. Field notes and selected samples of students' work were taken in addition to the recorded data.

Observation 5: Two independent expert assessors participated in all lessons. The first assessor, who had pedagogical and mathematical background, observed differences in the intended teaching approaches. She took written structured notes for all lessons (Appendix 8). The second assessor was not informed about the differences in the teaching approaches. He had an extensive mathematical background and he was regularly asked about any apparent differences in the teaching approaches. His opinion was used as a barometer with respect to how the delivered instruction related to the intended one. On a regular basis students in both groups were also asked about the type of difficulties they had. An additional indicator of the characteristics of the delivered Program was one participant's (Tom's) opinion. He was regularly interviewed after every session.

At the end of the Program both observers were asked to present their written opinion about the Program and to express their personal preferences with regard to the teaching approaches.

Phase 3

Two tests, identical with the pre-tests, were administered at the end of the Program.

Phase 4

Individual interviews with students who participated in the problem-posing class were conducted. Some data from the *Challenge Problems* was collected.

Phase 5

At the conclusion of the Program, many parents asked the researcher to continue to work with their children. In 1996, a total of eighteen students drawn from both groups were enrolled for one semester in the third level of the *Mathematics Challenge for Young Australians Program —the Neother Level.* Some data from this *Challenge Stage* and students' work were collected.

Instruments

The problem-posing and the problem-solving tests were each administered for 20 minutes during normal classroom work. Subjects were asked to work on the tests on an individual basis.

The Problem-posing Test

The design of the problem-posing test was based on the premise that it should include *free, semi-structured* and *structured* problem-posing situations. Two independent examiners were used to validate that the situations chosen could be classified as problem-posing situations on the basis of the definition provided in Chapter 1. The classification of each problem as free, semi-structured and structured was validated in a similar way.

In the first problem-posing-situation (see Figure 3.1, Item 1) students were asked to make up as many problems as they could on the basis of the given computation. The problem by itself implies a natural question "What is the value?". By suggesting that students should pose their own problems, the problem statement was changed in a way which was likely to prompt students to reflect in at least two other ways: to add data, to model a situation based on the given calculation, or to rearrange the given calculation in identical or non-identical forms.

The second problem-posing situation (see Figure 3.1, Item 2) was validated by an independent assessor as *semi-structured.* Students were given a sequence of six symbols, four of which were integer numbers. Two of the elements in the sequence were replaced with another symbol. Students were asked to suggest meanings for the missing elements, and to construct mathematical problems by using one of these meanings. It was up to the student to decide how many problems to pose. The wording of the statement did not place emphasis on the number of problems to be posed.

The third problem-posing task in Set 1 (see Figure 3.1, Item 3) was described as *free*. It required students to pose a problem similar to one which the students enjoyed solving, and invited them to explain why they liked it and how they created it.

MATHEMATICS QUESTIONS: SET 1

Question 1: Make up as many problems as you can using the following calculation:
$3 \times 25 + 15 + 5 - 4$.

Question 2 Given that: 1 2 3 * 5 *.
a) What could the meaning of sign "*" be?
b) Can you make up a (some) problem(s) using one of these meanings?

Question 3: Give an example of a problem similar to one you enjoy solving.
a) Explain why you like it and how you created it.

Figure 3.1. Mathematics Questions: Set 1.

The additional questions were aimed at helping students to reflect further on the features of a specific problem based both on their previous experiences, and on their own understanding of the strategy used for generating a problem with a structure similar to a problem they enjoy solving.

The Problem-solving Test

The problem-solving test comprised seven interrelated items (see Figure 3.2). The design process was based on the premise that the test should include items for testing students' specific concept skills as well as their abilities to apply this concept for solving problems based on real-life situations.

The test had two interrelated parts. In the first part, which consisted of Items 1 to 5, some basic skills needed for an application of the concept of percentage were tested. Students' responses were marked with 0 for an incorrect response and 1 for a correct response.

MATHEMATICS QUESTIONS: SET 2

Circle the right answer:

Question 1: $^{2}/_{3}$ of 15 is:

A) 6; B) 10; C) 15; D) 5.

Question 2: $^{2}/_{5}$ of a specific number is 10. Which is the number?

A) 50; B) 100; C) 25; D) 4.

Question 3: 120% of 50 is:

A) 62; B) 60; C) 600; D) 620.

Question 4: 30% of a specific number is 21. Which is the number?

A) 630; B) 141; C) 70; D) 63.

Question 5: Which of the following has the same value as 1994/1995 ?

A) $\frac{1994 - 2}{1995 - 2}$; B) $\frac{1994 + 1}{1995 - 2}$; C) $\frac{1994^2}{1995^2}$; D) $\frac{3 \times 1994}{3 \times 1995}$.

Question 6: If a discount of 20% off the market price of a jacket saves you $15, how much will you pay for the jacket?

Solution:

Question 7: A jacket has been discounted twice: first with 15% off and then with 20% off of the new price. What was the initial price of the jacket, if its price now is $136?

Solution:

Figure 3.2. Mathematics Questions, Set 2.

The second part consisted of two interrelated questions (Items 6 and 7). The aim of the questions was to help students illustrate the extent to which they could use these skills for solving practical problems with different levels of difficulty. The mathematical context of the Items 6 and 7 was chosen to allow students to connect with ease the problem context with a situation from their every-day life. Both problems could be solved by applying three different solution approaches — logical, algebraic or geometrical. Students were required to write only one solution for each problem.

Administration of the Tests

The tests were administered at the beginning and at the end of the Program during the normal school hours — by the researcher for Groups A and B and by the students' teachers for Group C. Students worked individually on each test for 20 minutes. It was felt that the problem-posing situations chosen would provide appropriate environments which would allow every student to reflect on the situations by posing at least one problem. The problem-solving test comprised types of problems which were adapted from problems published in students' and teachers' support materials. Students were under no obligation to the researcher, and no pressure was placed on them to submit their written responses to the tests.

CHAPTER FOUR

THEORETICAL FRAMEWORKS

Introduction

This chapter introduces the frameworks which were developed by the researcher to guide the implementation of the different problem-posing categories in the project classroom.

The first framework describes the problem-posing situations derived from the literature and foreshadows the problem-posing categories which the researcher attempted to use and extend during the study.

The second framework comprises the questions developed by the teacher to prompt students to reflect on particular problem-solving situations. The questions were designed to incorporate "hidden" problem-posing tasks and were aimed to help students to focus their attention on some characteristics of problem or solution structures before, during or after solving a particular problem.

The third part of this Chapter presents the conceptual framework of the instructional approach employed in the project classroom which will be referred to as the "open problem-solving approach." This approach attempts to incorporate problem posing as a means for facilitating students' problem solving when problem structures, solution structures and problem-solving activities are *open.*

Development of Framework to Describe Problem-posing Situations

Application of problem-posing situations as a part of problem-solving environments required the development of a framework to describe the problem-posing categories which were used in the project classroom. The role of such a framework was to guide both the choice of the category of problem-posing situations and the design process as well.

Problem-posing Situation Categories

The problem-posing situation categories were developed on the basis of an analysis of the literature on the types of problem-posing situations which have been used as research tools or which have been recommended as appropriate for use in mathematics classrooms. Three problem-posing situation categories were then defined —*free, semi-structured* and *structured.*

1. Free problem-posing situations. In a free problem-posing situation, students are asked to generate a problem from a given, contrived or a naturalistic situation. In order to prompt students to reflect on specific actions, or to recall particular previous experiences, students can be given some additional directions.

The literature review established that a number of *free* problem-posing situations had been used by researchers: (a) Problems written for a friend (Ellerton, 1988; Richardson & Williamson, 1982); (b) Problems from data (*Shell Centre*, University of Nottingam, 1991); (c) Problems I like (*Euler Student Notes, 1995*); (d) Problems I enjoy solving (*Euler Student Notes, 199*[illegible] Problems which involve the use of a specific concept(s) (Kennedy. 198[illegible]); (e) Problems about a particular topic

(Anderson & Sullivan, 1995; Kennedy, 1985); and (f) Problem posing based on the use of a specific mathematical method (Polya, 1957).

Thus, *free* problem-posing situations would include those addressed directly to problem posers, or which place problem posers in situations in which they are forced to consider the person/people for whom they were posing the problem. Guidance such as: "Make up a difficult problem," "Pose a problem that you would like to see in a mathematics competition paper," "What kind of problems do you expect to find in your mathematics test?" "Pose a problem to be solved by your teacher," or, simply "Make up any problem you like," are aimed to help students to mathematise their previous experiences from a specific perspective.

Students in a mathematics classroom might also be asked to pose problems associated with the topic which is being studied at the time. Students could be invited to suggest problems which involve the use of a specific concept or solution method. Examples of free problem-posing situations which were developed to be used in the project classroom are presented in Figure 4.1.

> *Example 1:* Make up some problems which relate to the right angled triangle.
>
> *Example 2:* Describe a real-life problem which can be solved by using the concept of the Highest Common Factor of two numbers.
>
> *Example 3:* Give an example of a problem which can be solved by finding the Least Common Multiple of two or more integers.

Figure 4.1. Examples of free problem-posing situations which involve the use of a specific concept.

Example 1 (see Figure 4.1) illustrates a situation which was designed to prompt students to pose a problem about a right-angled triangle. Examples 2 and 3 describe situations in which students were asked to pose problems whose solutions incorporate, in some way, finding the least common multiple or the highest common factor of two or more integers.

*2. **Semi-structured problem-posing situations.*** In semi-structured problem-posing situations, as was stated in Chapter 1, students are given a *situation* in which they are invited to explore and formulate a problem which would draw on the knowledge, skills, concepts and patterns gained from their previous mathematical experiences.

In order to help define structured problem-posing situations, two new definitions relating to the structure of a problem and the structure of a solution will be introduced. The *structure of a problem* refers to the key elements of the problem which contain the given, the operations and the goals. The *structure of the solution* refers to the key elements of the solution presentation which contain the main steps of the solution approach, and a justification for the applicability of the algorithms used.

Several types of semi-structured problem-posing situations were derived from the literature review: (a) Problem posing based on situations with missing elements in the problem structure (Caldwell, 1984; Kruteskii, 1976); (b) Problems which are *similar* to a previously solved problem (Hachimoto, 1987); and (c) Problems with surplus or insufficient information in their structures (Krutetskii, 1976).

In fact, semi-structured problem-posing situations can range from situations incorporating missing elements in particular problem structures (the *Given*, the *Obstacles,* the *Goal,* or a *combination* of some of these) to posing sequences of interconnected problems. The premise behind the design of semi-structured problem-posing situations which was applied in the study was to help students to focus their attention on both *problem* and *solution* structures.

A. Problem-posing situations based on a specific problem structure. Problem-posing activities can be based on situations which incorporate either insufficient or surplus information in the elements of their structures. Students can be presented, for example, with unfinished problem structures and asked to suggest problems which can be created on the basis of the information given. Examples of problems with an *Unstated Goal* which were used in the study are presented in Figure 4.2.

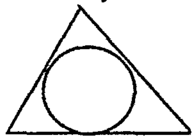
> *Example 1:* Last night there was a party and the host's doorbell rang 10 times. The first time the doorbell rang only one guest arrived. Each time the doorbell rang after that three more guests arrived than had arrived on the previous ring.
> *Ask as many questions as you can. Try to put them in a suitable order.*
>
> *Example 2:* Consider the following infinite sequence of digits:
> 1234567891011121314 . . . 9991000 1001. . .
> Note that it is made by writing the base ten counting numbers in order.
> *Ask some meaningful questions.*

Figure 4.2. Examples of problem-posing situations based on a problem with an *unstated Goal.*

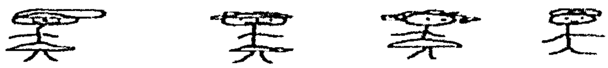
Such unfinished problem structures might be given either by a picture, calculation, equation or inequality. In Figure 4.3 situations based on unfinished problem structures presented by a calculation, a diagram, and a picture are illustrated.

In the light of the definitions given in this study, *open problems* (see for example Ellerton & Clarkson, 1996; Pehkonen, 1995; Silver, 1995), can be regarded as semi-structured problem-posing situations when they require the child to suggest other problems or generalisations.

Example 1: Make up some word problems which can be solved by using the calculation:
$2 \times 15 + 10 \div 2 - 4$.
Example 2: Describe the picture below by extracting all given information and state some meaningful questions:



Example 3:
Given:



+1 +2 +3
Sally ⇒ Beth ⇒ Ruth ⇒ Edmund
58 altogether
What might the problem be about?

Figure 4.3. Examples of problem-posing situations which involve unfinished problem structures presented by a calculation, a diagram, and a picture.

Students could also be asked to find the surplus information in a situation based on a specific problem structure and to pose problems by using selected sub-sets of the information given.

B. Problem-posing situations based on a specific solution structure. Problem-posing activities can also be based on part of a particular solution. The solution might incorporate missing or surplus information in its structure. For example, students might be asked to restate a problem when *only part* of its solution is given (see Figure 4.4). A problem solution which is not written precisely could provide a starting point for involving students in useful discussions.

Identify the main idea of Peter's solution of a problem, which is presented below. Try to suggest a possible problem statement:
You need 12 books from every language. There are 6 languages, so $6 \times 12 = 72$. $72 + 1 = 73$. Because the French, German, Italian books don't have 12 then the number is less. Take away the difference between 12 and those languages and it equals 65.
F = 1,
G = 5,
I = 2
8 $73 - 8 = 65$.

Figure 4.4. Restating a problem on the basis of a part of its solution.

Students might be also presented with *part* of the problem structure and a *set* of possible answers. Figure 4.5 illustrates teaching materials of some of the problem-posing situations used in the study. The example presents a situation based on missing elements in the *Given* and in the *Obstacles*. In this case students were asked to finish the problem structure if only one of the answers should be answer to the problem.

> *How will you finish the problem if you want only one of the given answers to be the answer to the problem?*
>
> Take any .... Write it down twice to make a .... digit number. This number always will have among its factors:
>
> A) 11; B) 101; C)1001; D)10001.

Figure 4.5. Restating a problem on the basis of a part of its structure and a set of possible answers.

Other semi-structured problem-posing situations from which data were generated in the study are presented in Figure 4.6. These illustrate the posing of a class of problems related to a specific solution method — such as the use of the Pigeon-hole Principle, and permutations or combinations — when this activity is combined with presenting a problem structure with missing elements.

> *Example 1: Finish the problem situations below so that the solution method implies the use of the Pigeon-hole Principle:*
> a) There are 5 pigeons in .....pigeon-holes. Show that there is a pigeon-hole with at least two pigeons.
> b) There are ......in my class. Why were at least 2 two students born on the same day of the week?
>
> *Example 2: Finish the problem situations below so that the solution method implies the use of permutations:*
> Two girls and four boys are standing in a line ....

Figure 4.6. Teaching data for posing problems based on the use of specific solution methods.

Students' work on semi-structured problem-posing situations can also be supported by specific instructions such as: "Make sure you include all arithmetic operations," or "Try to make up a problem about the radius of an inscribed circle," or "Pose a problem using the notion of a prime factor," and so on.

3. Structured problem-posing situations. In a *structured* problem-posing situation, as was stated in Chapter One, problem-posing activities are based on a specific *problem* or a written *solution*. Students are invited to generate new problems which are derived from a given problem or solution.

Structured problem-posing activities have been recommended by many educators. Polya (1957) mentioned three approaches for constructing a new problem from a given problem:

> Firstly, keep the unknown and change the rest (data and the condition); or secondly keep the data and change the rest (the unknown and the condition); or thirdly change both the unknown and the data. (p. 78)

In other words, Polya's recommendations address the possibilities of varying the elements in the structure of a problem.

Brown and Walter (1990, 1993), who also designed an instructional problem-formulating approach based on the posing of new problems from already-solved problems, recommended that systematic variations of the conditions or the goals of a specific problem could be used to initiate problem posing.

Kilpa (1987) argued that, when students attempt to solve problems, there are two phase uring which new problems can be created.

> As a mathematical model is being constructed for a problem, the solver can intentionally change some or all of the problem conditions to see what new problem might result. After a

problem has been solved, the solver can look back to see how the solution might be affected by various modifications in the problem. (p. 127)

Kilpatrick's ideas can in fact be linked to the "looking back" phase of Polya's description of the process of problem solving.

The structured problem-posing situations which were developed in this study can be divided into two sub-categories: (a) Problem-posing situations based on a specific problem (Polya, 1957; Walter & Brown, 1983); (b) Problems to *fit* a given computation (Hart, 1981); and (c) Problem-posing situations based on a specific solution (Kilpatrick, 1987; Polya, 1957). Examples will be presented to illustrate some of the structured problem-posing situations which were used in the study.

A. Problem-posing situations based on a specific problem. Situations which fall in this first sub-category of structured problem-posing situations have been designed to help students *to understand the problem structure*. Students would normally be involved in problem-posing activities from this sub-category mainly before or after solving a problem. For example, students could be asked to pose a series of additional questions which follow *directly* from the given information in a particular problem or by adding some data to pose questions and to put them in a suitable order (see Example 1, Figure 4.7).

Some integers are arranged in the way shown below:

1
2 3 4
5 6 7 8 9
10 11 12 13 14 15 16
17 ... 25

a) What would be the third number from the left of the 89th row of the accompanying triangular number pattern?

A) 8103 B) 6982 C) 10681 D) 7747 E) 7924

b) State other meaningful questions.

Figure 4.7. Example of a problem-posing situation based on posing additional questions which follow directly from the *Given*.

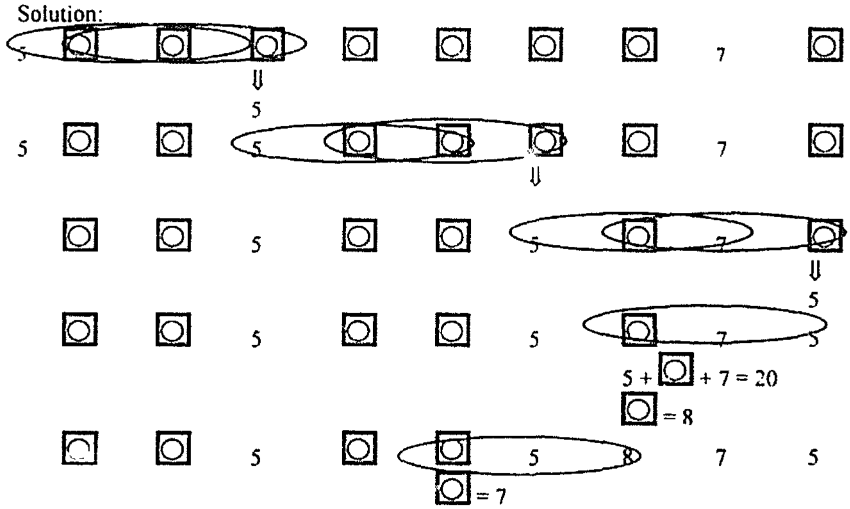
In other cases, students could be asked to add new data and then to pose additional questions.

Students could also be invited to suggest a problem which *resembles* a given problem but which *might have* a different solution method. For example, problems which are the inverse of a given problem could also be formulated. Helping students to understand the interrelationships between the problem statement and its solution is regarded in this study as an important instructional goal.

B. Problem-posing situations based on a specific solution. This sub-category of structured problem-posing situations includes situations designed to help students *to understand structures of the solution approaches used.* In some cases, as shown in Figure 4.8, it is appropriate for the problem solution structure to be given by using a series of pictures. By presenting a solution approach through a series of pictures, some students might be able to understand better the main features of the solution approach.

Problem: Ten positive integer numbers are arranged in a line. Find the missing elements if the sum of any three consecutive numbers is 20.

Solution:



Answer:

5 8 7 5 8 7 5 8 7 5

Figure 4.8. Presenting a problem solution structure through a series of pictures.

Restating the problem on the basis of its *entire* written solution is a problem-posing situation, and can be based on analysing a written solution or a series of pictures (diagrams). Students could be asked to formulate a problem whose solution matches the written one or one shown by a series of pictures (diagrams).

Improving the language characteristics and the logic of a written solution and determining and formulating, as independent problems, the main steps in a specific solution approach are other problem-posing activities in which students might be engaged. An example is given in Figure 4.9.

During, before and after solving a specific problem, students could be asked, on a regular basis, to suggest data in the problem which affects the solution or the solution approach in a particular way (Figure 4.10).

Read the solution of the problem given below and formulate the main idea. In what ways can this solution be improved?

The leftmost digit of a six-digit number N is 1. If this digit is removed and then written as a rightmost digit, the number thus obtained is three times N. Find N.

Solution:

		2.....	
1 A B C D E	3 × E = * 1	1 A B C D 7	3 × D + 2 = *7
× 3	E = 7	× 3	D = 5
A B C D E 1	⇒	A B C D 7 1	⇒
1 2		..2.....	
1 A B C 5 7	3 × C + 1 = * 5	1 A B 8 5 7	3 × B + 2 = *8
× 3	C = 8	× 3	B = 2
A B C 5 7 1	⇒	A B 8 5 7 1	
1 A 2 8 5 7	3 × A = * 2	1 4 2 8 5 7	
× 3	A = 4	× 3	
A 2 8 5 7 1	⇒	4 2 8 5 7 1	

Figure 4.9. Determining and formulating the main steps in a solution approach.

In the project classroom students could be also asked to pose a problem which could be solved by using more than one approach. They could be asked, for example, to compare different solution approaches and to suggest other problems, which look different, but which could be solved by using the same approaches.

In the problem below add or change some information so, that the solution approach is affected:
In how many ways can 2 girls and 3 boys stand in a line?

Figure 4.10. An example of a problem-posing situation aimed at involving students in exploring the interrelationships between the problem statement and solution or solution method.

Principles Underlying the Design Process of Problem-posing Situations

The process of the design and extension of problem-posing categories which will be used in the project classroom will be based on some basic principles which were derived from the literature.

After analysing the research literature about possible implications of problem posing for school mathematics (Brown & Walter, 1983, 1990; Kilpatrick, 1987; Krutetskii, 1976; Silver, 1993), and after taking into account the difficulties which such application might face (Krutetskii, 1976; Mousley, 1990; Pehkonen, 1993), the following three basic principles were postulated. These principles then formed the basis of the design for all problem-posing situations:

1. Problem-posing situations should corresponded to, and arise naturally out of, pupils' classroom problem-solving mathematical activities.

2. Problem-posing situations should corresponded to the pupils' previous problem-solving and problem-posing experiences.

3. Problem-posing situations should be generated from sources and materials normally used in mathematics classrooms, including textbook problems, by modifying and reshaping the language and task characteristics.

The corner stone of the design process is the notion that all problem-posing situations should be a part of the problem-solving classroom environment. They should aim at prompting students to reflect on their own problem-solving performance and to assess the features of the mathematics they are learning.

Problems in the *Euler Student Notes* were presented as closed problems and the first step for the researcher (and for any teacher who wishes to adopt problem posing as a regular classroom activity) was to develop a strategy to make them open. Many researchers and mathematics educators such as Pehkonen (1993) and Hopkins (1995) have recognised the importance of teachers' ability to pose open problems. Hopkins (1995) wrote:

> To retain the power of the open approach and to increase the knowledge base of the students, the challenge is to develop a teaching style which preserves pupils' involvement in the problem whilst concentrating the work on the syllabus content. (p. 41)

Indeed, this was the biggest challenge which faced the researcher, and the most interesting and enjoyable part of the application of problem-posing situations in the project classroom.

In Figure 4.11 the main categories of problem-posing situations which the researcher attempted to use and extend are presented. The framework was designed to embrace problem-posing activities, mathematics curricula and problem solving in mathematics classrooms.

The main aim of the framework was to guide the design process of problem-posing situations and help the teacher in making appropriate choices for problem-posing categories, according to the instructional goals. Krutetskii's system of problems used for investigating the structure of students' mathematical abilities has been decided to be used in order the initial framework to be enriched and extended. In fact, although Krutetskii's (1976) major focus was problem solving, and his insights into the relationship between problem solving and problem posing have inspired the researcher to reflect on how his ideas could be applied to link both problem solving and problem posing. The author believes that there is much to be gained by invoking the ideas of Krutetskii (1976), and in particular, by extending Krutetskii's problem-solving categories into the realm of semi-structured and structured problem posing.

Problem-posing categories:	Problem-posing situations:
Free	Problems written for a friend; Problems from data; Problems I like, Problems which involve the use of a specific concept or mathematical method.
Semi-structured	***Problem posing situations based on a specific problem structure:*** Problems which fit given computations; Problems which are similar to a previously solved problem; Open-ended problems; Mathematical investigations. ***Problem posing situations based on a specific solution structure:*** Problem posing which involves the use of a specific mathematical method within a given problem structure.
Structured	***Problem-posing situations based on a specific problem:*** Problem variations; Reformulations. ***Problem-posing situations based on a specific solution:*** Restating a problem on the basis of its solution.

Figure 4.11. The framework for problem-posing situations developed in the study.

Development of Framework for Teacher's Questions in the Project Classroom

The importance of teaching students to ask good questions has been recognised in mathematics education research (Clarke & Sullivan, 1991b; Polya, 1957; Sternberg, 1987). A classroom environment in which for example, students feel free to ask questions and to discuss mistakes which are made in formulating and solving problems, might help them to start to understand the problem and solution structures.

Recognition of the Importance of the Teacher's Questions

The teacher plays an important role in supporting students' efforts to make meaningful conjectures and to discover their own problem-solving approaches. Bruner (1961) recognised the difficulties involved in designing an environment which will lead to students making a "discovery" when he wrote that "there is a vast amount of skilled activity required of a 'teacher' to get a learner to discover on his own." In fact, helping students to make discoveries on their own, requires the teacher to design suitable sets of interconnected activities and questions, and these are very likely to differ from student to student.

Anderson and Sullivan (1995) suggested that teachers should plan and use *preliminary prompts* and *extension questions* to provide structure which would assist in making educationally rich situations. More recently, Bruner (1996) has stated that the art of framing challenging questions is undoubtedly as important and as difficult as the art of giving clear answers. Recognising the importance of the teacher's role, he noted that the "art of cultivating such questions, of keeping good questions alive, is as important as either of those" (p. 127).

In fact, in the current study, prompting students to respond to the problem being solved was regarded an essential part of the teacher's preparation for the lessons in both groups. Applying two different teaching approaches required the researcher to create two different sets of questions. According to Doyle and Carter (1982), the ways in which students work or respond to their teacher's (or other students') questions, depend on what the questions are and the nature of work they have been asked to do. The questions given to the problem-solving group (Group B) were consistent with Polya's (1957) recommendations. Incorporating problem-posing situations was supported through the development of a set of questions which involved "hidden" problem-posing activities. Some of the questions which were used by the teacher in the project classroom, were designed to assist students to reflect on specific problem-posing and problem-solving situations from a given perspective. The sub-headings of the discussion which follows relate to the anticipated instructional goals.

Teacher's Questions which Involve "Hidden" Problem Posing

Problem-posing situations can be presented to the students as written or as verbal prompts, according to the nature of work students are asked to do. The researcher designed a number of questions which were designed to incorporate problem posing in a way which would help the students to reflect on their experience when they attempted to solve mathematics problems.

1. Teacher's questions for helping students to focus their attention on the language characteristics of the problem. The first set of questions given to students was aimed at helping students learn to focus their attention on the language characteristics of the problem. Students, for example, could be asked questions such as those presented in Figure 4.12.

Most of these questions require students to focus their attention on some of the elements of the problem statement (such as unknown words, key-words, mathematical concepts, etc.) or the interrelationships between them within the overall wording of the problem.

How can we restate the problem?
What are the unknown words?
What are the key words?
How could the problem be made easier to understand?
How can we reformulate the problem statement so it is shorter?
How can the problem be made clearer?

Figure 4.12. Examples of teacher's questions aimed at focussing students' attention on the language characteristics of a problem.

By asking students to focus their attention on the language characteristics of the problem, it was hoped that students would reflect via their own problem-posing activities according to their understanding.

2. Teacher's questions for helping students to focus their attention on the problem structure and its features. The next set of questions was aimed at helping students to focus their attention on the structure of the problem and its features. The teacher asked questions such as those shown in Figure 4.13.

This sequence of questions could be easily extended by taking into account the characteristics of the specific problem-posing situations involved which relate to the problem structure.

What is the problem about?
[2] *What are the data?
*What are the restrictions?
What might the question be?
What other questions might be asked?
Is the reformulated problem the same as the initial problem?
What other physical situations would give rise to the same types of data?
What changes in the numerical situations could lead to a similar problem?
What changes in the numerical situations could lead to a different problem?
What changes in the physical situation could lead to a similar problem?
What changes in the physical situations would result in an easier problem?
What changes in the problem format would give rise to a more difficult problem?.

Figure 4.13. Examples of the teacher's questions aimed at helping students to focus their attention on elements in the problem structure.

3. Teacher's questions for helping students to focus their attention on the solution structure. As was mentioned earlier, encouraging students to learn to pay attention to the solution structure was regarded as an important part of their mathematical culture. Questions such as those listed in Figure 4.14 became a normal part of the project classroom environment.

What problems similar to this unsolved problem can we pose?
What are the main stages of the solution approach?
What changes in the problem can change the solution approach?
Could you suggest another problem with the same solution approach?
*Use the idea of the solution approach to give an example of another problem of the same type.
Think of familiar problems in which this idea might be applicable.
Think of a situation (problem) in which this approach would not be applicable.
What changes in the problem will increase (decrease) the number of the solutions?
How can the problem be reformulated so that the solution approach will be changed?

Figure 4.14. Examples of the teacher's questions for helping students to focus their attention on the solution structure.

Although problem posing was used as an inseparable part of problem solving, not all problems posed by the students during this study were solved. In many cases problem posing was used as an activity to help students to understand

[2] All questions which are literally taken from Polya (1957) will be designate with an asterisk.

some of the features of the mathematical concepts involved and of the problem or solution structures.

4. Teacher's questions used before, during or after solving a problem. Teacher's questions, involving "hidden" problem posing, can also be classified depending on whether the student is trying to solve, is in the process of solving, or has already solved, the problem. When the teacher directed questions to the whole class, she tried to express them in a general form, while at the same time taking into account the characteristics of the task and the instructional goal she wants to achieve. Some examples are presented in Figure 4.15.

What am I going to ask (can you see the pattern)?
Could you tell me what I am going to write (can you see the pattern)?
*Look at the unknown and give me an example of a familiar problem having the same unknown.
What conditions are sufficient to determine the unknown?
How many ways do you know for determining that unknown?
*Could you restate the problem?
*Could you restate it still differently?
Could you tell me a familiar problem with the same unknown?
*Could we derive something useful from the data?
What kinds of problems can we pose from the data given?
Which elements from the data can be interpreted differently?
Could you restate the problem in your own words?

Figure 4.15. The teacher's questions on the basis of "hidden" problem posing which can be asked *before* solving a problem.

At the same time the teacher tried to connect these questions with specific problem-posing activities which are likely to reduce the difficulty of the task or the solution idea and to help students to proceed with the problem solution. Examples of such questions are given in Figure 4.16.

*Give me an example of a related (or similar) problem.
Could you tell me another problem with the same structure?
Could you tell me another problem with the same method of solution?
What can I find from this?
What gives us the reason to apply this method of solution?
Think of familiar ways of finding such an unknown.
Suggest a procedure which might be successful.
Could you change the unknown or the data, or both if necessary, so that the new unknown or data are nearer to each other?
How can we make the problem easier and solve the new one?
Could you introduce some auxiliary elements in order to make the problem applicable to a range of situations?
What arguments should we provide? Why is that true?
What follows from here?
When is this theorem (relation) true?
Does the result satisfy all the given conditions ?
How many solutions does the initial problem have?
Which of the solutions to the final version of that problem are solutions to the initial problem?
Is the reformulated problem the same as the initial problem?

Figure 4.16. The teacher's questions incorporating problem posing which could be asked *during* students' attempts to solve a problem.

After solving a particular problem students could be prompted to explore the generalisibility of the solution approach, to make changes to the problem statement and to predict changes in the solution (see Figure 4.17).

Use the idea of the solution to give me an example of another problem of the same type.
Could you tell me another problem with the same structure which uses the same mathematical relationships?
Could you tell me another problem with the same method of solution?
Could we assume that this is always true?
What could change the method of solution for this problem?
Could you give me some concrete examples for applying this theorem?
Could you give me some concrete examples for applying this method/approach?
Could you tell me the main steps of the solution?

Figure 4.17. The teacher's questions which include problem posing which could be asked *after* solving a particular problem.

It should be emphasised that attempts to orchestrate the problem-solving environment by incorporating new activities means that the teacher must react immediately, and respond appropriately yet thoughtfully. Given that the Program involved mathematically able students, one of the essential skills needed by the teacher was that of being able to generate and provide appropriate questions and prompts which were consistent with the goals of the study.

Development of Framework for Open Problem-solving Approach

Theoretical Background

Design of the problem-posing situations developed in the study, the ways of interactions between problem solving and problem posing, and the implementation of problem posing-activities in the mathematics classroom have been based on previous research by Vygotsky (1978), Krutetskii (1976) and Doyle (1983).

Vygotsky (1978) introduced the notion of the zone of proximal development in an effort to deal with two practical problems in educational psychology—the assessment of children's intellectual abilities and the evaluation of instructional practices. He defined the zone of proximal development as "the distance between the child's actual development level as determined by independent problem solving" and the higher level of "potential development as determined through problem solving under adult guidance or in collaboration with more capable peers." Thus, from a Vygotskian perspective, the role of instruction is to provide environments which can help students extend the boundaries of their independent problem solving.

In this study, problem posing was incorporated in the mathematics classroom both as a *tool* for diagnosing some characteristics of students' learning and as a *means* for helping pupils to solve mathematical problems. The application of problem posing in order to facilitate problem solving is one of the major foci taken up in this study.

Conceptual Framework for the Open Problem-solving Approach

The conceptual framework of the teaching approach used in the project classroom, presented in Figure 4.18, has been developed on the basis of a review of the relevant literature, and on the premise that problem posing can take on a central role when the problem structure, the solution structure and students' problem-solving activities are open.

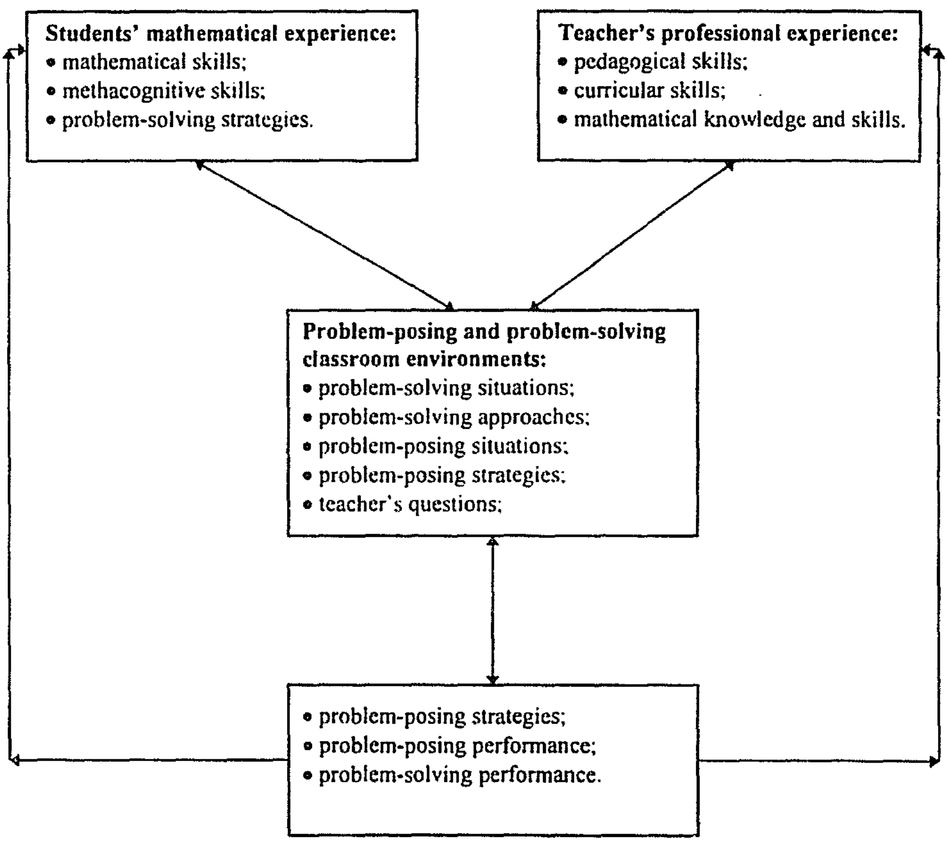


Figure 4 18 The conceptual framework of the open problem-solving approach.

The application of problem-posing in a problem-solving environment is coloured by students' previous mathematical experiences and the teacher's ability to make appropriate links between the level of mathematical understanding students have achieved and the objectives of a particular teaching session. At this level the the aim of the study is to design and to explore a framework of appropriate problem-posing situations (see Chapter 6) and explore students' problem-posing strategies (see Chapter 7).

The central part of the conceptual framework which underlies the open problem-solving approach is the notion of problem-posing as a means of instruction and as an inseparable part of problem-solving classroom environments. In this thesis, the teaching approach will adopt several different models to describe the interaction which could occur between problem-solving and problem-posing activities (see Appendix 9). A range of modes for the application of specific problem-posing situations in particular problem-solving contexts will be adopted (see Chapter 6).

The third major area of the conceptual framework covers the effects which an open-problem-solving approach might have on students' mathematical problem-posing and problem-solving performances (see Chapter 8 and Chapter 9). The extent to which some individual students would respond to the problem-posing situations and the characteristics of the strategies employed was one of the important aspects investigated in this study.

The conceptual framework of the open problem-solving approach therefore attempted to take account of the central role played by problem posing when students attempt to solve mathematics problems.

CHAPTER FIVE

DATA ANALYSIS PROCEDURES

In this chapter procedures used for analysing the data collected during the study will be presented. Two main procedures underlie the process of data analysis for this study—general and specific. The specific procedures will be presented under major headings which relate directly to the research questions.

General Procedure

The main categories, identified by the research questions, served as a basis for the process of data analysis. The data collected during the study were divided into the following groups according to the initial source:

- Data obtained from lesson transcripts;
- Data obtained from the tests;
- Data obtained from individual students' worksheets;
- Individual interviews;
- Solutions to *Challenge Problems*;
- Data obtained from independent observers;
- Teaching materials developed specifically for the study (revision papers, individual worksheets, problems and problem chains written on folios, hints to the *Challenge Problems*);

- Other data including students' individual results on the *Challenge* and the *Enrichment Stages*; individual results on *the Australian Mathematics Competition.*

The following steps were adopted in preparing the data for analysis, and for the subsequent analysis:

1. Data from all categories were divided into observation units, which were connected to a particular research question (Schoenfeld, 1985, p. 292);

2. Data describing problem-posing or problem-solving actions, were divided into episodes;

3. All units were coded and placed in groups, according to preliminary definitions given for problem-posing situations, problem-posing strategies, problem-posing performance and problem-solving performance (see Chapter 1);

4. Data which had not been categorised was set aside for later analysis;

5. After additional analysis of the common characteristics of the data in each preliminary category had been carried out, precise definitions and criteria for separating the categories were developed;

6. Reclassification of the data into new categories based on the more precise definitions was made;

7. Relationships between the categories were examined for possible combination, sub-division, extension and redundancy;

8. After two months had elapsed, all data were re-read and the categories were redeveloped.

Specific Procedures Followed for the Analysis of Problem-posing Situation Categories

Specific procedures were developed for the process of data analysis for each research question.

In order to develop a classification for the problem-posing situations observed in the study, the researcher adopted the following path-analysis:

1. At the beginning of the study, on the basis of the review of the literature the researcher developed an initial framework to describe the problem-posing situations she intended to apply in the project classroom (Stoyanova, 1995);

2. Throughout the study a journal containing self-observation notes and strategies used for designing problem-posing situations was kept. These notes were organised as "observation units" for the purpose of analysis;

3. All observation units on problem posing were placed in preliminary categories, according to the initial framework;

4. Preliminary categories were analysed and refined;

5. From these analyses and refinements, new categories were defined;

6. A more precise classification of the categories was made;

7. Data from observation units, teaching materials and students' work which was not coded was re-read, analysed and placed into appropriate categories;

8. Relationships between the problem-posing categories were examined for possible combinations, extensions, sub-divisions or redundancy;

9. The modes of interaction between problem posing and problem solving were examined and described according to their instructional goal;

10. Two months after the above analyses had been carried out, the categories were redeveloped;

11. An independent observer was given 30 percent of the data and was asked to validate the classification of the problem-posing categories.

Specific Procedures Followed for Analysis of Students' Problem-posing Strategies

The study explored the characteristics of students' problem-posing strategies in free, semi-structured and structured problem-posing situations. The main aim was to monitor the range of problem-posing strategies used by students. The other aim was to study the effects (if there were any) of problem-posing environments on students' problem-posing strategies.

The framework for data analysis for students' problem-posing strategies was based on the assumption, that *if a student's work presents evidence in the form of mathematical problems which are not identical to the initial source of problem posing, then a student has applied specific action(s) to link the problem-posing task and the written product.* In other words, it is assumed that making meaningful changes to the initial situation imply an application of a problem-posing strategy.

The following inductive steps were carried out by the researcher so that students' problem-posing strategies could be identified and classified:

1. Students' written responses on the tests were divided into four groups acccording to the structures of the problem-posing products: correct, intermediate correct, not correct responses and responses which should be excluded from further analysis;

2. Problem-posing products, classified as correct, intermediate correct or not correct responses, were read and an initial account of the actions associated with a problem-posing process was made;

3. Data which were not categorised were set aside for later analysis;

4. Problem-posing products were re-read and the actions were broken into further sub-sets, according to the differences in problem-posing products;

5. All problem-posing sub-sets were used to form an initial framework to describe problem-posing strategies. Two sets of actions were placed in the same class when the analysis of problem-posing products, based on the written evidence, showed that the goals of subjects' actions were linked to the initial source of problem posing in a similar way;

6. Preliminary definitions of the problem-posing categories identified were given. The definitions were designed to embrace the common characteristics of individual paths of actions used by students when posing a problem under specific conditions;

7. After one month had elapsed, the problem-posing tests completed by students were re-examined. Any new categories which emerged were added to the framework;

8. A more precise classification was made;

9. Data from problem-posing tests, which were not coded, were re-read, analysed and replaced into appropriate categories;

10. Additional data from tape-transcriptions, researcher's journal entries, students' individual written work, and individual interviews with students, were analysed and the framework refined;

11. Relationships between problem-posing categories were examined for possible combinations, extensions, sub-divisions and redundancies;

12. After two months had elapsed, all data were re-read and the categories were redeveloped. The two sets of categories were consolidated;

13. About 100 products, referred to as problem-posing products, were included in the process of validation of the categories. An independent assessor was invited to read the data and to validate the categories developed to describe students' problem-posing strategies.

Specific Procedures Followed for Analysis of Students' Mathematical Performance

As a part of the research design, two sets of mathematical questions were administered to the students from both Groups A and B as pre-tests and post-tests (see Chapter 4). It was envisaged that this would enable the researcher to detect any major changes in students' problem-posing and problem-solving performances.

Scheme for Assessing Students' Problem-posing Performance

1. A rationale for a development of a scheme for assessing students' problem-posing performance. From the literature review it is evident that previous research provides insufficient information about the processes involved in students' problem posing, and about the characteristics of students' problem-posing products. According to Silver & Cai (1996)

> Research on children's problem posing has tended to focus only on small numbers of subjects and to provide only a fairly superficial analysis of the posed problems, if any

> analysis. . . . If progress is to be made in understanding the nature of mathematical problem posing, or if rigorous attempts are to be made to study the instructional impact of interventions related to mathematical problem posing, then better analytic techniques must be developed to study problem posing by elementary and middle school students. (pp. 522-523)

Thus the need to develop better techniques for assessing students' problem-posing products has been recognised.

*2. **Definitions.*** In this study a scheme for assessing problems posed and solved by students was developed by adapting and extending problem-posing schemes proposed by Balka (1974) and Leung (1993). The scheme also took into account the problem-solving scheme used by Stacey et al. (1993) who proposed the following five aspects for measuring students' problem-solving performance: (a) *correctness of the answer* (Was the answer correct?); (b) *method used* (How good was the approach used?); (c) *accuracy* (Were the calculations free of errors?); (d) *extracting information* (Was the problem understood?); and (e) *quality of explanation* (Was the thinking explained clearly?).

The proposed scheme for assessing students' problem-posing performance takes into account the fact that, in this study, problem-posing activities were an inseparable part of students' problem-solving activities. At the same time, consideration needed to be given to the specific characteristics of the problem-posing and problem-solving products.

The assessment scheme for problems posed by students is necessarily connected with the type of problem-posing category involved, as well as with the characteristics of the problems actually posed. Problems posed by students under the conditions of *free, semi-structured* or *structured* problem-posing situations were assessed according to the following five *common* aspects: *accuracy*, *correctness*, *originality*, *level of difficulty* and *type of the problem*.

Accuracy refers to the precision of the mathematical language used. Three levels are defined: precise, partially precise and not precise.

Correctness is related to *problem structure correctness*. Problems posed by students are assessed as correct, partially correct or not correct.

Originality assessed the quality of the problem structure by taking into account the extent to which the formal structures of the posed problems related to student's problem-solving experience. A problem is regarded as original when its structure is invented by the student; partially original if it is a well-known problem, but its structure is a discovery for the student; and not original if the problem can be linked *directly* to student's mathematical experiences.

Level of difficulty of the posed problems refers to the complexity of the problem solution structure needed for the posed problem. Problems posed by students were assessed as difficult, partially difficult and not difficult.

It should be emphasised that, because students posed mathematical problems from a range of topics, with different formats and on the basis of problem-posing situations from different categories and types, it was necessary for the coding scheme to be set up in fairly *general* terms.

The types of problems posed by students were categorised as algorithmic, logical or generalisible, according the type of knowledge underlying the solution process. A problem was regarded as *algorithmic* when its solution involved a well known algorithm (algebraic, arithmetic or geometrical) included in the school curricula or the Program content. A problem was categorised as *logical* when its solutions required inductive or deductive logical reasoning. When the problem generalised a pattern it was referred to as a *generalisable*.

Because conditions underlying the process of problem posing are likely to colour the problem posed, additional criteria were added for semi-structured and structured problem-posing situations. Two new characteristics were used for assessing problem-posing products in semi-structured and structured situations — *fluency* (number of correct problems related to the problem-posing situation) and *flexibility* (number of different types of problems generated).

Scheme for Assessing Students' Problem-solving Performance

The problem-solving skills involving a basic use of the concept of percent were assessed through Items 1 to 5 (see Figure 3.7). The two levels of responses were matched to a scoring scheme of 1 (for correct response) and 0 (for incorrect response).

The characteristics of problem-solving approaches used by students when solving word problems based on an application of the same concept in real-life situations were assessed through Items 6 and 7 (see Figure 3.7). The problem-solving products were coded according to the scheme provided below which is an adaptation of the scheme used by Stacey, Groves, Bourke and Doig (1993, pp. 2-3).

The assessment included four aspects of the problem-solving product: *understanding, correctness, accuracy and originality.*

Understanding referred to the *understanding of the problem structure* on the basis of the choice made by the student of an appropriate solution strategy. A problem was classified as understood, partially understood or not understood.

The *correctness* was associated with the *correctness of the result.* The results for each item were determined as correct, partially correct or not correct.

Accuracy referred to the *mathematical accuracy of the written solution.* The problem solving products were divided into three groups — precise, partially precise or not precise.

Originality — the fourth aspect assessed in the solution provided by the students — referred to *the originality of the solution strategy.* This was connected with the elegance of the solution strategy by itself and how it relates to the student's previous experience. The problem solutions were classified as original, partially original or not original.

The three levels in every assessment category were matched to a scoring scheme of 3, 2 and 1. For example, when the problem-solving product was assessed as an original one, then it was scored with 3 points, and if it was categorised as not original, the score was 1.

In addition to these four characteristics of students' problem-solving products, the differences between the solution approaches, if any, at the beginning and at the end of the study, were regarded as one of the most important aspects of students' mathematical performance.

CHAPTER SIX

CLASSIFICATION OF PROBLEM-POSING SITUATIONS USED IN THE PROJECT CLASSROOM

Introduction

The literature search revealed that the types of problem-posing situations used as a means of instruction and as a research tool for investigating students' understanding of mathematics is limited (Kilpatrick, 1987; Silver, 1993). The need for a systematic account of problem-posing situations in which students could be involved in mathematics classrooms has been recognised by mathematics educators and researchers (Pehkonen, 1995; Silver, 1993).

The aim of this chapter is to present a classification of problem-posing situations used in the project classroom within the framework described in Chapter 4. The types of problem-posing situations were identified from analysis of the project classroom data, on the basis of the procedure which was described in Chapter 5.

In the mathematics classrooms problem posing can be applied as a goal or as a means of instruction (Kilpatrick, 1987). In this study the role chosen for problem posing was as a supporting activity to students' problem solving. Designing a range of problem-posing activities which could be embodied in school problem-solving environments was an important part of the study.

The varieties of problem-solving contexts in which problem posing took place in the project classroom will be illustrated by examples of the teaching materials which were developed specifically for this project, and by selected samples of students' responses for some but not all of the problem-posing situations described. It is believed that this will help to illustrate key features associated with the application of particular types of problem-posing situations within a wide range of different problem-solving contexts.

Important aspects of the application of problem posing were the anticipated links between the use of a particular problem-posing activity and the instructional goals of the sessions. The problem-posing situations designed for use in the project classroom, aimed to assist students to perceive: (a) the features of the problem structure: (b) the features of the solution structure; and (c) the interrelationships between the problem and solution structures.

Terminology

To help students understand and distinguish the differences and similarities between two problem structures, new terms such as *similar*, *same* and *identical* problems were introduced by the teacher and regularly used for prompting specific students' actions. It should be noted that the term *problem* was used as a class of equivalence. Two problems were referred as the *same* when they had the same *content* (the mathematical substance) but differed in their *context* (the characteristics of the physical situation of the problem) (Kilpatrick, 1984). When problems had the same *content* and differed in their *syntax* (the language characteristics) it was said that they were *identical*. Two or more problems were called *similar* when they had some similarities in their content. For example, problems which differed only in their numerical information were referred to as *similar*. In these definitions an emphasis was placed on the problem *content*, even though it is clear that two *identical* problems could differ markedly according to *other* criteria (e.g. difficulty).

The participants in the study were not given any precise definitions about the range of possible problems they might pose. Generally, students were encouraged to use a word or a group of words which they thought best described their ideas. When relevant, the teacher asked them to present more detailed explanations of the meanings of the terms they were using.

Students from both classes were engaged in solving problems whose structures were represented in different ways. In such cases it was said that the problems were presented in different *formats*. The study incorporated problems with: *true-false* format, *answer* format, *multiple-choice* format, *fill-in-blank* format, *matching* format and *solving* format. A problem with a *solving* format is defined as one which requires the problem so*lution* to be written precisely.

In order to help students to perceive even the most salient features of a particular problem structure, the problem-posing categories in the project classroom comprised free, semi-structured and structured situations. The classification of possible problem-posing situations will be presented within the framework illustrated in Figure 4.1.

Free Problem-posing Situations

The aim of designing free problem-posing situations was to place students in situations in which they would be prompted to pose problems which in turn, would reflect specific perspectives of their mathematical experience.

On many occasions students from the project classroom were invited to make up problems such as one which they thought was difficult, problems which they wanted to see in a Mathematical Olympiad, or a problem which they wanted to be solved by their teacher (see selected samples of students' problem-posing products in Appendix 10).

The problems posed by students under the category of free problem-posing situations were aimed to help the teacher to become aware, in a sensitive way, of the diversity of the difficulties students might have experienced in perceiving particular types of problems. At the same time, students' problem-posing products served as a mirror in which students' understanding of particular concepts was reflected: those of linear equations, permutations, or application of specific theorems, for example. When students with higher mathematical aptitudes were invited to pose problems, some of them tried to make up problems which they did not know how to solve, and their problem-posing products provided an environment for involving all students in solving more complex problems. Free problem-posing situations which were used in the study are presented in Figure 6.1.

Posing problems which were found to be interesting;
Posing problems about a particular topic;
Posing problems for a mathematics competition;
Posing problems on every-day-life contexts;
Posing problems from data;
Posing problems with given answers;
Posing problems written to be solved by the teacher;
Posing problems which were found to be difficult;
Posing problems which involved a use of a specific mathematical concept(s);
Posing problems which involved a use of a specific mathematical method;
Situations based on posing problems which involved an use of a specifics solution method.

Figure 6.1. List of *free* problem-posing situations used in the study.

Classroom and individual discussions provided a natural atmosphere for prompting students to express their opinions by making up problems under free problem-posing situations (see also Chapter 9). In order to help students structure their knowledge in a particular learning area, the teacher encouraged pupils to exhibit their understanding by posing problems of the same type which they learned to solve. Observations from the classroom suggest that free problem-posing situations are likely to provide a non-stressful environment for most of the students and a starting point for involving some of the students in deeper inquiry.

The nature of the application of free problem posing in the project classroom will be illustrated with three types of problem-posing situations: posing problems which involve (a) the use of a specific concept(s); (b) a particular solution method; and (c) an artificial operation.

1. Problem-posing situations which involve the use of a particular concept. Problem solving environments provide a natural atmosphere for involving students in posing problems which incorporate a specific mathematical concept, notion, or a rule. In the project classroom this activity was used to prompt students, in a natural way, to pose examples which would illustrate their knowledge about a specific concept such as right-angled triangles, diofantine equations or tangents.

Problem-posing situations of this type were mainly formed as verbal prompts, when the teacher asked the students to present special cases or to illustrate their understanding of problems which they knew how to solve.

Observations from the project classroom suggest that this type of activity provides an educationally rich environment in which the students and the teacher play equal roles. On the other hand by having had students express their understanding, the teacher became aware of the difficulties students had expressed.

2. Problem-posing situations which involve the use of a particular solution method. This is one of the problem-posing activities recommended by Polya (1957) as well as others (Kilpatrick, 1987; Koenker, 1958; Peretz, 1985). In the project classroom students were frequently asked to suggest problems which could be solved by a particular solution method (see also Chapter 9). In some cases the situation was presented to the students by a suitable written prompt which they had to incorporate into the problem posed. A sample of problems posed by students which involve the use of the Pigeon-hole Principle, working backwards, and the Least Common Multiple are given in Figure 6.2.

Posing problems which involve the use of a specific solution method was not an isolated activity in the project classroom. In many cases, in addition to solving the problem, students were asked to justify their suggestions and to connect the use of a specific method with some features in the problem structure.

Example 1 (Martin): If a bag contains 9 blue marbles, 7 red marbles, and 10 black marbles. What is the least amount of how many you have to pull out to ensure that I have 8 of one kind?

Example 2 (Nelly): If there is a girl at every 3rd desk, a cockroach on every 4th desk, beetle on every 5th desk, which is the next desk that will have all three?

Example 3 (Nick): Tommy was walking home from school when he stopped by an apple thee to fill up his basket with apples. Later he ran into his friend Sasha and gave him $^1/_4$ of his apples plus 2. After that he met his brother Alex and gave him $^1/_2$ of the remaining apples plus 2. Later he ran into his other brother Michael and gave him $^1/_2$ of his remaining apples plus 3. By the time he came home he had eaten 2 more apples and had 1 apple left. How many apples did Tommy pick from the apple tree?

Figure 6.2. Examples of problems posed by students which involve the use of solution methods such as the Pigeon-hole Principle, the Least Common Multiple and working backwards.

3. Posing problems for formation of a mathematical operation. In mathematics, definitions play a key role in the process of building scientific structures. In the project classroom, students had the opportunity to construct functions and to pose "operation" problems. This was one of the rare problem-posing activities in which mathematical notations were changed into words.

> The *operation* "super product" for any two numbers *a* and *b* calculates $a - 2b$.
>
> a) What is 2 * 3?
>
> b) If $a * 3 = 6$, what is *a*?

Figure 6.3. Teaching material which illustrates an *operation* problem.

At the beginning, the teacher illustrated an *operation* problem (see Figure 6.3). The examples were written in generalised, abstract form and suggested applications which are close to the school curriculum — calculating and solving linear equations. Students then were invited to solve operation problems presented in a multiple-choice format, to construct their own *operations,* and to suggest meaningful applications (see Figure 6.4, also Worksheet 16B, Appendix 4).

> *Problem posed by Carol and Nora:* $a * b = a \times b - a + b$. What is 6 * 2?
>
> *Problems suggested by Martin:*
> a) If $a * b = (a^2 \times b)^3 + (b^4 - a)^2$, what does 5 * 7 ?
>
> b) $a * b = (a+b) \times \frac{ab}{a} - (\frac{ab}{1} - b)$
>
> What does 3*4=
>
> *Problem created by Gregory:*
> $a * (2 * 3) = (a * 2) * 3$, where $a * b = {}^1/_{ab}$. What is *a*?

Figure 6.4. Problems posed by students which involve a formulation of a mathematical operation.

Classroom and individual discussions based on the characteristics of the problems posed by students were designed to help them focus on the way in which the problems were created, and to help them discern the key features of how the problems were applied. The properties of the associative and the distributive laws when used in specific cases were discussed (see the problem posed by Gregory in Figure 6.4). The mistakes which could be made when problems — such as "If $a * b = {}^{1}/_{ab}$, what is 2*(3*4)?" — were outlined.

The problems posed by students suggest that students in the project classroom tended to imitate the structure and the type of the problem category which was illustrated initially. However, most students tried to increase the problem difficulty by suggesting more complex operations.

Semi-structured Problem-posing Situations

This problem-posing category was designed to assist students to perceive pertinent features in the structures of particular problems or solutions. The premise behind the use of semi-structured problem-posing situations was that, by involving students in exploring situations based on problem or solution structures, it would help them to understand the salient relationships between the *Given*, the *Obstacles*, the *Goal* and the solution approach used. The framework makes use of two sub-categories: (a) problem-posing situations based on a specific problem structure; and (b) problem-posing situations based on a specific solution structure.

A. Problem-posing Situations Based on a Specific Problem Structure

The semi-structured problem-posing situations based on a specific problem structure ranged from situations with either missing elements or surplus information in the *Given*, *Obstacles* or the *Goal* to posing sequences of interrelated problems. Figure 6.5 presents a list of semi-structured problem-posing situations based on a specific problem structure which were used in the study. Five of these categories will be discussed in further detail in the section which follows.

A. Problem-posing situations based on a specific problem structure:

Problem posing based on a problem structure with an unstated *Goal*;
Problem posing based on a problem structure with missing elements in a combination of the *Given*, the *Obstacles* and the *Goal*;
Problem posing based on a problem structure with surplus information:
  Situations with surplus information in the *Given*,
  Situations with surplus information in the *Obstacles*,
  Situations with surplus information in a combination of the *Given* and *the Obstacles*;
Posing problems on the basis of different interpretations of a mathematical concept;
Posing problems which have more than one solution;

Figure 6.5. List of the semi-structured problem-posing situations based on a specific problem structure which were used in the study.

1. Problem posing based on a problem structure with an unstated Goal. The major aim of this problem-posing situation was to help students discover possible connections between given numerical facts and the mathematical relationships in the information given (if there are any), and to predict a meaningful *Goal* which follows from the information given. The presentation of problem-posing situations which are based on problems with an unstated goal statement to a group of students invariably leads to useful discussion.

The basic assumption here is that when students attempt to pose a question to an unfinished problem structure, they will pose one which makes sense to their own level of understanding and conceptual development (Yackel, Cobb, Wood, Wheatley & Merkel, 1990).

Problem-posing situations based on problems with unstated question were created by omitting the *Goal* (the question or the goal statement) in the structure of a *closed* problem. In the project classroom, when students were presented with a new problem of this type, the teacher would frequently ask the students to formulate a question which would follow *directly* from the information in the problem statement (see Krutetskii, 1976). The teacher encouraged students to make guesses with regard to what the question might be about, and to justify their predictions.

Problem structures with an unstated *Goal* were also presented in multiple-choice question format. Multiple-choice question structures with unstated questions required students to respond not only by posing a meaningful question, but also by linking the solution of the problem posed with exactly one of the elements in a set of possible answers. The goal of such a problem-posing situation was to assist students to focus their attention on the interrelationships between the changes in a problem structure and the numerical value of the solution. Although this type of problem-posing situation at first sight appeared to be quite simple, some situations required comprehensive reasoning skills.

At the beginning of the study, in order to enable students to make an easy start, the level of difficulty of the problem-posing situations was reduced by including "None of them" as one of the possible answers. Later, when students had more experience in solving problems from this type, multiple-choice question problems with more missing elements in their structures were added. Problem-posing situations of this type were created by omitting only some of the elements of the conclusion, or by omitting part of the data in the *Conclusion* and in the *Given*. The first example in Figure 6.6 illustrates a structure in which students had to decide which of the given answers matches a specific term in the arithmetic sequence. In the second example the word "term" is omitted, and students have to consider whether the conclusion is about a *term*, a *difference* or about a *sum of* several consecutive terms. In the third example, students are asked to define an *artificial operation* (a function of two variables) and also to state a *suitable question*.

> *Example 1:* The first term of an arithmetic sequence is 1. The second term is 10.
> The.........................term is:
> A) 18; B) 16; C) 9; D) 19;
>
> *Example 2:* The first term in an arithmetic sequence is 1. The fourth is 10.
> The.......................is:
> A) 13; B) 29; C) 22; D) 100.
>
> *Example 3.* For integer numbers a and b we define $a*b$ as $a*b$ =.........................
> Then......................................equals:
> A) 2; B) 4; C) 6; D) None of them.

Figure 6.6. Examples of multiple-choice format structures with unstated questions.

Observations from the project classroom suggest that involving younger students in defining a function of two or more variables (Example 3, Figure 6.6), combined with designing specific applications of this new operation, is likely to help students to understand the features of this concept.

> *Without adding more information formulate a meaningful conclusion for the problem situations listed below:*
> A) There are 3 pigeons in 2 pigeon holes. Then..
>
> B) Mrs. Simpson has three children. Then..
>
> C) In my maths class I have 27 students. Then..
>
> D) This week Carol has been to the library 8 times. Then..................................
>
> E) There are $k+1$ pigeons in k pigeon-holes. Then ....

Figure 6.7. Problem-posing situation based on a problem with an unstated question.

Some other examples of the teaching materials developed which involved problem-posing situations based on the formulation of goal statements are given in Figure 6.7. In the examples shown in Figure 6.7 students had to formulate the *Goal* in the form of a convincing argument without including additional information.

> *Finish the problem formulation below:*
>
> From the set of digits (1, 2, 3, 6, 7, 9):
>
> A) How many .... digit integers can be formed?
>
> B) How many ....digit integers can be formed?
>
> C) How many .....digit integers can be formed?
> (*Assume that no digit may be used more than once*)

Figure 6.8. Example of a problem with a set of sub-problems with missing elements.

Figure 6.8 shows an example from the teaching materials developed for the project classroom which illustrates a *set of sub-problems* which have missing elements in their structures.

> Last night there was a party and the host's doorbell rang 10 times. The first time the doorbell rang only one guest arrived. Each time the doorbell rang after that three more guests arrived than had arrived on the previous ring. (*Ask some meaningful questions.*)
>
> ***Martin and Irene:*** In the first group, there was one child; the second group, 2 children; the third, 3 children; etc. How many children were at the party after 10 groups?
> If every 2nd child brought a bird, and every 4th child brought a bird and a cat, how many birds were at the party after 10 rings of the doorbell? And how many cats were there?
>
> ***Nelly:*** Every 5th person is a child and every child brings a dog. There is a room in the house especially for dogs. The room can house 15 dogs. How many times does the doorbell ring if the room is full of dogs? And if the first time 1 child arrives and brings his dog and if 4 more people arrive each time the doorbell rings, how many people did arrive after the 15th doorbelling.[3]
>
> ***Carol:*** How many people were at the party after the bell rang 10 times? If 75% were children how many were adults?
>
> ***Edward:*** How many guests will there be on the 14th ring? If ≈ 25% of the guests like beer and the rest have wine and each person takes 1 pint of beer, how many pints of beer should be bought?

Figure 6.9. Students' responses based on a problem structure with an unstated goal which require a multiple response.

[3] The problems are literally presented from students' works.

Figure 6.9 presents students' responses to the situation presented earlier (see Figure 4.2, Example 1). It this case students were given an opportunity to add information and finish the problem structure in *more than one* way. To help the reader to understand students' responses, the problem structure is presented again in Figure 6.9. After posing a number of questions, students were invited to present their ideas and answer to their peers' questions. Some of the problems posed were selected by the students and solved by the whole class.

Another way of applying problem posing by formulating a goal statement involved presenting students with a problem structure and asking them to pose a *series* of questions. For some problems, students were also asked to put the questions in a suitable order. Figure 6.10 presents some typical student responses. This type of problem-posing situation provided a natural starting point for the teacher to involve students in solving problems which were beyond the level of the initial problem. In addition to written exercises, students were actively involved in discussion about the problems, and were asked to explain what they had suggested and why.

Consider[4] the following infinite sequence of digits: 1234567891011. . . 10011002. . .
Note that it is made by writing the base ten counting numbers in order.
Ask some meaningful questions. Put them in a suitable order.

Student 1:
a) Find the 100 000th digit,
b) Write the same sequence using base (2) and find the 100th digit.

Student 2:
a) What is the 1 000th digit?
b) What is the 150th number which contains the digit 0?
c) What is the total of the digits from 1 to 50?

Figure 6.10. Students' responses to a problem which requires multiple goal statements.

[4] The situation was designed on the basis of a problem from the 1995 "Mathematics Contest" - Junior Level.

The observations suggest that problem-posing situations of this type are likely to help students both to make connections between the information given and possible questions, and link their previous experience with the content which is being learned.

The teacher provided support for the students by asking questions which had general forms like: "How can we finish the problem?", "What kind of questions can we ask using the information given?" or "Can we ask something else?" Sometimes the questions took the form of an open invitation: "Write down all problems you can pose about this situation." In some cases, the boundaries of the situation were extended by asking students to add *new* numerical data or *new* relationships.

2. Problem posing based on a problem structure with missing elements in a combination of the Given, the Obstacles and the Goal. Out-of-school problems arise from situations which often contain incomplete information in more than one element of the problem structure. The first step involved in solving a real-life problem is to give it an initial formulation as a problem.

One aim of giving students experience in finishing the structure of a problem by revealing the missing elements — for example, incomplete numerical information or mathematical relationships — was to help students to focus their attention on possible interrelationships between the elements in the problem structure. Another goal was to present a problem-posing situation which imitates a structure close to a real-life situation which students might encounter out-of-school, and to give students experience in finding ways to approach such problems.

The design strategy used by the researcher to develop problems with unfinished structures was one of omitting a specific numerical fact or a mathematical relationship from a given problem, or by omitting one or more groups of words. By adopting these strategies, it was felt that the range and scope of possible ways in which students might respond would be maximised.

In the project classroom, students were frequently asked questions such as: "What initial solutions could be found?" or "What other information might you need to solve the problem?" The teacher suggested that students should try to extract as much as possible from the facts provided in the given problem. Students were then asked to write down or suggest other problems which might be posed using the given information.

Example 1:

1). The product of $(x + \square)^2$ is:

A) $x^2 + 25$; B) $x^2 - 25$; C) $x^2 + 10x + 25$; D) $x^2 - 10x + 25$.

2). The product of $(x - \square)^2$ is:

A) $x^2 - 25$; B) $x^2 + 25$; C) $x^2 - 10x + 25$; D) $x^2 + 10x + 25$.

3). The product of $(x + 5)(x + \square)$ is:

A) $x^2 + 5x$; B) $x^2 + 2x$; C) $x^2 + 5x + 7$; D) $x^2 + 7x + 10$.

Example 2: Take any .... Write it down twice to make a .... digit number. This number will always have among its factors:

A) 11; B) 101; C) 1001; D) 10001.

Example 3: Which digit has to be into the $\square$ in order the equality $5^{\square^{4^{5}}} = 5$ holds?

A) 2; B) 5; C) 10; D) 0.

Figure 6.11. Examples of multiple-choice questions with unfinished structures.

The first example given in Figure 6.11 shows a set of problem-posing situations in which students were asked to connect the missing element in the problem structure (in the *Given*) with one of the possible answers. The second example required students to find a pattern related to the product of a one, two, three or four digit number with one of the given numbers and the order of the digits in the final product. Example 3 in Figure 6.11 presents a problem in which none of the given answers is correct. In this case, students were asked to suggest appropriate changes in the problem statement so that its solution matches exactly one of the possible answers.

More complex situations, which involve missing elements, were also developed for the project classroom (see Figure 6.12).

Match each of the elements of Group A with exactly one of the elements of Group B and form a meaningful problem:

Group A	Group B
A) *Suppose we have .... natural numbers.*	E) *There are at least............of them born in the same day of the month.*
B) *There are ...... students in my class today.*	F) *It is possible to choose 5 of them whose .... is divisible by 5.*
C) *There are only .... students in my class.*	G) *Show .... number is 10 or more.*
D) *Ten numbers are chosen at random. Their sum is 82.*	H) *There are at least............of them with the same first initial.*

Figure 6.12. A combination of problem-posing activities based on a problem with missing elements in its structure.

These were used mainly for individual work with some students. Students were asked to match the information in Group A and Group B, and then to formulate the goal statement.

3. Problem posing based on a problem structure with surplus information. In school mathematics students very rarely have the opportunity to meet a problem which contains surplus (contradictory or not) information and on its basis to formulate well-structured problems.

The aim of including in the study problem-posing situations which have surplus information in their structures, was to provide students with an opportunity to explore problem structures which relate in more than one way to the question asked. It was envisaged that students would engage in analysing possible relationships between subsets of the data, ask questions about the relevance of the data, and pose well-structured problems which might help to make sense of the data. Students would therefore gain experience in understanding the interrelationships between the different elements which make up the problem.

Problem-posing situations which contain surplus information were created by the teacher in several ways:

- by changing the original question posed and replacing it with one which can be answered with only a part of the given information (situations with surplus information in the *Given* or in the *Obstacles*);
- by adding numerical data or mathematical relationships which do not change the problem structure — that is, the additional data or relationships are *irrelevant* to the problem content; and
- by adding information which is contradictory to the problem structure.

Situations with surplus information were used in the project classroom, for example, when two or more basic theorems had to be "bridged." After proving that the rule $a + b = c + 2r$ holds for any right angled triangle (a, b and c are the lengths of the sides and r is a radius of the inscribed circle), problem-posing situations with surplus information took place naturally. Students were involved in solving problems about finding the radius of the inscribed circle of a right angled triangle when problem-posing situations with surplus information were given. Teaching materials used in the project classroom (see Figure 6.13) illustrate the nature of the problem-posing situations used in this study.

> *Example 1.* Two of the sides of a right angled triangle are 12 *cm* and 5 *cm* and the length of the hypotenuse is an integer number. Find the radius of the inscribed circle.
>
> *Example 2.* One of the sides of a right angled triangle is 5 *cm*, and the other two are integer numbers. Find the radius of the inscribed circle, if the perimeter is 31 *cm*.

Figure 6.13. Examples of teaching materials which include situations with a surplus information.

Example 1 in Figure 6.13 presents a situation with surplus information which provides a broad basis for further explorations. Some of the data in this situation — for example the fact that *the hypotenuse is an integer number* or that *one of the sides is 12 cm* — are *irrelevant* to the problem. If "5 *cm*," however is dropped from the problem, then in this case, the new problem will have more than one solution. In Example 2 (Figure 6.13), the additional information (the perimeter is 31 *cm*) is contradictory to the other data.

Students' problem-posing products were also used as a source of problem posing. On many occasions, this approach was used to promote discussions with individual students or with the whole class.

The episode below presents one of the classroom discussions about an ill-structured problem posed by one of the students. At the time of the discussion the author of the problem was not in the classroom, so students had to explore all possible ways for correcting the problem without being able to ask the author for his opinion.

Episode "Solving a problem posed by Nelly"

T: This is Nelly's problem [reads]. *There are 62 dog houses and in each dog house they speak a different language. I need 92 dogs in each dog house, how many dogs need to come* *to the doghouse to be sure that at least one doghouse has at least 92 dogs in it.* OK, do you understand the problem?

Because nobody answered the teacher's question, it was rephrased.

T: Do you understand the problem is the first question. Is it clear?
Nora: It's a little bit confused.
T: You think that it's a little bit confused, who thinks that it's a little bit confused . . . all of you? OK, what is the confusing part of the problem, I agree, what is the confusing part? . . . What is the confusing part, Nora?
Nora: I just don't get it!
T: You don't get it!
Nora: The ways it's worded, it's just . .
Sara: She's repeating herself!
T: She's repeating herself. How is she repeating herself?
Sara: Because she says at the beginning of the sentence that she needs 92 dogs,
T: In each doghouse . . .
Sara: Yeah, and then she states "at least one doghouse has at least 92 dogs in it," now that is like, talking about the same thing.
T: What would be the proper problem?
Sara: You mean like with those . . .
T: Yeah, just rephrase it quickly. There are sixty . . .
Sara: There are 62 doghouses and in each doghouse the dogs speak a different language... If I need 92 dogs in each doghouse, how many dogs will I need to fill the 62 doghouses?

Sara finished the problem by giving it a simple wording. Gregory also wanted to make some comments:

T: OK, what's your question; how will you finish the problem Gregory?
Gregory: You have to decide which parts she actually wants.
T: Well what do you want to do?
Gregory: If there's 92 dogs in each doghouse, well then it's 92 times 62, that's the finish.
Sara: But that's got nothing to do with the Pigeon hole.
Gregory: Yeah, there are 62 pigeon holes, you try to put 92 dogs in each doghouse, at least. So you need at least 92 times 62 dogs.
Sara: Maybe she wants to say how many dogs will you need so that one doghouse will be full . . .
T: Maybe she wants that . . . This type of problem is an example of not well-structured problems. It's called an ill-structured problem . . . And usually such problems can be corrected in more than one way.

Some of the advanced students were involved in problem-posing activities based on a situation with a multiple-choice question format which required deeper and more precise analysis of the problem structure. An example of a such a problem is presented in Figure 6.14.

Problem: A certain number has exactly eight factors, of which 49 and 55 are two. The number is:
A) $7^2 \times 5 \times 11$ B) $5 \times 7 \times 11$ C) $5^2 \times 7 \times 11$ D) $5 \times 7 \times 11^2$

Figure 6.14. An example of a problem with surplus information.

The following episode presents the individual discussion which took place with one of the students called "Norm" for the purpose of this study. When Norm faced an ill-structured problem for the first time, he immediately pointed out what he believed to be a *mistake*.

Episode "The problem with a mistake"

T: Norm, have you finished?
S: There's a mistake with this problem.
T: Pardon?
S: Number 11. A certain number has exactly 8 factors, and 49 and 55 are two, so this one has 12 factors, see 3 times 2 times 2, 12 factors, this one is 2 times 2 times 2, that's 8, this is 3 time 2 times 2, that's 12, and 2 times 2 times 3, that's 12. So that means this is the one that must be right. But now can we have the factor of 49 with that?
T: I don't know. . .
S: We can get 55, with 7 to the power zero. . .
T: OK, what is the reasoning behind that, if my number has 49 as a factor it should have what?
S: 7 to the power of 2.
T: Yes. . .
S: But look, this has to have 8 factors, exactly 8.
T: Exactly 8.
S: This one[A] *has 12, this one* [C] *has 12, this one* [D] *has 12. . .*
T: And this one has. . .
S: 8. 2 times 2 times 2.
T: Yes. But this can't be our number.
S: Yeah. So none of these, these can't be our number either, because they have 12 factors.
T: Yes, you are right. Can you tell me how the problem should be changed, in order for this [B] to be the right answer?
S: Um. . . we can take away the 49.
T: OK, take it, and instead of 49, what else could there. . .
S: 35.
T: Why is 35 a *good* number?
S: Because you can have five and, see, I mean all of these will have 35, and all of them will have 55.
T: Yes, and if I have here 35 and here 55 everything will be OK?
S: Yeah, and then this one [B] *will be right.*

My next questions were aimed to help Norm to focus his attention on exploring some possible changes in the given structure:

T: Tell me now another two numbers which go. . .
S: 77.
T: OK, 77 and which one?
S: And the one that's 35 and 11; 385.

We went back to the *mistake* in the problem:

T: If I want 49 and 55 to be my numbers, how should I change the answers . . .?
S: If you want them to be 49 and 55?
T: Yeah.

S: But you . . . then you would change the 8 factors . . .You don't want 8 factors you want 12 factors.
T: OK, say 12 factors.
S: Then that's [A] *the answer. . .*

Norm was now ready to accept the fact that there were other ways in which the problem could be changed so that it would be well-structured:

T: There are several possibilities to correct this problem . . . The first one is what?
S: To change 49.
T: To change it to a good number, to 35. OK. Or, . . .
S. Or to change the factors [the number of factors].
T: Or to change. . . this [I underlined both 49 and 55] and this [the number of factors] and then the answer will be?
S: That [B] . . .

The last episode shows that Norm quickly made a connection between the new changes in the problem structure and the set of answers. After the lesson Norm asked the teacher why we were solving such problems. (He was asking about the problem reproduced in Figure 6.14). The teacher replied with a question "What do you think?". He thought for a few seconds before answering: "If I solve a problem, I just will solve it, but if it has a mistake it makes me think." Then the teacher said: "And this is a good reason, I hope."

4. Posing problems on the basis of different interpretations of a mathematical concept. In addition to asking students to suggest problems which involve the use of a specific mathematical concept (a prime number, the least common multiple, etc.) students were encouraged to interpret a mathematical concept (or a given symbol) within a specific situation in different ways and to illustrate their interpretations by posing some examples. It was expected that an involvement of students in constructing mathematical concepts and relationships on the basis of a given situation could assist them in problem solving. Participants in the project classroom were encouraged to give different interpretations, for example, of the elements in given geometrical figures and to suggest examples of specific applications. For instance, after showing that the line segments *AC* and *CB* (see Figure 6.15) have equal length, students were invited to interpret the segment *MN* in different ways (as a tangent, as a side in the triangle *MNC*, and as the sum of two segments). The experience in constructing *links* between the elements of geometrical figures in a way which makes sense to the students was expected to assist pupils when confronted with similar problem-solving situations.

What is the role of the segment *MN* in the following picture?

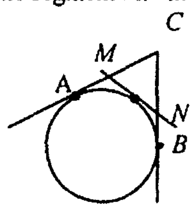


Figure 6.15. Examples from teaching materials used as an instructional prompt for posing problems based on interpreting the segment *MN* from different perspectives.

After this initial interpretation, students were encouraged to help the teacher to pose and solve problems which involve using these perspectives. Through such activities, the teacher helped students to focus their attention on different applications of the concept of tangents that they might meet when solving mathematical problems. Such activities provided a helpful background for connecting the basic problems into a chain and solving much more difficult problems, including problems from national mathematics competitions.

5. Posing problems which have more than one solution. In school mathematics, most of the problems are closed and the problem structure usually implies exactly one solution. This could lead to limiting students' experiences in solving problems which have more than one solution or which do not have a solution at all. At the beginning of this study, some students' reactions indicated that they had the impression that every problem always has a solution which is unique. For example, when the problems shown in Figure 6.16 was solved at the beginning of the study it was quite natural for some students to suggest one or two solutions, but not to determine all solutions.

Substitute the sign " * " with a digit in the number 512 *, so that the number is divisible by:
a) 2;
b) 3;
c) 5.

Figure 6.16. Example of a problem which has more than one solution.

It appeared to be difficult for some students to recognise the difference between the *number* of the solutions and the notion of *all solutions*. By changing 512* to 511* and asking the question for which values of the "*" the number is divisible by 25, students were presented with problems whose solution was "The problem does not have a solution."

The situations used in the study, which required students to explore all possibilities in a problem structure, were created by omitting some mathematical restrictions of the problem statements or by replacing some of the digits in a specific number with a symbol and asking a *suitable* question. In addition to Question 2 from Set 1 of the Mathematics Questions, on several other occasions, students were prompted to find multiple problem solutions by considering changing the problem in order for different cases to be considered. An example of students' work is presented in Figure 6.17 (see also the "Episode with Nelly" in this Chapter).

In the first example Nora increased both the numbers of the boys and girls. Martin put the restriction on the girls. Then the solution had to take into account two cases: when girls are occupying the first and the last two places or the first two places and the last place.

Example 1:
In how many ways could 2 girls and 4 boys stand in a line if the girls insist in occupying the middle two places?

Problem posed by Nora:

In how many ways can 12 boys and 10 girls stay in a line if the girls occupy the middle two places?

Example 2:
In how many ways could 3 girls and 4 boys stand in a line if the boys insist in occupying the first and the last places?

Problem posed by Martin: In how many ways could 3 girls and 4 boys stand in a line if the girls insist in occupying the first and the last places?

Figure 6.17. Problems with more than one solution posed by students.

It was observed during this study that, on many other occasions students naturally "posed" problems which do not have a unique solution. When a problem with missing elements in its structure was posed, the teacher drew students' attention to the fact that more than one case needed to be considered.

By designing problem-posing situations based on problems which do not have a unique solution or indeed any solution, students were given an opportunity to gain experience which resembled every-day life situations more closely. In real-life contexts most problem-posing situations can be resolved in more than one way and one has to choose the *optimal* solution.

B. Problem-posing Situations Based on a Specific Solution Structure

For professional mathematicians, the development of new techniques of investigations, and of comparing and analysing the elegance of particular solution approaches, are tasks which are equally as important as formulating or solving problems.

In the project classroom, students were encouraged to perform activities which were similar to those in which professional scientists are involved. For example, students were engaged in: (a) formulating different solution approaches and comparing the elegance of their solution ideas; (b) investigating the precision of particular solution structures; (c) constructing problems with interrelated solution structures; and (d) writing precise solutions.

At the beginning of the study it was observed that most students encountered difficulty when a precise solution was needed. The meaning of the words "precise solution" was not clear to some students, and they asked for some explanations and examples. This observation lead to the decision that special attention should be paid to the range of ways in which the problem solution could be presented. The problem-posing activities which were based on a particular solution ranged from improving the language and the logic of the presentation to designing assessment schemes and applying them for marking complete as well as incomplete solutions. The aim of involving students in posing problems based on a particular solution structure, therefore, was to help them to improve their written mathematical performance.

The problem-posing situations presented to the students comprised solutions with insufficient and surplus information. On many occasions students were invited to write their solutions on folios which provided a natural starting point. In the example given in Figure 6.18 students were presented with five different solution approaches to a given problem and asked to suggest changes which would make the solutions clearer and more precise. The solutions were selected from those written by a group of different students during small-group works several weeks earlier.

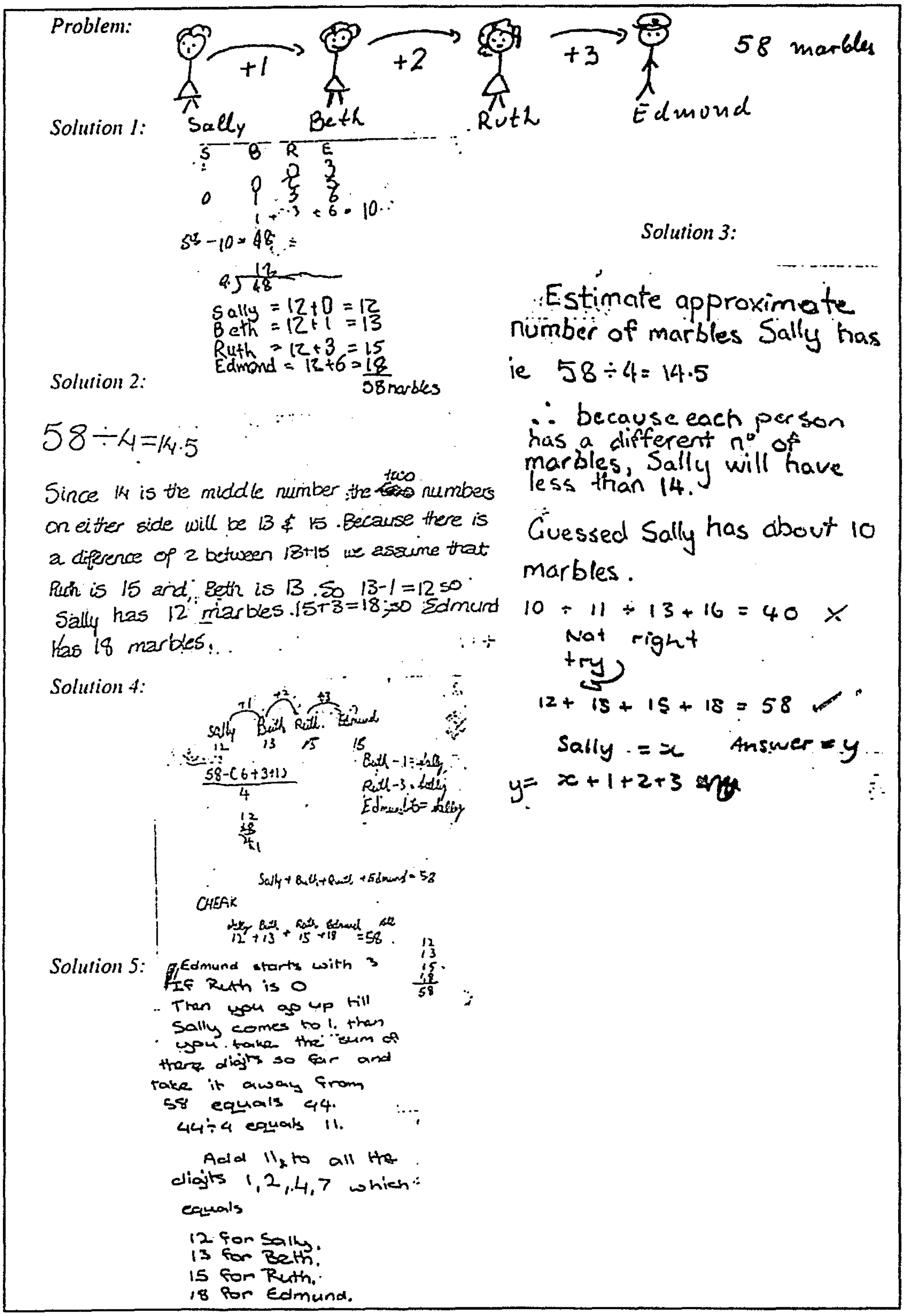


Figure 6.18. Teaching data used for helping students to improve the characteristics of a written solution to a given problem.

As has been already been stated, the introduction of most of the problem-posing situations needed to be done in a sensitive manner so no students felt they were being criticised. It was difficult for some of the students to feel that they could share their work with the whole class when they knew that their written explanations were not precise enough. When Tom's mother, a professor of mathematics, attended one of the sessions, she mentioned with surprise: "I did not expect to see my son feeling free to make mistakes and to make comments on them."

Structured Problem-posing Situations

The structured problem-posing situations which were used in the project classroom are presented in Figure 6.19. They are divided into two sub-categories: (a) problem-posing situations based on a specific problem; and (b) problem-posing situations based on a specific solution. Each of these sub-categories will be discussed in the section which follows, and examples provided to illustrate the key features of each.

A. Problem-posing Situations Based on a Specific Problem

1. Posing problems by varying the mathematical vocabulary of a problem. Research has shown that a lack of understanding of specific mathematical vocabulary can affect a student's ability to solve a particular problem (Adelula, 1990; Ellerton, 1988; Mousley & Marks, 1991). It was anticipated that in many cases it would be quite unlikely for a student to guess the meaning of specific notation or of terms used in mathematics. Furthermore, as in normal language usage, some terms in mathematical language have synonyms. Students needed to be familiar with mathematical language used in school mathematics textbooks in order to understand the mathematical problems they encounter.

A. Problem-posing situations based on a specific problem:

Posing problems by varying the mathematical vocabulary of a problem;
Problem posing by presenting a specific problem in students' own words without changing the nature of the problem;
Posing problems by varying the semantic structure of a problem;
Posing multiple goal statements on the basis of a well-structured problem;
Posing problem chains-problem series, problem fields and problem cycles;
Posing problems which are variations of a given problem;
Presenting a problem statement "briefly."

B. Problem-posing situations based on a specific solution:

Formulating the main solution idea;
Restating a problem on the basis of its solution;
Posing problems with unrealistic solutions;
Problem posing established on the basis of a problem with several solution approaches;
Posing sets of problems which might have a common solution approach;
Posing sets of problems which resemble a given problem but have different solution approaches.

Figure 6.19. List of structured problem-posing situations used in the study.

Problem-posing situations of this type were introduced into the project classroom by inviting students to replace a word(s) with another word or group of words *without changing* the *mathematical meaning of the problem.* The example in Figure 6.20 illustrates a typical exercise used by the teacher to help some students become familiar with expressions which they were likely to meet when solving problems involving the use of the Pigeon-hole Principle.

Say differently without changing the meaning of the following groups of words:

more than 7;	less than 6;
7 or more;	0, 1, 2 or 3;
4 or more;	3 or less;.
at least 4;	at most 5;
minimum of 2;	maximum of 5;
not less than 3;	not more than 5.

Figure 6.20. Examples of teaching materials for helping students to extend their mathematical vocabulary when solving problems involving the use of the Pigeon-hole Principle.

The example in Figure 6.21 illustrates teaching material which was used to assist students to extend their vocabulary when they solved problems which involved the concepts of prime and composite numbers.

Which of the expressions have the same meaning?
a) 6 is divisible by 2;
b) 2 divides 6;
c) 6 can be divided by 2;
d) 2 goes 3 times in 6;
e) 6 is three times greater than 2;
f) the factors of 6 are 1, 2, 3 and 6.

Figure 6.21. Example of teaching materials for helping students to extend their mathematical vocabulary.

Similar exercises were also used in individual discussions with those students who experienced difficulty using or understanding appropriate vocabulary. In addition, students were encouraged to suggest problems which incorporated the mathematical vocabulary that they preferred to use.

The example shown in Figure 6.22 illustrates another classroom application for posing problems which involve changing the mathematical vocabulary of the problem. Before solving a problem, students' attention was focused on the ways in which a specific word(s) can be replaced with one or more words without changing the nature of the problem (see Figure 6.22).

In the problem given below replace the underlined group of words with a another word without changing the nature of the problem.

Which of the following numbers is <u>midway between</u> $^{1}/_{5}$ and $^{13}/_{25}$?
A) $^{17}/_{27}$; B) $^{7}/_{15}$; C) $^{3}/_{5}$; D) $^{9}/_{25}$; E) $^{8}/_{25}$.

Figure 6.22. Example of a problem-posing activity which involves a change in mathematical vocabulary of a problem.

In this case, students suggested that the underlined words be replaced with phrases such "half-way between," and "in the middle between." One of the students (Martin) suggested the mathematical term "the average of."

Another approach used to prompt students to use specific mathematical vocabulary was that of presenting students with a series of problem structures which incorporated missing elements in the *Given*, the *Obstacles* and/or the *Goal*, and to ask them to finish the problem formulation. The instructional goal was to help students to become more familiar with the nature of specific mathematical vocabulary used in a new topic. At the same time, students had to pay attention to specific features of the problem structure.

Two episodes which show Irene and Samantha posing problems involving the use of the Pigeon-hole principle illustrate the key role of language in understanding the problem. In the episode with Irene, she was asked to pose a problem similar to the problem posed by Martin which had just been solved: "What is the minimum number of people needed so that at least two of them are likely to have the same first initial?"

Episode with Irene

T: Can you make a problem similar to that, Irene? Did you get the idea?
I: Kind of.
T: Give me a problem similar to that?
I: Um 15 people . . .
[Irene stopped. As Martin she also had difficulties with constructing relationships in a suitable context. I decided to help her with part of the information she needed.]
T: OK 15 people, and how many apples?
I: Apples, 16.
T: 16 apples what can you claim?
I: That one person will have at least 2 apples. At least one person will have at least 2 apples.
T: At least one person will have 2 apples. Why did you say at least one person?
I: I was just saying it the same as the one on the board.

The episode shows that, although Irene was using "appropriate" language (in the sense that she was imitating the language used in the conclusion of the problem written on the board), she did not understand the meaning of the words *at least*. But Irene was able to *explain* the solution of the problem correctly. In the same lesson, Samantha was presented with a more open structure and was asked to go further. She posed the problem confidently:

Episode with Samantha

T: Now, say we have 30 days in a month. How will I go further? Samantha, can you guess what I am going to say?
S: Yeah.
T: What I'm going to say?
S: If there are 30 days in a certain month, and if you have 31 people,
T: Uh-huh, and if we have one more, 31 people, what can we claim?
S: Then you can guarantee that there'll be at least 2 people born on the same day.

In the project classroom students were also given an opportunity to express the meaning of selected mathematical vocabulary — concepts and algorithms — in their own words. For example, they were presented with a picture, specific numbers, and a question related to the picture shown in Figure 6.23.

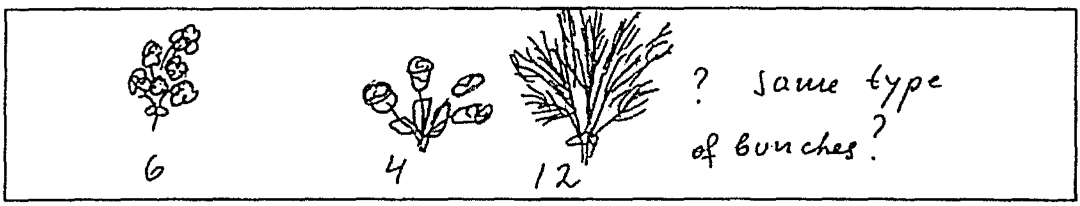


Figure 6.23. Example of a problem-posing situation which involves the use of a specific mathematical concept.

In the example shown in Figure 6.23, students were invited to interpret the picture and the meaning of "*same type of bunches*" by using appropriate mathematical language. The conversation between the teacher and one of the students, Nelly, is presented below:

Episode with Nelly

T: Nelly, could you tell me what kind of a problem I have written here? . . . These are flowers!

N: 6 of that first type, 4 in the second bunch.
T: Oh, it's not a bunch, this is another type of flower.
N: 12 from the 3rd type.
T: And what do I have to do?
N: Oh, from those flowers, how many bunches can you make which have the same number of flowers from each type.

In this case, Nelly interpreted the meaning of the *same type of bunches* as *the same number of flowers from each type* and solved the problem by using the concept of the *highest common factor* (HCF), but her problem formulation did not require finding the *biggest* number of bunches. Because of that, the teacher then asked how could students solve the problem if they did not know the concept of HCF, and Nelly was able to suggest an appropriate algorithm.

The independent observer's notes, presented below, summarise this part of the lesson.

> HCF [*the highest common factor*] from prepared overhead. Bunches of flowers problem [made clear now by changing wording]—asked students to describe/explain what the question meant—Nelly explained it well, then Elena asked Carol to repeat the question in her own words, she could do that well too, and all students seemed to understand the question and helped give the answers. Then the students helped to create another question. Carol answered it [using HCF theory]. Elena asked how they would have solved the question if they didn't know about HCF—students were stumped for a while, but Nelly managed to explain [really made them think about what HCF meant, rather than just knowing it was a HCF problem and solving it without thinking about the meaning].

The episodes also indicated that even when students were able to present precise solutions, they did not always have an understanding about the salient features of the mathematical vocabulary. By encouraging them to interpret mathematical concepts in their own words, the teacher tried to help students increase the personal meaning of these concepts.

In order to assist students to focus their attention on the vocabulary of a problem and to reflect on problem-posing situations from a specific perspective, the teacher used different approaches for prompting problem-posing actions. For example, students were asked questions such as: "How can I say . . . differently?", "What do you understand by . . .?", "What do you think is the meaning of . . .?" or the more general question "What is the problem about?" In other cases problems with a *fill-in* format were designed or the *key* words were underlined.

During the preparation for the sessions the researcher also made a careful analysis of the vocabulary of the problems and tried to "predict" the types of difficulties the students might encounter. In many cases suitable "preventing" problem-posing activities (such as those shown in Figures 6.20 and 6.21) were designed for individual work with students.

2. Problem posing by presenting a specific problem in students' own words without changing the nature of the problem. In addition to presenting problem-posing situations in which students were asked to make changes to the mathematical vocabulary used in specific problems, students were also asked to present entire problem statements in their own words *without changing the nature of the problem.* The aim of asking students to reformulate a problem in this way was to involve students in activities which might help them to perceive a range of characteristics of a problem such as: the language, the mathematical content, the *Given*, the *Goal* and the *Obstacles*. The following example is provided to illustrate how teacher's and students' problem-posing activities were linked in one of the instructional sessions. The aim was to help students to revisit the algorithm for finding the Least Common Multiple of two numbers.

Activity 1. Students were presented with a definition of the Least Common Multiple [LCM]of two integers.

Activity 2. Students were invited to solve a simple basic problem: *Find the least common multiple of 360 and 240.*

Activity 3. The problem was reformulated by a student:
Which is the smallest integer divisible by 240 and 360?

Activity 4. The problem was reformulated by the teacher and a student:
Construct the smallest number divisible by 240 and 360.

Activity 5. The teacher changed the nature of the problem:
Pose a number divisible by both 240 and 360.

Activity 6: A student formulated a sub-problem:
What is the prime decomposition of 240 and 360?

Activity 7. A student reformulated the problem using mathematical vocabulary which "suggests" a solution approach:
What are the prime factors of 240 and 360?

Activity 8: Students applied the algorithm for finding the prime decomposition of a number.

Activity 9. Individual discussions with students. Students were encouraged to formulate a step of the algorithm as a simple problem:
Find the smallest number divisible by:
a) 2^4 and 2^3;
b) 3 and 3^2;

Activity 10. Individual discussion with a student who suggested the next step of the algorithm:
A number divisible by 2^4, 3^2 and 5 needed to be constructed.

Activity 11. The teacher invited students to explain why the number $2^4 \times 3^2 \times 5$ is indeed the smallest number divisible by 240 and 360.

Activity 12. Students were invited to formulate a procedure for finding the LCM of two numbers.

Activity 13. Students were asked to take an example and check the algorithm. A student suggested a simple problem: *Find the LCM of 48 and 36.*

Activity 14. Students were asked to predict whether the algorithm could be generalised for more than two numbers. Then they are invited to suggest a problem and to check the prediction made: *Find the LCM of 240, 360 and 48.*

Activity 15. The teacher constructed a real-life problem and invites students to suggest solutions:
Betty, David and Rebecca attend maths classes at the Uni. Betty attends every second lesson, David- every fourth and Rebecca - every sixth. If they are in the class today, in how many days' time they will be in the class together again?

Activity 16. The teacher invited students to help her construct a more complicated real-life situation which solution involves an use of the LCM.
Today three ships leave Perth harbour for Darwin. The first ship can make the trip Perth - Darwin in 2 days, the second in 4 days and the third in 6 days. Assume continuous round trips. If the ships leave Perth today in how many days time will they next leave Perth together again?

Activity 17. Students were invited to solve the problem on an individual basis.

Activity 18. The teacher challenged the students by suggesting a more difficult problem. She added a new concept - *remainder* - and demonstrated how a problem which involved the new concept could be created:
What is the smallest number, which if it is divided by 2, 3 and 4 gives a remainder of one?

Activity 19. Students solved the problem. Then the teacher invited them to reformulate the problem in way which fit a calculation used in the solution.
If 1 is subtracted from a number, the remaining number is divisible by 2, 3 and 4. What is the smallest number with that property?

Activity 20. The teacher invited students to solve a problem from a mathematics competition in Bulgaria:
At a parade, the general wanted his soldiers to go in front of the Queen in lines of equal groups. He tried to make groups of 12, 11, 10, 9, . . . , 2, but always one soldier was left. At the end they had to go one after another. Find out what the smallest number of the soldiers could be.

Activity 21. Students were asked to solve questions which involve the concept of LCM, and to pose a problem whose solution will involve LCM in its solution (see Worksheet 10, Appendix 4).

This session continued with individual student work, and a classroom discussion on the features of the solution ideas in Worksheet 10 (Appendix 4), and on problems posed by students.

The sequence of problem-posing activities described is just one of the *models* of interaction between problem posing and problem solving which were used in the Program[5] for introducing a new concept by presenting the definition to the students and involving them in various problem-posing and problem-solving activities.

In addition to asking students to present the problem statement in their own words, the teacher frequently asked questions such as: "Could you explain what the problem is about?", "How do you understand the problem?", "What is your own version of the given problem?". More specific question such as "What do we know and what has to be found?" were also asked.

In other cases, the problems posed by students were presented to the class, analysed and reformulated if this was necessary (see Chapter 9).

[5] see Chapter 10 for more details.

3. Posing problems by varying the semantic structure of a problem. Posing problems with the same mathematical content as a given problem, but which utilises different semantic structures, was another problem-posing situation used in the project classroom. At the beginning of the study the teacher used problems which have different semantic structures for *repeating* the problem statement in ways which would give the students a hint for formulating a solution idea.

> *Example 1:*
> *A.* One pencil and 1 rubber cost $3.00. The difference between the price of 2 pencils and the price of 1 rubber is $1. What is the price of 1 pencil and 1 rubber?
>
> *B.* One pencil and 1 rubber cost $3.00. A rubber cost a $1 less than the price of 2 pencils. What is the price of 1 pencil and 1 rubber?
>
> *Example 2:*
> *A.* Amanda and Greg have altogether 300 cents. If Amanda's money is doubled, she will have 100 cents more than the pocket money Greg has. How many cents do each of them have?
>
> *B.* Amanda and Greg have altogether 300 cents. Greg needs a dollar to have twice as much money as Amanda has. How many cents do each of them have?

Figure 6.24. Teaching materials used to help students identify the differences between problems with the same mathematical model.

Students were also asked to make changes to the problem so that the mathematics involved did not change in nature, and so that the mathematical relationships would be presented in a way which would make the problem easier to understand. This activity was used mainly when some students had difficulties understanding a particular problem. Students were then asked to rephrase the problem, to interpret the mathematical relationships differently and to explain what the problem was about. Thus this type of problem-posing activity was aimed at extending students' experience in identifying similarities and differences between two isomorphic problem structures. The examples presented in Figure 6.24 illustrate teaching materials used by the researcher to illustrate two sets of problems which adopt similar semantic structures for the two respective questions.

Additional teaching materials, used to help Karel to pose problems with the same semantic structures are presented in Chapter 9.

On many occasions during the study, the teacher asked students to suggest a suitable context for word problems which were based on equivalent forms of linear or simultaneous equations. Observations in the project classroom suggest that activities of this type are likely to help students to distinguish between isomorphic and similar problem structures. When Nora was asked whether she understood the problem she had just posed, she answered: "I cannot pose a problem if I do not understand what I am doing."

4. Posing multiple goal statements on the basis of a well-structured problem. Another application of problem posing which was used in the project classroom included posing *additional* questions which must be associated with a given well-structured problem. Thus this problem-posing situation involved formulating other *Goal* statements.

For example, after solving the problem presented in a multiple-choice format (see Worksheet 20, Problem 6 in Appendix 4), the teacher involved students in the construction process of the following "triangular" arrangement:

1
2 3 4
5 6 7 8 9
....

Then students were asked to find out possible ways for describing the pattern and to state additional questions about the given situation. Some episodes of the classroom discussion are presented below.

Episode with Martin, Sara and Carol

T: What other questions could we ask about this triangle?
Martin: What is the 4th number on the left . . .
T: What is the 4th number from the left, on which line?
Martin: 21st line.
T: Other questions?
Sara: What is the 5th number in the 10,000th row.
T: What is the 5th number in the 10,000th row? [Sara's question was not precise, but nobody noticed that]. I asked Carol to explain the solution.
Carol: 9,999 squared plus 5.
T: Yes, but when you speak about the 5th number, you said plus 5, you meant which number, from this side or from this side? [Sara anwered quickly to the question I was asking Carol].
Sara: From the left

The teacher then invited the students to ask other types of questions. Irene, Nora and Tom suggested the different goal statements.

Episode with Irene, Nora and Tom

Irene: How many digits would there be in the pyramid if there were 152 rows?
Nora: How many numbers would there be if there are 110 rows?
Tom: How many numbers are there on the 60th row.

These questions were in fact, similar to the types of problems which the researcher had prepared in advance, on worksheets. Students were then given worksheets and asked to solve the problem, and also if they could write down other questions. The additional types of questions posed by Gina are illustrated in Figure 6.25. In the second example Gina had added additional structure — namely, "every second number is turned into a negative."

Situations of this type require a careful analysis of "boundaries" of the basic problem. The problem should allow students to connect their previous experience with the mathematical content of the problem being solved. It is difficult to generalise how one might design situations from this type which are educationally rich. It seems clear, however, that every problem, to some extent, would allow at least some of the students to make up meaningful questions which somehow relate to the structure of the given problem.

Example 1: What is the sum of all numbers in the 101th row?

Example 2: What is the sum in the 9th row if every second number is turned into a negative?

Figure 6.25. Extending a problem structure by posing multiple goal statements.

5. Posing problem chains — problem series, problem fields and problem cycles. In the study, solving a particular problem was not an isolated activity. The researcher tried to involve students in various activities which had been developed in order to help students to connect the solution of the problem which has been just solved, for example, with previously solved problems.

In this study the term *problem chain* will be used to describe a sequence of problems. Two or more problems will be referred to as belonging to the same chain if they are somehow connected. Three sub-categories of problem chains were applied in the study: problem *series*, problem *fields* and problem *cycles*.

A. Problem series with gradual transformation from concrete to abstract. During the study students were involved in posing problem series by gradual transformation from concrete to abstract. Some of these problem-posing activities involved the use of a set of similar problems which had been placed in order, according to their level of difficulty. The aim was to prompt students to make meaningful generalisations by exposing them to problems in the series which have increased levels of difficulty.

Two strategies for posing problems which involve a gradual transformation from concrete to abstract were applied in the study. The first one was based on posing a sequence of problems which have the same algebraic presentation, and inviting students to extend the sequence in order to enable them to predict the generalisation. The examples presented in Figure 6.26 were designed to assist students to grasp the features of the sequence, to continue it by posing some specific examples, and to generalise the idea and express it by using mathematical symbols.

$(x+1)^2 =$	$(2x+1)^2 =$	$(2x+3y)^2 =$
$(x+2)^2 =$	$(2y+2)^2 =$	$(^1/_2x + {}^1/_3)^2 =$
$(y+2)^2 =$	$(3y + {}^1/_2)^2 =$	$(3x + {}^1/_2y)^2 =$
$(y + {}^1/_2)^2 =$	$(x+y)^2 =$	$(2x^2 + 5z + 1)^2 =$

Figure 6.26. Teaching material designed to present students with a sequence of problems with the same algebraic structure.

The observations conducted in the project classroom suggest that, when students are involved in posing examples in which specific elements in a problem structure are varied, some students can grasp the common elements in the problem structure and can express these elements in terms of mathematical symbols.

As the students worked through examples such as those presented in Figure 6.26, the teacher prompted students to reflect by suggesting problems with the same structure which lead to an abstract generalisation. Students were asked questions such as: "Are there any common elements between the problem statements?", or "Tell me another problem which relates closely to the given problems," or "How can I continue this sequence?".

Suggest some problems on the basis of the rule $\square^2 - \bigcirc^2$:

Student 1: $3^2 - x^2$;

Student 2: $y^2 - 2x^2$;

Student 3: $5^2 - 7^2$;

Student 4: $2x^2 - 3^2$.

Figure 6.27. A sample of a problem series posed by students on the basis of a given rule.

It is important to note that not every unfinished problem structure provides a good basis for constructing problem fields. In his work, Zimmerman (1991) described the main characteristics of problems which he considered suitable for constructing problem fields, and these were taken into account in this study.

The *inverse* activity, in which students were asked to pose *specific* examples by using a *general* rule, was used by the teacher to gain an insight into the types of mistakes students make. Problems posed by students were then used as a starting point for discussion with the whole class (Figure 6.27).

Another variation of this approach, which appeared to be very difficult for most of the students, was that of presenting students with a general principle and asking them to suggest problems whose solution ideas might match. For example, the following rule was presented in the *Euler Student Notes* (1995):

> If one operation can be done in n different ways and if in every case a second operation can be done in m different ways, the two operations can be performed in succession in $n \times m$ ways. (p. 50)

However, none of the students was able to suggest a problem which illustrated this general rule. In fact, this was the only problem-posing situation used in the study which seemed to be inappropriate, and can probably be attributed to the fact that students had had no experience in solving problems using the Multiplication Principle.

A different series of interconnected geometry problems was described by Sharigin (1990). Although the structures of the problems in the series are not similar, they are connected and play a basic role in the solution of other problems. In the study, this type of problem-posing situations was used mainly for focusing students' attention on constructing examples similar to the *basic* problems from a specific domain or on posing problems similar to the ones which were likely to be *important* for future work.

B. Posing problem fields. Problem fields are chains of problems in which the problems are connected in some way (Pehkonen, 1992, p. 6). Pehkonen mentioned several strategies used by prospective teachers to create a specific problem field. For example, specific problem fields can be created: (a) by changing the mathematical operation; (b) by placing algebraic expression in some places; (c) by changing the required sum to a different value; (d) by changing the dimension from two to three; or (e) by changing the final question. In other words, the problems in a specific problem field can be obtained by a systematic variation of the elements in the structure of a particular question.

In this study, students were prompted to pose problem fields by asking them to construct problems based on specific unfinished problem structures (semi-structured situations). For example, Figure 6.28 describes a problem in which students were invited to find particular elements, and to pose additional questions.

Consider the sequence 1, 2, 3, 4, 5, . . . N. If $N = 200$, how many digits have been used? Other questions?

Additional questions posed by students:

Student 1: Which digit is on the 147th place?

Student 2: If the last number is 999, how many 3's in total have been used?

Student 3: If the last number is 200 how many prime numbers are there?

Student 4: If the last number is 250 how many numbers of this sequence are divisible by 2, 3, and 4 but are not divisible by 5?

Figure 6.28. Problem fields posed by students on the basis of an open structure.

*C. **Problem cycles.*** Problem sequences, called by Dorofeev (1983) ***problem cycles***, are those in which every problem is a sub-goal in the solution path of a larger (the goal) problem. Problem cycles are usually presented in mathematics literature as a problem with several sub-problems. Problem cycles have been successfully used in examination papers for specialised secondary schools and tertiary institutions in Bulgaria and Russia.

An easy way to pose problem cycles is to look back at the goal-problem and invite students to determine the main steps of the solution idea. Each step is then formulated as an independent problem.

In addition to formulating the main solution idea, students in the project classroom were also asked to formulate the main steps in a given algorithm (or solution approach) and to express these in concise terms. The activity was introduced to the students by presenting the structure of the solution of a particular problem by a diagram. Later in the Program, solutions which need other formats - for example, written or a combination of written and a diagram — were introduced by the teacher and analysed by students.

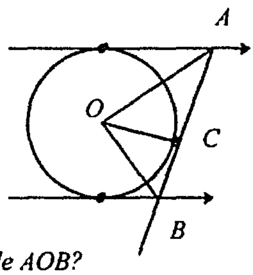
The aim of using problem cycles was to provide structured support for students as they tackle more difficult problems. In this study, when the teacher expected a problem to be difficult for most of the students, she gave them part of the problem structure and invited them to suggest meaningful questions (see Figure 6.29). In this way, the problem "Show that the triangles *ABO* and *OBC* are similar" (see the first diagram in Figure 6.29) became an easy one for most of the students when it was posed later in the session.

Two parallel lines are tangents to a circle with centre O, and a third line, also tangent to the circle, meets the two parallel lines at A and B. Draw the diagram.

Ask some meaningful questions:

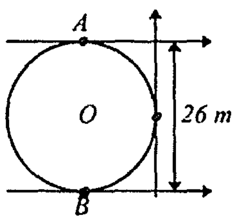
Students' responses:

Student 1:



What is the angle AOB?

Student 2:



If the line segment AB is 26 m long, what is the circle's area?

Figure 6.29. Problems posed by students which were used as a starting point for solving a difficult problem.

Making a chain of interrelated simple problems which might help the students acquire the mathematical skills needed for solving the goal problem was also used in the project classroom. After the students had attempted to solve a problem which most of them had found difficult, the teacher invited them to identify the main solution steps and encouraged students to formulate these as independent problems. It was anticipated that, by giving pupils the freedom to ask questions about a specific situation, and to order these questions logically, this would help them to understand the interrelationships between the structure of the given problem and specific "sub-problems."

6. Posing a series of word problems on the basis of equation, inequality or system of simultaneous equations. Problem-posing activities which involved students in exploring the relationships between a problem and solution structures were introduced sensitively and gradually in the project classroom, as already explained in general terms in Chapter 4.

Teaching students to solve word problems is a compulsory element of school curricula in many countries. The *inverse* activity, posing classes of word problems based on the same mathematical model is not a common practice in mathematics classrooms.

The instructional goal of this problem-posing activity was to extend students' experience in transforming (connecting) an abstract mathematical relationship to a range of real-life situations within different contexts. It was anticipated that, through the use of such problem-posing situations, students would be able to attach personal meaning to the connections between mathematical methods and their applications.

In the study this problem-posing activity took place in various ways. In some cases, after solving a problem, the teacher changed the mathematical model and invited students to suggest a suitable wording. In other cases, students were given a diagram and were asked what the problem might be about. During the study, on a regular basis, participants in the project classroom also were asked to suggest a problem which might corresponded to a given set of data (see Figure 6.36 and also Chapter 9).

7. Posing problems which are variations of a given problem. The posing of problems which are variations of a specific problem plays a significant role in the work of professional scientists. These are problems obtained from a particular problem by varying some of the information given. In other words, a particular problem might have one or more variations and the problems posed might be similar to the given problem, but the conjectures made are not necessarily true.

It was anticipated that involving students in posing problem variations of a given problem would help them to learn to identify the similarities and differences between two problem structures. The two problems shown in Figure 6.30 represent one of the first types of inverse problems used in the study.

Problem 1: A total of 675 digits was used for numbering the pages of a book. How many pages did the book contain?

Problem 2: A book contains 268 pages. A total of how many digits was used for numbering the pages of the book?

Figure 6.30. Example of inverse problems used in the study.

A natural way for posing problems in the project classroom which related in some way to a given problem, was after a new formula had been introduced. Then, by varying the set of given elements and the goal statement, students posed and solved a class of interrelated problems. The examples in Figure 6.31 illustrate some algebra problems in which students had to pose a range of questions using the two basic formulae for an arithmetic sequence: (a) for the sum of the first n terms of a sequence: $S_n = 0.5\,(2t_1 + n \times d)(n + 1)$, and (b) for the nth term $t_n = t_1 + (n - 1)d$.

a) In which place will the number 99 be in the arithmetic sequence: 3, 6, 9, . . . ?

b) In the arithmetic sequence in which $t_9 = 96$ and $t_{10} = 99$, what is the first term?

c) What is the sum of the first five terms in the arithmetic sequence in which $t_1 = 9$ and $t_5 = 17$?

Figure 6.31. Problems posed by students which solutions might involve the use of two interrelated formulae.

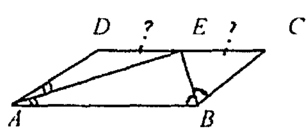
At the beginning of the study when students were faced with constructing a problem which was the inverse of a given problem, it was difficult for some students to identify differences in the problem structures. For example, one of the Year 9 students (we will name him Peter) recognised that he was not able to see any difference in the problem statements in the problems presented in Figure 6.32 "because the solution ideas were the same."

> *Problem 1:* Let *AB* be a chord of a circle with centre *O*, and *M* be a point on the chord *AB*. If *M* bisects *AB* then *ON* is a perpendicular to *AB*.
>
> *Problem 2:* Let *AB* be a chord of a circle with centre *O*, and *M* be a point on the chord *AB*. If *OM* is perpendicular to *AB* then *M* bisects *AB*.

Figure 6.32. Examples of inverse problems which appeared to be difficult for one of the students to recognise as different problems.

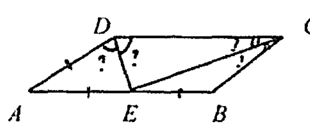
In the example provided in Figure 6.33, students solved Problem 1 and formulated the inverse one (Problem 2), which was in fact one of the *Challenge Problems.* The results showed that after such preparation, 50 percent of the students in Group A were able to submit correct solutions to Problem 2. In contrast, in Group B, where students were not involved in formulating the inverse problem, only 35 percent of students submitted correct solutions to Problem 2.

Problem 1.



Given	Show that:
ABCD - a parallelogram	*E* is a midpoint of *DC*
BE - bisector of $\angle B$	
AE - bisector of $\angle A$	
$E \in CD$	

Problem 2:



Given	Show that:
ABCD - a parallelogram	*DE* bisects $\angle D$
$DC = 2\,AD$	CE bisects $\angle C$
E - midpoint of *AB*	

Figure 6.33. Examples of inverse problems solved and discussed with students.

Later in the study, the teacher introduced how both problems could be written as one by using the term "if and only if," after the solution of *Challenge Problem 11* (Problem 2) had been submitted. The other reason for delaying the introduction of this term was the expectation that an earlier introduction might confuse most of the Year 8 students.

8. Presenting a problem statement "briefly." A problem-posing activity, which also relates to identifying key features in the problem structure, is that of constructing a "brief" representation of a given problem. When presented with this problem-posing activity, students were involved in separating the numerical information, the relationships and the goal from the problem statement and presenting them using suitable mathematical symbols.

Example 1:

Given:	*Goal:*
2 skirts (ice cream);	How many combinations?
3 blouses (lollies)	
4 pairs of shoes (cups)	Answer: 120

a) Why is 120 the right answer?
b) What might be the meaning of the words in the brackets?

Example 2:

Given:	*Goal:*
2 boys	How many groups can be made
3 girls	if there are 1 boy, 1 girl and 1 teacher
4 teachers	in a group?

Figure 6.34. Problems with "brief" presentations of their structures.

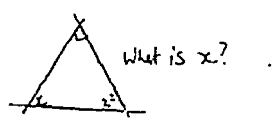
The aim was to focus students' attention on the key elements of the problem — mathematical concepts and relationships — and to ask the students to define the problem statement in terms of the elements: *Given, Obstacles and Goal.* At the beginning of the study the teacher "translated" students' explanations (about the

Given, the *Obstacles* and the *Goal*) on the board by writing the elements in the problem structure "briefly." For example, in Figure 6.34 the teaching material used for introducing problems on combinations is presented.

Later, some geometry problem statements were also presented to the students with a "brief" structure and students were asked to determine the *Given*, the *Obstacles* and the *Goal*.

When the teacher was confident that students had gained the skills necessary to enable them to summarise problem statements effectively, they were asked to write problem statements in brief form when a new problem was verbally presented. Examples of some "brief" versions of problems posed by students are presented in Figure 6.35. Example 1 was posed by Norm without specific prompting. The second example shows a problem posed by Norm when he was prompted to incorporate a triangle in the question.

Example 1:



Example 2:
Make a problem on the basis of [*a triangle*]

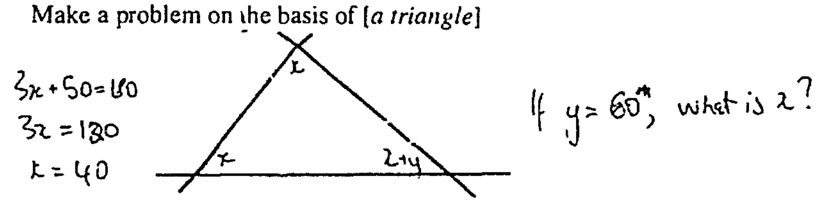


Figure 6.35. Problems (from the domain of geometry) which were posed by Norm and presented "briefly."

On many occasions, for example when some students expressed difficulties in understanding a problem, the teacher asked them to read the problem statement sentence by sentence and to explain what was *Given* and what was the *Goal*. The

B. Problem-posing Situations Based on a Specific Solution

The basic purpose of the problem-posing activities based on a specific solution (see Figure 6.37) was to help students to form generalised perceptions about the structure of the mathematical methods incorporated in the Program and the characteristics of different possible approaches used for solving a particular problem.

> Formulating the main solution idea;
> Restating a problem on the basis of its solution;
> Posing problems with unrealistic solutions;
> Problem posing established on the basis of a problem with several solution approaches;
> Posing sets of problems which might have a common solution approach;
> Posing sets of problems which resemble a given problem but have different solution approaches.

Figure 6.37. List of problem-posing situations aimed at assisting students to understand the problem-solving approaches and mathematical methods used in the Program.

The activities aimed at prompting pupils to stand back from a specific solution approach and to analyse its features, to consider its basic applications, and possible limitations and extensions. Through the teacher's questions, students focused their attention on the main steps involved in a particular approach, and identified the basic skills needed in order to apply a mathematical method properly. Students were also asked to give examples of situations in which the approach could or could not be applied, and to justify their predictions.

Problem-posing activities which were based on a specific written solution were designed to help students to grasp the structure of a particular solution and the mathematical approach used. The classroom and individual discussions were focused at: (a) formulating the main solution idea; (b) restating a problem on the basis of its solution; (c) posing problems with unrealistic solutions; (d) posing problems which can be solved by using several solution approaches; (e) posing sets of problems which might have a common solution approach; and (f) posing sets of problems which resemble a given problems but have different solution approaches.

1. Formulating the main solution idea. As a way of looking back at the solution, students were asked to formulate the main solution idea in their own words (see Worksheet 19, Appendix 4). The aim in this case, was to help students to connect their approach to the solution of a particular problem with their previous problem-posing experience (see Chapter 9 for more examples). Students were presented with a question which had a simple statement formulation (see Problem 1 in Figure 6. 38). After solving Problem 1, it was not difficult for one of the participants in the classroom to suggest that the same solution idea could be applied to a problem which involved the use of a quadratic equation (Problem 2, Figure 6.38). In fact Problem 1 was a step of the solution to Problem 2. At the same time, this problem provided the basis for solving a whole class of interrelated problems.

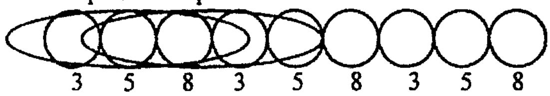
Problem 1: If $a + b = 5$ and $a \times b = 2$, calculate:
a) $^1/a + {}^1/b$
b) $^1/a^2 + {}^1/b^2$.

Problem 2: Without solving the equation $x^2 - 5x + 2 = 0$, calculate:
a) $^1/x_1 + {}^1/x_2$;
b) $^1/x_1{}^2 + {}^1/x_2{}^2$.

Figure 6.38. Example of problems which involve the same solution idea taken from teaching materials developed for the project classroom.

2. Restating a problem on the basis of its solution. Restating a problem on the basis of its solution is one problem-posing activity which has not previously been identified and described in the mathematics education research literature. Work on this type of problem-posing situation started in the project classroom when students were presented with a problem solution which students were told was written as part of a folio some time ago by one of their peers. Students were asked to suggest a problem statement whose solution *would match* the given one.

The picture represents a problem solution and its answer. What might the problem be about?



Students' suggestions:

Irene: This could be about students who are standing in a line. Three, five and eight cold be their ages.
T: And what is the question?
Irene: What is the total of their ages?

Nora: It could be about apples eaten at breakfast, lunch and dinner.
T: In how many days:
Nora: Four.
T: And you are asking about. . . ?
N: How many apples were eaten?

Figure 6.39. Problems posed by students on the basis of series of pictures representing the solution.

In other lessons for Group A, students were presented with solutions which they had not seen before, and they were asked to formulate a suitable problem. When it was appropriate, the problem solution was given by a series of pictures and students were asked to make guesses concerning what problem this series might represent. Some examples of students' interpretations of one such problem-posing situation are shown in Figure 6.39.

Another variation of this problem-posing situation was based on "decoding" a written algorithm. For example, students were asked to suggest what the given written explanations might be about, to determine the main steps involved in the solution, and to express them in concise terms. The notes taken by an independent observer and her impression of a part of a lesson which involved students in various formulating and reformulating problem-solving activities are given below. The discussion was based on four different algorithms for finding the highest common factor of two or more numbers.

> Showed prepared overhead–Highest Common Factor (HCF), 4 ways to find HCF. Instead of simply explaining (as she did for the first class [Group B]) Elena asked the class [Group A] "What do you think I have written here, what do I want to tell you, what's it all about?" etc. ie, asked students to explain what was meant by four different ways, students showed much more interest and understanding than the first class [Group B].

These observations suggest that there is a link between engaging students in "discovering" and formulating the main steps of an algorithm and the development of students' understanding of the features and the elements in an algorithm (solution) structure.

*3. **Posing problems with unrealistic solutions.*** Researchers have shown that some students do not interpret the solution of a mathematical problem as one which may have a real-world application.

Problem-posing activities based on situations which do not have real-life meaning were introduced in the project classroom when students were solving word problems. After solving a problem students had to justify which of the solutions of a particular equation are solutions to the mathematical model of the problem. In addition, the teacher asked some students to make changes to the problem statement so that the solution of the modelling equation *had no real meaning*.

The goal of this problem-posing situation was to provide students with an opportunity to explore the connection between the solution of a mathematical problem and its possible interpretations in real life situations. Results obtained in the project classroom suggest that students seem to develop better understanding of real-life interpretations of solutions when they have been involved in analysing the similarities and differences between the solution of a mathematical model and the solution of the modelling equation.

In addition to activities such as these, students were asked to interpret the applicability of some solution methods to real-life situations. For example, after solving a problem posed by one of the students, the participants in the project classroom were asked to explain how they would solve the problem if they did not understand the meaning of highest common factor. Here is Nelly's explanation of her "practical" solution:

Episode with Nelly (with reference to bunches of flowers)

T: If you didn't know the words highest common factor, how would you solve the problem?
Nelly: You just see if they all divide by 2, and then divide again if you can . . .
T: Uh-huh, you say that 6, 28 and 14 are all divisible by 2, it means that you can make 2 bunches. OK, is that enough?
Nelly: You look at the numbers you get if you divide them by 2.
T: If you divide our numbers by 2 you will get 3, 12 and 7. And after that?
Nelly: There is no number . . .
T: Which is common.
Nelly: But if there is you divide again, and keep on going.
T: Oh wonderful, say we have the numbers, 12, let me take such an example, 12, 36 and 60. The question is how many bunches can I make?

Nelly explained the solution idea once again and justified that, in this case, exactly 12 bunches with the same number of flowers can be made.

T: This is another nice way! I hope that you won't work in flowers, but if you work there you can solve the problem very easily!
Martin: Florists don't usually worry how many flowers are in the bunches . . .

The last episode suggests that some students are likely to see potential applications of the mathematical methods used mostly within the context of the original problems which were used to illustrate the application.

*4. **Problem-posing situations established on the basis of a problem with several solution approaches.*** Mathematical problems which can be solved in several ways are referred by some authors as "open" problems (Nohda, 1995; Silver, 1995; Stacey, 1995). In this study, as has already been discussed in more detail earlier (see Chapter 2), problems which invite several solution methods are generally considered to be *problem-solving* rather than *problem-posing* situations. The research literature contains many examples of problem-solving research studies, and the identification of different solutions is emphasised in high level mathematics competitions, such as the International Mathematical Olympiad. Thus, encouraging students to present more than one solution idea was regarded as an important and useful problem-solving experience.

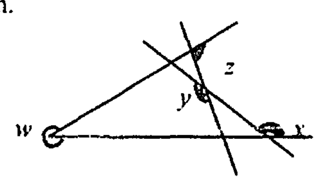
In this study the problem structure, students' activities involved in problem solving, and the solution method were all regarded as *open*. This understanding led to the fact that many traditional problem-solving activities were taken up in a non-traditional applications. For example, asking students to create a problem which can be solved by using different approaches took place naturally. After students had solved a particular problem in several ways, they were then asked to pose a problem similar to the one they had just solved. The applicability of the different solution approaches was then discussed.

In other cases, students were presented with a set of questions in multiple-choice format (see Worksheet 18, and also Figure 6.40).

Example 1: A rectangle has a perimeter of 20 cm and an area of 21 cm^2. What are its dimensions, in centimetres?

A) 1 and 20; B) 4 and 4; C) 9 and 2; D) 3 and 7; E) 6 and 3.5.

Example 2: Four straight lines intersect as shown.



The value of $x+y+z+w$ is

A) 360; B) 630; C) 450; D) 540; E) 720.

Figure 6.40. Problems which can be solved in more than one way.

After determining the different solution strategies used by students, possible changes in the problem format and content which will preserve or change some of the solution strategies were suggested and discussed.

5. Posing sets of problems which might have a common solution approach. During the study students were also involved in activities which required them to suggest problems *similar* to a given problem and to predict possible links between this similarity and an expected solution approach. In asking students to formulate a problem which might have a common solution approach with a *given, unsolved* or *solved* problem, the aim was to focus students' attention on those elements in a problem structure which were likely to be relevant to the use of a specific solution approach.

Before solving a particular problem and when it was appropriate, students were asked questions such as "Can you suggest an approach for solving the problem which you might expect to work?" or "What kind of methods have you used to solve similar problems?" or "Could you suggest a problem which might have the same solution method?".

Figure 6.41 provides an example of situations in which students posed problems which can be solved using permutations. The students were asked to work in pairs on a worksheet which presented the following problems (see Figure 6.41.

1. In how many ways can 3 students stand in a queue?

2. In how many ways can ...... students stand in a queue?

3. In how many ways can 10 students stand in a queue?
A) 100 000; B) 3828900; C) 3628800; D) 50; E) 1.

Figure 6.41. Examples of problems which can be solved by the same method.

After a discussion about the solution of the first problem and students' problems based on some open structures, the teacher asked the question "In how many ways can 4 boys and 3 girls stand in a queue?". The problem does not have any instructional value by itself if it is used alone. Tom answered immediately "In 7! ways." Then the teacher posed a question which looked similar but had an additional restriction — "the boys insist in occupying the first and the last places" (see Example 2, Figure 6.17). The problem was not difficult for the class and Carol suggested a precise solution. The teacher continued by asking the same question and Martin changed the problem slightly (see Example 2 in Figure 6.17).

Irene said that the problem could be solved in the same way. She did not realise, initially, that there were two possibilities — the girls can occupy the first and the last two places and the first two and the last place.

After *solving* a particular type of problem the teacher would then often invite pupils to suggest a problem which they felt was likely to have the same solution method as the problem that was *just* solved and to present arguments justifying the predictions made or to solve the problem.

Problem 1: Calculate: 1 - 2 + 3 - 4 + 5 -. . . + 999 - 1000.

Students' suggestions:

Student 1: 1 + 2 - 3 + 4 - 5 + . . .- 999 + 1000;
Student 2: 2 - 4 + 6 - 8 + 9 - . . . - 998 + 1000;
Student 3: 1 + 3 + 5 + . . . + 999 - 2 - 4 - 6 - . . . - 1000.

Figure 6.42. Problems posed by students which have the same solution method.

The two examples presented in Figure 6.43 give students' suggestions for changes to the problem just solved, which would preserve the solution method. As can be seen from Figure 6.43 some of the students' suggestions had a fairly general form, while in other cases they simply varied some of the elements in the problem structure without taking into account the relationships between the elements. However, all problems posed could be described as having their own instructional value — they were owned by students and provided a basis for further classroom and individual discussions.

Example 1: In each of the ten boxes there is a digit—two of them are shown. When the digits in three successive boxes are added, the total is always 20. What digits are in the other boxes?

5								8	

Student 1. If the sum is 25, the numbers can be 8, 8 and 9.
Student 2. If the sum is 17, the numbers can be 7, 6 and 4;
Student 3. Instead of 10 boxes, you can have 12 and you could have groups of 4 or 5.
Student 4. Instead of numbers you could have x and y and you could ask about the values of x and y.

Example 2: Four friends are racing together down a flight of stairs. *A* goes 2 steps at a time, *B* 3 at a time, *C* 4 at a time and *D* 5 steps at a time. The only steps which all four stepped on are the top one and the bottom one. How many stairs in the flight were stepped on exactly once?

Student 1. You could alter the number of people and for the new people you could add a rule.
Student 2. You could increase or decrease the number of people.
Student 3. You could have set amount of steps. A group of friends go up 500 steps. How many steps are stepped on once?

Figure 6.43. Problem posing which involve the use of a specific solution method.[6]

The observations indicate that when students are involved in *constructing* problems *similar* to a given problem which can be solved by a specific solution method, they seem to develop a better understanding of the elements in the problem structure which may or may not be relevant to the use of the specific solution method.

[6] Examples 1 and 2 are identical to two of the *Challenge Problems.*

6. Posing sets of problems which resemble a given problem but have different solution approaches. The teacher prompted classroom and individual discussion about the expected solution method by posing problems which have *isomorphic* or *similar* structures. Some problems *look* the same as other problems but they may be *different* and may involve *different* solution approaches. This problem-posing category was aimed at extending students' experience in recognising differences in problem structures which might lead to different solution approaches.

Initial classroom work in this area started with the teacher drawing students' attention to similarities in sets of problems which contained "problems from different types" (Krutetskii, 1976). The teacher created such problem-posing situations for the students from the project classroom by changing numerals, mathematical relationships, and key words, so that the solution approach was affected.

The situation shown in Figure 6.44, for example, is slightly different from the situation shown in Figure 6.8. The restriction in this problem is that zero cannot be a first digit.

From the set of digits (0, 2, 3, 6, 7, 9):

A) How many 2-digit integers can be formed?
Answer:
B) How many even 2-digit integers an be formed?
Answer:
C) How many ....-digit integers can be formed?
Answer:
D) How many ....-digit integers can be formed?
Answer:
E) How many ....-digit integers can be formed?
Answer:

(*Assume that no digit may be used more than once.*)

Figure 6.44. Problem-posing situation with some restrictions incorporated in the data.

After the lesson the researcher wrote in her diary: "All students asked standard questions. Only Martin went beyond by writing "How many integer numbers between 2000 and 7000 can be formed?"

In the project classroom, the teacher also suggested that students try to isolate basic facts from irrelevant details in the problem structure and to identify possible changes which might affect the solution approach. For example, problem situations which involved the use of permutations or combinations were found to provide a useful environment for introducing restrictions which might lead to a change in the solution method.

Figure 6.45 presents students' suggestions when they were asked to create a problem similar to the problem presented in Figure 6.44. Changes in a problem which would lead to a change of the solution method were then discussed.

How many numbers can be made if the digits 1, 2, 3, 4 and 5 should be used only once.
Suggest changes to the problems which would lead to a change in the solution approach.

Students' suggestions:

Student 1: You could change the numbers.

Student 2. Zero could be one of the digits.

Figure 6.45. Investigating changes in the problem structure which lead to changes in the solution approach.

The suggestion made by the first student was very broad because not every set of numbers will lead to a change in the solution approach. The change proposed by the second student leads to a change in the solution approach, because "0" cannot be the first digit.

The example presented in Figure 6. 46 shows changes made by students in the problem format when students were asked to suggest how the problem solution strategies could be narrowed. The second version of the problem does not allow application of the "checking the answers" strategy.

Version 1:
Peter's age on his birthday in 1986 was equal to the sum of the digits of the year of his birth. Peter was born in
A) 1966; B) 1967; C) 1965; D) 1976; E) 1964.

Version 2:
Peter's age on his birthday in 1986 was equal to the sum of the digits of the year of his birth. What year was Peter born in?

Figure 6. 46. Investigating changes in the problem format which lead to changes in the solution approach.

The observations suggest that when students are involved in activities which require distinguishing similarities in problem structures and analysing the interrelationships between problem structures and solution approaches, they seem to develop better understanding about the particular mathematical approach.

Conclusion

In this chapter a classification of free, semi-structured and structured problem-posing situations used in the study has been presented. The categories were developed using grounded theory techniques (Glaser & Strauss; 1967) which drew on: (a) the initial framework derived from an analysis of the literature related to students' problem posing (see Figure 4.11); (b) a range of data gained from the project classroom; and (c) the anticipated instructional goal with respect to students' problem solving.

The results of this study suggest that a wide range of interrelated problem-posing situations can be incorporated as a part of students' problem-solving activities. The classification of problem-posing situation categories presented in Figure 6.1, Figure 6.5, Figure 6.19 and Figure 6.37 form a broad basis for constructing quality problem-posing tasks. The idea of linking problem posing to the features of the problem structure was to a large extent inspired by Krutetskii's work. However, during the study this idea was naturally tied to the features of the solution structure as well and the final classification presented in Appendix 10 is one of the important results of this study.

The author believes that it is impossible to categorise *all* problem-posing situations in which students could be involved. Clearly this must be the case if it is accepted that there are no limits to human creativity. However, this chapter defines broad categories of problem-posing situations which represent a first step in helping to re-orient the current problem-solving mentality in mathematics education towards a more balanced perspective which integrates problem-posing structures.

CHAPTER SEVEN

CATEGORIES OF STUDENTS' PROBLEM—POSING STRATEGIES

Introduction

The strategies used by Years 8 and 9 students in response to the instrument, which included a free, a semi-structured and a structured problem-posing situation, were classified in three main categories termed: *reformulation, reconstruction and imitation.* An additional category, which was termed *invention,* was identified when data from the project classroom sessions were analysed. The analysis procedure adopted has been described earlier in the thesis in Chapter 5.

This chapter outlines the features of the four categories identified and presents selected prototypic examples of students' problem-posing products.

Initial Procedure for Analysis of the Problem-posing Products

Students' problem-posing products were initially divided into three groups — correct responses, correct intermediate responses and problem-posing products which should be excluded from further analysis.

Correct Responses

Students responded to the problem-posing test in various ways according to the nature of the problem-posing tasks. Some problem-posing products were presented precisely, in the form of well-structured problems. These were classified as *correct* problems. Figure 7.1 represents some of Gloria's responses which were classified in this category.

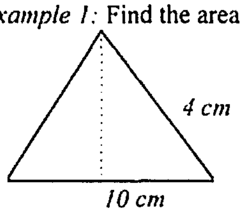
Example 1.
a) What is the answer of the calculation [3 × 25 + 15 ÷ 5 - 4] ?
b) What would the answer be if the "-" was a "+" and "+" was a "-"?

Example 2.
How many "*" would there be between the numbers 1 and 35, if "*" stands for a composite number.

Figure 7.1. Problems posed by Gloria identified as "correct" problems.

Responses which contained a picture or a diagram as part of a problem statement were also accepted as correct problems when the diagram and the written expressions contained enough information for determining the goal statement (Figure 7.2).

Example 1: Find the area of the triangle



Example 2: Make 3 triangles using 3 sticks

Figure 7.2. Problems posed by students identified as correct problems in which a part of the problem statement was presented as a diagram or picture.

Correct Intermediate Responses

Some students listed possibilities for different arrangements of the elements in a problem, or constructed problem situations, assuming that the goal was the value of the mathematical expression or the values of numbers replaced with asterisks. These responses were described as *correct intermediate* results. In Figure 7.3 illustrative examples taken from Ani's responses to each of the problems in Mathematics Questions Set 1 are presented. The *Goal* statement in all examples is transparent.

Example 1: 3 + 25 ÷ 15 × 5 - 4
Example 2: 3 6 9 * 15 *
Example 3: 0 20 * 60 80 *

Figure 7.3. A sample of problems posed by Ani identified as correct intermediate results.

Another class of problem-posing products was also referred to as *correct intermediate*. These were the problems posed by students which contained surplus or insufficient information. Although some problems were not written precisely, they contained important information about the problem-posing strategies developed by students.

Example 1:	a) [(3 × 25) + (15)] ÷ (5 - 4) b) (3 × 25) + [(15) ÷ (5 - 4)]
Example 2:	[2*a* (5*a* + 19*]*

Figure 7.4. Problems with surplus and insufficient information defined as correct intermediate results.

Problems with surplus information posed by Christine are presented in Example 1, Figure 7.4. The second example in Figure 7.4 illustrates a problem with insufficient information posed by Harry.

Problem-posing Products Excluded from Further Analysis

For a small number of responses, the decision was made that they should be excluded from further analysis. These included examples in which students did not attempt a response or problems which did not provide enough written evidence to allow the researcher to make judgements about the students' actions. Several students posed problems for which they admitted that they "remembered from the book" or that they "didn't create because they read it somewhere" (see Example 1 in Figure 7.5).

Example 1: * A book has 200 pges how many times is the digit 7 printed on it's pages?

Example 2: If 1. child and 2 adults each have pizza .but the adult has twice as much as the child show much pizza does adult get and how much pizza does 1 child gets from 5555555555 slices of pizza.

Figure 7.5. Examples of problem-posing products which were excluded from further analysis.

Problem-posing products which did not relate to the problem-posing situations presented were excluded from further analysis. Example 2 illustrates a word problem posed by Merry in response to the semi-structured problem-posing task. In this case no links between the problem-posing product and the content in the initial situation could be found.

Definitions

Reformulation Strategy

When the problem-posing actions of students resulted in a rearrangement of the elements in the problem structure in ways which do not change the nature of the problem, the problem-posing strategy is defined as *reformulation*. In other words the problem-posing products are the *same* or *identical* to the given problem and differ from the initial problem only in the presentation of the information in the problem statement.

Reconstruction Strategy

A problem-posing strategy will be referred to as *reconstruction* when the problem-posing product is obtained by modifications made to the initial problem and when these modifications change the nature of the problem. Thus the problem-posing products relate, in some way, to the given problem but differ from it in content.

Imitation Strategy

A problem-posing strategy will be referred to as *imitation* when the problem-posing product is obtained from the given problem (or situation) by the addition of a structure which is *relevant* to the problem, and the problem-posing product resembles a *previously encountered/solved* problem. In other words, the imitation strategy takes into account two important issues: the problem-posing product has an extended structure and the student has encountered these types of problems before.

Invention Strategy

On a number of occasions during the Program students created mathematical problems which could not be linked to their previous mathematical experiences. A problem-posing strategy was referred to as *invention* when the new problem students created was *different* from the ones already solved and students *did not know* how to solve the new problem immediately.

Problem-posing Strategies Used by Students in a Free Problem-posing Situation

Students from Group C were invited to pose problems on the basis of the following problem-posing situation which was validated as structured:

> Make up as many problems as you can using the following calculation: $3 \times 25 + 15 \div 5 - 4$.

When students were presented with this particular structured problem-posing situation they responded in a variaty of ways to obtain *new* problems. Their suggestions ranged from changing the order of the numbers and the operations in the given calculation to posing problems by extending the structure of the given problem. After analysing students' written responses, the strategies identified were classified into three categories: (a) reformulation; (b) reconstruction; and (c) imitation.

Reformulation

In a structured problem-posing situation students used a range of actions to pose problems which were the *same* or *identical* to the given problem.

In this category, it has been possible to identify the following different sub-categories of *reformulation*:

- Rearrangement of numerical information;
- Adding irrelevant structure;
- Replacing mathematical operations in equivalent forms;
- Replacing numerical information with equivalent expressions;
- Combinations of some of the above sub-categories;
- Interpreting the calculation in a real-life context.

Examples of these sub-categories will be presented in Figures 7.6 to 7.11.

1. Rearrangement of numerical information. Students rearranged the numerical information in the initial problem in such a way that, although the problem-posing product seemed different, in fact, it was a problem which was identical with the initial problem.

Example 1: $3 \times 25 - 4 + 15 \div 5$

Example 2: $15 \div 5 + 3 \times 25 - 4$

Example 3: $-4 + 15 \div 5 + 3 \times 25$

Example 4: $15 \div 5 - 4 + 3 \times 25$

Example 5: $25 \times 3 + 15 \div 5 - 4$

Figure 7.6. Problem posing based on the use of the commutative law.

The examples presented in Figure 7.6 illustrate how students applied the commutative law to obtain problems identical with the given problem. The problem-posing products can be obtained by changing the positions of some groups of numbers in the initial problem. Applying the commutative law for the addition operation, is in fact, an action which does *not* lead to a different problem. Example 5 illustrates the commutative law for the multiplication operation which was applied by Simon to obtain a "new" problem.

Changing the places of groups of numbers and variables in a specific problem and justifying (when appropriate) that the problem obtained was identical with the given one, was an action which was an inseparable part of students' work when they were involved in solving equations or inequalities, proving identities, analysing the problem statements of word or geometry problems, and so on. It was also observed that rearranging the information in a problem statement was used by students when they were asked to present a specific problem in their own words (see Chapter 6).

2. Adding irrelevant structure. Students also generated problems by introducing additional elements to the problem structure, such as one, two or more pairs of brackets. For example, some pupils used brackets to pose problems identical with the initial one. Figure 7.7 shows students' problem-posing products incorporating one or two pairs of brackets which are *irrelevant* to the problem structure. In these cases the brackets are used in inappropriate ways, suggesting that students who posed these problems have a limited understanding of the hierarchy of mathematical operations.

Example 1: $(3 \times 25) + (15 \div 5) - 4$

Example 2: $25 \times 3 + (15 \div 5) - 4$

Example 3: $(3 \times 25) + 15 \div 3 - 4$

Example 4: $3 \times 25 + (15 \div 5) - 4$

Example 5: $(3 \times 25) + [(15 \div 5) - 4]$

Example 6: $(3 \times 25) + 5(3 \div 1) - 4$

Example 7: $(3 \times 25 + 15 \div 3) - 4$

Example 8: $(3 \times 25) + (15 \div 3 - 4)$

Figure 7.7. Examples of students' responses showing the use of brackets which does not change the problem.

3. Replacing mathematical operations with equivalent forms. A few students retained the identity of the problem by presenting some of the mathematical operations in an equivalent form.

Example 1: $3(25) + {}^{15}/_{5} - 4$

Example 2: $3(25) + 15 \div 5 - 4$

Example 3: $75 + {}^{15}/_{5} - 4$

Example 4: $3(25) + 3 - 4$

Example 5: $3 \times 25 + 3 - 4$

Figure 7.8. Examples of students' responses showing retaining the identity of the problem by presenting some of the mathematical operations in an equivalent form.

In Figure 7.8 students' work was based on the presentation of multiplication and division in equivalent forms. Examples 3, 4 and 5 in Figure 7.8 in fact represent intermediate results when the value of $3 \times 25 + 15 \div 5 - 4$ was calculated.

4. Replacing numerical information with equivalent expressions. A few students tried to pose problems identical with the given problem by replacing some of the numbers with the result of two arithmetic operations (see Figure 7.9). In such cases, students tried to present the problem content in a more complex form by preserving the problem identity.

Example 1: $(2 + 1) \times 25 + 15 \div (7 - 2) - 4$

Example 2: $(2 + 1) \times (16 + 9) + (3 \times 5) \div (25 \div 5) - 4$

Figure 7.9. Replacing numbers with equivalent expressions.

5. Combinations of two or more sub-categories. Students also tended to apply two or more problem-posing actions in their formulation of the given mathematical problem. Examples of students' problem-posing products defined under a reformulation strategy, which produced a problem identical with the given problem by combining two or more problem-posing actions, are presented in Figure 7.10.

Example 1: $3 \times 25 - 4 + 15 \div (2 + 1)$

Example 2: $-4 + (2 + 1) \times 25 + (10 + 5) \div 5$

Example 3: $(15 \div 5) - 4 + (3 \times 25)$

Example 4 : $-4 + (3 \times 25) + (15 \div 5)$

Example 5: $-4 + (25 \times 3) + 15 \div 5$

Figure 7.10. Example of problem-posing strategies identified as reformulation.

6. Interpreting the calculation in a real-life context. The final group of problems defined under *reformulation* can be described as problems in which students made connections between a mathematical expression and a real-life situation. These have been categorised as *reformulation* because the product differs from the initial problem only in the presentation of its structure. Figure 7.11 provides examples of students' interpretations of the basic calculation in real-life contexts. In the first two cases the students had expressed, to the teacher, their frustration in trying to find a suitable context in which to pose problems.

The problem-posing products presented by the students who had expressed difficulty in finding an appropriate context suggest that they were attempting to interpret the structure of the whole calculation as a sequence of interrelated real-life situations.

Example 1: There was once three Boogie monsters who ate 25 cookies each in the morning and a total of 15 altogether in the afternoon. The number of cookies altogether which went to the cookie monster had to all share one cookie. How many pieces was the cookie cut up into?

Example 2: If I have 3 children and I need to pay them $25 each for pocket money, and $15 extra each for chores. How much money do I have to spent altogether?

Example 3: I bought three $25 items of clothing and gave my 5 brothers and sisters $15 between them and lost $4. How much money:
a) did I start with?
b) did my brothers and sisters get each?

Example 4: Cameron had 3 guitars which had 25 strings on each, but as a birthday present he was given 15 spare strings. So, he decided to sell the spare strings to 5 other people. While selling the strings he lost 4. How many strings does he have left concluding the ones on the guitars?

Figure 7.11. Interpretation of a given mathematical expression as a life situation[7].

Observations from the project classroom based on data obtained from students' worksheets and the lesson-transcripts revealed that, depending on the type of problem, students react naturally by changing the language characteristics of the initial problem statement without changing the nature of the problem. For example,

[7] The problems are literally presented from the student's papers.

when students were asked to describe the problem or explain what the problem was about, their responses included rearranging the order of the information in the initial problem (see Chapters 6 and 9), or replacing some of the words with ones more familiar to them (see Chapter 6), or just extracting the wording which contained the mathematical substance of the problem (see Chapter 6).

Changes which led to changes in the *nature* of the problem were not regarded as reformulations. Some of the strategies used by students in the reconstruction of the problem are presented in the next section.

Reconstruction

When the reconstruction strategy was employed the problem-posing product resembled the initial problem but differed in its content. Five sub-categories of *reconstruction* strategy were identified:

- Changing the order of the numerical information;
- Changing the order of the operations;
- Changing the numbers;
- Regrouping the problem information by using brackets;
- Presenting mathematical operations in equivalent forms;
- Taking sub-structures.

Examples relating to these sub-categories will be given in Figures 7.12 to 7.20.

1. Changing the order of the numerical information. Students applied a *reconstruction* strategy to obtain problems from the initial problem when they changed the order of the numbers but keeping the order and the types of the mathematical operations. Figure 7.12 presents some examples of students' responses of this type. In fact, all examples presented illustrate problem-posing products which are *similar* to the given problem but which differ from the initial problem in their content.

Example 1: 25 × 3 + 5 ÷ 15 - 4

Example 2: 25 × 3 + 5 ÷ 4 - 15

Example 3: 3 × 25 + 5 ÷ 15 - 4

Example 4: 3 × 25 + 15 ÷ 4 - 5

Example 5: 3 × 5 + 25 ÷ 15 - 4

Example 6: 3 × 5 + 15 ÷ 25 - 4

Example 7: 3 × 5 + 25 ÷ 4 - 15

Example 8: 5 × 4 + 3 ÷ 25 - 15

Example 9: 4 × 3 + 25 ÷ 15 - 5

Figure 7.12. Examples of applying a reconstruction strategy in which the order of the numerical information was changed.

2. Changing the order of the operations. In other problem-posing products, the order of the operations was changed while the numbers and their order were kept the same.

Example 1: 3 + 25 ÷ 15 - 5 × 4

Example 2: 3 ÷ 25 - 15 × 5 + 4

Example 3: 3 - 25 × 15 + 5÷ 4

Example 4: 3 × 25 + 15 ÷ 5 - 4

Example 5: 3 × 25 + 15 ÷ 4 - 5

Example 6: 3 × 25 + 5 ÷ 15 - 4

Figure 7.13. Reconstruction strategy achieved by changing the order of the operations and preserving the numbers and their order the same.

Examples of one student's problem-posing products in this sub-category are shown in Figure 7.13. In this case, the student had tried to pose other examples which resembled the initial problem but differed from it in the way the operations and the numbers were combined.

*3. **Changing the numbers.*** Students also posed new problems by changing the numerical information and retaining the same operations and their order (Figure 7.14).

Example 1: 4 × 7 + 1 ÷ 2 - 100

Example 2: 2 ÷ 1 - 15 × 7 + 40

Example 3: 4 ÷ 2 + 25 × 6 - 14

Figure 7.14. Reconstruction strategy of changing the numbers *and* the order of operations.

The second and third examples in Figure 7.14 show the application of a reconstruction strategy in which both the numbers and the order of the operations are changed.

4. Regrouping the problem information by using brackets. Students also made changes to the initial problem structure by imitating some traditional classroom activities — solving problems with brackets — and they created possibilities by using one, two or more pairs of brackets to obtain *different* problems. Figure 7.15 illustrates some typical examples of problems posed when students inserted additional structure (brackets). All examples shown in Figure 7.15 were posed by Blair. Two of them, Examples 4 and 5, contain surplus information. Thus, some of the brackets were not used in appropriate ways.

Example 1: $3 \times 25 + 15 \div (5 - 4)$

Example 2: $3 \times (25 + 15) \div 5 - 4$

Example 3: $3 \times (25 + 15) \div (5 - 4)$

Example 4: $3 \times [25 + (15 \div 5)] - 4$

Example 5: $3 \times \{25 + [(15 \div 5) - 4]\}$

Example 6: $3 \times [(25 + 15) \div 5] - 4$

Example 7: $3 \times [(25 + 15) \div (5 - 4)]$

Example 8: $3 \times \{(25 + [15 \div (5 - 4)]\}$

Figure 7.15. Reconstruction involving systematic grouping based on the use of brackets.

5. Presenting a mathematical operation in an equivalent form. Some students combined the use of brackets with the representation of division and multiplication in an equivalent form (see Figure 7.16).

Example 1: $\frac{3(25 + 15)}{5} - 4$

Example 2: $\frac{3 \times 25 + 15}{5 - 4}$

Example 3: $\frac{3(25 + 15)}{5 - 4}$

Figure 7.16. Reconstruction based on changes made to the problem by using brackets and representing division in an equivalent form.

Other students, as shown in Example 2, Figure 7.16, simplified the representation of the problem structure by replacing the use of brackets and division with a fraction. Observations from the project classroom suggest that students' ability to represent a specific problem structure in equivalent forms and to recognise isomorphic problem structures is very likely to be linked to their problem-solving performance.

*6. **Taking sub-structures.*** Problems were also obtained by selecting sub-structures of the given calculation. For example, some students posed simple calculation problems by taking some of the numbers and one or two of the given operations (see Figure 7.17). Examples 1 to 7 in Figure 7.17 were drawn from Peter's work. He posed a total of 80 problems by taking different sub-structures of the content of the initial problem. In the last two examples a part of the information in the problem was used for constructing two fractions and an equation.

Example 1: $3 \times 25 + 15$

Example 2: $15 \div 5$

Example 3: $5 - 4$

Example 4: 3×25

Example 5: $3 - 4$

Example 6: $3 \div 5$

Example 7: $25 + 4$

Example 8: True or false: $\frac{3-4}{5} = \frac{25}{15}$

Example 9: $\frac{x}{5} \div \frac{5}{3} = 8\frac{1}{3}$

Figure 7.17. Reconstruction based on taking sub-structures.

*7. **Combinations of two or more strategies.*** Some students combined two or more consecutive strategies and obtained new problems. For example, in some cases *both* the order of the operations *and* the order of the numbers were changed (see Figure 7.18). All problems shown in Figure 7.18 differ from the initial problem in the ways in which the numerical information and the operations are related. In other cases students used brackets and changed the order of the numbers while keeping the mathematical operations the same. Examples of such students' responses are illustrated in Figure 7. 19.

Example 1: $5 \div 15 + 4 - 3 \times 25$

Example 2: $4 + 15 \div 5 - 3 \times 25$

Example 3: $25 \times 3 \div 5 + 15 - 4$

Example 4: $15 \div 5 - 4 + 3 \times 25$

Example 5: $-4 \div 5 + 15 + 3 \times 25$

Example 6: $15 - 4 \div 5 + 3 \times 25$

Figure 7.18. Examples of a reconstruction strategy obtained by changing the order of the operations *and* the numbers involved.

Example 1: $3((-4 + 15)\,25) \div 5$

Example 2: $3(15 \div 5) + (25 - 4)$

Example 3: $(25 + ((15 \div 5)\,3) - 4$

Example 4: $((25 + 15) \div 5 - (-4 \times 3)$

Example 5: $(-4 + 25) \times 3 + (15 \div 5)$

Figure 7.19. Reconstruction based on changes made to the numerical information by introducing brackets and changing the order of the numbers while keeping the mathematical relationships the same.

In fact, all problems included in Figure 7.19 differ from the initial problem in their content and they also include additional information (the brackets) which is *relevant* and changes the nature of the given problem.

The next group of problem-posing products represent a combination of three basic sub-categories. In those cases students obtained new problems by changing the order of the numbers and the order of the operations, and by presenting the division or multiplication in equivalent forms. Examples are given in Figure 7.20.

Example 1: $\frac{15 \times 3 - 25}{5 + 4}$

Example 2: $\frac{25 \times 4 - 3}{15} + 5$

Example 3: $\frac{25 - 4 \times 15 + 5}{3}$

Example 4: $\frac{(-4 + 3)\,25}{5} + 15$

Example 5: $3(-{}^{4}/_{5}) + 15 \times 25$

Example 6: $\frac{(15 - 4) + (3 \times 25)}{5}$

Figure 7.20. Reconstruction achieved by changing the order of the operations, the order of the numbers and presenting operations in equivalent forms.

Observations from the project classroom data obtained from students' work and the lesson-transcripts showed that students used reconstruction strategies in structured problem-posing situations regardless of the *format* of the problem. For example, in Chapter 6 examples in which additional questions were added to a problem presented in a multiple-choice question format were presented. In a few cases, some students have posed a problem which was inverse to the given problem (see also Chapter 6). Most students obtained new problems by changing the numerical information or by including additional structure which was *relevant* to the problem content.

Imitation

Students employed the *imitation* strategy when problem-posing products were obtained from the given problem by *adding* a structure which was relevant to the problem and the problem-posing product resembled a *previously encountered* or *solved* problem.

The following two problem-posing sub-categories were identified under the imitation strategy:

- Formulating life-situations by interpreting the division operation as a ratio;
- Extending the problem structure by changing the *Goal*.

These sub-categories are illustrated with examples presented in Figure 7.21 to 7.22.

1. Interpreting the division operation as a ratio. Some students interpreted division as a ratio and then they posed word problems based on the use of this new interpretation in a real-life context. The first example shown in Figure 7.21 was posed by Brad. The author of the second example is Nelly. Both students were among those participants in the study who have shown high mathematical performance on the *Challenge Problems.*

Example 1: If the above ratio [3 × 25 + 15 : 5 - 4] is used to make a miniature of a famous painting, which has an original size of 50 *cm* × 60 *cm*, what size will the miniature be? *Example 2:* If a model of a dog is 5 *cm* with that ratio (90:1) what is the size of the real dog?

Figure 7. 21. Imitation strategy employed by students for interpreting division as a ratio.

2. Extending the problem structure by changing the Goal. A few students extended the structure of the given problem by constructing a new goal statement. All authors of the examples shown in Figure 7.22 were participants in the Program. Students changed the structure of the given problem by extending the goal statement in such a way that the initial problem became a step of the solution process of the *new* problem. The problem-posing products resemble types of problems which were solved during the instructional sessions of the Program.

Example 1: What is the prime factors of the answer to this calculation.

Example 2: Around which two digits could you place brackets so that the answer [of the calculation $3 \times 25 + 15 \div 5 - 4$] is minimal?

Example 3: Write the prime factorisation of the result of this [$3 \times 25 + 15 \div 5 - 4$] calculation.

Example 4: Around which two digits could you place brackets so that the answer is 80?

Example 5: What is the last digit of $3 \times 25 + 15 \div 5 - 4$?

Figure 7.22. Imitation by posing specific examples involving the use of mathematical concepts learnt in the Program.

Strategies Used by Students in a Semi-structured Problem-posing Situation

Students from Groups A, B and C were also presented with the following problem-posing situation which had been validated as semi-structured:

Given that: 1 2 3 * 5 *. (a) What could the meaning of sign "*" be? (b) Can you make up a (some) problem(s) using one of these meanings?

Problems posed by students in response to this semi-structured situation demonstrate that most students are likely to have a natural capacity to interpret a given situation from their own particular perspectives and to pose problems using these interpretations. All students' problem-posing strategies in response to the semi-structured situation were classified as *imitation*. The following problem-posing sub-categories of imitation strategy were observed.

- Interpreting the asterisks as terms in the arithmetic sequence: 1, 2, 3, ..., n,... .
- Interpreting the asterisks as "missing terms" in other arithmetic sequences;

- Interpreting the asterisks as "missing terms" in a particular sequence of numbers;
- Interpreting the asterisks as an arithmetic operation and a goal statement;
- Interpreting the asterisks as missing digits in a specific number.

Examples of students' responses within these sub-categories are presented in Figures 7.23 to 7.30.

1. Interpreting the asterisks as terms in the arithmetic sequence 1, 2, 3,. . . , n, . . . Some students made the assumption that 123*5* were the first six terms of an arithmetic sequence and interpreted the meaning of the asterisks respectively as the digits 4 and 6. Two types of problems can be identified in this sub-category. The first type includes simple *calculation* problems in which 4 and 6 are elements in the problem content. In Figure 7.23 Graham's and Anny's basic problems are presented. Gregory also posed simple calculation problems by using all of the digits given (see Example 3, Figure 7.23).

Example 1: 4 + 6 = 10

Example 2: 4 × 8 = 32
6 × 4 = 24

Example 3:

12	123	456
+ 54	+ 456	− 123
56		

Figure 7.23. Basic calculation problems by using digits 4 and 6.

The second type of problem can be described as an *equation* problem. It involves presenting "4" and "6" as solutions or as coefficients of a linear or a quadratic equation (see Figure 7.24).

Example 1: a) * × 5 = 20
b) 20 ÷ * = 5
c) * (3 + 1) ÷ 4 = *
d) 4 * = 16
e) 2 × * = 8

Example 2: a) * = 5 - 1
b) * = 1 + 3
c) * = 1 + 5

Example 3: a) $4x = 21$
b) $\frac{2(4+a)}{6} = \frac{21+18}{a}$
c) $\frac{x}{4} = 2$

Figure 7.24. Basic equation problems by using digi's 4 and 6.

2. Interpreting the asterisks as "missing elements" in an arithmetic sequence. Some students interpreted the asterisks as missing elements in an arithmetic sequence. Then they posed their own problems which required finding the missing terms in particular arithmetic sequences. The first example in Figure 7.25 represents Kathryn's work. She created two arithmetic sequences with positive differences. In the second example George made up an arithmetic sequence with a negative difference.

Example 1: What are the missing numbers in the patterns:
a) 2, *, 6, 8, 10, 12
b) 0, *, 8, 12, 16, 20

Example 2: 10, 9, *, 7, 6, *, 4, *, 2, *.

Figure 7.25. Problems posed by students illustrating the use of pattern in an arithmetic sequence.

3. Interpreting the asterisks as "missing elements" in a particular sequence of numbers. Some students extended the structure of the given situation by constructing sequences to illustrate their own patterns.

What are the missing numbers in the patterns:
a) 8, 4, 2, *, .5, *.
b) 10, 20, *, 80, 160, *, 640.

Figure 7.26. Problems posed by students illustrating the use of pattern in a geometrical sequence.

In Figure 7.26 two examples of geometrical sequences created by Cheryl are presented. All terms in the first sequence can be obtained by halving the previous term ($a_{n+1} = \frac{1}{2} a_n$, $a_1 = 8$, where $n = 1, 2, 3, ...$). The first term in the second sequence in Figure 7.26 is 10. The other terms can be obtained by applying the rule:

$a_{n+1} = 2\, a_n$, where $n = 1, 2, 3, \ldots$

Several students imitated the structure of the initial situation by constructing their own patterns, creating number sequences and then stating a meaningful question.

What are the missing numbers in the patterns:
a) 12, 30, 84, *, 732, *, 6564.
b) 6, 11, 21, 41, *, 161, 321, 641, *.
c) 1, 2, 4, 7, *, *
d) 1, 2, 3, 2, 5, 2, *

Figure 7.27. Problems posed by students illustrating the use of their own pattern for creating number sequences.

The first two examples in Figure 7.27 were posed by Edrida. She applied the rule $a_{n+1} = 3a_n - 6$, $a_1 = 12$ for the first sequence and the rule $a_{n+1} = 2a_n - 1$, where $a_1 = 6$, for the second example. The third sequence problem was created by Chris. He made it up by applying the rule: $a_{n+1} = a_n + n$, $a_1 = 1$. The last number sequence was posed by Bao. She interpreted the initial set of numbers as a sequence which was an alternate combination of two sequences: 2, 2, 2, . . . and 1, 3, 5, . . . and posed a question by extending the structure of the semi-structured situation presented in the test.

4. Interpreting the asterisks as an arithmetic operation and a goal statement. The examples presented in Figure 7.28 present selected students' problems in which the first asterisk was interpreted as an arithmetic *operation* and the second as a *goal* statement. The first example shown in Figure 7.28 illustrates Jammy's calculation problems in which the first asterisk was interpreted as an arithmetic operation and the second as "=."

Merilyn also interpreted the first asterisk as a "+" and the second as "=" but she went on and posed a word problem to fit this calculation (see Example 2, Figure 7.28). The third example in Figure 7.28 was created by Bao. She interpreted the initial set of digits and symbols as a calculation problem in which some of the numbers and the operations were missing and then, by working backwards, she posed her own problem.

Example 1:
a) 123 ÷ 5=
b) 123 × 5=
c) 123 + 5 =
d) 123 + 5=
e) 123 × 5=
f) 123 ÷ 5=

Example 2: There were 5 boys and each of them had 123 marbles. How would they have altogether?

Example 3: 1 * 5 * 10 * 45 * 3
Find the meaning of "*". Each "*" represents a symbol.
Solution :- 1 x 5 + 10 = 45 ÷ 3

Figure 7.28. Problems posed by students illustrating the interpretation as an operation.

5. Interpreting the asterisks as missing digits in a specific number. A few students interpreted the initial set of digits and asterisks as a specific number. They then formulated problems by adding a restriction for the second asterisk. For example, in their problems Helen and Carol wanted the number to be divisible by four and eight respectively (see Example 1, Figure 7.29). George's and Tim's restrictions involved the sum of all number digits to be 24 and 20 respectively.

In other cases the restrictions added to the missing digits of a number were "borrowed" from types of problems solved before. The examples shown in Figure 7.30 comprised two problems in which the "missing numbers have to be filled in."

The first example illustrates Clara's "fill in the missing digits" problem. The problem shown in the second example in Figure 7.30 was posed by Mark. This problem requires two different cases to be considered.

Example 1:
a) What digits could be placed in the position of "*" so that the number created is a multiple of 4?
b) What digits could you substitute for "*" so that the number is divisible by 8?

Example 2:
a) If the sum of these six digits is equal to 24, what are the possibilities for the numbers replacing "*"?
b) When you add up the numbers the sum is 20?

Figure 7.29. Examples posed by students to illustrate an interpretation of the initial situation as a specific number.

Fill in the missing number.
Example 1:

```
  5*
+ *6
  89
```

Example 2:

```
  2*943
  36*7*
+ 5*184
  *91*9
```

Figure 7.30. Examples of problems in which the missing digits have to be found.

Strategies Used by Students in a Free Problem-posing Situation

The instrument also included a situation which had been validated as a free problem-posing situation. Students were asked to provide responses to the following question:

> Give an example of a problem similar to one you enjoy solving. Explain why you like it and how you created it.

Responses provided by students to this situation were classified only in one category which was defined as *imitation.*

Imitation

A problem-posing strategy was referred to as *imitation* when the structure of the problem-posing product was *isomorphic* to the structure of a *previously encountered/solved* problem. In other words, problems posed by students were all *similar* to previously solved problems.

In this category, it has been possible to identify the following different sub-categories of *imitation:*

- Direct modelling by constructing problems which are similar to previously solved problems;
- Direct modelling by constructing problems which can be solved by a specific solution method;
- Constructing "money" problems based on real-life contexts;
- Working backwards;
- Direct construction.

Examples of these sub-categories will be presented in Figures 7. 31 to 7.36.

1. Direct modelling by constructing problems which are similar to previously solved problems. Some students constructed their problems directly, by imitating problems similar to ones already solved in mathematics classrooms. The problem-posing products, defined under this sub-category, included algorithmic, algebraic, geometrical and logical types of problems. Selected sample of students' responses are presented in Figure 7.31.

Algorithmic problems constructed by students

Example 1: $8 \div (4 \times 2) + 6 - 2 =$

Example 2: $(3 + 4) \times (3 + 4) =$

Example 3: Find the mean (average) of: 6, 9, 12, 15, 22.

Algebraic problems constructed by students

Example 4: $3(2b + b) + 15 = (4b - 2)3 - 12$

Example 5: $\dfrac{2}{(2x + 1)(3n - 2)} + \dfrac{3}{(5x - 4)(2x + 1)}$.

Example 6: $6a + 9 > 11$

Logical problems constructed by students

Example 7: Spider climbs up a 10 *m* Drain each day. It climbs up 3 *m* but always gets washed back 2 *m*. How many days will it take to climb out?

Example 8: How many times does the two clock hands make a straight line in 1 hour?

Example 9: A computer printed out 2 000 numbered pages but did all the number 'ones' wrongly. How many digits were printed wrongly, and how many pages had wrong digits on them?

Figure 7.31. Imitation by construction of a problem similar to a previously solved problem.

Example 1 was created by chaining basic mathematical questions which Nick had done before. Rennie posed her problem (Example 2 in Figure 7.31) by "substituting the numbers in an example from the book." In other words she posed a *special case* to illustrate the distributive law: $(a + b)(c + d)$. Carol posed her "spider" problem by changing the context of a previously solved problem. She replaced the *frog* and *dwell* respectively with *spider* and *drain*. For her problem (Example 8 in Figure 7.31) Helen admitted: "I just created by the one we had in our maths exam, but it was different. It said how many 90^0 angles are there in 12 hours?" In fact, the problem posed by Helen was one step of the solution process of a problem which she admitted she had solved in school.

2. Direct modelling by constructing problems which can be solved by a specific solution method. A special case of the use of the imitation strategy arises when a problem-posing product is associated with a possible application of a specific solution method. The examples presented in Figure 7.32 illustrate problems posed by Carol (Examples 1 and 2) in which she has posed problems which could be solved by two particular approaches learned in the Program.

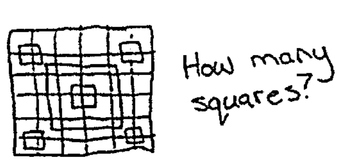
Example 1:

$1996^{3^{9^{8^{7}}}}$ What is the last digit?

Example 2: There are 8 boys and 14 girls in a line. The boys have to fill the middle two places. How many ways can you have the people in the line?

Example 3: A book has 948 pages. How many digits will be used to number all pages?

Example 4:



Example 5: Susan invested \$10 000 in the stock market, at a rate of 11.5% p.a., for 15 years. How much interest did she earn?

Figure 7.32. Problems posed by students which involve the use of particular solution methods.

In the third example Valerie has constructed a problem whose solution method involves systematic counting. The author of the fourth example, Tom, wrote: "You can create these problems if you know how to solve them." For the problem shown in Example 5 in Figure 7.32, John admitted: "I knew I had to have the amount of money × [*times*] the rate of interest × [times] how long the money was in the bank."

*3. **Constructing word problems based on real-life contexts.*** The work presented by students in response to a free problem-posing situation suggests that, in many cases, problems posed are coloured both by what students are currently learning *and* by their every-day-life experiences. This was demonstrated when students posed *word* problems in order to illustrate specific algorithms which they have been learning in school or to mathematise an encountered life-situation.

Example 1: Larry bought 50 TV's for $100 each. Then he sold them for $152 each. What was the profit?

Example 2: A pair of shoes costed $100.-, but I had a discount card saying 15% off all shoes. So I had to take 15% off, and I only paid $85.

When I went to the shoe shop it really happened (I saved $15)

Figure 7.33. Examples of word problems posed by students.

*4. **Working backwards.*** Students also created problems similar to ones they enjoy solving by working backwards. The first example in Figure 7.34, presents a problem which is similar to a problem solved in the Program. The student (Rob) concerned made up the problem by taking three consecutive numbers, adding them up, and then posing a question which is the same as in a problem encountered before. In the second example, Andy imitated the structure of a problem which was he had solved before. He "worked the answer first" and then decided on a suitable wording for a sequence of interrelated events. In the same way, Jim made up his "matches" problem (see Example 3 in Figure 7.34). He constructed a rectangle, then he took three matches away and asked a question about the shape which would be obtained.

Example 1: Three consecutive numbers, x, y, z, add up to 243. Find the values of the pronumerals.

Example 2: A man stands in the middle of a ladder. He climbs up 7 rungs to paint the wall but he runs out of paint so then he goes down 17 rungs. Then he goes up 5 rungs, back down 2 then up 16 rungs to the top. How many rungs are there?

Example 3:  can you make this shap By ~~movin~~ taking out 3 matcher 

Figure 7.34. Problems which were posed by using a "working backwards" strategy.

5. ***Direct construction.*** A few students admitted that they created their problems by "just writing down anything that comes to mind." In those cases, the solutions provided indicate that the authors were familiar with the solution methods involved.

Example 1: Tom's brother is three times as old as Tom. In four years time Tom's brother will be double the age of Tom. Find out both Tom and his brother's ages.

Example 2: If a dog runs from A to B at 10km/hour and then runs back at 14km/hour at what average speed did he run for the entire journey.

Figure 7.35. Problems posed by Betty and Nikol by using direct construction.

Students' responses incorporated problems which were direct recalls, or were posed on the basis of problems encountered before. In many cases, students recognised that the example posed was similar to a problem which they had seen in mathematics textbooks, or to one which they had solved before. In such cases students mapped the problem structure from the example onto their posed problem and changed the numbers. In fact some students admitted in their explanations that their example differed from the original problem which they had seen only by the numbers used.

Invention Strategy used by Participants in the Program

A problem-posing strategy was referred to as *invention* when the new problem students created was *different* from the ones already solved and students *did not know* how to solve the new problem immediately.

The problem-posing instrument did not require students to solve the problems posed and it was not possible to conduct individual interviews with all 112 students immediately after the test had been completed. In order to reveal how problem-posing products related to students' problem-solving experiences, data from the project classroom — students' written work on free, semi-structured and structured problem-posing situations and tape-recordings of the individual discussions — were collected during the study. Data analysis revealed, that in a few cases, some students posed problems which could not be linked to their previous mathematical experiences. The *new* problems incorporated specific structural elements which students had created by themselves — the problems were therefore new for the authors. Some problem-posing products involved the use of solution approaches which students had not encountered before. In both cases, however, students had tried to create a *new* problem and the solution approach was beyond students' previous problem-solving experiences.

In this category it was possible to identify the following different sub-categories of *invention*:

- Posing related problems;
- Direct modelling;
- Extending the structure of a problem/situation;
- Generalisations.

The study has revealed that students employ a range of actions for inventing new problems. Examples of students' work under the invention strategy are presented in Figures 7.36 to Figure 7.42.

Posing Related Problems

Students posed new problems on the basis of a given problem by varying the elements in the problem structure. As was already mentioned in Chapter 6, Martin re-discovered one of the *Challenge Problems* by changing the places of the *Given* and the *Goal* in the problem he was solving.

Some students posed more *difficult* problems by increasing the complexity of the structure of the problem which had been just solved. The problem shown in Figure 7.36 was constructed on the basis of a problem solved in the Program. The author of the last example shown in Figure 7.36, Norm, admitted that the problem posed related to the "sausage problem" solved in the classroom, but he recognised that "he does not know how to solve the problem although he could possibly understand its solution."

> *Example 1*:
> ***The initial problem:***
>
> "What is the last digit of $6^{2^{2^{3}}}$?"
>
> ***Problems posed by students:***
>
> ***Carol:***
> What is the last digit of $4^{2^{2^{3}}}$?
>
> *Nelly:*
> What is the last digit of $1995^{2^{2^{3}}} - 7^{2^{2^{3}}}$?
>
> *Example 2:*
> ***Initial problem:***
> *Seven* sausages are to be divided *equally* among *five* people.
> What is the smallest number of pieces of sausage necessary to make this possible?
>
> ***Problem posed by Norm:***
> There are 30 Alan Bonds. They have to pay off 80 bills. If they share the bills, what is the least amount of total bills? (If 2 Alan Bonds share one bill, it is counted as 2 bills.).

Figure 7.36. Invention by posing problems which relate to a specific problem.

Direct Modelling

Data analysis revealed that some students proceeded directly to pose problems which related to life situations drawn from their every-day experiences. The examples shown in Figure 7.37 were posed respectively by Eddi and Sarah at the beginning of the study. Eddi tried to mathematise a real-life situation by increasing the complexity of the problem structure and the problem context. Sarah posed her problem by constructing a number and then trying to provide a description of the relationships observed between the digits in the number posed.

Example 1 (Eddi):

There are three horses in a field. If Mr. Ed eats 6 bales of hay, Black Beauty eats 175% more hay than Mr Ed and Phar Lap eats 260% more then Black Beauty, how many bales are eaten per day?

If one bale of hay costs $5·63, what is the overall cost for a year?

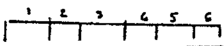
If 4m² of wheat will make one bale, how many bales would a 50ha field make?

If there is one wheat seed per 0·25cm², how many wheat seeds are needed for the 50ha field?

Example 2 (Sarah):

Find the 6 digit numeral from the clues given below:

- Each digit is different.
- The 1st digit is odd, less than the 2nd digit and more than the 4th digit.
- The 2nd digit is three times the 5th digit.
- The 3rd digit is an even square number and is less than the 1st digit
- The 5th digit is half of the fourth.
- The 6th digit is half of the 3rd digit and is and even number less than the 1st digit.



I invented a 6 digit number and made some rules.

Figure 7.37. Invention by describing a real-life situation in the form of well-structured problem.

Extending the Structure of a Problem/Situation

Students invented new problems by extending the structure of particular problems and also of situations. The following examples illustrate problem-posing products which were created by adding structure to a given problem and it was clear that the students were not able to solve their own problems. For example, the problems shown in Figure 7.38 illustrate problem-posing products made up by adding structure to a well-known problem.

Last night there was a party and the host's doorbell rang 10 times. The first time the doorbell rang only one guest arrived. Each time the doorbell rang after that, three more guests arrived than had arrived on previous ring.
Ask as many questions as you can. Try to put them in a suitable order.

A) Every 5th person is a child and. every 2nd child brings a dog.
B) There is a room in the house especialy for dogs.
C) the room can house 15 dogs how many times does the doorbell ring if the room is full of dogs. And if the first time 1 child arriver. and brings' his dog and if 4 more people arrive each doorbell ring than last doorbellring. . . . . .

Figure 7.38. Invention by adding new elements to the structure of a well-known problem.

Posing problems which could be solved by applying a specific solution method used in the problem which had been just solved, also led to creating types of problems which were new for the students. The example provided in Figure 7.39 was posed by Martin when he created a problem which involved the use the restriction principle (see Appendix 4).

> *The initial problem:*
>
> From the set of digits {0, 2, 3, 6, 7, 9}:
>
> *The problem posed by Martin:*
>
> e) how many ....4....... digits can be formed?
> Answer:

Figure 7.39. Invention strategy identified when Martin was asked to pose a problem which could be solve by using permutations.

Generalisations

On several occasions during the Program a few students made conjectures in the form of generalisations. In the first example shown in Figure 7.40, Samantha was able to solve her own problem after she got help from the teacher. She had tried in fact, to find all non-empty sub-sets which contain different elements of a set of four elements.

> *Example 1* (Samantha): How many different groups can be formed from a group of 4 people?
>
> *Example 2* (Karel): If a number has exactly three factors then it is a prime number squared.
>
> *Example 3* (Brad): If a number has an odd number of factors it is squared.

Figure 7.40. Theorems which have been "re-discovered" by students.

The invention strategy also included problems produced when students rediscovered a well known mathematical rule and formulated it as their own problem (see Example 2 and 3 in Figure 7.40).

Conclusion

This study investigated the types of problem-posing strategies used by Years 8 and 9 students under free, semi-structured and structured problem-posing situations. The strategies identified were classified on the basis of: a) the set of actions used by students to obtain the problem-posing products from the initial problem-posing source; and b) evidence about possible relationships between the problem posed and the student's previous mathematical experience. Four main categories of problem-posing strategies were revealed: *reformulation, re-construction, imitation and invention.*

Students seem to have a natural capacity to pose problems on the basis of a given calculation. The problem-posing actions employed by the students at the beginning of the study did not depart from the types of problems traditionally solved in mathematics classrooms. At the end of the Program a few participants involved in the Euler Level posed problems by extending the structure of the given problem. None of the problems posed in response to the structured problem-posing situation included in the test was classified as an invented problem. However, on a few occasions during the Program some problems posed under a *structured* problem-posing situation were classified as *new* (invented) problems.

All students' problem-posing strategies in response to the semi-structured situation were classified as *imitation.* Semi-structured problem-posing situations used in the project classroom nurtured the creation of several *new* problems. It was observed that, by the end of the study, the preference of most students in the project classroom had changed from working with structured to semi-structured problem-posing situations.

All students recognised that problems posed under *free* problem-posing situations were the *same* or *similar* to ones they had seen before in textbooks or which had been solved in school. The study indicates that it is very likely that the process of free problem-posing can be linked to the level of students' problem-solving performance on the topic area within which the situation is created.

CHAPTER EIGHT

THE EFFECTS OF STUDENTS' EXPERIENCE IN PROBLEM POSING ON THEIR MATHEMATICAL PERFORMANCE

The focus of this chapter is on the effects of an open problem-solving approach on selected aspects of students' problem-solving and problem-posing performances. Four ways of measuring students' problem-solving performance were utilised. The structure of the discussion is based on mathematical performance sub-categories defined in Chapter 1 and on the assessment schemes described in Chapter 5. The results of the participants in the project classroom, on a number of mathematical performance sub-categories, are compared and contrasted with those of students who were exposed only to problem-solving activities. The sub-headings relate directly to the research questions formulated in Chapter 3.

The Effects of Students' Experience in Problem Posing on Their Problem-solving Mathematical Performance

Four performance sub-categories were created to describe different ways of measuring students' problem-solving performance. These were performances on:

- mathematical skills — tests results;
- solving application problems — tests results;
- results on the solutions to the problems on the *Challenge Stage* 1995 (six *Challenge Problems*), and to the problems on the *Enrichment Stage* 1995 (sixteen *Challenge Problems*);

- individual achievements on the Australian Mathematics Competition — 1995 and 1996.

In addition, independent observers' impressions about changes in students' mathematical performance will be reported.

Performance Sub-category: Mathematical Skills — Tests Results

Students' mathematical skills were assessed with a test based on applying the concept of *percent*. The Mathematics Questions Set 2, which students undertook at the beginning and at the end of the study, included five problems in multiple-choice format and two word (process) problems of different levels of difficulty for which students were required to present complete written solutions. All problems in Set 2 (see Figure 3.2) related to the application of the concept of *percent*. In particular, this concept was used in problems designed to test some representative basic calculation skills, and for solving two word problems.

The mathematical context of the word problems (see Items 6 and 7) was chosen so that the concept of percent was applied to situations likely to be familiar to students. In other words, it was anticipated that the contexts of these items were ones which students would have encountered in their every-day lives.

Item 1: $^{2}/_{3}$ of 15 is:
A) 6; B) 10; C) 15; D) 5.

Item 3: 120% of 50 is:
A) 62; B) 60; C) 600; D) 620.

Figure 8.1. Items 1 and 3 from the Mathematics Questions Set 2.

The basic mathematical skills which students needed for solving the word problems were tested with items 1 to 5. Students were presented with five questions in a multiple-choice format and asked to circle the correct answer. Items 1 and 3 required the students to calculate the value of a particular fraction or percent of a given number (see Figure 8.1).

Item 2: $^{2}/_{5}$ of a specific number is 10. Which is the number?
A) 50; B) 100; C) 25; D) 4.

Item 4: 30% of a specific number is 21. Which is the number?
A) 630; B) 141; C) 70; D) 63.

Item 5: Which of the following has the same value as 1994/1995 ?
A) $\frac{1994-2}{1995-2}$; B) $\frac{1994+1}{1995-2}$; C) $\frac{1994^2}{1995^2}$; D) $\frac{3 \times 1994}{3 \times 1995}$;

Figure 8.2. Items 2, 4 and 5 from Mathematics Questions Set 2.

Items 2 and 4 from Mathematics Questions Set 2 required students to apply reasoning which was the reverse of what was needed for solving Items 1 and 3: given a particular fraction or percentage, students were required to find the number (see Figure 8.2). Item 5 called on students' skills in recognising an extension of a specific fraction.

Students' solutions to Items 1 to 5 was given a score of 0 when the answer was not correct, or 1, for a correct response. The percentage of participants in Group A and B, who gave correct responses at the beginning and at the end of the study are shown in Table 8.1.

An analysis of the data from individual responses on these test items suggests that all participants in both groups, at the beginning and at the end of the study, had the computational skills needed for solving the word problems (Items 6 and 7).

Table 8.1 also reveals that, in both groups, the level of students' mathematical skills for solving basic problems which involved the concept of percent was, on the whole, higher at the end of the study, although the program did not include application problems of this particular concept.

Table 8.1.
The Mean Scores in Percentages of Correct Responses for Group A and Group B on the Mathematical Skills Items in Pre- and Post-Tests

Item	**Group A**		**Group B**	
	Pre-test	**Post-test**	**Pre-test**	**Post-test**
Item 1	100	100	100	100
Item 2	100	100	100	100
Item 3	100	100	100	100
Item 4	92	100	86	98
Item 5	69	85	73	90

Problem-posing activities aimed at helping students to improve their performance on solving a particular type of mathematical problems were frequently used during the study (see Chapters 6 and 9). Observations made by the independent observer in the project classroom, tape-transcriptions and compiled students' individual worksheets suggest that students' experience in posing and solving a particular type of problems there affects, in a positive way, their skills in solving problems from the types they learnt to pose. The episodes presented in Chapters 6 and 9 suggest that students' experience in posing problems of a specific type helps them to solve problems with isomorphic and similar structures. The following excerpt is taken from one of the independent observer's notes in the project classroom on 25th of May, 1995:

> 5.25pm: Solutions to algebra questions. Similarly to the first class, asked which rules were used to find the answers. But then [different to first class] the students were asked to invent their own questions, similar to the one they had completed, and have other students answer them; very successful and one girl [Chermaine] who did not understand at first, caught on after many problems had been invented.

Thus problem-posing activities incorporated as part of solving a particular type of problem are likely to help students to grasp the formal structure of the problem and the structure of the solution method as well.

Performance Sub-category: Solving Application Problems - Test Results

Students' problem-solving skills for resolving real-life situations related to the concept of percent were examined through Items 6 and 7 from the Mathematics Questions Set 1. The first item presented a situation which students might face in an every-day life context:

Item 6: If a discount of 20% off the market price saves you $15, how much will you pay for the jacket?

Devising a solution to this problem could be approached in one of several possible ways. For example, students could apply the skills already tested in Item 4 in order to find the initial price of the jacket and then calculate the final price by subtracting the discount. Or students could find directly the new price of the jacket, which is four times greater than the amount of the discount made.

In Table 8.2 the mean percentage scores for students' solutions to Item 6 in the Mathematics Questions Set 1 are presented. All scores were obtained in accordance with the assessment schemes introduced in Chapter 4.

Table 8.2.
The Mean Percentage Score Results for Group A and Group B for Item 6, Mathematics Questions Set 1

Assessment Aspects	**Group A**		**Group B**	
	Pre-Test	**Post-Test**	**Pre-Test**	**Post-Test**
Correctness	87.5	100	79	93.9
Originality	71	76	66.7	69.7
Accuracy	71.5	80	63.7	69.7

As can be seen from Table 8.2, first, at the end of the study students from both groups showed an improved problem-solving performance on Item 6. And second, the relative increase in problem-solving performance on Item 6 is greater for Group A than for Group B.

It should be noted that a decision was made not to conduct individual post-test interviews with students. First, research investigations carried out by other researchers, for example by Silver et al. (1996), showed that students written work can be successfully interpreted. And second, there was an expectation that any verbal prompts, although carefully selected, might affect students' problem-solving performance on the post-test.

Item 7, which is shown below, was regarded as the most difficult in Mathematics Questions Set 2 because of the complexity of its solution and therefore it was placed at the end of the test.

Item 7: A jacket has been discounted twice: once with 15% off and twice with 20% off of the new price. What was the initial price of the jacket, if its price now is $136?

Again it was anticipated that students would apply the mathematical skills tested in Items 1 to 5. The problem could be approached in several ways, such as working backwards, solving an equation, etc. The mean percentage scores for students in Groups A and B are shown in Table 8.3.

Table 8.3.
The Mean Percentage Score for Students in Group A and Group B on Item 7 in the Mathematics Questions Set 1

Assessment Aspects	**Group A**		**Group B**	
	Pre-Test	**Post-Test**	**Pre-Test**	**Post-Test**
Correctness	47.6	85.7	45.3	69.7
Originality	43	76.2	39.3	54.3
Accuracy	43	71.4	39.3	54.5

Table 8.3 suggests first, that at the end of the Program students from both Group A and B showed greater gains in problem-solving performance on Item 7. And second, students who experienced an open problem-solving approach have produced slightly larger improvements in problem-solving performance on Item 7 than students who did not.

These conclusions were supported by the facts that at the beginning of the Program, 13 percent of the students from Group A and 27 percent of those from Group B presented a full solution for Item 7. At the end of the Program, these figures were respectively 71 percent of students in Group A and 58 percent of the students from Group B.

In addition, *changes* in the quality of *solution approaches* used by students to solve this problem were noted. Two students from Group A (Nora and Karel) and two from Group B (Hillary and Dan) presented solution ideas which were to some extent "better" from the ones used at the beginning of the study. Figure 8.3 includes examples of Nora's solutions which show how the student approached the problem differently in the pre-test and post-test.

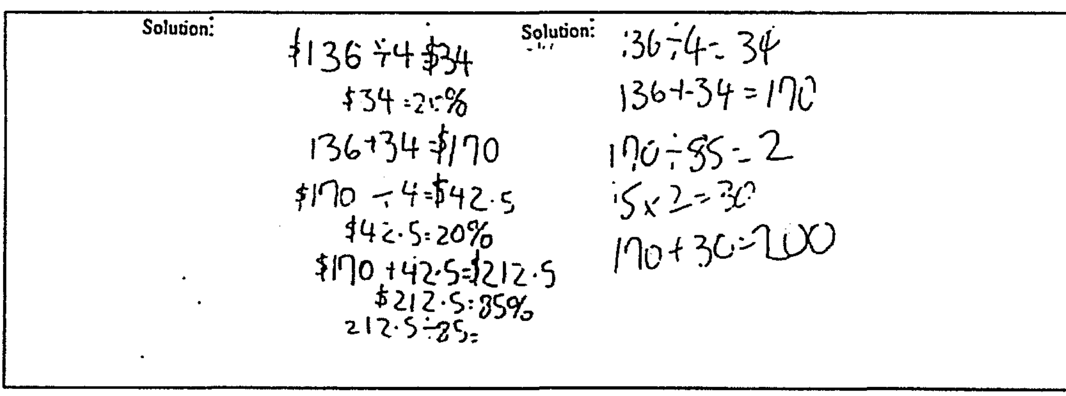


Figure 8.3. Examples of Nora's solutions on Item 7 which show a change in the solution idea.

Performance Sub-category: Results of Solutions to the Challenge Problems

In addition to the results on the pre- and post-tests, students' solutions to the 1995 Challenge Stage (six *Challenge Problems*), 1995 Enrichment Stage (sixteen *Challenge Problems*) of the Euler Program, and to problems from the Australian Mathematics Competition (AMC) (in 1995 and 1996) in which students participated on a voluntary basis, provided additional assessment data related to students' problem-solving performance.

Table 8.4 presents the mean scores on the *Challenge Problems* for students in Groups A and B, who submitted their solutions, shown over a period of one year on the *1995 Challenge Stage* and *1995 Enrichment Stage*:

Table 8.4.
Mean Scores (out of possible 24 for the Challenge Stage and out of possible 64 for the Enrichment Stage) and Percentage of the Students in Group A and Group B who Received Certificates for the Solutions to the Challenge Problems

Group/ Certificate	1995 Challenge Stage	1995 Enrichment Stage
Group A		
Mean score	13.5	33.4
Certificates received by students at the end of the Program:		
Excellence (Top 8%)	29	29
Merit (Next 17%)	14	14
Achievement (Next 25%)	29	29
Group B		
Mean score	14	32.2
Certificates received by students at the end of the Program:		
Excellence (Top 8%)	27	27
Merit (Next 17%)	18	18
Achievement (Next 25%)	27	27

Although all *Challenge Problems* are an important part of the Enrichment Programs, the submission of their solutions was not compulsory for the students. The solutions involved the use of mathematical concepts and solution techniques which are not part of the school curriculum. In order to support equally students from both groups in their work on these problems, the researcher prepared some written hints for both Groups A & B (see Appendix 7). The assessment scheme was provided by the organisers of the Euler Program. The solution to every problem was given a score between 0 and 4 points, according to the correctness of the main stages of an appropriate solution idea.

The mean score obtained by students in Group A (33.2) on the *Enrichment Stage* was slightly higher than that of students in Group B (32.2), but the overall percentage of students from both groups who received certificates for *excellence, merit* and for *achievement* was about the same. Two of the participants in the study (one from Group A (Samantha, Year 9) and one from Group B (Hillary, Year 8)) had the highest performance in the State on the *Challenge Problems*.

There was a clear difference in the quality of solutions provided by students from both groups to the *Challenge Problems* for the *Challenge Stage* and the *Challenge Problems* for the *Enrichment Stage*. At the end of the Program students from Group A and B provided solutions which were more precise and more importantly, which were beyond their problem-solving performance at the beginning of the Program. Although some individual students improved their problem-solving performance on the *Challenge Problems* for the *Enrichment Stage* profoundly (for example Brad, Norm, Hillary, Dora, Rebecca (Group B) and Tom, Karel, Samantha, Nora, Martin, Carol, Nelly (Group A) however, the researcher found these data insufficient for drawing inferences about the project classroom. First, despite the scoring schemes, there may have been variations among the examiners in interpreting these schemes. Second, not all participants in the study submitted solutions to the *Challenge Problems*. Third, although the content of the Program was not part of the school curriculum content, it was possible that some students may have had help from other sources such as parents, friends, teachers. Finally, if a student presented a solution *different* from the one provided by the organisers, then this would imply the need to use a different scoring scheme (and necessitate idiosyncratic decisions on the part of examiners).

Performance Sub-category: Results on the Australian Mathematics Competition

During the Program, the students participated in one or two papers for the *Australian Mathematics Competition* (AMC) organised from the Australian Mathematics Trust. This is the most popular competition in Australia with more than 500 000 participants. The competition involves solving 30 multiple-choice problems in 60 minutes. In 1995, 14 percent of participants in Group A and 27 percent of participants in Group B were among the best 100 out of 9164 contestants in the State. One of the students in Group B, let us call him Norm, obtained the highest result for AMC in the State in 1995.

One year later, 55 percent of the students from Group A and 46 percent of the students from Group B were among the hundred best-performing students in the State on the AMC.

Independent Observers' Impression

The following excerpts have been taken from the notes made by the independent observer who had mathematical and pedagogical background, about her impression of changes in students' performance in the two groups during the course of the project:

Group A:

Most students in this class appeared to make substantial improvements in their abilities throughout the year, eg Nicki, Carol, Tom and Irene—at first they contributed little, often looked puzzled or couldn't provide answers when called on, etc; but by the end of the year were confident in sharing correct and useful answers.

Also students initially were not good at responding to the types of questions Elena asked, but by the end of the year were able to contribute greatly to class discussion.

Group B:

Obviously some improvement made over the course of the year by most students; although little improvement by some who tended to be easily distracted and talkative. However improvements in abilities and confidence certainly not as marked as in the second class [Group A].

Thus, according to the independent observer, first, students who were exposed to an open problem-posing approach (Group A), showed greater confidence in their approach to problem solving at the end of the study. And second, the open problem-solving approach appears to create an environment which nurtures appropriate discourse. This discourse differs from that in a traditional classroom and students need to develop particular skills for responding to questions incorporating "hidden" problem posing.

At the end of the Program, on putting a question to the second observer concerning his view about both groups, he said: "Students from Group A are better at problem solving."

It is recognised that there are, from a qualitative perspective, limitations which are likely to make it difficult to compare some data from the two classrooms. Nevertheless, the results presented in this chapter demonstrate first, that an open-problem-solving approach helps students to improve their confidence and subsequent performance on *specific* problem-solving tasks from the type students learnt to pose. And second, students who were exposed to an open-problem-solving approach produced higher achievements on a number of performance sub-categories than students who did not.

The Influence of Students' Experience in Problem Posing on Their Problem-posing Performance

The influence of students' experience in problem posing on their problem-posing performance will be discussed according to the following problem-posing performance sub-categories defined in Chapter 5: (a) language accuracy; (b) correctness; (c) level of difficulty; (d) fluency; and (e) flexibility.

The design of Mathematics Questions Set 1 (see Figure 3. 1) included a free, a semi-structured and a structured problem-posing situation. The main goal was for the students to have an opportunity to reflect on the process of problem posing under environments created from different problem-posing categories.

In the first problem-posing-situation, students were asked to make up as many problems as they could on the basis of the calculation "$3 \times 25 + 15 \div 5 - 4$."

In the second problem-posing situation students were given a sequence of 6 symbols (1 2 3 * 5 *), four of which were integer numbers. They were asked to: (a) suggest meaning for the missing elements; and (b) construct mathematical problem(s) by using one of these meanings. The wording of the statement did not place emphasis on the number of problems to be posed.

The third problem-posing situation required students to pose a problem similar to one the students enjoy solving, and invited them to explain why they liked it and how they created it.

Performance Sub-category: Language Accuracy

The precision of the language used by students to formulate problems was one of the aspects of problem posing which was regarded as an important characteristic of students' problem-posing performance.

The results presented in Table 8.5 suggest that students from both groups at the end of the Program tended to present more accurate formulated problems.

Table 8.5.
The Mean Percentage Score Results for Group A and Group B on Language Accuracy Shown on Pre-Tests and Post-Tests

Language	Structured		Semi-Structured		Free	
Accuracy	Pre-Test	Post-Test	Pre-Test	Post-Test	Pre-Test	Post-Test
Group A	57	81	33.3	52	38	62
Group B	69.7	75.7	48.3	81.7	63.6	75.7

In addition to the data provided by the problem-posing pre- and post-tests, data from the project classroom were collected throughout the year. A distinction was made between problems in which the language of the formulated problems was not precise, and those which involved precise and appropriate use of mathematical terms. At the end of the study, the problems formulated by some students in a structured, semi-structured or in a free problems-posing situation were constructed more precisely and used language which included more appropriate mathematical terms.

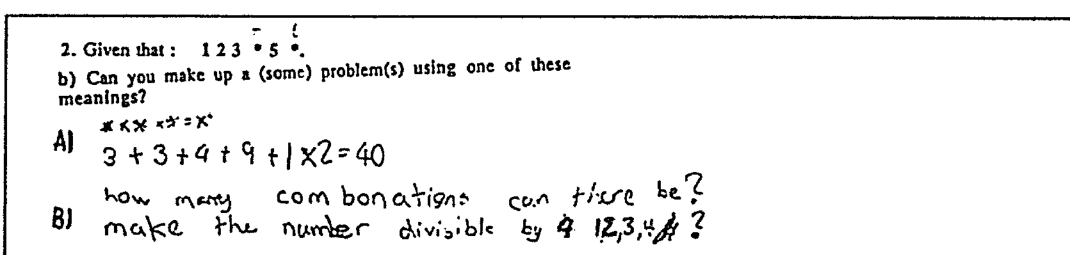

2. Given that: 1 2 3 • 5 •.
b) Can you make up a (some) problem(s) using one of these meanings?
A) 3 + 3 + 4 + 9 + 1 x 2 = 40
B) how many com bonations can there be? make the number divisible by 4 1,2,3,4,5 ?

Figure 8.4. Problems posed by Tom at the beginning and at the end of the study.

The examples presented in Figure 8.4 show problems posed by Tom at the beginning and at the end of the study which reveal a difference in the language accuracy.

Performance Sub-category: Correctness

Students from both Groups A and B showed an improvement when the correctness of the problems posed at the beginning and end of the study - posed under free, semi-structured and structured problem-posing situations - are compared. From Table 8.6 it appears that students' exposure to an open problem-posing approach has influenced, in positive ways, the correctness of students' problem-posing products when semi-structured and free problem-posing situations were adopted. The results of Group B show that problem solving had, by itself, a positive effect on the correctness of the students' problem-posing responses in two of the three problem-posing situation categories.

Table 8.6
The Mean Percentage Score Results for Group A and Group B on the Correctness of Problem-Posing Products

Correctness of the Problem	Structured		Semi-Structured		Free 3	
	Pre-Test	Post-Test	Pre-Test	Post-Test	Pre-Test	Post-Test
Group A	71.4	81	38	81	43	80.1
Group B	90.7	75.7	63.3	84.7	78.9	87.7

The classroom observations suggest that there is a link between the type of problems students pose and the *correctness* characteristics. For example, when students imitate a problem structure by posing problems similar to a given problem, or when they illustrate the use of a concept by constructing specific examples, it appears more likely that they will pose a correct problem. When problem posing involves interpreting an equation and presenting the mathematical relationships through a real-life situation which can be modelled by the given calculation, then students are more likely to experience some difficulties. This could be one of the reasons for the lower mean score on the correctness of problem-posing products shown by Group B at the end of the Program.

Performance Sub-category: Level of Difficulty

Assessing the problem-posing product difficulty was another aspect considered in this study. The difficulty of the problem refers to the complexity of the problem solution structure needed for the posed problem and takes into account whether the solution method is familiar or not familiar to the student. In other words, when students posed complex problems by imitating previous classroom experiences the problem item was assessed as "not too difficult." In all cases when students had had little or no experience solving the type of problems they had posed, the problem was assessed as "difficult."

The level of difficulty is deliberately oriented towards the complexity of the solution structure rather than to the *problem* structure. The complexity of the problem structure plays an important role for *understanding* the problem. The problem-solution complexity in this study refers not to the number of the steps in a specific solution, but rather to the complexity of the mathematical idea involved in obtaining the solution.

Table 8.7 suggests, first, that the level of difficulty of problems posed by students from Group B was not hampered by the Program. Second, the open problem-solving approach implemented in the project classroom had a positive effect on the level of difficulty of problems posed by students from Group A.

Table 8.7.
The Mean Percentage Score Results for Group A and Group B on the Difficulty of the Problem-Posing Products

Level of Difficulty of the Problem	**Structured**		**Semi-Structured**		**Free 3**	
	Pre-Test	**Post-Test**	**Pre-Test**	**Post-Test**	**Pre-Test**	**Post-Test**
Group A	38	43	23.7	66.7	28.7	62
Group B	51.3	44.7	36.3	60.3	51.3	60.3

The major difference between the problems posed by students from the project classroom at the beginning and at the end of the Program was the complexity of the solution idea involved. At the end of the Program, students in the project classroom tended to pose more problems which involved a prediction based on a general idea. In fact there was a clear trend that problems created at the end of the study required the use of more complex mathematical concepts and associated solution methods than the problems posed at the beginning of the Program.

Performance Sub-category: Fluency

Fluency was the term adopted to refer to the number of all-correct responses given by a student with respect to a particular problem-posing situation (see Chapter 5). This characteristic was required for Item 1 (validated as a structured situation) and Item 2 (validated as a semi-structured situation) in the Mathematics Questions Set 2. The mean scores of students on these two situations on the pre-test and post-test are presented in Table 8.8.

The results shown in Table 8.8 suggest first that, for structured problem-posing situations, students from Group A posed an average of nearly two responses.

Second, at the end of the study there was a relatively large increase of the average of correct responses for students from Group A on the semi-structured problem-posing situation. Third, at the end of the study, students from both Groups A and B presented more correct responses on semi-structured than on the structured problem-posing situation.

Table 8.8.
The Mean Score for Students from Group A and Group B on Fluency Shown on Pre-Tests and Post-Tests

Fluency	Structured		Semi-Structured	
	Pre-Test	Post-Test	Pre-Test	Post-Test
Group A	1.86	1.43	0.57	2.29
Group B	2.18	1.36	0.81	1.36

Table 8.8 illustrates also that at the end of the Program students' fluency on the structured problem-posing situation for both groups A and B decreased. These results can be easily explained with the trends observed in both classrooms. At the end of the study most students in Group A and a number of students in Group B tended to pose problems from different categories rather than problems with isomorphic structures.

Performance Sub-category: Flexibility

Flexibility is a characteristic of the problem-posing product which refers (see Chapter 5) to the number of different problem categories posed by students when presented with specific problem-posing situations.

Table 8.9 presents the mean scores of the participants of the study on the pre-test and post-test.

Table 8.9.
The Mean Score Results on Flexibility Shown by Group A and Group B on Pre-Test and Post-Test

Flexibility	Structured		Semi-Structured	
	Pre-Test	Post-Test	Pre-Test	Post-Test
Group A	1	1.43	0.43	1.57
Group B	1.18	1.18	0.73	1.09

The difference in the number of problem categories within a specific problem-posing situation is the other major change observed for students' problem-posing performance on structured and semi-structured situations when the beginning and end-of-study data are compared. Although students from Group B showed no improvement in their flexibility for structured situations, the Program had a positive effect on their abilities to pose more problem categories on semi-structured situations. Students from Group A, who were exposed to the open-problem-solving approach, posed far more problems categories at the end of the Program when compared with the beginning of the study.

Table 8.10.
Combined Mean Results for Group A and Group B on the Structured and Semi-structured Problem-posing Situations on the Pre- and Post-Tests

Combined	Group A		Group B	
Results	Pre-Test	Post-Test	Pre-Test	Post-Test
Language	43	65	59.6	77.8
Correctness	50.8	81	76.7	92.3
Originality	31.7	58.7	68.7	80.8
Difficulty	30.2	57	46.5	55.6
Fluency	1.2	1.8	1.41	1.33
Flexibility	0.71	1.5	0.86	1.09

The positive influence of the open problem-solving approach on students' problem-posing performance is also supported by the combined results shown in

Table 8.10. These results illustrate the mean score for Groups A and B on each of performance sub-categories on the structured and semi-structured problem-posing situations. For example, 43 percent on the language performance sub-category for students in Group A means, that if students' results on semi-structured and structured problems-posing situations on the Pre-test are combined then the average score is 43 percent.

The results in Table 8.10 suggest that students in both Groups A and B showed an increase in problem-posing performance at the end of the Program. However, the relative increase in problem-posing performance is much greater for students from the project classroom (Group A) than for students exposed only to problem-solving activities (Group B).

Observations made in the project classroom are consistent with the results presented in Table 8.10. At the end of the Program, most participants from the project classroom seemed to feel free to expose their problem-posing performance. They tended to provide examples from different categories. In addition, students demonstrated an increased attention to the language, they seemed to feel free to expose their understanding and to make conjunctures beyond their problem-solving experience.

Although some severe limitations inherit from the use of a qualitative perspective, the results of this study demonstrate that the relative increase in problem-posing performance is much greater for Group A than for Group B.

CHAPTER NINE

CASE STUDIES

Introduction

Case studies of two students — Karel and Samantha — which were developed during the study will be introduced in this chapter. These case studies will illustrate the broad range of classroom contexts in which problem-posing activities were used to help these two students to reflect upon their problem solving through problem posing. Excerpts, presented in chronological order, will illustrate the nature of the students' work. Their problem-solving and problem-posing performances will be assessed from several different perspectives.

First Meeting with Karel and Samantha

In December 1994, it was just before Christmas when a colleague of mine asked me to meet a seven-year-old boy whose parents thought he was mathematically gifted.

I met Karel and his family a month later, on 23rd of January. It was about 10am when the parents came with their two children: a girl, Samantha, aged 13 years, and her brother, Karel. I gave a booklet with some interesting geometry problems to Samantha and started to work with Karel.

Case Study One: Karel – An Individual Profile

In order to gain an idea of the level of his mathematical performance and of the problem-solving strategies he could apply, I decided to involve Karel in solving some interrelated problems. Although his father was sitting next to us and he observed all my work with Karel, it seemed that the child felt comfortable with his presence. We started with a simple problem:

> *Problem 1:* Five apples have to be divided between two people, one should have an apple more than the other. How many apples will each of them get? How should that be done?

Figure 9.1. An illustration of the first problem solved by Karel.

I illustrated the verbal presentation of the problem statement by the picture in Figure 9.1 and asked Karel to answer my questions. The answer Karel gave to the first question came immediately: "Three and two." He drew a line between the third and the fourth apple. Karel did not pay attention to the second question. The practical way the division could be done was probably not important to him. In order to prompt the child to pay attention to the ways in which the division of the apples could be done, I decided to design a similar problem in which the number of the objects was greater.

> *Problem 2:* Eighteen apples have to be divided between 2 people. The second person should get 4 more apples than the first. How many apples will each of them get? How should that be done?

Karel wrote: 18 - 4 = 14; 14 ÷ 2 = 7. Then he added "The first will get seven, the second, seven plus four, eleven." Karel was also able to verbalise the solution by explaining the practical meaning of the operations: "Take away four, the rest should be divided into two groups of seven each. Then the first gets seven, the second person gets seven plus four apples."

I wanted him to solve the problem using another algorithm and I asked: "What about if I do not want to divide the apples into two groups?" He answered: "Give four to the second person and start to give one apple to each of them until the apples are finished."

My next question was about how he would share 15 apples between two people if the second person should get one more than the other person, his answer was correct. "Should I give one apple to each of them until the apples are finished?" I asked. "You could give them two apples, it will be quicker," he answered. At that stage he did not recognise that the "*second* person should get one apple more" was a restriction in the problem statement, and did not suggest from which person the division process should start.

I gave him a problem in which this restriction was more clearly presented:

> *Problem 4:* Karel, imagine now, that there are three people, You, your sister and my son - Nick and you have 18 apples. The second person should get one apple more than the first, the third person–one more than the second. [I used the same language for the relationships and I made a picture describing the mathematical content of the problem]. How many apples will each of them get? How will you divide the apples?

This time Karel had to find the answer, explain to me the way the division could be done, and convince me that the answer he got was the right one. As usual, after explaining what the problem was about, I drew a picture. I hoped that it might prompt him to show the algorithm first and then mathematise it.

He solved the problem using the picture: one apple to the second person, two to the third and the rest should be divided equally between them: five each. He drew a line after the third apple and two more lines to divide the rest of 15 apples into three groups of five apples in each. Then "The first person gets five, the second six and the third seven apples," Karel said.

I asked Karel to write some words for me to explain to me how I should divide the apples. In his solution he added information (shown here in italics) which was not in the problem statement:

> First you give one apple to my sister, *because I am the youngest.* So she must have one apple more than me and Nick must have one apple more than my sister *because he is the oldest.* So he must have 2 apples because my sister had one. There were 18 apples and now there are 15. Then you just divide [15 ÷ 3 = 5] the apples among the three people.

The next problem did not have a solution: There are 18 apples. The second person gets one apple more than the first, and the third person gets two more apples than the second person. But this time I just wrote the number 18 and presented the relationships diagrammatically as in Figure 9.2.

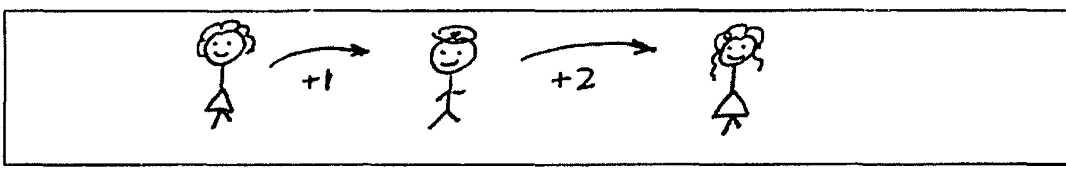


Figure 9.2. An illustration to the problem.

Karel started to share the apples and said: "I cannot divide fourteen apples into three groups, I need fifteen apples. Nineteen is a *good* number, not eighteen." "Can you tell me another *good* number?" I asked. "Twenty-two" was the immediate answer. But Karel could not explain to me why this was a *good* number.

The next problem situation we made together. It was more complicated and is shown in Figure 9.3. This time we decided to have a basket with some apples and Karel suggested that there should be four people. The second person should get one apple more than the first, the third person one more apple than the second, and the fourth person two more apples than the third person.

Figure 9.3. First problem made with Karel.

He solved the problem by applying the same solution strategy — working backwards. We need to have a difference of 7 apples. After that he wrote 12 ÷ 4 = 3 (he took a number which is divisible by 4) and added 3 respectively to 1, 2, and 4. Under the figures he wrote 3, 1 + 3 = 4, 2 + 3 = 5 and 4 + 3 = 7, explaining to me that these numbers are the apples which everyone should get in this case.

T: Tell me another good number of apples, Karel?
K: 15.
T: Could we have less than 15 apples?.
K: Yes, 7.
T: What about the number of the apples which everyone will get when we have only 7 in the basket?

He wrote 7 - 7 = 0; 0 ÷ 4 (stopped and after a while he wrote) = 0. Then he proceeded and wrote confidently: 1 + 0 = 1; 2 + 0 = 2; 4 + 0 = 4.

I continued to question him:

T: Could we have a smaller number of apples than 7?
K: No, we cannot.
T: What about a bigger number?
K: Yes. [He wrote several numbers: 7, 11, 15, 19, 23 and 27].
T: Why is the difference always 4?
K: The number of the people is 4.

This explanation was enough. I decided to stop, because Karel was starting to look tired. "We are finished" I said to the father. We discussed with Karel's parents their plans, and I invited Karel to the mathematics classes at Edith Cowan University. In one of these classes, there were some younger students from Year 5 and I was hoping that there would be a place for him in one of the junior groups.

"What about his sister?" the father asked me. I had not expected any questions about the other child and I did not have any idea about the level of Samantha's mathematical experience. But the classes in the University were free and I had decided to keep the doors of my classes open for all children who wanted to be challenged. "OK, she is welcome to come to my class," I said.

9th February, 1995

When my first lesson with the participants in the study at Edith Cowan University had finished, I saw Karel and his parents patiently waiting for me. "There is no place in the Year 5 class for Karel, his father said." I did not know what to say. My class was a big group with mathematically able Years 8 and 9 students, and Karel was only in Year 3. I would have to help build all of the additional mathematical language and skills that he would need to understand the program content. But the small boy was looking at me with his big eyes, expecting my decision. His sister was in my class, and there was no place for him in the other group. I did not have the moral right to disappoint the child and I said: "OK, I will find one place on the first row for you. Will you come, Karel?" He could not hide his smile and just said: "Yes."

At the next session, in the first row, I had my youngest-ever student in my teaching experience — Karel Chun.

Pre-test Results

In the pre-tests Karel attempted only the simplest problems. He gave correct responses to the first three questions from the Mathematics Question Set 2 (see Figure 9.4) and managed to solve Item 6 correctly.

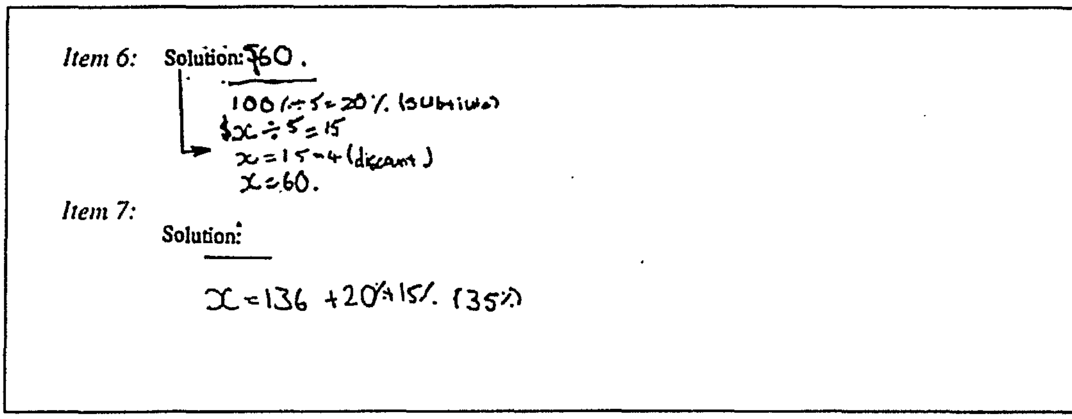


Figure 9.4. Karel's responses to the Mathematics Questions, Set 2.

On the last problem, Item 7, he made a logical mistake, assuming that the total discount would be 20% + 15% = 35%.

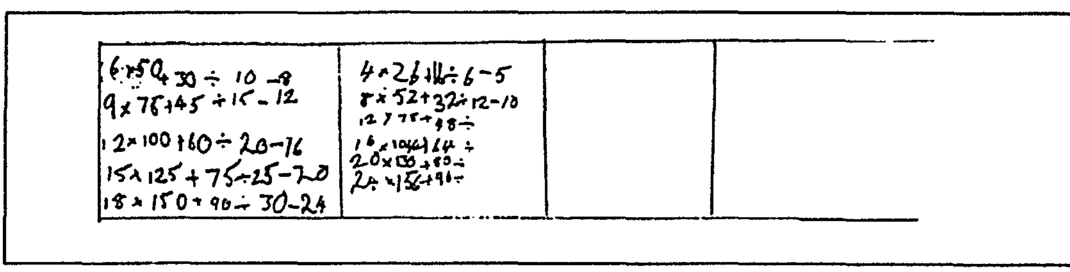


Figure 9.5. Karel's responses to Question 1 from the Mathematics Questions, Set *l*.

Karel attempted only the first task in the problem-posing test, the Mathematics Questions Set 1, (see Figure 9.5). He posed one category of seven correct examples by keeping the initial order and type of the operations the same but changed the numbers. The other problem-posing situations were not attempted.

6th April, 1995

Prime numbers, factors, indices, even and odd numbers — all of these concepts were new for Karel. At the beginning of the lesson he was able to continue the sequence of examples of prime numbers:

T: Which numbers are prime, Karel?
K: . . .
T: Prime numbers are 2, 3, after that?
K: 5,
T: 5, after that?
K: 7,
T: 7, 11,
K: 13, 17.
T: Is 1 a prime number?
K: No.
T: No, why Karel?
K: . . .
T: Because 1 is divisible only by 1. It doesn't have exactly 2 different factors.

The question about why 1 is not a prime number was difficult not only for Karel. He had to recall the key relationships in the definition of a prime number and apply these to the new situation.

Several times during the same lesson I asked Karel to guess and predict the *goal* on problems with the unstated question I was using. All questions involved new mathematical concepts:

T: Which are the numbers of which 2^3 is divisible, Karel?
K: 4.
T: 4. Good number, another one?
K: 2, 8.
T: 8. Another one?
K: . . .
T: OK, I'll write this, 2, 2 to the power of 2, 2 to the power of 3, [I wrote the factors of 2^3 as a sequence of powers with basis of 2: 2, 2^2, 2^3] and there's something missing in this sequence, what is that?
K: . . .
T: Which is the number of which every number is divisible?
K: . . . 1?
T: 1, OK, 1, which can be written as 2 to the power of what?
K: Zero.
T: I've written this 1 like 2 to the power of 0, [I wrote $1 = 2^0$] why? Karel, can you guess?
K: . . .[Karel did not answer].

Later during that session Karel was able to find the number of factors of 3^{11}, and with some hints he then constructed an example in the form a^{11} which has more than 12 factors. He posed a question about the number of factors of 10^{11} by imitating the structure of the previous examples, but could not present any arguments to support his conjecture. Obviously in all cases he was grasping the formal structure very easily and was able to construct an example which belonged to the same class of problems, but was not able pose an element which belonged to a particular class on the basis of the description of that class.

On the individual worksheet paper Karel solved correctly 13 out of 18 questions; one question was not attempted (see Worksheet 9B). A discussion with him during the individual work period showed that he was able to apply the concept of prime numbers for recognising a prime number among a set of integers, to construct examples of numbers which have three factors, and to make a meaningful conjecture that a number has three factors *if and only if* it is a prime number to the power of 2. He also gave some arguments based on a specific example why a non-prime number to the power of two will have more than three factors.

T: What about here, Karel? Write a number which has 3 factors.
K: 9
T: Why does 9 have 3 factors?
K: 3, 9 and 1.
T: Another number?
K: 9, 25, 49 and 121.

To make the next step was quite difficult for both of us. He noticed that all of these numbers were odd, and that they were squared. Finally, the conjecture came:

T: 121, 49, . . . and when a number has exactly 3 factors?
K: When it is odd number . . . a prime number to the power of 2.
T: Is 81 a good number?
K: No.
T: No, because. . .
K: Because it's also divisible by 3.

The language was still not precise, but this was a big step for Karel. This was at about the time that he started to use other strategies for solving multiple-choice questions in addition to his "guess" and "check which of the answers works" strategies. To the question "What is the most important part of problem solving to you that you try to understand and remember after solving a particular problem?" he answered: "The solution."

18th August, 1995

After three months' experience in my class Karel was treated in the same way as the other students. It was no longer necessary to help him acquire the mathematical skills and language he needed to understand the program content. The next episode recounts a discussion with Karel about Worksheet 20 (see Appendix 4). I wanted to focus Karel's attention on the interrelationship between the problem structure and the solution idea.

> T: Do you understand what the first problem is about? [I showed him the problem about the value of: 1 - 2 + 3 - 4 + 5 - 6 + . . . - 998 + 999 - 1000 + 1001 and read it] 1 minus 2, plus 3, minus 4 plus 5, minus 6, and after that we have. . .?
> *K: Plus 7.*
> T: Plus 7, minus 8, plus 9, . . . and so on. We have to find the sum, but, see how many numbers I have. . . A lot!
> *K: One thousand and one.*

It was not difficult for Karel to work out that the first 1000 numbers can be paired and the sum of every pair is minus 1. His answer of 501 was correct. I invited Karel to make up a problem which could be solved by applying the same solution method:

> *K: 0 minus 1 plus 2*[He wrote 0 - 1 + 2 and stopped].
> T: Minus three... Which should be the last one?
> [He wrote: - 1000 + 1001 - 1002 + 1003] . . . Uh-huh, why did you write, here minus, here plus? [I showed the signs in front of 1000 and 1001]. How did you work out that in front of 1000 you will have minus?
> *K: Oops!*
> T: Oops, what does it mean oops?

K: Mistake.
T: Mistake, it should be . . .?
K: Plus.
T: Plus, why?
K: This one here was minus 1000, and if we start with zero, we have to . . .[He stopped again].
T: You have to move them, you have to have the opposite. Wonderful. It's plus. . .
K: Plus, minus, plus, minus.

I asked Karel to suggest another problem in which he could use the same solution idea. He said:

K: You reverse the numbers.
T: Which one?
K: Instead of going from 1 to 1000 or whatever number you go from the big number to the smaller number.
T: Oh, write it here for me please. . . [he wrote 1005 - 1004 + 1003 - 1002 + . . .] The last number will be which one?
K: Minus 2 plus 1.

He explained that the solution would be found by grouping in pairs, and that every pair would have a sum of 1. The result of 502 +1 = 503 was the correct one. The next problem Karel solved was:

Find the sum of all the two digit numbers greater than 10 such that the tens digit is one less than the units digit.

T: Any unfamiliar words in this problem?
K: No.
T: No. OK. Can you give me an example of such a number?
K: 23.
T: Which is the smallest one?
K: 23 . . . no 12.
T: And which is the biggest one?
K: 98.
T: 98 or 89?
K: 98.
T: Why? 12, 23, 34. [Karel started to read the problem statement again.]
K: They're two digit number. . .
T: Such that the tens digit is one less than . . .
K: 89.

To find the sum he wrote all numbers 12, 23, 34, 45, 56, 67, 78, and 89 and paired them in four sums of 101. He applied the same solution idea (making pairs of equal sums) but he provided no justification regarding why the idea worked.

On several other occasions I invited Karel to pose a problem similar to a given problem and to explain the solution idea. For example, after solving the problem "What is the last digit of 3^{10}?" he posed the problem "Find the last digit of 4^{12}." which was similar to the problem he had just solved. He then explaind how to solve the problem he had posed.

7th September, 1995

A month later, when I presented the class with problem situations which had more than one solution, I tried to help Karel to understand the difference between the *number* of the solutions and the *nature* of the solutions. In one of the problems on the worksheets, the goal statement focused on the number of the solutions (see Problem 1, Worksheet 24, Appendix 4):

Substitute the symbol "*" with a digit in the number 123*7*. So that the number is divisible by:
a) 2;
b) 5;
c) 2 *and* 5;
d) 2, 5 *and* 3.
How many solutions are there in each case?

Using the appropriate mathematical language Karel determined that for option *c)* there were two possibilities for the last digit and ten for the fourth; altogether "fifty," he said, but then he corrected himself and said "twenty solutions."

19th October, 1995

During one of the next sessions, a similar problem (see Worksheet 27, Problem 10) was presented to the class, but in this case, the students were asked to write down all solutions:

T: OK, tell me about [problem] number 6 Karel.
*K: Substitute the symbol, with a digit in the number 973*1*, so that the number divisible by. . .?*
T: 2, 5 and 3. What is the meaning of that Karel?

K: You have to put a number so that the, you have to put 2 digits in, so that the number's divisible by 2, 5 and 3.
T: OK. . . Did you solve it?
K: Yes.
T: OK could you explain the solution to me please? Why did you put zero at the end? [He had written the number 973 710].
K: Because it's divisible by 2 and 5.
T: Because it's divisible by 2 and 5 then the last digit should be a zero. So, for the last digit we have only one possibility. What about the other digit, the middle digit? How many possibilities do we have?
K: 10.
T: 10, why?
K: Coz you can have zero as well.
T: Yes the middle digit can be 0, 1, 2, up to 9, but will we get in all cases a number divisible by 3?
K: Some of them. . .
T: Some of the numbers will be divisible by 3. This one is divisible by 3. Which is the other number which is divisible by 3? You could have 7 [I pointed at the digit 7 Karel had written], and what is the next possible digit here?
K: 4.
T: 4, and the next possible digit?
K: 1.
T: 1. How many solutions does the problem have?. . . How many right answers does the problem have?
K: 3.

Like some of the other students, Karel needed some verbal prompts to find out all of the solutions in this case. But he was able to distinguish between the notions of *number* of solutions and the *nature* of the solutions.

9th November, 1995

The following problem was presented to the students in the last session of the Program (see Worksheet 29, Appendix 4):

A pencil and a rubber cost 25 cents. Seven pencils and 4 rubbers cost $1.30.

a) How much should Greg pay for 2 pencils and 2 rubbers?
b) What will be the price of 1 pencil?
c) What will be the price of 1 rubber?
d) How much should Ben pay for 3 pencils and 2 rubbers?
e) *Other questions?* [italic added]

Some of the students solved the different parts by using simultaneous equations, others used only a linear equation, and some, as Karel did, just guessed the prices and checked them in the problem statement. The following excerpt is taken from a discussion about part b).

> T: How did you find what is the price of one pencil? Just guessed it? [Karel admited that shaking his head]. Good guess! But give me some reasons. Can you give me some reasons Karel? ... Yes why, why the pencil costs a little bit more? . . . Four rubbers and 4 pencils will cost how much? Two pencils and 2 rubbers cost 50c, 4 pencils and 4 rubbers will cost?
> *K: One dollar.*
> T: A dollar. So 7 pencils and 4 rubbers cost $1.30. So, what is the price of one pencil?
> *K: 10.*
> T: 10 cents, excellent. Now can you try to write the solution?

A few minutes later he showed me his written solution (see Figure 9.6).

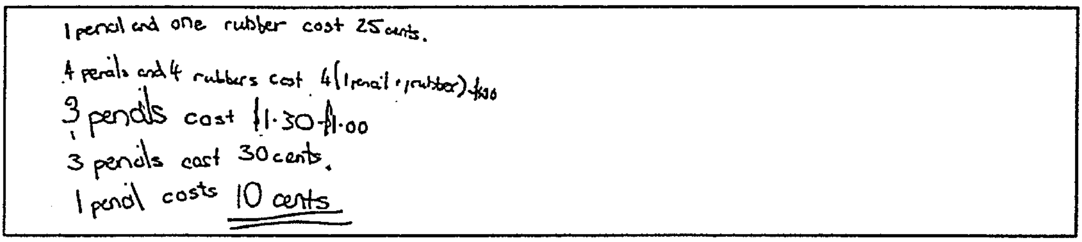


Figure 9.6. Karel's last solution in the Program.

Although Karel had the necessary skills to solve the problem using simultaneous equations (he had solved some word problems by using this technique), it was probably more natural for him to solve it logically.

Two new girls from one of the local government schools who had also been involved in the Euler Program attended the last session. The girls were in Year 8 and were the youngest participants in the *Academy for Young Mathematicians,* conducted by the University of Western Australia. Because they found that the level of the Academy was not suitable for them (most of participants were in Year 11), I invited them to come to the last session at Edith Cowan University. They admitted that their solution for the pencil and rubber problem was based on a trail and error approach. It is relevant to note that I had to explain to them, also what the difference was between a solution and an answer.

Problem-solving Performance Profile

At the beginning of the year Karel received more attention than the other students. In many cases I posed some problems specially for him the aim being to help him develop some basic computational skills.

1. ***Performance on the Challenge Problems.*** Karel was the youngest Western Australian participant in the Euler Program and he received a certificate of excellence for his solutions to the *Challenge Problems.* Some of the solutions were written with a precision which suggested that the author had appropriate understanding of the mathematical concepts and methods he was using.

Classroom observations showed that Karel learned to pay increasing attention to the wording of the problem statement. For example, when Nelly presented her problem to the class (see Chapter 9) Karel asked "What period of time does the dog go 9 metres or the rabbit go 7 metres?"

He did not have any difficulty applying a theorem or an algorithm for solving "standard" questions. As can be seen in the example presented in one of the episodes in this chapter (see 6th April, 1995), it is apparent that, with some prompts, he could make generalisations.

2. ***Problem-solving strategies.*** At the beginning of the study he used to attempt most of the multiple-choice questions problems by "trial and guess," or by "checking the answers in the problem statement." In fact, although he was not able to solve a particular problem, he used his own approach to determine the right answer. Throughout the study he solved problems using a range of different strategies: making algebraic models, listing possibilities, applying inductive extensions and making deductive conjectures.

3. Problem-solv'ng performance. Test results. At the end of the study, on the problem-solving post-test (Mathematics Questions Set 2) he provided correct solutions on 6 out of all 7 items. The only mistake was made on Item 5. His solution for the last Item of the problem-solving test was particularly elegant (see Figure 9.7).

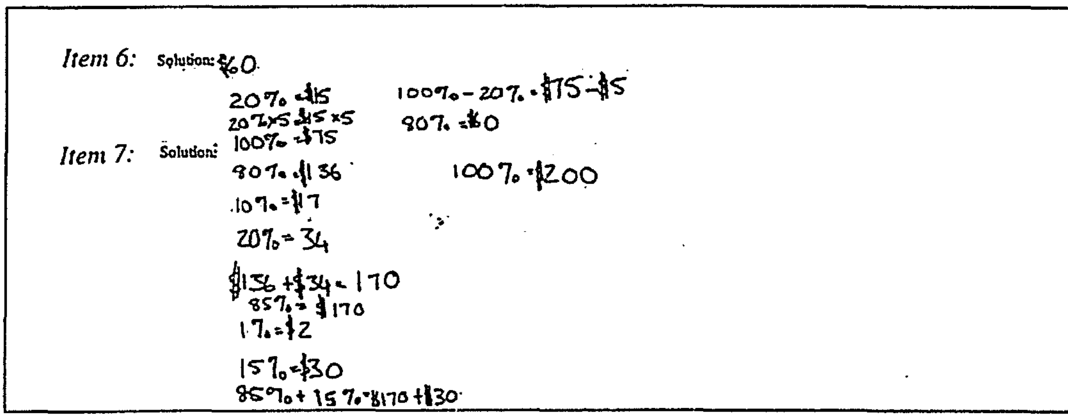


Figure 9.7. Karel's solutions for Items 6 and 7 in the Mathematics Questions, Set 2.

As can be seen from Karel's work, although the mathematical language of his solutions was not precise - for example he wrote 20% = 34 instead of 20% *x = 34, where x is the price of the jacket before the second discount* - the solution methods and the calculations were correct.

Problem-posing Performance Profile

On many occasions during the study Karel demonstrated that he could grasp the structure of a problem or a structure of a solution very easily and could imitate them by constructing problems which were similar to the given problem.

Not all of the problems which Karel posed was he able to solve. Some of Karel's problems contained surplus information, because (according to his sister) "he wanted to blend in the content" which he had been learning. For example, in response to being asked to make up a problem from the domain of geometry, Karel posed the problem presented in Figure 9.8. The problem was set as part of work which was completed at home.

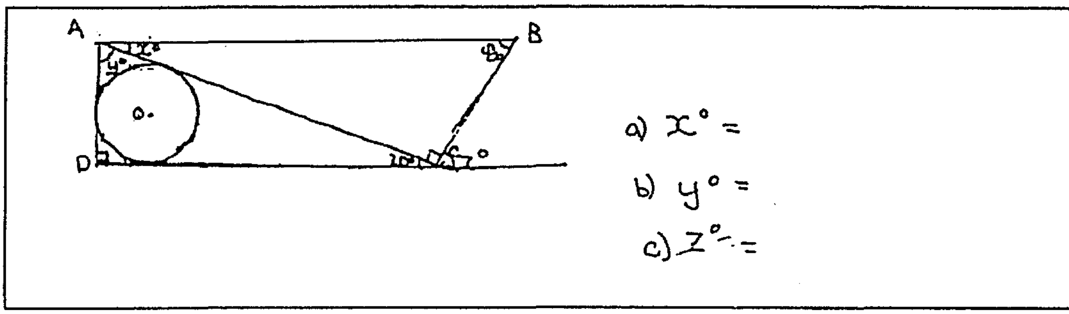


Figure 9.8. Karel's problem which involves some concepts of the domain of geometry.

The problem shown in Figure 9.8 contained surplus information in its structure. I asked Samantha:

T: Who made up this problem, did he make it alone?
Samantha: With Dad.
Karel: No I made up the problem; he helped me with the solution.
T: Can you explain quickly what the problem is about? This is the problem, and what is given Karel, what do you know about this picture? ...
Karel: The 2 right angles.
T: You have one right angle D, and another one here. And something else Karel? You have 2 right angled triangles, this one and this one. And something else? This angle here is 20 degrees, this one is?
Karel: 50.
T: And what is the question about?
Karel: Find x, y and z.
T: There is a circle there, why?
Samantha: Told you so!
T: What?
Samantha: I told him that circle serves no purpose, why put it there?
T: Why is the circle there Karel?
Samantha: He said he wanted it to blend in with what we're learning right now.
T: . . .But it looks nice, and you can extend the problem Karel, and your homework for the holiday will be how to extend this problem to use the circle. You want to have the circle but you have to use this circle somehow.

After the holidays he proudly presented the revised problem (see Figure 9.9) to the class. At the end of the Program Karel was also invited to complete the same problem-posing test which he was given at the beginning (see Mathematics Questions, Set 1).

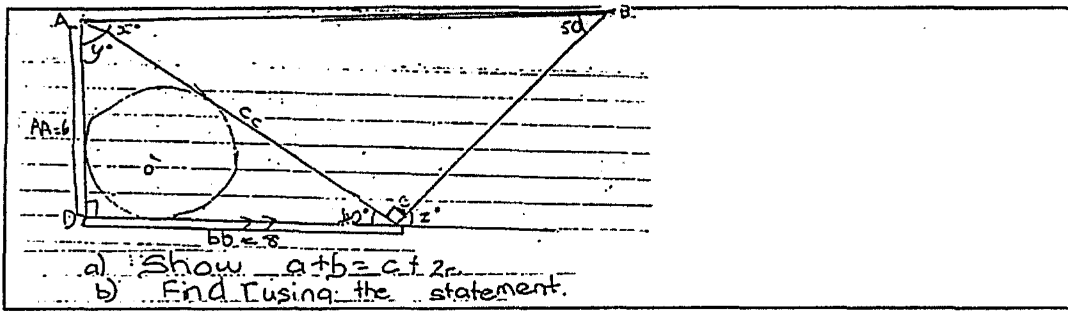


Figure 9.9. Karel's revised geometry problem.

1. Problem-posing post-test results. The first problem-posing situation was not attempted. The semi-structured situation was interpreted using previous experience. Karel posed a problem similar to one he had solved in the project classroom (see Figure 9.10).

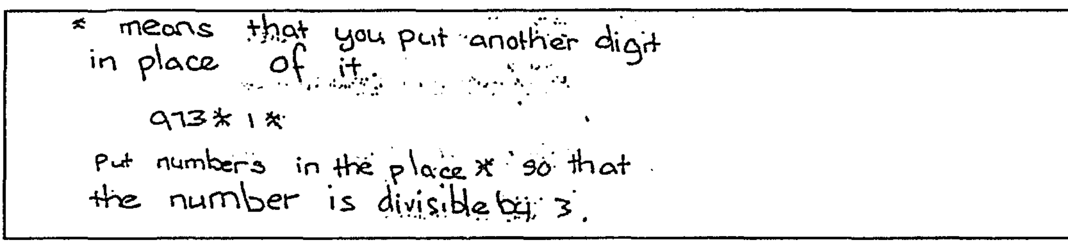


Figure 9.10. The problems posed by Karel in response to the semi-structured problem-posing situation on the problem-posing post-test.

The problem posed by Karel in response to a of semi-structured problem-posing situation was juged to have precise language, to be correct and difficult.

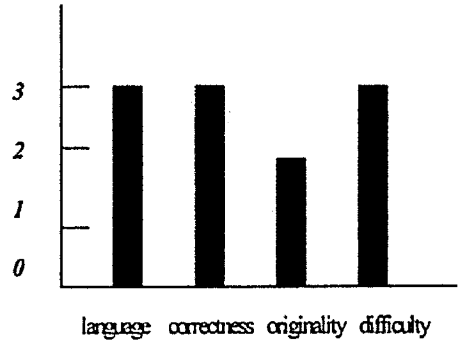
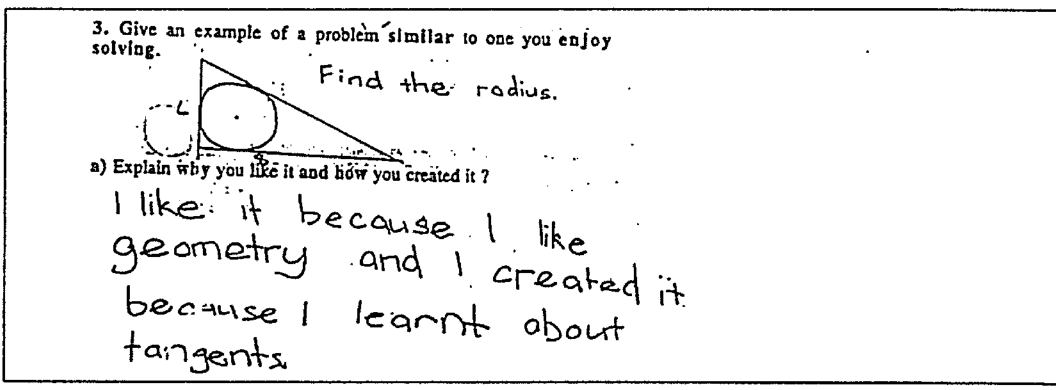


Figure 9.11. Assessment of Karel's response to a semi-structured problem-posing situation.

In the free problem-posing situation the content of the posed problem was from the domain of geometry. He imitated previous experience by constructing a problem in which a part of the problem statement was presented by a figure (See Figure 9.12).



3. Give an example of a problem similar to one you enjoy solving.

Find the radius.

a) Explain why you like it and how you created it ?

I like it because I like geometry and I created it because I learnt about tangents.

Figure 9.12. The problems posed by Karel in response to the free problem-posing situation on the problem-posing post-test.

The problem posed by Karel in response to a free problem-posing situation was assessed as correct, it used precise mathematical language and was judged to be difficult (see Figure 9.13).

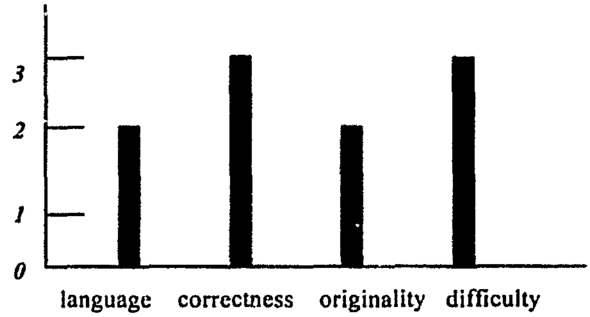


Figure 9.13. Assessment of Karel's response to a free problem-posing situation.

*2. **Problem-posing strategies.*** Throughout the study it was observed that Karel used several different problem-posing strategies. For example, when a problem was given, he was able to pose similar problems by varying the numerical information in the problem statement. Later he made changes to the problem structure which preserved the solution method by extending the number of the elements in a sequence, or by reversing the order of the elements in the sequence. In many cases he demonstrated that he could pose problems similar to a given problem by imitating the problem structure.

In a problem situation with an open structure, in addition to asking questions which follow from the data, he posed questions by adding some data and by connecting the goal statement with all data given. Some of the problems posed by Karel were beyond the level of his problem-solving skills, and he was often unable to solve the problems he posed.

Case Study Two: Samantha - An Individual Profile

Samantha, Karel's sister, was a Year 9 student in one of the private catholic schools in Perth. At the time of our first meeting, in January 1995, when she and her parents visited me, I gave her a booklet of geometry problems while I started to work with her brother. Because their father did not mention anything about Samantha in our phone conversation, I was not expecting any questions about her possible enrolment in the Program. As already mentioned earlier in this chapter, however, it transpired that Samantha became one of the students in the project classroom.

Problem-solving Performance

Studious, quiet and shy, at the beginning of the Program, Samantha very rarely demonstrated any initiative to share her solutions or ideas with the other students or even to ask questions.

1. Performance on the Challenge Stage Of the six *Challenge Problems*, submitted at the Challenge Stage, in 1995, Samantha presented a complete solution only for Problem 2 (see Table 9.1). Problem 4 was not attempted and her solutions to the other problems were not precise.

Table 9.1.
Samantha's results on the Challenge Stage in 1995

Challenge Problem	No 1	No 2	No 3	No 4	No 5	No 6
Results	3	4	2	0	3	2

2. Problem-solving tests results. On the problem-solving pre-test, Samantha showed that she had all of the mathematical skills needed to solve the application problems (Items 6 and 7 from the Mathematics Questions Set 2). Problem 6 was solved by modelling the situation using the ratio concept ($^{15}/_{x} = {}^{20}/_{100}$). Although the answer was correct, as was the case with most participants in the study, she did not take time to write a precise solution.

Although Samantha's solution to Item 7 on the pre-test demonstrated that she had a good understanding of the problem structure, she did not pose a correct mathematical model, consistent with her lack of experience in applying sound mathematical techniques. She wrote $136x = 80\ 000$ (instead of $^{136}/_{x} = {}^{80}/_{100}$), although she had already made use of the correct model in the solution she presented for Item 6 (See Figure 9.14).

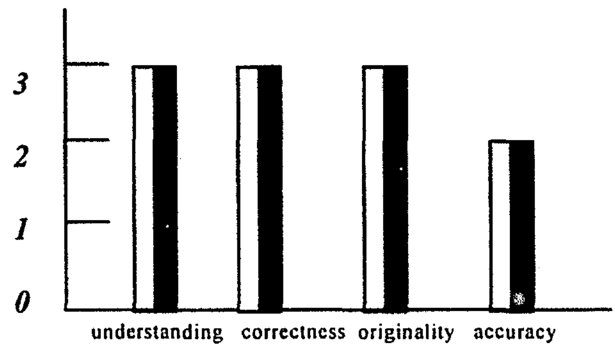


Figure 9.14. Samantha's scores on Item 6 on the pre-test (unshaded) and post-test (solid) in Mathematics Questions, Set 1

The changes observed for Samantha's attempts at Item 7 on the pre- and post-test are shown in Figure 9.15.

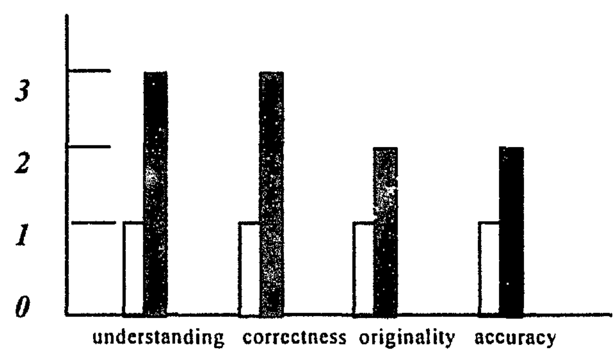


Figure 9.15. Samantha's scores on Item 7 on the pre-test (unshaded) and post-test (solid) in Mathematics Questions, Set 1.

The solution idea Samantha presented for Item 7 on the post-test was the *same* as that on the pre-test, but in the post-test she made no mistakes in her written explanations. Her written solutions were correct, but as at the beginning of the Program, no explanations were provided for the way in which the modelling equation had been created, and restrictions on the values of the variables were not mentioned. The scores obtained on the solutions of Item 7 on the pre-and post-tests are shown in Figure 9.15.

*3. **Observations from the project classroom.*** In the individual discussions with Samantha I tried to draw her attention to a range of the features involved in the process of problem solving by prompting her to reflect on my questions with a problem-posing activity. For example, when a new problem was presented to the class, I asked her to *extract the important* information in the problem statement (see Worksheet 19, Problem 1, Appendix 4), as illustrated by the following excerpt:

T: What is the important information in the problem?
Samantha: The fact that the number with the deleted digit, it becomes 7 times smaller.

In other cases, I tried to encourage her to make proofs - for example, of some theorems from the domain of geometry - by presenting arguments about why a specific statement is *always* true and formulating a general statement from this (see Figure 6.16).

T: Samantha, why is it always true, that the perimeter of the triangle *MNC*, which is *MC* plus *CN* plus *MN*, is always equal to *AC* plus *CB*?
Samantha: Because if AM is equal to PM. . .
T: They are tangents, and?
Samantha: Then PN will be equal to BN.

The *elegance* of a particular solution idea was an important aspect of the individual discussions with Samantha. When the mathematical model of a word problem included one equation instead of two simultaneous equations, for example, Samantha was able to associate the beauty of the of the model with its simplicity (see Problem 2, Worksheet 29, Appendix 4).

T: Why is this equation so beautiful, Samantha?
Samantha: You only have one variable.

The next episode provides an insight into how problem posing was used in helping Samantha to build an understanding of the interrelationships between the problem structure and the solution method. After solving a typical basic problem which involved the use of combinations, I asked students to pose problems similar to the one already solved *and* I specified that the problem had to be solved in the same way:

T: There are 2 boys, 3 girls and 4 teachers. I want you to make a problem similar to one already solved which could be solved by the same solution method.
Samantha: If you have 1 boy, 2 girls and 3 teachers,
T: And the question will be what?
Samantha: How many groups you can make with that requirement?
T: [To help the other students to remember the problem, I repeated it].
Ah, you have this number of boys, 2 boys 3 girls and 4 teachers, and the question is how many groups can you have with 1 boy 2 girls and 3 teachers. And what will be the answer?

Samantha had posed a question by immitating the structure of a problem which was just solved — she changed the numerical information, but she hesitated when asked to solve the problem. I proceeded with some hints:

T: 1 boy can be chosen in how many different ways?
Samantha: 2.
T: 2 girls can be chosen out of 3 in how many different ways?
Samantha: 3.
T: And 3 teachers can be chosen out of 4 in how many different ways? I will write 4 times 3 times 2, over how many factorial? ... I have to divide by what? *Samantha: 3!*
T: And the answer is
Samantha: 4.
T: And the final answer for your question is what?

Although initially she was not able to provide the solution without any help, when she was asked whether she needed some additional examples of problems from the same type, she confidently answered: "No."

On many occasions during sessions in the project classroom, Samantha was invited to make suggestions about the applicability of a particular mathematical approach to other situations. The following dialogue, which took place near the end of th^ Program, illustrates how Samantha was unable to imitate a particular problem structure and construct a problem similar to a given problem which might be solved by the same solution approach.

T: What is the solution to problem 2?
Samantha: Number 2 says find the sum of all the two-digit numbers greater than 10 such that the tens digit is one less than the units digit. . . So all the first digit . . . would be 12, then 23, then 34, 45, . . . ,89.
T: And you have to add them up. How did you add them?
Samantha: I just added them all together.

I hoped that, by now asking her to pose a similar problem, she would see the relationship between the elements in the sequence and to predict a generalisation:

T: Are you ready to say another problem similar to this one in which we can use the same solution idea?
Samantha: With a 3 digit number.
T: 3 digit number and the pattern will be what? Find the sum of all 3 digit numbers, in which. . .
Samantha: The first digit is one less than the second digit and the second digit is one less than the third digit.
T: [It was my turn to reflect on the problem statement]... And one example is this, 123, the next one will be?
Samantha: 234.
T: The next one?
Samantha: 345.
T: And so on, OK thanks a lot.[8]

Later in the Program, I involved Samantha in activities in which she had to focus her attention on the formal structure of a problem solution and determine the key-element in it. For example, she gave the following interpretation of the steps involved in finding a solution to Problem 2 (see Figure 6.36):

T: What are the main steps of the solution, where are they shown?
Samantha: Where you've got the 3 brackets . . . [I wanted a more precise answer.]
T: Here, or here or here. Which is the main step which you have to understand and after that the solution is clear?
Samantha: They're the same.

Samantha demonstrated that she had understood that the solution idea can be presented by a sequence of problems with the same formal structure.

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[8] Neither Samantha nor some of the other students made any comments about the fact that the numbers 12, 23, 34, 45, . . ., 89 represent elements in an arithmetic sequence (in which $a_1 = 12$ and $d = 11$).

At the end of the Program, Samantha was one of the few students who demonstrated an ability to grasp the structure of a solution method and to provide arguments about key aspects of the method and its applicability to other situations.

T: What do you remember after solving a problem?
Samantha: How to solve it.
T: How to solve it, and especially what Samantha? What do you mean by how to solve it?
Samantha: The way you took to find the answer.
T: The way, and what is the way? ... Say it somehow, after that we will refine it.
Samantha: Well if you know how to solve . . . after that to apply the same principles to solve another problem.

During the Program, Samantha appeared to have developed a *sense* for the structure of the solution method.

4. Problem-solving performance on the Challenge Problems. In addition to her improved performance on Item 7 in the post-test, Samantha showed a significant improvement by presenting full solutions to all 16 *Challenge Problems.* The score she obtained gained her first place among the Western Australian participants in the Euler program. On the six Challenge Problems in 1996, a year after our first meeting, she submitted three full and precisely written solutions (see Table 9.2).

Table 9.2.
Samantha's results on the Challenge Stage in 1996

Challenge Problem	No 1	No 2	No 3	No 4	No 5	No 6
Results	2	4	3	4	4	2

A few months later, in July 1996, she was amongst the top 60 participants from Western Australia in the Westpac mathematics competition. An invitation to represent Perth in the Tournament of the Towns came as a recognition for achievements such as these.

Problem-posing Performance

Samantha's problem-posing performance was assessed on the problem-posing tests (see Mathematics Questions, Set 1) and additional data were collected through her individual written work and during classroom observations.

1. ***Problem-posing tests.*** On the pre-test, Samantha attempted the first problem situation by using a variable to denote the value of the calculation and both attempts to solve the equation were unfinshed. Samantha's results on both pre- and post-test are shown in Figure 9.14. On both problem-posing tests (as shown in Figures 9.16, 9.17 and 9.18), all problems posed by her were correct. However, the quality of the problems she posed on the post-test was higher than those she posed on the pre-test. The language used was more precise, the solutions involved more complex ideas and the problems comprised different categories.

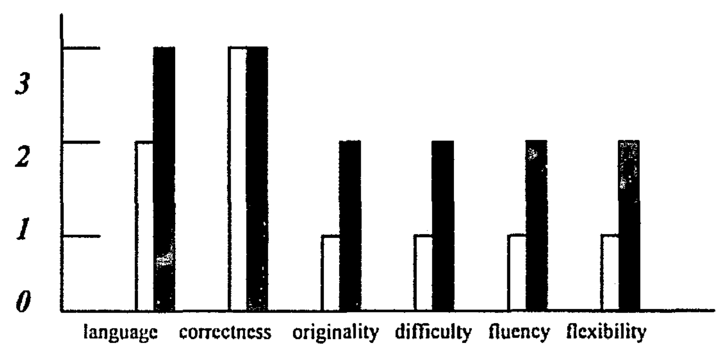


Figure 9.16. Samantha's problem posing results on the structured situation in Mathematics Question Set 2 (pre-test (unshaded) and post-test (solid)).

In the semi-structured and free problem-posing situations Samantha constructed problems which appeared to make direct use of her previous experiences. She recognised that she liked problems which "involve discussion, logic and provide a challenge." The diagram in Figure 9.17 presents the changes between the beginning and the end of the Program in Samantha's problem-posing performance for semi-structured situations.

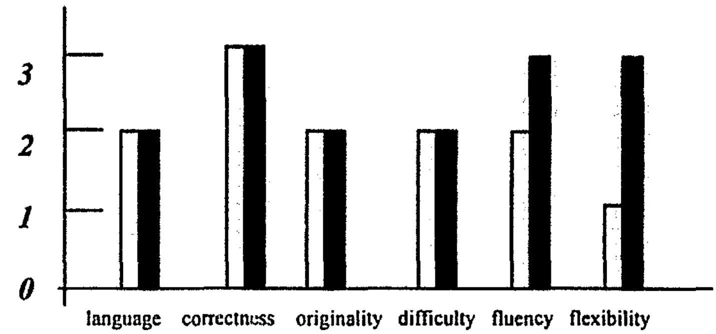


Figure 9.17. Samantha's results in the pre-test (unshaded) and post-test (solid) for semi-structured situations.

At the end of the Program, the problems which Samantha posed in questions involving semi-structured situations in Mathematics Questions Set 1, differed from those posed at the beginning of the study in their fluency and flexibility. She was able to pose more problems from more different categories. The changes are reflected in Figure 9.17.

In contrast, there was little difference between Samantha's scores on free problem-posing situations in the pre- and post-test, as shown in Figure 9.18.

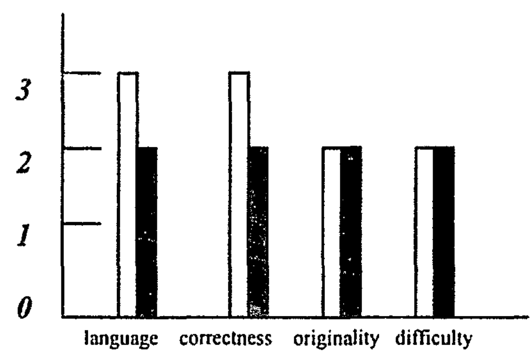


Figure 9.18. Samantha's results in the pre-test (unshaded) and post-test (solid) for free problem-posing situations

In both pre- and post-tests, Samantha constructed mathematical problems whose structure imitated those of previously-solved problems.

*2. **Classroom observations.*** Although the test results suggest that the precision of the language used by Samantha in the problems she had posed had improved only in a structured problem-posing situation, observations in the project classroom indicated that the precision of the language she used for formulating problem statements had improved and that the explanations she provided for her written solutions contained more detail and were accurate. By the end of the Program, she had also developed a "sense" for the interrelationships between elements in the problem structure (see, for example, the episode on page 230). This episode showed that for Samantha, it seemed clear that every element in a problem statement "should serve a purpose."

The following episode took place when Samantha had posed a problem which involved the use of combinations. As her teacher, I tried to see if she could formulate another question based on the following situation: "There is a group of four people. Ask some meaningful questions which involve the use of combinations." The following dialogue took place:

T: Let me solve another problem like that. What is another question for these four people, Samantha?
Samantha: How many different ways the group can be divided?

As I wanted to prompt her to use another expression for formulating the goal statement in an equivalent way, I asked:

T: How many groups. . .[but she repeated the question in the same form].
Samantha: How many different ways the group can be divided?

My strategy did not work and I asked a more direct question:

T: OK what do you mean by your question Samantha?
Samantha: Like into 1 and 3, or 2 and 2' . . .

As in other examples quoted, although Samantha did not know how to solve the problem, when she was presented with a solution, she was able to evaluate its appropriateness.

The following episode demonstrates how problem-posing activities were used in the project classroom to help students to connect the formal structure of a problem with the main features underlying the solution idea. The teacher posed a problem similar to one solved several months earlier and asked students to predict the answer:

T: Do you remember a problem like this one, what is the last digit of $3^{9^{17^{0^{199}}}}$?
Carol: Yes.
T: Yes, what is the last digit?
Carol: 0.
Samantha: 1.
T: 0 or 1? What is the last digit? ... 1, why?
Samantha: Because when you get to the power 0, anything to the power of 0 is 1.
Carol: 0 to any power is 0.

The predicted results were wrong and it was clear that, in this case, the girls did not take into account the order of the operations and the definition of a^0 (when $a = 0$, the meaning of a^0 is not defined). After a short discussion, the question "Is it possible, and how if it is, could the value of the calculation be made 0 or 1?" was answered correctly.

Problem-posing Strategies

Throughout the Program, Samantha was involved in different types and categories of problem-posing situations.

On many occasions, when working on structured problem-posing situations, her reaction was *to change the numerical information and to preserve the problem structure*. The following episode shows that she applied the same strategy when she was presented with a problem with an open structure and asked to increase the problem difficulty.

T: I have several coins. ... Questions? Tom?
Tom: What's the least number of coins you need to make $26.95.

T: Think about a harder question. Samantha?
Samantha: How many different ways you can make $50?

In another context (see Worksheet 23, Problem 1, Appendix 4) instead of finding a number on the 100th row as required by the question, she suggested another question, with identical format, about finding the 5th number on the 10,000th row. Although the question she posed was a "standard" question, she had already started to pay more attention to the precision of problem formulation. I asked one of the students to solve Samantha's problem. Carol responded:

Carol: 99,999 squared plus 5.
T: Yes, but when you speak about the 5th number, you said plus 5, you meant which number, from this side or from this side?

Samantha reacted immediately to my question and answered instead of Carol: "From the left."

Samantha used the same problem-posing strategy to respond to the teacher's question about what kind of changes in a problem statement might *preserve* the solution idea. The episode below presents some comments on the solution idea for one of the six *Challenge Problems,* which students solved on an individual basis at the beginning of the Program:

T: You have 4 friends on the bottom stage together and next they are together on the top stage. The question is how many steps are stepped on . . . When will this idea work again?
Samantha: You want me to say another problem?
T: Yes another problem similar but with the same idea for the solution.
Samantha: You could alter the number of people.

Samantha continued to apply her favourite strategy "Change the numbers" throughout the year for posing problems which were similar to a given problem. The problems which she posed involved strategies such as *reformulation, re-construction, imitation* and *invention* and there was a clear shift towards posing more complex problems from different problem categories.

CHAPTER TEN

DISCUSSION

Introduction

The main goal of this study was to design and implement a range of problem-posing situations as part of mathematics classroom problem-solving environments and to explore the effects of these environments on students' problem-solving and problem-posing performances and on their problem-posing strategies.

In order to meet the goals of the study, three interrelated frameworks were designed and then implemented, extended and explored. In the first part of this chapter some important features of the implementation of the initial frameworks are discussed. The second part focuses on some characteristics and the categorisation of problem-posing strategies which students exhibited. The effects of the open problem-solving approach on students' mathematical performances are discussed in the third part. A section on some of the challenges of this study completes the discussion chapter.

Discussion of the Frameworks Developed in this Study

Problem-posing classroom environments can be based either: (a) on problem-posing situations included in students' textbooks and teachers' or students' support materials; or (b) on teacher-designed support materials for the students which meet some specific individual instructional goals. The success of either of these approaches will vary with the students' capacity to respond to the problem-posing activities presented to them.

This study aimed to consider not only the role of the teacher as an instructor and problem poser, but also the role of students as equal partners in the learning process. Classroom work also incorporated small-group problem-posing activities. Research has shown that collaboration between students with different abilities is likely to provide a supportive atmosphere for working on specific tasks (Bennett & Dunne, 1992; King, Barry, Maloney & Tayler, 1993, 1994). It had been expected, therefore, that small-group work would provide an environment in which the extent to which students develop their own problem-posing strategies and the identification of the key characteristics of these strategies could be observed.

Classification of Problem-posing Situations

Although the use of problem-posing situations has been recommended in many curriculum documents, such recommendations did not provide any information on how these situations might be designed, or about possible ways in which problem posing might interact with other classroom activities. Therefore, before undertaking research into the application of problem posing as a means of instruction in the project classroom, a framework to describe problem-posing situations was designed (Stoyanova, 1995; Stoyanova & Ellerton, 1996), and strategies for the researcher to help students to reflect on their solution attempts via specific problem-posing actions, were developed.

An analysis of the type of problem-posing situations used as instruments in the research literature, and of those recommended for use by classroom teachers, revealed that these relate very closely to: (a) the openness of the task structure; (b) the openness of the student's activity; and (c) the nature of the problems used for generating these situations. These observations suggested that it was a sound decision to choose Krutetskii's system of problems as the basis for the design of a range of structured and semi-structured problem-posing situations.

In a mathematics classroom however, it is not always appropriate for problem posing to be based on a specific problem. Many of the situations which a student might face out of school could have a structure which has few, if any constraints (in other words a *free* structure), or a structure which has to satisfy some preliminary requirements (a *semi-free* structure). At the same time problem-posing activities needed to be linked to instructional goals of the lesson and to provide support for the students during *all stages* of the process of solving particular problems.

The framework designed at the beginning of the study comprised three categories of problem-posing situations — free, semi-structured and structured. Specific examples of problem-posing situations from each category were derived from the literature, and were used to establish the initial framework (see Chapter 4). This framework was extended in an iterative way as the study progressed, and the classification of problem-posing situations presented in Chapter 6 was developed from an analysis of the teaching data and tape-transcripts of the discussions in the project classroom. By presenting the classification according to the instructional goals in which the situations would be applied, possible applications of the classification in other classroom settings should be simplified. In fact, the presentation of the classification was aimed to foreshadow a range of possible implementations of problem posing in a variety of classroom problem-solving contexts.

As discussed in Chapter 1, most of the structured and some of the semi-structured problem-posing situations were inspired by Krutetskii's system of problems which he used as an instrument to study the structure of students' mathematical abilities. Krutetskii's system by itself, according to Kilpatrick (1987), could have its own instructional value. It appears logical to suggest that, if Kruteskii's system was to be included as the basis of activities aimed to support students' problem-solving activities, then the types of problems in this system might help students to analyse problems and their solution structures, or to examine a set of problems from contrasting perspectives.

The main value of the problem-posing classification developed in this study is not only in the diversity of the problem-posing situations described and implemented, but also in the simplicity of the initial framework. It needs to be noted that the framework places *equal* emphasis on both the problem and the solution structures. In addition, through the experience of designing free, semi-structured or structured situations, the researcher was aware, to some extent, of the *diversity* which might be expected in problems posed by students.

The framework also helped the researcher to embrace the traditional mathematics curriculum in which most of the problems are closed, and to design, from this base, a series of problem-posing activities in a range of problem-solving environments. In Chapter 6, various possible applications of problem posing in the project classroom were described, but the author believes that their number could be extended considerably.

The Nature of Communications in the Project Classroom

An inseparable part of establishing effective problem-posing environments is the way in which the teacher communicated with the students, the types of questions asked and the responses given to the students. In the present study it was anticipated that the questions asked would, to some extent, determine students' actions (Doyle & Carter, 1982). It was expected, therefore, that the ways in which students could work or would respond to their teacher's (or other students') questions would depend on what the questions were and the nature of work they were asked to do (Doyle, 1983). The initial preparation of the materials for the study, therefore, also included the design of a system of verbal prompts for the students in both classes. The use of appropriate questions was regarded as crucial to the effectiveness of both teaching approaches.

The teacher's questions used for the students in the problem-solving class (Group B) were designed according to Polya's recommendations (1957). For the group involved in problem-posing *and* problem-solving activities (Group A) questions which incorporated "hidden" problem posing were created — in other words, it was expected that students would reflect on these questions via problem-posing activities. The main aim of these "verbal prompts" was to assist students to understand the problem and the solution approach via a formulation of problems or situations which had particular features. The underlying expectation behind this approach was that, if a student tried to create a problem, then she or he is likely to try to make some effort to understand the structure of the problem or the structure of the solution idea first. "Hidden" problem-posing questions were asked in order to prompt actions before, during, or after solving a specific problem (see Chapter 4 and Appendix 9). Of the many classroom examples of students responding to "hidden" problem-posing questions at the beginning of the study it is evident that students created problems only after the teacher made the initial step of choosing, for example, a suitable context (see Chapter 6).

The teacher's questions which incorporated "hidden" problem-posing activities were classified into three basic categories according to the type of support offered to the student: (a) understanding the formal structure of the problem; (b) understanding the formal structure of the solution approach (method); and (c) understanding the interrelationships between the data and the solution approaches used. Some researchers who have investigated exceptional intellectual performance in a wide variety of domains have found that the questions asked are at least as important as the questions answered (Albert, 1983; Sternberg & Davidson, 1985; Todd, 1987). On the other hand, Silver (1987) recognised that "an environment in which students feel free to ask questions and make comments is essential for the successful introduction of open-ended problems" (p. 35). Although Silver is referring to a context in which students were asked to solve problems which were *open-ended*, his words are equally applicable to the environment nurtured in this study. For a classroom environment which was aimed at involving students in an analysis of problem structures and solution approaches, it was essential that students felt free to ask questions and to make comments.

It is also well known that students tend to pay more attention to that part of a task which is likely to be evaluated (Doyle & Carter, 1982). Thus, it was expected that the incorporation of an evaluation of some problem-posing products might stimulate students to focus their attention on key features in the structure of the problem-posing situation and the characteristics of the product. For example, when students were asked to present their work to the class, they seemed to enjoy the challenge of answering questions and explaining the origin of created problems.

The nature of communication between the teacher and the students in the project classroom played a critical role for introducing effectively what is termed in this study, an *open problem-solving approach*. By asking specific questions, the teacher tried to prompt students to focus their attention on particular characteristics of the problem or solution structures. For every session, the researcher prepared individual worksheets for the students which they were invited to solve on an individual basis or working in pairs (see Appendix 4). In addition to the problems, a range of problem-posing situations of different types and categories were included in the worksheets for the students in Group A. By asking questions incorporating "hidden" problem posing, the teacher intentionally influenced the nature of the classroom discussions. In addition to making comments on different solution approaches used by students, students from the project classroom were involved in problem-posing activities aimed to catalyse their understanding of specific mathematical concepts, algorithms, solution approaches, or "types of problems" (see Chapter 6 and Chapter 9).

The list of *specific* questions incorporating "hidden" problem posing could be extended far beyond the one presented in Chapter 4. Such extension would depend on the type of problems and the features of problem analysis in which the teacher wanted to involve students. However, the list of questions presented can be used as a basis for the development of further questions.

The teacher did not introduce all specific "hidden" problem-posing questions immediately. Because students were unfamiliar with being placed in situations which required them to pose problems, inducting students in a classroom culture which encouraged this approach had to be done gradually and supportively. Initially, the researcher asked some rhetorical questions and demonstrated how responses could be created. Step by step, students were involved in constructing more complex problems and teacher's questions became more complicated and demanding. It should be mentioned that students' experience in working on related-task settings seemed to influence the starting point for the introduction of problem posing. For example, at the beginning of the study, on several occasions some students, who were good problem solvers, did not manage to provide any responses to algebraic and geometrical problem-posing tasks with open structures. Some of them admitted that they did not understand what had to be done. Thus, the prior experience in problem posing for specific types of problem-posing situations seems to have an influence on how students tackle problem-posing when they first encounter it.

It should be emphasised that, by including "hidden" problem-posing questions, the researcher helped students to *start* to reflect on problem-posing activities. In all cases when students expressed some difficulties, she was able to react and support their initial attempts to pose well-structured problems. The general form of the teacher's questions incorporating "hidden" problem posing was aimed not to direct students to solve a particular question, but rather to help students to cultivate mental habits for guiding their own progress when solving a particular problem.

Open Problem-solving Approach

The term *open problem-solving approach* has been used in this thesis in an attempt to capture the nature of the applications of problem-posing situations as a tool for providing supportive environments for students' problem solving. The following comments have been included here to describe some of the key features involved in applying, in the project classroom, the framework of the open problem-solving approach presented in Chapter 4.

The cornerstone of the application of this framework was that the teaching approach must be based on the *openness* of the *problem* structure, *solution* structure and students' mathematical *activities*. In other words, students were involved in solving both well-structured problems and problem situations. They were also encouraged to use different approaches, to comment on their elegance, to pose problems which illustrate the use of a particular approach, and to predict what aspects of the problem content of a set of questions make it possible for a particular solution method to be used. In the project classroom students were also involved in various mathematical activities which ranged from resolving problem-posing situations based on variations of the structure of a given problem or a solution, through finishing the structures of mathematical situations including unfinished solutions, to reflecting on mathematics via creating problems based on a range of preliminary conditions.

Problem posing can be applied in mathematics classrooms as an *isolated* activity (either as a goal or as a means of instruction), or as an *inseparable* part of the classroom mathematical activities. According to Silver (1993) the application of problem posing in mathematics classrooms is associated with its potential to improve students' problem solving. In this study, the aim of all problem-posing activities was to facilitate, in a sensitive manner, students' attempts at solving mathematical problems. In other words, problem-posing situations were used as *heuristics*. It should be emphasised that explicit training of problem posing was not involved in the open-problem-solving approach adopted in this study.

Linking Students' Problem Posing and Problem Solving in the Project Classroom

The focus of the discussin in the next paragraphs will be on some additional features of the implementation of problem posing in the project classroom.

In the project classroom setting, it was helpful for the researcher to consider possible *models of interaction* to describe the characteristics of the sequence of possible interactions between problem posing and problem solving. Design of the model needed to take account of: (a) the features of the mathematical content of the Program; (b) students' problem-solving experiences; (c) students' problem-posing experiences; and (d) the instructional goals of the unit.

At the beginning of the study, on the basis of the literature review, the researcher developed a general model in which the possibility of interaction between problem posing and problem solving was taken into account. As the study progressed, this model was enriched through systematic observation in the project classroom. The refined model presented in Appendix 9 was developed in an inductive way as the study progressed and was used by the researcher to gain a general view of the structure of a complete sequence of both her and her students' problem-posing activities within the context of a particular session (a model of interaction). The general model takes account of the fact that problems may arise from every-day-life situations, from modifications of specific tasks, or can just be posed by an individual. Problem-posing situations can therefore occur before, during or after solving the given task (Silver, 1993).

It is widely accepted in the mathematical community that the first step of the problem-solving process is understanding the problem statement. At this stage problem posing aimed at helping students to understand the problem statement, the mathematical concepts used and to connect the problem being solved with appropriate previous experience.

Understanding and Exploring Problem Structures via Problem Posing

When problems were given to students, the researcher attempted to focus students' attention on various features of the problem structure. As a first approach to having students start to pose problems, students were asked to *reformulate a specific problem using their own words* without changing its mathematical nature. Students were also asked to look for key words in the mathematical vocabulary of a problem and to replace them with synonyms, or a group of words so that the meaning of the problem would become clearer. The types of situations varied and involved students in activities such as: (a) *listening* (when the teacher demonstrated specific examples); (b) *interpreting* (the students interpreted a problem statement with their own words); (c) *rearranging the information* (this is the case when students suggested problems which had a structure either isomorphic or non-isomorphic compared with the structure of the given problem); (d) *reformulating* (the goal was to reformulate the problem statement in order to clarify the problem, but in the reformulation the problem structure was to stay the same); and (e) *presenting the problem statement in a "brief" form* (extracting the information which constituted the *Given*, the *Obstacles* and the *Goal*). As the study progressed, students were led to predict and to justify their predictions that two identical problems might differ in their *syntax* — length, grammar, synonyms, sequence of information, numerals or symbols and clauses — or that two *similar* problems might relate in their content and structures in various ways.

For fostering students' understanding of the problem *content,* a wide range of problem-posing situations were designed. When a new problem was introduced, the researcher intentionally asked questions incorporating "hidden" problem posing, such as: "What might the question be?" or "How could the problem be finished?" Problems with surplus or insufficient information were also used as starting points for discussion with individual students about ways in which the mathematical content could be reconstructed.

Involving students in Group A in *posing* or *analysing problem sequences* with problems from the same or different types (Krutetskii, 1976) aimed to help pupils understand the features of a problem's structure, and to enable them to recognise problems of the same type more easily when they encountered them again. Students posed and analysed problems which *looked* different but whose content was the same. Then students justified their predictions about whether the problem could or could not be solved by the same method. By changing numerical information or mathematical relationships students posed problems which *looked* alike, but because they had different content, students were not always able to solve the problems by the same solution method. Thus problem posing was used to help students develop an internal *sense* of the possible links between the problem elements and the solution approach.

Embracing Students' Current and Previous Mathematical Experiences

Observations made in this study suggest that semi-structured problem-posing situations are likely to provide the most suitable and educationally rich environments which can help students to embrace their current and previous mathematical experiences. Problem-posing situations based on an unfinished problem structure[8] and involving students in finishing it (*creating problem fields*), were used to provide opportunities for individual work and for involving students in solving and posing problems which were beyond their mathematical experience. In some cases students were invited to put the sub-problems posed in a suitable order, according to any perceived interrelatedness between them. Although the problem-posing situations were designed to require knowledge of simple mathematical concepts, at the same time, they needed to be educationally rich to allow students to pose problems and to make conjectures by connecting the different elements in the structure of the given situation with their previous mathematical experiences. Silver (1990) strongly recommended the use of non-goal-specific questions. According to Silver, these questions can help students to organise their knowledge more effectively and to acquire more useful problem-solving skills. The observations made in the project

[8] "Unfinished" does *not* mean that the problem is one with an unstated question (see definition in Chapter 4).

classroom suggest that there is a difference in the *quality* of problems posed by students depending on whether their knowledge within a specific topic domain is limited or extensive.

Exploring Problem Solution Structures via Problem Posing

Lester (1985) stated that understanding the structure of the solution approach is no less important than understanding the problem statement. In the project classroom a number of problem-posing situations were directed towards helping students develop an understanding of the structure, limitations and extensions of the solution methods learnt. For example, by looking back at the sequence and discussing the features of the solution path the researcher involved students in an analysis of the basic mathematical facts necessary for solving the goal-problem (see Chapter 6). As a way of "looking back" at the problem, students were also involved in investigating the structure of a particular solution approach, its features, limitations and extensions. The aim of having students distinguish between similarities in the problem *solutions* and similarities in problem *solution structures* was to extend their experience in understanding the main features of different mathematical methods.

According to pupils' preferences and mathematical abilities, students' individual work ranged from: (a) *guessing* (for example, on the basis of key words in the problem statement to predict a possible solution method); (b) *discovering* (defining the main steps of the solution structure and relating the problem to previously solved problems); (c) *inventing* (creating mathematical problems which might involve a specific solution method); (d) *investigating* (making changes to the problem statement which might/might not affect the solution method); and (e) *reconstructing* (posing a problem when the solution is given).

Many students, at the beginning of the study, when asked to solve a mathematical problem, simply wrote down the solution value (the answer) without providing any justification. Through posing problems with different formats, students came to understand not only how to choose or state the right answer, but how to present precise written explanations for their choice.

Students' ability to present a specific solution precisely was regarded as a significant component of students' mathematical culture in this study. One of the very first types of problem-posing activities in which students were involved was that of improving a written solution. In this way, careful attention was therefore given to the language, logic of the explanations, and the precision with which the mathematical ideas were presented.

Having the student identify the main steps involved in a particular solution approach was aimed at assisting students to improve their written mathematical skills, and at helping them aspire to develop an understanding of the culture of written mathematics. Having a "sense" for the solution structure was expected to result in better, more precise, written explanations. Such activity naturally took place when students were solving new types of problems. In the project classroom, students were asked to give an example of a problem whose solution might involve the same solution method as a given problem. They were then asked to solve the problem, to write the solution precisely and to point out the links in the problem content which might be associated with the use of the solution method.

During the Program students were involved in using different formats for the presentation of specific solution structures. In addition to writing a precise solution, students were encouraged to present problem solutions verbally or by a picture or a series of pictures.

When it was appropriate, students were asked to solve a problem in different ways. They were then asked to explain the features of different solution approaches, to discuss the elegance of solution ideas used and to give their preferences.

For individual work with the most advanced students, the teacher used problem-posing situations based on the "inverse" activity — on the basis of a written solution some pupils were asked to restate the problem or to finish the solution and to restate the problem or just to guess what the problem was about. At the end of the study students were presented with a written solution made at the beginning of the program by one of their peers and invited to guess/explain what the problem was about.

Exploring the Limitations and Extensions of Problem and Solution Structures via Problem Posing

Students were presented with ill-structured problems and asked to find the "mistakes" (they ranged from computational to logical) in specific multiple-choice question problems. The problem structures were analysed and explored, and "correct" versions suggested. Similar activities, based on finding the "mistakes" in written problem solutions and on suggesting a "better" problem solution structure, were also used.

Recognising the conditions under which a particular approach can be applied, discovering its limitations, or selecting more effective approaches from several alternatives have been regarded as key elements of successful problem solving (Lester & Groves, 1977; Polya, 1957). As a first step in that direction an open discussion about the reasons for choosing a particular approach was initiated in the project classroom. By prompting students to give examples of other problems to support or oppose the use of a specific solution strategy, some of the limitations were discussed. The main goal of these activities was to help students to see the interrelationship between a specific solution approach and the key elements of a particular problem structure.

Greeno (1977) acknowledged that a problem has been solved with "good understanding" only when the problem solver recognises the relationship of the solution to some general principle. Greeno's statement was interpreted for the purposes of this study in several ways. For example, in some cases the problem statement was presented in ways which the student might meet in mathematics textbooks such as: "Solve the equation," "What are the roots of the equation?" or "Find when the two mathematical expressions given below have the same values." The interrelationships between a specific solution approach and different problem formulations were used to help students to extend their mathematical vocabulary.

Two other categories of problem-posing activities were used after students had had experience in applying a specific mathematical method. First, students were asked to pose problems which *looked* as if they would have the same solution approach (but it was possible that they would not have the same solution approach) and second, to pose problems which one would expect could *not* be solved by the same method. Activities such as these were aimed at extending students' experiences in analysing the elements of problem structure which might determine the solution method. The expectation was that when faced with a similar problem, students could then imitate these activities and limit the choices of possible solution approaches.

Students were also involved in solving, and later in the study, in constructing problem chains. These problem-posing situations were aimed at helping students to perceive the problem structure, to choose an appropriate series of connections and to apply it for: (a) producing interrelated problems; (b) making generalisations; and (c) for solving non-trivial problems. The aim of these activities was to help students to *chain* a particular problem and its solution approach to their previous mathematical experience. The research literature is silent about the effects of solving or posing problem chains on students' mathematical performance.

Discussion of Students' Problem-posing Strategies

The identification of students' problem-posing strategies and the framework which describes the categories that emerge from a detailed analysis of these strategies demonstrate how the problems posed by students are directly related to the initial situation under which the problem has been posed and on students' previous experiences. The case studies suggest that the mathematical content and the activities in which students were involved in a classroom *colour* the problem-posing products. The observations also showed that students with lower levels of mathematical performance are likely to prefer working on *structured* problem-posing situations. These students very rarely used the freedom provided by the semi-structured or free situations to reflect beyond the school curriculum.

On the other hand, students with higher levels of mathematical performance tended to prefer problem-posing situations with semi-structured and open structures.

They also tended to avoid "standard" answers and rarely posed a number of problems of "the same type" when presented with structured situations. The issue of the extent to which problem posing can be considered as an index of one's problem-solving ability was first raised by Kilpatrick (1987). The observations in this study indicate that, at the beginning of the study, most students posed problems which they knew how to solve. In other words, the problem-posing products did not represent *problems* for the authors. As the study progressed, students started to feel free to pose more complex questions. In some cases the authors admitted that they had not solved the problem yet, but indicated that, if a solution was provided, then they would be able to understand it. On a number of occasions, some students recognised that they understood what the problem was about, but that they could not solve it, "because it is very difficult."

The problems posed by students included algorithmic, algebraic and generalisable types of problems from the domains of Arithmetic, Algebra and Geometry. The problems posed ranged from direct recall of problems posed in mathematics classrooms, through imitating problem structures, to posing questions which incorporate concepts from different learning areas and solutions which involve new (for the student) solution methods.

Data from the project classroom, which included the results of tests and classroom observations, indicated that, in free problem-posing situations, students with lower levels of mathematical performance tended to respond with a problem which was a direct recall of one already solved or one which had a very simple structure. On the other hand, some students with higher mathematical performance constructed examples using their own ideas about the formal structure of the problem and tended to pose problems whose solutions were beyond their problem-solving abilities. These students were also able to imitate the structure of a problem by posing problems whose structure was isomorphic to those in different mathematical contexts. Those who showed lower levels of mathematical abilities would tend to recall a problem which they had solved before, or they posed a problem similar to a solved problem by changing part of the numerical information. These students were therefore able to avoid taking the risk of making up problems which they did not know how to solve.

The author believes that different problem-posing strategies might involve different cognitive processes. She expects that the type of problem-posing strategies used by students might depend on the mathematical background, the nature of the problem-posing task, the students' prior experiences in related-task settings and some personal characteristics, such as creativity.

The case studies also suggest that problem-posing skills, as with all other skills, could be developed and nurtured. At the end of the study, students exposed to an open problem-solving approach were observed to pay more attention to the *quality* of problems posed and to problem difficulty. There was a strong tendency for students to pose problems by using *imitation* and *invention* strategies, and to pose problems from different categories rather than to pose problems by reformulation or reconstruction or to pose more problems from the same category.

The Effect of an Open Problem-solving Approach on Students' Mathematical Performance

The results of this study suggest that the open problem-solving approach created environments which assist students to develop their problem-solving and problem-posing performances. It also appears likely that students' long term engagement in problem-solving and problem-posing activities would benefit students' mathematical performance and the quality of their problem-posing and problem-solving products.

The findings and the observations made in the project classroom are consistent with previous research in the field of problem posing. First, students seem to have a natural capacity for posing mathematical problems and for producing multiple solutions. However, problem-posing activities which take place in multiple problem-posing task environments are coloured by the problem-posing category, students' knowledge, skills and students' problem-solving and problem-posing experiences in related task settings (Leung, 1997).

Structured problem-posing categories based on problem types with which students have had extensive problem-solving experience are likely to provide educationally rich environments for generating new problems by employing reformulation and reconstruction strategies. Therefore, an instructional approach based only on structured problem-posing situations, for example Brown and Walter's (1983) "What-if" and "What-if-not" approach, might be implemented successfully when problem-posing situations are based on problems which involve the use of concepts and solution approaches with which students have had extensive mathematical experience. According to Sweller and his colleagues (Owen & Sweller, 1985; Sweller, Mawer, & Ward, 1983; Sweller, 1992, 1993) "non-goal specific problems" provide environments which help students to develop knowledge that is better organised and skills that are more useable. The observations made in this study support the conjecture that semi-structured problem-posing situations nurture environments in which students can embrace their previous mathematical experience within the structure of a given situation. Observations from the project classroom do not support the vision that conventional problems cannot provide educationally rich environments for organising students' skills and knowledge. However, a teacher's ability to *catalyse* and *nurture* a particular problem structure by asking appropriate "hidden" problem-posing questions and involving students in useful discussions can play a vital role towards achieving such goals.

Semi-structured problem-posing categories seem to provide appropriate cognitive support for most students as they attempt to make links between their current and previous mathematical experiences when using imitation and invention strategies. These situations are likely to be particularly fruitful when students have prior experience in working on similar problem-posing tasks (Silver, 1993). When students have a sound mathematical background in the content area on which the specific problem-posing situation is based, then it is likely that some would attempt to pose more problem categories when presented with a particular problem-posing task rather than create problems which have isomorphic structures (Leung, 1993).

The study suggests that, in contrast to language education, the use of free problem-posing situations in mathematical instruction seems to make it difficult for most of the students to begin the task. However, when free problem-posing situations are used in appropriate ways, students' problem-posing products can provide a useful insight into the type of difficulties students have expressed and into students' level of understanding of mathematics (Ellerton, 1986).

There is an expectation that collaboration among peers might influence the quality of the problem-posing product. Observations from the project classroom do not provide full support for this expectation. Most of the participants in the project classroom seemed to prefer working on an individual basis during the productive phase of the problem-posing process. Collaboration among peers in a problem-posing environment was seen in the project classroom as an activity which could *assist* students to reflect on the *quality* of the problem-posing product and on the ways in which problem-posing products could be solved or linked to previously solved problems rather than as a way of generating "better" quality initial problems. This finding is compatible with the result reported by Silver et al. (1996) that there were no significant differences between the problems generated by subjects who worked as individuals and those who worked as pairs.

In her study Ellerton (1986) reported that "more able" students posed problems by using more complex numbers which required more operations for solutions, than did their "less able" peers. This study suggested that this is likely when students' mathematical knowledge or problem-posing experience within a specific domain is limited. When students have extensive problem-posing and problem-solving experiences within a particular topic area then their perceptions of "the level of problem difficulty" seem to be different. Observations from the project classroom indicate that students' perceptions of problem difficulty is likely to reflect both their problem-solving and problem-posing experiences in related-task environments. Students with a high level of mathematical performance at the end of the study tended to pose problems in the form of conjecture, problems which require the use of particular solution approach in new contexts, or problems which involve the use of more complex solution structures. The perception of problem complexity/difficulty for those students seemed to be linked neither to the verbal nor to the computational complexity of a problem or its solution, but rather to an extension of the problem structure by integrating concepts from different domains whose solutions require more complex forms of mathematical cognitive activity.

In his work Sweller (1992) claimed that "goal-free problems require less time to solve than equivalent conventional problems" (p. 53). The project classroom observations suggest that this is indeed likely to be the case when students solve their *own* problems when the imitation strategy has been employed. A possible explanation might be that in those cases the authors do not need to spend time understanding their own problems. In fact, at the beginning of the study, most participants tended to pose problems which they knew how to solve.

A research investigation carried out by Sullivan, Bourke and Scott (1995) found that a statistically significant greater proportion of the students in a Year 6 class provided correct responses to an open-ended problem from a specific topic in geometry than they gave on a closed problem from the same domain. This finding is consistent with the observation in the project classroom that problem-solving and problem-posing might involve different cognitive processes. However, this study does not support the observation made by Sullivan et al. (1995) that, on the post-test, most students had reverted to giving just one answer. Eight months after the post-test, when students from the project classroom were invited to pose problems on a semi-structured situation, all of them provided responses from at least two categories.

According to Ellerton and Clements (1996) school children find it difficult to respond in divergent and creative ways to open-ended task situations. They claimed also that, in fact, when students are presented with an open-ended question, students need to be able both to pose a correct problem and then to solve the problem posed. This conclusion is consistent with the observation made by Sullivan (1995), and is in contrast with Owen and Sweller's (1885) findings. In his work Sullivan claimed that open-ended questions (a type of semi-structured problem-posing situation) required higher levels of thinking skills than did well-structured problems. On the other hand Owen and Sweller concluded that open-ended questions reduced the cognitive load.

This study suggests that discussion about cognitive load needs to take account of the problem-posing strategies involved. Problem posing is very likely to reduce the cognitive load when students create problems by employing reconstruction or imitation strategies. In contrast, when students' mathematical experience within a particular domain is limited, or when students have used the invention strategy to pose a problem, then most students would find it difficult to respond on any problem-posing category or to solve the question which was invented.

It should be noted, once again, that the scoring schemes developed by the researcher in order to evaluate students' problem-solving and problem-posing products were designed specifically for this study. The literature review revealed a lack of schemes appropriate for assessing students' problem-posing products. In fact, the evaluation schemes constructed by some researchers, Leung (1993) and Balka (1974) for example, are limited only to specific types of problem-posing tasks. The nature and the diversity of the problems in the project classroom required the development of a basic structure which could assess problem-posing products regardless of their nature and complexity. The results of the coding process have shown that under these schemes 98% of all problem-posing products could be classified. It cannot be assumed, however, that the classification presented will cover all possible problem-posing products.

There is also a limitation concerning the evaluation of the problem-posing products in terms of students' previous experiences. Although the content of the program was different from the school curricula, it is possible that out-of-school experience may have influenced students' mathematical performances.

The Challenges of This Study

The Challenges of the Research Design

It should be emphasised that the researcher needed to take several risks in this study. First, at the beginning of the Program it was an open question whether students would be willing to participate in the classes for several months and to attend, on a regular basis, the instructional sessions which involved "unusual" classroom activities.

Second, no research findings to date have discussed to what extent students are likely to respond to different problem-posing situation categories and how new environments might affect students' attitudes. Although no negative effects of problem-posing activities on students' attitude have been reported so far, "non-traditional" classroom activities needed to be introduced in the project classroom gradually and sensitively.

Third, the question whether the Euler Program could be adapted successfully for a mixed group of Years 8 and 9 students, was also open. The fact that four of the students had already attended the Euler program in the previous year, and that a Year 3 student was going to participate in the classes, were additional challenges.

And fourth, the researcher had also to take the risk that it was up to the students whether or not they took part in the research program, and whether or not they submitted their written work and their solutions to the *Challenge Problems.*

The Challenges of Intensive Classroom Communication

Classroom observations showed that an open problem-solving approach provided an atmosphere which was likely to *intensify* the communication between the teacher and the students: they shared their ideas, commented on their own mistakes and on the mistakes made by others, made predictions and guesses, and raised additional questions. An open problem-solving approach is also likely to provide an environment in which students can become engaged immediately in the lesson. In fact, problem-posing situations seem to assist both teacher and students to "personalise" the nature of the classroom learning environment.

During the study, on a regular basis, students from both groups A and B were asked about whether they were having any difficulties with the Program. One of the participants, Tom, a Year 8 student from one of the government schools in Perth, attended the problem-solving class during the first semester (Group B), and in the second semester, the project classroom sessions (Group A). Tom was interviewed on a regular basis to compare both approaches. After every session Tom was asked about his perceptions of the lesson, what he liked and what he did not like, and was prompted to support his opinion with some examples. Tom often admitted that he liked the second group more, because "the atmosphere is different." After several sessions Tom was asked to comment on any differences between the two classes. He said: "The second [Group A] is more intensive, but I understand more easily."

During the study every attempt was made to introduce problem posing in a sensitive way. The researcher paid particular attention, for example, to the feelings of students about whether they felt comfortable having their work shown to the whole class. For some students, like Nelly for example, the changes were quite dramatic. The independent observer described her impressions about Nelly's reactions when she was involved for the first time in problem posing in the following way:

> Elena made them [the students] make up their own questions to do with LCM: eg Nicki (usually very quiet) made up problem to do with LCM of 3, 4 and 5—if there is a girl at every 3rd desk, a cockroach on every 4th desk, beetle on every 5th desk, which is the next desk that will have all three? . . . Students enjoyed the novelty of creating their own problems, and clearly they understood the concepts to be able to create and answer their own problems.

No evidence was found that students with a high problem-solving performance might find some types of problem-posing situations easy, boring or not challenging.

CHAPTER ELEVEN

IMPLICATIONS

Implications of the results of this study for further research investigations, for the teaching and learning of mathematics, for preservice and inservice teacher education, and for curriculum design policy, will be discussed in this chapter.

Implications for Further Research Investigations

The implications of this study for further mathematics education research can be summarised under two headings: (a) Problem posing as a research tool; and (b) Problem posing as an instructional tool.

Problem Posing as a Research Tool

The classification of problem-posing situations developed in this study provides mathematics education research with a tool for gaining insight into different aspects of students' problem posing. Because the aim of the design of the framework was to assist students to develop an understanding not only about the problem structure, but also about the structure of the solution method, and because of the interrelationships between the elements of the problem and the solution structures, the research undertaken has extended the boundaries of traditional approaches to research on students' understanding of mathematics. The following three areas of problem posing warrant further investigation.

1. Classification of problem-posing situations. The problem-posing situations used in this study were designed and developed on the basis of the initial framework presented in Chapter 4. Further research investigations are needed to throw light on possible extensions, adaptations and other areas of application of the framework.

The framework also provides a basis for extending the problem-posing situations proposed in this study and for the development of new problem-posing situations. Possible new problem-posing situations might include, for example, taking into account particular features of a specific classroom environment. One of the observers in the study admitted that he had tried and used some of the problem-posing situations within a tertiary setting. Research into the application of the framework in other settings — for example, tertiary or early–childhood — should be conducted.

The study has presented details of a classification of problem-posing situations used as a *means* of instruction. Application of problem posing as a *goal* of instruction in which students could be involved during classroom work might lead to significant changes in the initial framework and to the types of problem-posing situations.

The extent to which the problem-posing classification set out in this thesis could be adopted and extended to other school subject areas — for example science and language education — needs further exploration. Incorporation of problem-posing situations in language education, for example, might involve an emphasis on the use of free and semi-structured situations in studies of literary works, and the use of structured situations when applied to classroom contexts in which the basic grammar rules are applied.

2. Students' problem-posing strategies. Problem-posing strategies used by students in the study led to the development of a framework to describe and categorise these strategies. This framework needs further exploration and specification, including, for example, research into whether the strategies are specific when applied to particular content areas, and whether they vary with the student's age, experience, motivation and ability. Possible links between the type of problem-posing categories and the characteristics of students' problem-posing products need to be explored. Both the definition of "quality" and what factors influence the invention of "quality" mathematics problems should be regarded as problematic.

3. The open problem-solving approach. In this study problem-posing activities were an integral part of students' problem solving. This required the researcher to develop a range of questions which provided support for students in reflecting about their problem solving via problem-posing activities.

Although the study has foreshadowed and modelled some possible ways (modes) of application of problem-posing situations in mathematics classrooms, further research is needed.

Within a specific classroom environment, different sets of problem-posing activities could be embodied by using different models for interaction between problem-solving and other classroom activities. Characteristics of possible models for the interaction between problem posing and problem solving in different classroom settings and topics areas need further investigations.

Problem Posing as an Instructional Tool

Applications of the problem-posing classification and an open problem-solving approach, should be researched in various dimensions. The following suggestions outline the scope of further questions and issues which need to be investigated:

1. In relation to students' mathematical understanding. The effect, on students' mathematical understanding, of encouraging students to pose problems needs to be investigated. For example, teaching students to pose problems by applying a *particular* problem-posing strategy might affect some *specific* aspects of their problem-posing or problem-solving performances. If, for example, students are taught to construct problems with the same mathematical model but in different contexts, then this might have a positive effect on students' performance in solving word problems of the same type.

In this study it is foreshadowed that a *particular* problem-posing situation could be integrated as part of problem solving under *different* modes for applications, according to its instructional goal. Do, for example, different modes for application have different effects on specific aspects of students' mathematical performance? What modes are appropriate for students with low or high mathematical aptitudes?

Observations from the project classroom indicate that there may be a link between the appropriateness of problem-posing situations and the level of students' mathematical performance. For example, some structured problem-posing situations might be particularly useful for working with students with poor mathematical skills. Students with more extensive mathematical skills may be able to benefit from semi-structured or free problem-posing situations. Further work is needed to clarify this. Insight into the effects of applying a *particular* problem-posing category to students with different levels of mathematical performance could throw light on possible classroom applications of problem-posing approaches.

Further investigations are also needed to explore students' preferences for working on specific types of problem-posing situations, and in how these relate to students' mathematical understanding.

Some problem-posing situations may have a greater influence on student's understanding of mathematics and on their mathematical performance than others. Questions as to whether this is indeed the case, and whether this depends on a student's age need to be addressed.

The extent to which the mathematical content of problem-posing situations, and the types of problems on which problem-posing activities are based, catalyse students' mathematical performance are issues of fundamental importance for mathematical instruction.

2. In relation to the quality of students' problem-posing products. Research to date is silent about identifying factors which lead to the posing of good quality questions. Key questions for mathematical instruction, for example, relate to identifying the types of problem-posing activities most likely to help students to produce good quality questions. Other questions which relate to the quality of problems posed by students and at the same time have relevance for the mathematics classroom, include:

- To what extent do the problem-posing strategies used by students relate to their level of problem-solving performance?
- What kind of instructional conditions which incorporate problem-posing activities are mostly to facilitate students' problem-posing performance?
- What kinds of environments are needed to facilitate students' reflections on specific problem-posing situations via a particular problem-posing strategy, such as invention for example?
- How do students' problem-posing strategies differ from those employed by teachers and professional mathematicians?
- To what extent does the framework developed in this study to describe students' problem-posing strategies provide a basis for developing a classification of students' problem-posing strategies in other subject areas?

Implications for the Teaching and Learning Mathematics

The main implication of this study for the teaching and learning of mathematics is that it provides frameworks (for both problem-posing situations and for students' problem-posing strategies) which can be readily used: (a) as a tool for designing problem-posing situations; and (b) as a means of instruction. These implications can be summarised under several sub-headings which relate directly to teachers' work in mathematics classrooms, as follows.

As a Tool for Designing Problem-posing Situations

The framework described in Chapter 4 and the features of the design process of problem-posing situations outlined in Chapter 6 provide teachers with a tool for designing problem-posing situations on the basis of mathematics textbooks problems. For extending students' experience in solving particular type of problem, for example, appropriate problem-posing activities could involve various types of reformulations, posing related problems, and creating and solving problem chains.

The classification of problem-posing situation categories can also be applied to the development of particular types of interrelated problem items, and for designing mathematical tasks with the *same* or *different* levels of difficulty (Stoyanova & Bana, 1997).

Students' capacity to pose mathematical problems could be used to provide an additional source of problems other than from mathematics textbooks. The framework presented in this study to describe students' problem-posing strategies provides background which may assist teachers in, for example, involving students in making up problem items with specific characteristics such as problems which are identical, similar, different, easier or more difficult then a given problem.

As a Tool for Diagnosing Students' Individual Difficulties

Students' work on specific problem-posing situations can provide information for the teacher about individual difficulties and the level of students' mathematical understanding which can be used as a starting point for further individual work.

Involving students in making problem variations — such as *identical* or *similar* problems — can be used for helping students to understand and analyse relationships between elements in the structure of a particular problem. At the same time, through such problem-posing activities, teachers may be able to support some students who are trying to overcome difficulties they have in solving particular types of problems.

The use of free problem-posing situations in the mathematics classroom may also be used as a diagnostic tool. For example, inviting students to pose a problem similar to the type of problems they have found difficult to solve, could provide information about the scope of problems which need additional practice and attention.

Observations from the project classroom suggest that a small number of students tend to pose problems which they are unable to solve, even though the problem structure has a clear meaning for them. These students' capacity for problem posing provides a natural starting point for involving students in mathematical investigations which extend the boundaries of the prescribed curriculum.

As a Means for Helping Students to Reflect on their Previous Experiences

The study suggests that structured problem-posing situations are appropriate for *all* students and allow them to reflect on specific actions based on their previous mathematical experiences. At the same time, some problem-posing situations in this category — such as posing inverse problems, counter examples or sequences of problems which relate to a given problem — can be used to stimulate students with high levels of mathematical understanding.

The study further suggests that the teacher can involve students in some structured problem-posing activities such as posing problems which involve the use of a specific solution method or problems which are inverse to a given problem, when students have some basic background skills and knowledge.

Semi-structured problem-posing situations can provide an environment in which students could link their current and previous mathematical experiences and could experiment with the application of different mathematical concepts and methods. The problems posed by students, may therefore, provide teachers with information about the level of understanding students have about these concepts (methods). This could then become a starting point for further enquires.

As an Instructional Environment in Which Students Could Monitor Their Own Learning

Students who are good problem solvers are more likely to attempt to pose problems which they cannot solve, but can understand and in most can evaluate a solution. Problem-posing activities can help to nurture students' motivation to pursue and solve difficult problems. For example, by asking students to pose problems from a specific learning area which they cannot solve, but which they understand, students could be involved in further explorations, according to their preferences. In fact, students might benefit if problem posing is introduced in mathematics classrooms as a type of skill which they have to learn and master.

As an Alternative Way of Assessing Students' Mathematical Performance

Students' mathematical performance is traditionally assessed by solving problem items from a particular learning area. An alternative, which places the student in a less stressful situation, is to ask the student to pose problems of a specific type which he/she can solve. Results of the study suggest that students with low and average mathematical aptitude tend to pose problems which they understand how to solve. Hence, instead of asking students to solve a problem, as an alternative, they could be asked to create problems of a specific type which they can solve and to show how they would explain the solution to someone who does not understand this type of problem.

Students' problem-posing products are not generally part of traditional assessment practice. Assessing the quality of the problems posed by students, taking into account key characteristics of the posed problem such as the language, correctness, originality, and level of difficulty, could be used as an alternative for estimating the level of their mathematical performance.

As a Means of Instruction Which Could Improve Students' Understanding of Mathematics

In most cases, during this study, problem posing was applied as an inseparable part of problem-solving activities and was aimed to facilitate students' problem solving. Possible applications include the following:

- The research described in this study has produced a system of problem-posing situations which could be readily applied to school mathematics education as a means of instruction for facilitating students' problem solving.

- Another relevant application of the system of problem-posing situations described in this study is that it could be used to differentiate between different aspects of problem posing. At the same time a teacher could work with students with different levels of mathematical ability. For example, while some of the students might be engaged only in the reformulation of a problem, others could analyse and explore the situation further by making sequences of interrelated problems.

- The teacher's knowledge about posing quality interrelated problems is an inseparable part of the preparation for mathematics lessons. The sequence, format and groupings of the problems in mathematics textbooks are not always appropriate for all classes. The categories and approaches developed in this study may assist teachers to develop mathematics problems and activities on their own.

- Models for the interrelationship between problem posing and problem solving are likely to vary according to the objectives of the lesson, the topic, and the teacher's and students' prior mathematical experiences. The study also provides some examples of the application of a range of modes in a classroom context which could be adopted for use in school mathematics classrooms.

As an Approach to Help Improve Teachers' Problem-posing Skills

Assessing students' knowledge is part of a teacher's work and plays an essential role in mathematical instruction. A teacher's ability to pose equivalent test items, to distinguish between similar and identical test items, and the quality of assessment all affect every student. The approaches described in this study for designing problem-posing situations and the categories of problems-posing strategies used by students provide a framework which would facilitate the development of identical, equivalent or similar test items.

One of the challenges of designing appropriate assessment instruments is the need to be able to pose problems. The problem-posing categories described in this study could be used towards achieving such a goal.

Implications for Preservice and Inservice Teacher Education

As a preparation for the day-to-day work of mathematics teachers, problem *posing* -- as well as problem *solving* -- should be an integral part of preservice and inservice teacher education (Leung, 1996). Lack of knowledge for example, about posing problems can affect not only the quality of assessing students' knowledge, but can also have a negative effect on classroom work if the teacher does not know how to adjust the main features of a given set of problems.

Prospective teachers can learn ways of creating sequences of algorithmic exercises in order to meet different instructional goals which are appropriate for classroom settings and which include students who have a range of mathematical abilities. Being able to present the same problem in different formats can enable a teacher to choose an appropriate level for presenting specific mathematical content and to reduce (or increase) the level of difficulty of the problems.

The study also describes strategies which can be used by teachers to pose identical, similar and interconnected series of problems to extend students' experiences by using different mathematical contexts. A teacher's knowledge and ability to present a problem structure within different contexts are most likely to help students to become intelligent users of mathematics in their every-day life.

Although the open-problem-solving approach used in this study needs further development and exploration, some of the ideas can be readily applied in mathematics classrooms and could enrich the kit of instructional tools of mathematics teachers.

Teacher's questions which incorporate "hidden" problem posing are an instructional prompt which is likely to help students to reflect on specific problems from a particular perspective. Useful prompts which are incorporated by the teacher into the classroom context might help students to focus on particular features of mathematical tasks.

Implications for Mathematics Educators at All Levels

The inclusion of problem-posing activities in mathematics classrooms has been recommended in the curriculum documents of several countries. This study provides an insight into how problem posing can be used as an inseparable part of students' problem-solving activities, and draws conclusions about the types of activities which might be appropriate. The study provides authors of mathematics textbooks with a point of reference for the development of problem-posing situations and problem items.

The different applications of problem posing described in the study illustrate a range of instructional goals which teachers might pursue in their mathematics classrooms. Although some mathematics educators might be aware of many of the activities described in the study (Silver, 1993), the ways in which they were embraced with problem solving might be used to help enrich standard textbook problems. The framework developed in this study to describe students' problem-posing strategies can also help to inform classroom practice.

Concluding Note

The research issues investigated in this thesis are important as *research*, as a *research tool* and as an *instructional approach to the teaching of mathematics.*

The study was inspired by previous research investigations conducted by Kilpatrick (1987) and Silver (1993). Via an open problem-solving approach students were involved, in natural ways, in discussing and solving complex, difficult and novel problems and solution methods.

This research has provided a glimpse of what might be possible when problem structures, solution methods, and students' activities are "open."

Beyond this, applications of problem posing and research opportunities which examine these applications, are limited only by one's creativity.

REFERENCES

Anderson, J., & Sullivan, P. (1995). Creating open-ended problems and mathematical investigations for your classroom. In A. Richards (Ed.), *Flair: Forging links and integrating resources* (pp. 29-35). Proceedings of the 15th Biennial Conference of the Australian Association of Mathematics Teachers. Darwin: Australian Association of Mathematics Teachers.

Australian Educational Council (1990). *A National Statement on Mathematics for Australian Schools*. Melbourne: Curriculum Corporation.

Baird, H. & White, R. (1984, April). *Improving learning through enchenced metacognition: A classroom study.* AERA Paper, New Orleans.

Baker, J. (1980). *Mathematics across the curriculum.* Milton Keynes: Open University.

Balka, D. S. (1974). Creative ability in mathematics. *Arithmetic Teacher, 21,* 633-636.

Bauersfeld, H. (1980). Hidden dimensions in so-called reality of a mathematics classroom. *Educational Studies in Mathematics, 11,* 23-41.

Beeridge, W. (1957). *The Art of Scientific Investigation.* London: Heinemann.

Bell, A., Fishbein, E., & Greer, B. (1984). Choice of operation in verbal arithmetic problems: The effects of number size, problem structure and context. *Educational Studies in Mathematics, 15,* 129-147.

Bennett, N., & Dunne, E. (1992). *Managing classroom groups.* Hemel Hempstead, UK: Simon and Schuster.

Bishop, A. J. (1988). *Mathematical enculturation.* Dordrecht: Kluwer.

Blum, W., & Niss, M. (1991). Applied mathematical problem solving, modelling, applications, and links to other subjects: State, trends and issues in mathematics education. *Educational Studies in Mathematics, 22,* 37-68.

Borasi, R. (1987). Exploring mathematics through the analysis of errors. *For the Learning of Mathematics,* 7(3), 2-8.

Brown, S. I. (1984). The logic of problem generation: From morality and desposing and rebellion. *For the Learning of Mathematics,* 4(1), 9-20.

Brown, S., & Walter, M. I. (1983). *The art of problem posing.* Philadelphia, PA: Franklin Institute Press.

Brown, S., & Walter, M. I. (Eds.). (1993). *Problem Posing: Reflections and applications.* Hillsdale, NJ: Lawrence Erlbaum.

Brueckner, L. (1932). Improving pupils' ability to solve problems. *NEA Journal, 21,* 175-176.

Bruner, J. (1961). *The process of education.* Cambridge: Harward University Press.

Bruner, J. (1996). *The culture of education.* Cambridge: Harward University Press.

Burjan, V. (1993). A plea for non-Olympiad type competitions. *Mathematics Competitions, 6*(1), 10-14.

Bush, W., & Fiala, A. (1986). Problem stories: New twist on problem posing. *The Arithmetic Teacher, 34*(4), 6-9.

Butts, T. (1980). Posing problems properly. In S. Krulik & R. E. Reys (Eds.), *Problem solving in school mathematics* (pp. 23-33). Reston, VA: National Council of Teachers of Mathematics.

Caldwell, J. (1984). Syntax, content and context variables in instruction. In Goldin, A. G., & McClintock, E. C. (Eds.), *Task variables in mathematical problem solving* (pp. 379-413). Philadelphia, PA: Franklin Institute Press.

Carpenter, P., Fennema, E., Peterson, L., Chi-Pang Chiang, & Loef, M. (1988, April). *Using knowledge of children's mathematics thinking in classroom teaching: An experimental study.* National Center for Research in Mathematical Sciences Education. Paper presented at the American Educational Research Association Annual Meeting in New Orleans, LA.

Charles, R., & Lester, F. (1982). *Teaching problem solving: What, why & how.* Palo Alto, CA: Dale Seymour.

Cheesman, J., & Doig, B. (1995). Up-front assessment: Open-ended questions. I ' Wakefield & L. Velardi (Eds.), *Celebrating mathematics learning* (pp. 484-490). Melbourne: Mathematical Association of Victoria.

Clarke, D. J. (1985). Classroom research: Generalising from case studies. In A. Bell, B. Low, & J. Kilpatrick (Eds.), *Theory, research & practice in mathematics education.* Collected papers from the 5th International Congress of Mathematical Education (pp. 523-540). Nottingham, UK: Shell Centre for Mathematical Education, University of Nottingham.

Clarke, D. J., & Sullivan, P. (1992). Responses to open-ended tasks in mathematics: characteristics and implications. In W. Geeslin & K. Graham (Eds.), *Proceedings of the 16th International Conference for the Psychology of Mathematics Education, Vol. I* (pp. 137-144). Durham, NH: University of New Hampshire.

Clarke, D., & Sullivan, P. (1991a). The assessment implications of open-ended tasks in mathematics. In *Reshaping assessment practices: Assessment in the* *mathematical science under challenge* (pp. 161-179). Melbourne: Australian Council for Educational Research.

Clarke, D., & Sullivan, P. (1991b). *Communications in the classrooms: The importance of good questioning.* Geelong: Deakin University.

Clarke, D., Sullivan, P., & Spandel, U. (1992). Student response characteristics to open-ended tasks in mathematical and other academic contexts. In B. Southwell, B. Perry & K. Owens (Eds.), *Space - The first and final frontier* (pp. 209-221). Proceedings of the 15th Annual Conference of the Mathematics Education Research Group of Australasia. Sydney: Mathematics Education Group of Australasia.

Cockcroft, W. H. (1982). *Mathematics counts.* Report of the Committee of Inquiry into Teaching of Mathematics in Schools. London: H. M. S. O.

Cohen, S. A., & Stover, G. (1981). Effects of teaching sixth-grade students to modify formal variables of maths word problems. *Reading Research Quarterly, 16*(2), 175-200.

Collins, A. (1988). Different goals of inquiry teaching. *Questioning Exchange, 2*(1), 1988.

Connor, W. L., & Hawkins, G. (1936). What materials are most useful to children in learning to solve problems? *Educational Method, 16,* 21-29.

Davis, R., Maher, C., & Noddings, N. (Eds.). (1990). Constructivist views of the teaching of mathematics. Reston, VA: National Council of Teachers of Mathematics.

Del Campo, G., & Clements, M. A. (1987). Elementary schoolchildren's processing of "change" arithmetic word problems. In J. C. Bergeron, N. Herscovics, & C. Kieran (Eds.), *Proceedings of 11th International Conference on the Psychology of Mathematics Education, Vol. 2* (pp. 382-386). Montreal: International Group for the Psychology in Mathematics Education.

Dillon, J. T. (1982). Problem finding and solving. *Journal of Creative Behaviour, 16,* 97-111.

Dorofeev, G. B. (1983). About the construction of cycles of interrelated problems. *Mathematika v Schkole, 6,* 34.

Doyle, W. (1983). Academic work. *Review of Educational Research, 53*(2), 159-199.

Doyle, W., & Carter, K. (1982). Academic tasks in the classroom. In M. Hammersley (Ed.), *Case studies in classroom research* (pp. 133-155). Milton Keynes, England: Open University Press.

Duncker, K. (1945). On problem solving. *Psychological Monographs, 58* (5, Whole No. 270).

Education Department of Western Australia. (1994). *Mathematics student outcome statements with pointers and work samples.* Perth: Author.

Einstein, A., & Infeld, L. (1938). *The evolution of physics.* New York: Simon & Schuster.

Ellerton, N. (1989). *The development of abstract reasoning in mathematics.* Unpublished doctoral thesis. Victoria University of Wellington, N. Z.

Ellerton, N. F. (1980). *An extension of Piaget's theory of cognitive development.* Research report, Victoria University of Wellington, N.Z.

Ellerton, N. F. (1986a). A window into children's perception of mathematics. In W. Caughey (Ed.), *From now to the future* (pp. 64-70). Melbourne: Mathematical Association of Victoria.

Ellerton, N. F. (1986b). Children's made-up mathematical problems: A new perspective on talented mathematicians. *Educational Studies in Mathematics, 17,* 261-271.

Ellerton, N. F. (1986c). Mathematics problems written by children. *Research in Mathematics Education in Australia* (December), 32-44.

Ellerton, N. F. (1988). Exploring children's perception of mathematics through letters and problems written by children. In A. Borbas (Ed.), *Proceedings of the 12th International Conference for the Psychology of Mathematics Education, Vol. 1* (pp. 280-287). Veszprem, Hungary: International Group for the Psychology of Mathematics Education.

Ellerton, N. F., & Clements, M. A. (1996). Researching language factors in mathematics education: The Australasian Contribution. In B. Atweh, K. Owens & P. Sullivan (Eds.), *Research in mathematics education in Australasia 1992-1995* (pp. 191-235). Campbelltown, NSW: Mathematics Education Research Group of Australasia.

Ellerton, N., & Clarkson, P. (1996). Language factors in mathematics teaching and learning. In A. J. Bishop, K. Clements, C. Keitel, J. Kilpatrick, & C. Laborde (Eds.), *International Handbook of Mathematics Education* (pp. 987-1033). Dordrecht: Kluwer.

Ellerton, N., & Clements, M. A. (1991). *Mathematics in language: A review of language factors in mathematics learning.* Geelong: Deakin University.

Ervanger, S. H. (1975). Case studies of children perceptions of mathematics - Part 1. *Journal of Children's Mathematical Behaviour, 1*(3), 157-283.

Evans, J. (1987). Investigations - the state of art. *Mathematics in School 16*(1), 27-30.

Evans, M., & Henry, B. (Eds.). (1995) *Euler student notes.* Canberra: Australian Mathematics Trust.

Evans, M., & Henry, B. (Eds.). (1995). *Challenge problems- Junior level.* Canberra: Australian Mathematics Trust.

Evans, M., & Henry, B. (Eds.). (1995). *Euler student problems.* Canberra: Australian Mathematics Trust.

Evans, M., & Henry, B. (Eds.). (1995). *Euler teacher reference notes.* Canberra: Australian Mathematics Trust.

Foddy, W. (1993). *Constructing questions for interviews and questionnaires.* Cambridge: Cambridge University Press.

Frankenstein, M. (1989): *Relating mathematics: A different third R - radical mathematics.* London: Free Association Books.

Gage, M. S. (1982). *A comparison of formatting and solving original mathematics word problems with solving ready made problems by community college students.* Doctoral dissertation, New York University.

Gagne, R. M. (1970). *The conditions of learning* (2nd ed.). New York: Holt, Rinehart & Winston.

Gagne, R. M., & Biggs, L. J. (1974). *Principles of instructional design.* New York: Holt, Rinehart & Winston.

Garofalo, J. & Lester, F. J., (1985). Metacognition, cognitive monitoring and mathematical performance. *Journal for Research in Mathematics Education, 16,* 163-176.

Geeslin, W. E. (1977). Using writing about mathematics as a teaching technique. *Arithmetic Teacher, 70,* 112-115.

Georgiev, B. C. (1988). Improving students' problem solving through the use of problem cycles. *Mathematika v Schkole, 1,* 77-78.

Getzels, J. W. (1979). Problem finding: A theoretical note. *Cognitive Science, 3,* 167-172.

Getzels, J. W. (1984). Problem finding and creativity in higher education. *The fifth Rev. Charles F. Donovan, S. J. Lecture,* Sponsored by the School of Education, Boston College.

Getzels, J. W., & Csikszenmichalyi, M. (1976). *The creative vision: A longitudinal study of problem finding in art.* New York: John Wiley & Sons.

Glaser, B., & Strauss, A. L. (1967). *The discovery of grounded theory.* Chicago: Aldine.

Goldin, A. G., & McClintock, E. C.(Eds.). (1984). *Task variables in mathematical problem solving.* Philadelphia, PA: Franklin Institute Press.

Goldman, A. M., & Zvavitch, L. I. (1990). Problem sequences in mathematics lessons. *Mathematika v Schkole, 5*, 19-22.

Gonzales, N. (1994). Problem posing: A neglected component in mathematics courses for prospective elementary and middle school teachers. *School Science and Mathematics, 94*(2), *78-84*..

Graham, V. G. (1978). *The effect of incorporating sequential steps and pupil-constructed problems on performance and attitude in solving one-step verbal problems involving whole numbers.* Doctoral Dissertation, Catholic University of America.

Greeno, J. G. (1980). Trend in the theory of knowledge for problem solving. In D. T. Tuma & F. Reif (Eds.), *Problem solving and education: Issues in teaching and research* (pp. 9-23). Hillsdale, NJ: Lawrence Erlbaum.

Greer, B., & McCann, M. (1991). Children's word problems matching multiplication and division calculations. In Furinghetti (Ed.), *Proceedings of the 15th Annual Conference for the Psychology of Mathematics Education* (pp. 80-87). Assisi, Italy: International Group for the Psychology of Mathematics Education.

Hadamard J. (1945). *An essay on the psychology of invention in the mathematical field.* Princeton, NJ: Princeton University Press.

Halford, G. S. (1987). A structure-mapping approach to cognitive development. *International Journal of Psychology, 22*, 609-642.

Halford, G. S., & Boulton-Lewis, G. M. (1992). Value and limitations of analogs in teaching mathematics. In A. Demetriou, A. Efkliades, & M. Shayer (Eds.), *The modern theories of cognitive development go to school* (pp. 183-209). London: Routedge.

Hart, K. (Ed.). (1981). *Children's understanding of mathematics: 11-16.* London: John Murray.

Hashimoto, Y. (1994). Theory and practice in mathematics education through two teaching methods (open-end approach and developmental treatment of mathematics problems) in Japan (pp. 159-163). In *Background Papers* for the ICMI Study Conference 8 - 11 May. College Park, MD: University of Maryland.

Hashimoto, Y., & Swada, T. (1984). Research on the mathematics teaching by developmental treatment of mathematical problems. In T. Kawaguchi (Ed.), *Proceedings of the ICMI-JSME regional conference on mathematical education* (pp. 309-313). Japan: Japan Society of Mathematics Education.

Hashimoto, Y., (1987). Classroom practice of problem solving in Japanese elementary schools. In J. P. Becker & T. Miwa (Eds.), *Proceedings of the U. S. - Japan seminar on mathematical problem solving* (pp. 94-119). Carbon Dale, IL: Southern Illinois University.

Herrington, T., Wong, K. Y., & Kershaw, L. (1993). *Fostering mathematical thinking and learning,* Adelaide: Australian Association of Mathematics Teachers.

Hoehn, L. (1991). Problem posing in geometry. *Mathematics Teacher 81*(1), 10-14.

Hopkins, C. (1995). Open teaching. *Mathematics Teaching, 150,* 41-43.

Hosmer, P. C. (1986). Students can write their own problems. *The Arithmetic Teacher, 34*(4), 10-11.

Hughes, T. P. (1983). *Networks of power* (pp. 25-27). Baltimore: Johns Hopkins University Press.

Immegart, G. L., & Boyd, W. L. (Eds.). (1979). *Problem finding in educational administration.* Lexington, MA: D. C. Heat.

Kantowski, M. G. (1974). *Processes involved in mathematical problem solving.* Doctoral dissertation, University of Georgia.

Kantowski, M. G. (1977). Processes involved in mathematical problem solving. *Journal for Research in Mathematics Education, 8*(3), 163-180.

Kantowski, M. G. (1980). Some thoughts on teaching for problem-solving. In: *NCTM Yearbook 1980* (pp. 195-203). Reston, VA: National Council of Teachers of Mathematics.

Keil, G. (1964). *Writing and solving original problems as a means of improving verbal arithmetic problem-solving ability.* Unpublished doctoral dissertation, Indiana University.

Kennedy, W. (1985). Writing letters to learn maths. *Learning* (February), 59-60.

Kilpatrick, J. (1967). *Analysing the solution of word problems in mathematics: An exploratory study.* Unpublished doctoral dissertation, Stanford University.

Kilpatrick, J. (1984). The variables in mathematical problem solving. In A. G. Goldin & E. C. McClintock (Eds.), *Task variables in mathematical problem solving* (pp. 461-472) Philadelphia, PA: Franklin Institute Press.

Kilpatrick, J. (1985). A retrospective account of the past 25 years of research of teaching mathematical problem solving. In E. A. Silver (Ed.), *Teaching and learning mathematical problem solving - multiple research perspectives* (pp. 1-15). Hillsdale, NJ: Lawrence Erlbaum.

Kilpatrick, J. (1987a). Problem formulating: Where do good problems come from? In A. H. Schoenfeld (Ed.), *Cognitive science and mathematics education* (pp. 123-147). Hillsdale, NJ: Lawrence Erlbaum.

Kilpatrick, J. (1987b). What constructivism might be in mathematics education. In J. C. Bergeron (Ed.), *Proceedings of the 11th International Conference of the*

International Group for the Psychology of Mathematics Education, Vol. 1 (pp. 3-27).

King, L., Barry, K., Maloney, C., & Tayler, C. (1993). *The MAKITAB small-group learning interaction analysis system.* Technical report. Perth, WA: Edith Cowan University.

King, L., Barry, K., Maloney, C., & Tayler, C. (1994, April). *Task-enhancing talk in cooperative learning.* Paper presented at the Annual Conference of the American Educational Research Association in New Orleans.

Kissane, B. (1988). Mathematical investigation: Description, rationale, and example, *Mathematics Teacher, 10,* 520-528.

Koenker, R. (1958). Twenty methods of improving problem solving. *The Arithmetic Teacher, 5,* 74-78.

Krulik, S., & Rundick, J. (1984). Helping teachers become teachers of problem solving. In H. Burkhart, S. Groves, A. Schonefeld & K. Stacey (Eds.), *An overview of problem solving in problem solving - A world view* (pp. 123-129). Adelaide: International Commission for Mathematical Education.

Krutetskii, V. A. (1976). *The psychology of mathematics abilities in schoolchildren.* Chicago: University of Chicago Press.

Lakatos, I. (1976). *Proofs and reformulations: The logic of mathematical discovery.* Cambridge: Cambridge University Press.

Lemonick, M. D. (1993). Fini to Fermat's last theorem. *Time,* July 5, p. 55.

Lester, F., & Groves, K. (1977). Key issues in mathematical problem solving research. *Research in Mathematics Education in Australia, Vol. 1* (pp. 189-197).

Leung, S. & Silver, E. A. (1997). The role of task format, mathematics knowledge, and creative thinking on the arithmetic problem posing of perspective elementary school teachers, *Mathematics Education Research Journal, 9*(1), 5-54.

Leung, S. (1993). *The relation of mathematical knowledge and creative thinking to the mathematical problem posing of prospective elementary school teachers on tasks differing in numerical information content.* Unpublished doctoral thesis, University of Pittsburgh.

Leung, S. K. (1996). Problem posing as assessment: Reflections and Reconstructions. *The Mathematics Educator, 1*(2), 159-171.

Leung, S. S. (1991). Building connections to the Pythagorean theorem: An example of teachers' treatment of textbook problems. In M. K. Heid & G. W. Blume (Eds.), *Making connections* (pp. 41-47). University Park, PA: Pennsylvania Council for the Teachers of Mathematics.

Leung, S. S. (1993). Mathematical problem posing: The influence of task formats, mathematics knowledge, and creative thinking. In I. Hirubashi, N. Nohda, K. Shigematsu, & F. L. Lin, *Proceedings of the 17th International Conference for the Psychology of Mathematics Education, Vol. 3,* (33-40). Tsukuba, Japan: University of Tsukuba.

Leung, S. S. (1994). On analysing problem-posing processes: A study of prospective elementary teachers differing in mathematics knowledge. In J. P. Ponte & J. F. Matos (Eds.), *Proceedings of the18th International Conference for the Psychology of Mathematics Education, Vol. 3,* (pp. 168-175). Tsukuba, Japan: International Group for the Psychology in Mathematics Education.

Leung, S. S. (1997). On the role of creative thinking in problem posing. *International Reviews on Mathematical Education, 97*(2), 48-52.

Lincoln, Y. S., & Guba, E. G., (1985). *Naturalistic inquiry.* Beverly Hills: Sage.

Ling, J. (1977). *Mathematics across the curriculum.* Published for the School Council by Blackie.

Lodholz, R. D. (1980). *The effects of student composition of mathematical verbal problems on student problem solving performance.* Doctoral Dissertation, University of Missouri, Columbia.

Lovitt, C., & Clarke, D. M. (1988). *The Mathematics Curriculum and Teaching Program activity bank* (2 volumes). Canberra: Curriculum Development Corporation.

MacGregor, M., & Stacey, K. (1995). The effect of different approaches to algebra on students' perceptions of functional relationships. *Mathematics Education Research Journal, 7*(1), 69-85.

Malone, J., & Miller, D. (1993). Communicating mathematical terms in writing: Some influential variables. In M. Stevens, A. Waywood, D. Clarke, & J. Izard (Eds.), *Communicating mathematics: Perspectives from classroom practice and current research* (pp. 177-190). Melbourne: Australian Council for Educational Research.

Mamona-Downs, J. (1993). On analysing problem posing. In I. Hirabayashi, N. Nohda, K. Shigematsu, & F. L. Lin (Eds.), *Proceedings of the 17th International Conference for the Psychology of Mathematics Education, Vol. III* (pp. 41-47). Tsukuba, Japan: International Group for the Psychology in Mathematics Education.

Marchall, S. P. (1995). *Schemes in problem solving.* New York: Cambridge University Press.

Mason, J. (1978). On investigations. *Mathematics Teaching, 84.* 43-47.

Mason, J. (1991). Mathematics problem solving: open, closed and exploratory in the UK. *International Reviews on Mathematical Education, 23*(1), 14-19.

Mason, J., & Davis, J. (1991). *Fostering and sustaining mathematical thinking through problem solving.* Geelong, Victoria: Deakin University.

Mayer, R. E. (1983a). *Thinking, problem solving, cognition.* New York: Freeman.

Mellin-Olsen, S. (1987). *The politics of mathematics education.* Dordrecht: Reidel

Meyer, M. (1983). *Meaning and reading: A philosophical essay on language and literature.* Amsterdam: Benjamin.

Moses, B., Bjork, E., & Goldenberg, E. P. (1990). Beyond problem solving: problem posing. In T. J. Cooney & C. R. Hirrsh (Eds.), *Teaching and learning mathematics in the 1990's* (pp. 82-91). Reston, VA: National Council of Teachers of Mathematics.

Mousley, J. (1990). Order in constructivism: teachers' roles in a problem-posing classroom. In K. Milton & H. McCann (Eds.), *Mathematical turning points: Strategies for the 1990's, Vol. 2* (pp. 418-426). Hobart: Australian Association of Mathematics Teachers.

National Council of Teachers of Mathematics. (1989). *Curriculum and evaluation standards for school mathematics.* Reston, VA: Author.

National Council of Teachers of Mathematics. (1990). *Teaching & Learning mathematics in the 1990's.* Reston, VA: Author.

National Council of Teachers of Mathematics. (1991). *Professional standards for teaching mathematics.* Reston, VA: Author.

Nohda, N. (1984). The heart of "open-approach" in mathematics teaching. In T. Kawaguchi (Ed.), *Proceedings of ICMI-JSME regional conference on mathematics education* (pp. 314-318). Tokyo: JSME.

Nohda, N. (1986). A study of "open-approach" method in school mathematics. *Tsukuba Journal of Educational Study in Mathematics, 5,* 119-131.

Nohda, N. (1988). Problem solving using "open-ended" problems in mathematics teaching. In H. Burkhardt, S. Groves, A. Schoenfeld & K. Stacey (Eds.), *Problem solving - A word view* (pp. 225-234). Nottingam: Shell Centre.

Nohda, N. (1991). Paradigm of the "open-approach" method in mathematics teaching: Focus on mathematical problem solving. *International Reviews on Mathematical Education, 23,* 32-37.

Nohda, N. (1995). Teaching and evaluating using "open-ended problem" in classroom. *International Reviews on Mathematical Education, 27*(2), 57-61.

Owen, E., & Sweller, J. (1985). What do students learn while solving mathematical problems? *Journal of Educational Psychology, 77*(3), 272-284.

Patton, M., Q. (1990). *Qualitative evaluation and research methods.* Newbury Park, CA: Sage.

Pegg, J., & Davey, G. (1991). Levels of geometric understanding. *Australian Mathematics Teacher, 47*(2), 10-13.

Pehkonen, E. (1989). Verwenden der geometrischen problemfelder. In E. Pehkonen (Ed.), *Geometrieunterricht 1989* (pp. 290-293). Bad Salzdetfurth: Franz-Becker Verlag.

Pehkonen, E. (1992). Using problem fields as a method of change. *The Mathematics Educator, 3*(1), 3-6.

Pehkonen, E. (1993). On teachers' criteria to assess mathematical activities. In I. Hirabayashi, N. Nohda, K. Shigematsu, & F. L. Lin (Eds.), *Proceedings of the 17th International Conference for the Psychology of Mathematics Education, Vol. 1* (pp. 220-227). Tsukuba, Japan: International Group for the Psychology of Mathematics Education.

Pehkonen, E. (1995). Introduction: Use of open-ended problems. *International Reviews on Mathematical Education, 27*(2), 55-57.

Perez, J. (1985). *Effects of student-generated problems on problem solving performance.* Unpublished doctoral dissertation, Colombia University.

Polya, G. (1954). *Mathematics as plausible reasoning.* Princetown, NJ: Princeton University Press.

Polya, G. (1957). *How to solve it ?* (2nd ed.). New York: Doubleday.

Polya, G. (1962). *Mathematical discovery* (Vol. I & 2). New York: John Wiley & Sons.

Powney, J., & Watts, M. (1987). *Interviewing in educational research.* London: Routledge & Kegan Paul.

Radatz, H. (1980). Students' errors in the mathematical learning process. *For the Learning Mathematics, 1,* 16-20.

Resnick, L. B. (1985). Cognition and instruction: Recent theories of human competence and how it is acquired. In B. L. Hammonds (Ed.), *Psychology and learning: The master lecture series, Vol. 4.* Washington, DC: American Psychological Association.

Resnick, L. B. (1987). *Education and learning to think.* Washington. DC: National Academy Press.

Resnick, L. B., & Klopher, L. E. (Eds.). (1989). *Toward thinking curriculum: Current cognitive research* (pp. 206-211). Alexander, VA: Association for Supervision and Curriculum Development.

Richardson, J., & Wiliamson, P. (1982). Towards autonomy in infant mathematics. *Research in Mathematics Education in Australia*, 109-136.

Romberg, T. A, Zarinnia, E. A., & Collins, K. F. (1990). A new world view of assessment in mathematics. In G. Kulm (Ed.), *Assessing higher order thinking in mathematics* (pp. 21-38). Washington, DC: American Association for the Advancement of Science.

Scharigin, M. (1989). Learning to solve problems in geometry. *Mathematika v Schkole, 2*, 87-101.

Schloemer, C. (1994). *Integrating problem posing into instruction in advanced algebra: Feasibility and outcomes.* Dissertation Abstracts International, AAC 95 21420, University of Pittsburgh.

Schoenfeld, A. H. (1984). Book review of Mason, J., Burton, L., Stacey, K.: Thinking mathematically. 1982. *International Reviews on Mathematical Education, 16*(3), 103-105.

Schoenfeld, A. H. (1985). *Mathematical problem solving.* Orlando: Academic Press.

Schoenfeld, A. H. (1989). Teaching mathematical thinking and problem solving. In L. B. Resnick & L. E. Klopher (Eds.), *Toward the thinking curriculum: Current cognitive research* (pp. 83-103). Alexander. VA: Association for Supervision and Curriculum Developing.

Schoenfeld, A. H. (1992). Learning to think mathematically: Problem solving, metacognition and sense making in mathematics. In D. A. Grous (Ed.), *Handbook of research in mathematics teaching and learning* (pp. 334-370). New York: Macmillan.

Schoenfeld, A. H. (1994). Mathematical thinking and problem solving. Hillsdale: Lawrence Erlbaum.

Sendov, B., Davidov, L., Georgieva, N., Stoyanova, E., & Vitanov, T. (1988). *High school mathematics education and training of mathematics and informatics teachers in Bulgaria.* Sofia: Ministry of Education.

Shell Centre, University of Nothingham. (1991). *Improving pupil's awareness of learning in mathematics*. Nottingham: Author.

Shimada, S. (Ed.) (1977). *Open-end approach in arithmetic and mathematics-A new proposal toward teaching improvement.* Tokyo: Mizuumishobo (in Japanese).

Silver, E. A. (1990). Contribution of research to practice: Applying findings, methods and perspectives. In T. J. Cooney & C. R. Hirsh (Eds.), *Teaching and learning mathematics in the 1990's* (pp. 1-2). Reston, VA: National Council of Teachers of Mathematics.

Silver, E. A. (1993). On mathematical problem posing. In I. Hirabayashi, N. Nohda, K. Shigematsu & F. L. Lin (Eds.). *Proceedings of the 17th International*

Conference for the Psychology of Mathematics Education, Vol. I (pp. 66-85). Tsukuba, Japan: International Group for the Psychology in Mathematics Education.

Silver, E. A. (1995). The nature and use of open problems in mathematics education: Mathematical and pedagogical perspectives. *International Reviews on Mathematical Education, 27*(2), 67-72.

Silver, E. A., & Adams, V. M. (1987). Using open-ended problems: Tips for teachers. *The Arithmetic Teacher, 34*(5), 34-35.

Silver, E. A., & Cai, J. (1993). Mathematical problem posing and problem solving by middle school students. In C. A. Maher, G. A. Goldin, & R. B. Davis (Eds.), *Proceedings of PME-NA 11, Vol. 1,* (pp. 263-269). New Brunswick, NJ: Rutgers University.

Silver, E. A., & Cai, J. (1996). An analysis of arithmetic problem posing by middle school students. *Journal for Research in Mathematics Education, 27*(5), 521-539.

Silver, E. A., & Mamona, J. (1989). Problem posing by middle school teachers. In C. A. Maher, G. A. Godlin, & R. B. Davis (Eds.), *Proceedings of the PME-NA 11* (pp. 263-269). New Brunswick, NJ: Rutgers University.

Silver, E. A., & Shapiro, L. (1990). Examinations of situation-based reasoning and sense making in students' interpretations of solutions to a mathematics story problem. In J. P. Ponte, J. F. Matos, J. M. Matos, & D. Fernandes (Eds.), *Mathematical problem solving and new information technologies* (pp. 113-123). Berlin: Springer Verlag.

Silver, E. A., Kilpatrick, J., Schlesinger, B. (1990). *Thinking through mathematics: Fostering inquiry and communication in mathematics classrooms.* New York: College Entrance Examination Board.

Silver, E. A., Mamona-Downs, J., Leung, S. S., & Kenney, P. A. (1996). Posing mathematical problems: An exploratory study. *Journal for Research in Mathematics Education, 27*(3), 293-309.

Simon, H. A. (1973). The structure of ill-structured problems. *Artificial Intelligence, 4*, 181-201.

Skemp, P. (1989). The constructivist perspective. In P. Ernest (Ed.), *Mathematics teaching: The state of the art* (pp. 151-152). London: Falmer Press.

Skinner, P. (1991). *What's your problem?: Posing and solving mathematical problems, K-2.* Portsmouth, NH: Heinemann.

Smilansky, J. (1984). Problem solving in the quality of invention. *Journal of Educational Psychology, 76*, 377-386.

Stacey, K. (1995). The challenges of keeping open problem-solving open in school mathematics. *International Reviews on Mathematical Education, 27*(2), 62-67.

Stacey, K., Groves, S., Bourke, S., & Doig, B. (1993). *Profiles of problem solving.* Hawthorn, Victoria: Australian Council for Educational Research.

Sternberg, R. (1987). Questioning and intelligence. *Questioning Exchange, 1*(1), 11-15.

Stone, J. (1994) The use of open-ended questions to cater for our gifted mathematicians. In A. Simic & C. McGrath (Eds.), *Developing excellence: Potential into performance* (pp. 211-213). Perth, WA: Australian Association for the Education of Gifted and Talented Children.

Stover, G. B. (1982). *Structural variables affecting mathematical word problem difficulty in sixth graders.* Unpublished doctoral thesis, University of San Francisco.

Stoyanova, E. & Georgiev, P. (1992, February). *The role of Bulgarian education system for preparing students for the life.* A paper presented at the UNESSCO International Conference for Preparing Children for the Life. Meinz: Germany.

Stoyanova, E. (1994). Some features of the system for working with gifted children in Bulgaria. In A. Simic & C. McGrath (Eds.), *Developing excellence: Potential into performance* (pp. 214-216). Perth, WA: Australian Association for the Education of Gifted and Talented Children.

Stoyanova, E. (1995, July). *Developing a framework for research into students' problem posing in school mathematics.* Paper presented at the 18th Annual Conference of the Mathematics Educational Research Group of Australasia, Darwin.

Stoyanova, E. (*In press*). Problem posing in mathematics classrooms. In N. Ellerton (Ed.). *Research in mathematics education: Some current trends.* Perth: Australian Institute for Research in Primary Mathematics Education.

Stoyanova, E., & Bana, J. (1997). Problem chains in mathematics classrooms *The Australian Mathematics Teacher 53*(4), 8-13.

Stoyanova, E., & Ellerton, N. F. (1996). A framework for research into students' problem posing. In P. Clarkson (Ed.), *Technology in mathematics education* (pp. 518-525). Melbourne: Mathematics Education Research Group of Australasia.

Stoyanova, E., & Ellerton, N. F. (1996, July). *Adaptation and extension of Krutetskii's system of problems: An application as problem-posing situations.* Paper presented at the 8th International Congress of Mathematical Education, Seville.

Strauss, A., & Corbin, J. (1990). *Basics of qualitative research: Grounded theory procedures and techniques.* Newbury Park, CA: Sage.

Sullivan, P. (1995). Content-specific open-ended questions: A problem-solving approach to teaching and learning mathematics. In J. Wakefield & L. Velardi (Eds.), *Celebrating mathematics learning* (pp. 176-180). Melbourne: Mathematical Association of Victoria.

Sullivan, P. A., Clarke, D. J., & Wallbridge, M. (1991). *Problem solving with conventional mathematics content: Responses of pupils to open mathematical tasks.* Australian Catholic University: Mathematics Teaching & Learning Centre. Research Report 1.

Sullivan, P. A., Clarke, D. J., Spandel, U., & Wallbridge, M. (1992). *Using content-specific open questions as a basis of instruction: A classroom experiment.* Australian Catholic University: Mathematics Teaching & Learning Centre. Research Report 4.

Sullivan, P., Bourke, D., & Scott, A. (1995). Open-ended tasks as stimuli for learning mathematics. In B. Atweh & S. Flavel (Eds.), *Galtha* (pp. 484-492). Proceedings of the 18th Annual Conference of the Mathematics Education Research Group of Australasia. Darwin: Mathematics Education Research Group of Australasia.

Sullivan, P., Bourke, D., & Scott, A. (1995). Open-ended tasks as stimuli for learning mathematics. In B. Athweh & S. Flavel (Eds.), *Galtha* (pp. 484-492). Proceedings of the 18th Annual Conference of the Mathematics Education Research Group of Australasia. Darwin: Mathematics Education Research Group of Australasia.

Sweller, J. (1984). Learning and problem solving: Desperate goals? In H. Burkhart, S. Groves, A. Schoenfeld & K. Stacey (Eds.), *An overview of problem solving in problem solving - A world view* (pp. 187-191). Adelaide: International Commission for Mathematical Education.

Sweller, J. (1992). Cognitive theories and their implications for mathematics instruction. In G. C. Leder (Ed.), *Assessment and learning of mathematics* (pp. 46-62). Hawthorn, VIC: Australian Council for Educational Research.

Sweller, J. (1993). Our cognitive architecture and its consequences for teaching and learni g mathematics. *Reflections 18*(4), 4-12.

Sweller, J., Mawer, R. F., & Ward, M. R. (1983). Development of expertise in mathematical problem solving. *Journal of Experimental Psychology, 112,* 639-661.

Tochikazu, I. (1993) A study on a treatments about mathematical problem posing. In I. Hirabayashi, N. Nohda, K. Shigematsu & F. L. Lin (Eds.), *Proceedings of the 17th International Conference for the Psychology of Mathematics Education, Vol. 3* (p. 239). Tsukuba, Japan: International Group for the Psychology of Mathematics Education.

Todd, D. K. (1987). A teacher's use of questions. *Questioning Exchange, 1*(2), 119-124.

Trowell, J. (Ed.) (1990). *Projects to enrich school mathematics.* Reston, VA: National Council of Teachers of Mathematics.

van den Brink,, J. F. (1987). Children as arithmetic book authors. *For the Learning of Mathematics, 7*, 44-48.

van der Brink, F. J. (1985, July). Discussion paper presented to the working group *"Principles of teaching design"* at the 9th International Conference for the Psychology for Mathematics Education, Noordwijkerhout, The Nederlands.

van der Heuvel-Panheizen, M. (1996). *Assessment and realistic mathematics education.* Utrecht University: Freudental Institute.

Vygotsky, L. (1978). *Mind and society.* Cambridge. MA: Harvard University Press.

Ward, M., & Sweller J. (1991). Cognitive load theory and the format of instruction. *Cognition & Instruction, 8,* 293-332.

Wertheimer M. (1945). *Productive thinking.* New York: Harper & Row.

Whitin, D. J. (1989). The power of mathematical investigations. In P. R. Trafton (Ed.): *New directions for elementary school mathematics* (pp. 183- 190). NCTM 1989 yearbook. Reston, VA: National Council of Teachers of Mathematics.

Wickelgren, W. A. (1974). *How to solve problems: Elements of theory of problems and problem solving.* San Francisco: W. H. Freeman.

William, D. (1994). Assessing authentic tasks: alternatives to mark-schemes. *Nordic Studies in Mathematics Education, 2*(1), 48-68.

Wilson, J. W. (1967). *Generality of heuristics as an instructional variable.* Unpublished doctoral dissertation, Stanford University.

Wilson, P. (Ed.), (1993). *Research ideas for the classroom: High school mathematics.* Reston, VA: National Council of Teachers of Mathematics.

Winograd, K. (1990). *Writing, solving and sharing original math story problems: Case studies of fifth grade children's cognitive behaviour.* Unpublished doctoral thesis, University of Northern Colorado.

Writz, R. W., & Kahn, E. (1982). Another look at application in elementary school mathematics. *The Arithmetic Teacher, 30*(1), 21-25.

Yackel, E., Cobb, P., Wood, T., Wheatley, G., & Merkel, G. (1990). The importance of social interactions in children's construction of mathematical knowledge. In T. Cooney (Ed.), *1990 Yearbook of the National Council of Teachers of Mathematics.*

Yerushalmy, M., Chazan, D., & Gordon, M. (1993). Posing problems: One aspect of bringing inquiry into classrooms. In J. Schwartz, M. Yerushalmy & B. Wilson (Eds.), *The geometric supposer: What is it a case of?* (pp. 117-142). Hillsdale, NJ: Lawrence Erlbaum.

Zimmermann, B. (1983). Problemlösen als eine Leitidee für den Mathematikunterricht. *Mathematikunterricht, 29*(3), 5-45.

Zimmermann, B. (1991). Öffene probleme für den mathematikunterricht und ein ausblick auf forschungs fragen. *International Reviews on Mathematical Education, 23*(2), 38-46.

Zuckerman, H. (1977). *The scientific elite.* New York: Free Press.

Appendix 1. **The Invitation Letter to the Parents.**

November
"Title" "Parents"
"Address"
"Suburb" "Postcode"

Dear "Title" "Parents",

We are pleased to advise you that your child, "Forename", has gained a place in the 1995 mathematical problem-posing and problem-solving class for students in Years 8 and 9. Classes will be held every week from February to the end of September, commencing on Thursday, 9 February at 4pm and finishing at 5 pm.

The class will meet at the Mount Lawley campus of Edith Cowan University, probably in Building 13. On the first class day students should assemble on the grassed area at the eastern end of Building 13 (see attached map).

Students should bring normal class materials such as pens, pencils, ruler, paper and a simple calculator. They should also have a file or folder in which to keep their paper and any materials which are given to them.

As you are probably aware, we have been unable to cater for all of the students who wanted to be in the classes. We have therefore established a list of reserves. Should your child find the class unsatisfactory for any reason, we would appreciate it if you could advise us of this so that the place can be allocated to another student.

Students in this class will be involved in the national programme, Mathematics Challenge for Young Australians. Classes will be conducted by the WA State Director for Mathematical Olympiad, Mrs Elena Stoyanova. Students will be given a range of problem-posing and problem-solving tasks. A number of the lessons will be audio- and video-recorded for a research study being undertaken by Mrs Stoyanova.

The aim of the research study is to design appropriate problem-solving environments for students, and to investigate the strategies students use when they pose and solve mathematics problems. It is hoped that the completed study will enhance the nature of the problem-solving activities which teachers use in mathematics classrooms. All information obtained during the Program will be kept strictly confidential. A report on the research will be sent to all students who take part in the study. No participants will be identified in any reports as the findings will be presented anonymously.

The Program consists of two parts.

(a) The first part is the Challenge Stage which runs for three weeks in March
(b) The second part is the Enrichment Stage which runs for April till October.

The cost for the Program is $25.

To confirm your child's place in the Program and the Research Study would you please complete the attached consent form and return it to us, together with a cheque for $25 made payable to Edith Cowan University. As we need to collect and send the fees to Canberra by the beginning of December, would you please ensure that the form and cheque reach us by December, 2, 1994.

We look forward to working with "Forename" during 1995.

Yours Sincerely

Dr Nathan (Norm) Hoffman

Elena Stoyanova
(WA State Director for the Mathematics Olympiad)

***Appendix 2.* Application and Consent Form for Participation in the Mathematics Challenge Program Through Edith Cowan University study**

APPLICATION AND CONSENT FORM FOR PARTICIPATION IN THE MATHEMATICS CHALLENGE PROGRAM THROUGH EDITH COWAN UNIVERSITY

NAME OF STUDENT:.. ....
(Please print) Surname Other names

ADDRESS: ....
....
....
.... POSTCODE: ....
HOME PHONE NO:..
DATE OF BIRTH:..
SCHOOL (1994):.. YEAR:....................................
SCHOOL (1995):.. YEAR:....................................

I WISH TO ENROL MY CHILD IN THE 1995 MATHEMATICS CHALLENGE PROGRAM FOR YOUNG AUSTRALIANS, AND I GIVE PERMISSION FOR MY CHILD TO TAKE PART IN THE ASSOCIATED RESEARCH STUDY.

If you or your child have any questions about the program of the Research Study which is called "Extending and Exploring Students' Mathematical Problem-posing Skills: A study of Year 8 and 9 students involved in the Mathematics Enrichment Stage of the Challenge Program for Young Australians," please contact the Principal Investigator Elena Stoyanova, in the Mathematics Education Department of the Faculty of Education, Edith Cowan University on 383 8200.

To be completed both by the child and by a parent/guardian:

I have read the information above and in the accompanying letter, and any questions I have asked have been answered to my satisfaction. I agree to participate in this activity, realising I may withdraw at any time.
I agree that the research data gathered for this study may be published provided I am not identifiable.

I attach my cheque for $25.00 (made payable to Edith Cowan University) being:
(I) $10.00 for the Challenge Stage, and
(ii) $15.00 for the Enrichment Stage.
Name of Parent:... Signature:..............................
Name of Child:.. Signature:..............................
Date:....................................

Please return to: Mrs. E. Stoyanova, Edith Cowan University, Department of Mathematics Education, Churchlands Campus, Pearson Street CHURCHLANDS 6018

Appendix 3. The Letter to the Chairperson of the Ethical Committee at Edith Cowan University

Elena Stoyanova

17 November, 1994

The Chairperson
Ethics Committee
Edith Cowan University

Dear Sir,

I felt it would be helpful to clarify the reason why the letter to parents and the consent form refer to both the Mathematics Challenge Program and the research study.

Participation in the Program and inclusion in the research study are closely linked. Students cannot participate in the Mathematics Challenge Program for Years 8 and 9 in Western Australia unless they also take part in the research study. It is not possible to take part in one without the other.

As you will note in the letter to parents, we are unable to offer every eligible Year 8 and 9 student a place in the Program. Therefore I anticipate no difficulty in filling the available places with students whose parents give consent for both the Program and the research study.

Please note that the permission letter and consent form to be sent to parents of Group C students will be identical to the ones accompanying this application, except that all reference to the Mathematics Challenge Program for Young Australians (and to fees charged) will be deleted.

I look forward to receiving approval to proceed with the study.

Yours Sincerely,

Elena Stoyanova

Appendix 4. A Sample of Individual Students' Worksheets Developed for the Study.[9]

Worksheet I

9. 2. 1995

1.
The sum of three odd consecutive numbers is 33. The smallest of the three is:
A) 11 B) 9 C) 10 D) 7 E) 13

2.
$(0.2)^3$ equals:
A) 0.06 B) 2.0 C) 0.008 D) 0.006 E) 0. 6

3.
The value of $9^4 \times 3^3$ is:
A) 27^7 B) 27^{12} C) 3^{11} D) 3^9 E) 9^{12}

4. The value of $(7^3)^5$ is:
A) 7^8 B) 7^{15} C) 35^3 D) 21^5 E) 7^8

5. What is the last digit of the number $(7^2)^7$?
A) 1 B) 3 C) 5 D) 7 E) 9

6. If X is a product of three consecutive integer numbers, then X is not always divisible by:
A) 1 B) 2 C) 3 D) 5 E) 6

7. The value of $\frac{1}{1995} + \frac{1996 \times 1994}{1995} - 1996$ is:

A) $\frac{1}{1995}$ B) -1 C) $-\frac{1}{1995}$ D) 1 E) 2

8. ♠ The value of 100! is the product of all integer numbers from 1 to 100 inclusive, i.e. 100! = 1 × 2 × . . . × 99 × 100. The maximum number of times that 2 will divide into 100! is:
A) 50 B) 100 C) 84 D) 97 E) 100

[9] The symbol "♠" has been used to designate problems from Australian Mathematics Competitions.

Worksheet 2

16. 02 1995

Indices

Definition: $a \times a \times a \times ... \times a = a^n$, n is a positive integer number (n = 1, 2, 3, ...).

1. Explain to your partner why the following examples are true:
a) $2^4 = 2 \times 2 \times 2 \times 2 = 16$
b) $(-2)^4 = (-2) \times (-2) \times (-2) \times (-2) = 16$;
c) $(0.5)^3 = (0.5) \times (0.5) \times (0.5) = 0.125$;
d) $(^1/_2)^3 \times (^1/_2) \times (^1/_2) = {}^1/_8 = 0.125$
e) $(0.1)^{1995} = 0.0...01$ (1994 zeros after the decimal point).

2. Write the following in index form:
a) $10 \times 10 \times 10 =$
b) $(-1) \times (-1) \times (-1) \times (-1) \times (-1) =$
c) $^2/_3 \times {}^2/_3 \times {}^2/_3 \times {}^2/_3 =$

3. Evaluate:

a) $3^3 =$	$(-3)^3 =$	$(-3)^4 =$	$-3^4 =$
b) $0.2^5 =$	$(-0.2)^5 =$	$(-0.2)^4 =$	$-0.2^4 =$

4. Find out the missing index in each of the following:
a) $3 \ \ = 27$; b) $10 \ \ = 10\,000$; c) $(0.1) \ \ = 0.0000001$

5. Without calculation compare the numbers:

a) $(^1/_2)^3$ and 0	$-^1/_2{}^4$ and 0	$(-^1/_2)^3$ and 0	$(-^1/_2)^4$ and 0
b) $(-5)^4$ and 0	$(-1995)^4$ and 0	$(-1995)^{1995}$ and 0	-1995^{2000} and 0

6. Explain to your partner why the calculations are true:
a) $(2)^5 + (-2)^5 = 0$; b) $1^{99} + (-1)^{99} = 0$; c) $(-5)^{1995} + 5^{1995} = 0$.

7. Give some examples for which the equality holds:
a) $(-1) \ \ = -1$
b) Finish the conjecture: If $(-1)^n = -1$, then n has to be an ....number.

8. Give some examples for which the equality holds
a) $(-1) \ \ = +1$
b) Finish the conjecture: If $(-1)^n = +1$, then n has to be an ....number.

9. Finish the conjectures and explain the differences between 7b) and 8b) and 9a) and 9b) respectively:
a) $(-1)^n = -1$ if and only if n is an ....number.
b) $(-1)^n = +1$ if and only if n is an ....number.

10. Fill in the blank squares with either = or ≠:
a) $2^3 \square 3^2$ b) $5^2 \square 2^5$ c) $4^3 \square 3^4$ d) $(-2)^{222} \square (2)^{222}$
e) $-184^4 \square (-184)^4$

Worksheet 3

23. 02. 1995

Find the Last Digit

1. Find out what the last digit is:

	a	a^1	a^2	a^3	a^4	a^5	a^6	a^7	a^8	a^9
0										
1										
2										
3										
4										
5										
6										
7										
8										
9										

2. Find out what the last digit is:

a) 1980^5 1981^5 12^{17} 123^{25} 124^{32} 25^{43}

b) 326^{33} 457^{78} 428^{198} 499^{45} $1980^5 \times 1981^5$

c) $123^{25} \times 124^{32}$ 3×326^{33} $428^{198} + 199^{45}$

Definition: $a^0 = 1$, for $a \neq 0$.

1. Explain to your partner which of the following terms does not have a meaning:

5^0; 123^0; 0^0; $(-3)^0$; $(\frac{1}{4})^0$

2. Evaluate:

a) $123^{1^{12}} =$

b) $(x-3)^{(x-5)}$, if $x = 5$.

c) $(x-4)^{(x-5)^{(x-6)^{(x+6)}}}$, if $x = 7$.

d) $(x-3)^{(x-3)}$, if $x = 3$.

Worksheet 4

1\. 03. 1995

1\. A total of 675 digits was used for numbering the pages of a book. How many pages did the book contain?

2\. Pose a problem using the idea of the solution of the above problem and try to solve it.

3\. A book contains 268 pages. A total of how many digits was used for numbering the pages of the book?

4\. Find out which digit will be on the 642-nd place in the sequence:
1, 2, 3, 4, 5,...

5\. Can you try to pose a problem for me similar to the one above?

6\. Consider the sequence 1, 2, 3, 4, 5, ... N.

a) If $N = 200$, how many digits have been used?

b) Which digit is in the 147th place?

c) If the last number is 999, how many 3's in total have been used?

d) If the last number is 200 how many prime numbers are there?

e) If the last number is 250 how many numbers of this sequence are divisible by 2, 3, and 4 but are not divisible by 5?

16. 03. 1995

OLYMPIAD PROBLEMS

1. Given that
$2.65 \times 1.32 = 3.398$.
The value of 0.03398 : 0.0132 is
A) 2.65 B) 0.265 C) 265 D) 26.50 E) 2 650

2. The value of
$1 - 2 + 3 - 4 + 5 - \ldots -798 + 799 - 800 + 801$ is:
A) 1 B) - 400 C) 801 D) 401 E) 1201

3. The value of
$1 + 2 + 3 + 4 + \ldots + 150 + 151$ is:
A) 98 B) 673 C) 11 325 D) 11 476 E) 10 325

4. The value of
$2 - 3 + 4 - 5 + \ldots -199 + 200$ is
A) 98 B) 75 C) 100 D) 101 E) 299

5. $3^{16} \div 3^{4}$ equals:
A) 3^4 B) 3^{12} C) 3^{20} D) 16/4 E) 3

6. $3^{(3)^2} \div (3^2)^3$ equals:
A) 1 B) 3 C) 3^3 D) 3^2 E) 3^4

7. The number of the digits in the product
$5^{15} \times 4^{8}$ is:
A) 16 B) 14 C) 15 D) 23 E) 24

8. If the following were re-arranged in order of magnitude, which would be the middle number:
$3\,(3^{10})$ $3(3^9) - 3$ 3^{10} $3 + 3(3)^9$ $3^9/3$
A) 3^{11} B) $3^{10} - 3$ C) 3^{10} D) $3^{10} + 3$ E) 3^8

9. $(\frac{1}{3})^3 - (\frac{1}{3})^4$ equals:
A) $^2/_{51}$ B) $^2/_{81}$ C) $^1/_{36}$ D) $^2/_3$ E) $-\,^2/_{81}$

10. Now, you are given the opportunity to pose a problem. Could you pose one difficult Olympiad problem for me, please?

Worksheet 6

23. 03. 1995

1. Which of the numbers 12, 372, 445, 171, 736, 3 672, 3 720, are divisible by:
a) 2; b) 3; c) 5; d) 4; e) 9; f) 10; g) 6 (*hint:* $6 = 2 \times 3$); h) 8.

2. Without calculating explain to your partner why the calculations below are not true:
a) $3 \times (17 + 1234) - 1245 \times 2 + 78603 - 171 = 87631$
b) $575 \times 13 - 105 \times 272 + 15720 - 1230 = 19\,752$.

3. Substitute the symbol "*" with a digit in the number 512 * , so that the number is divisible by:
a) 2; b) 3; c) 5; d) 4; e) 6; f) 8; g) 9; h) 10.

4. Replace the symbol * in the product
$5 \times 7 \times {*} {*} \times 17 \times 13 \times 101$, so that the last digit of the product is :
a) 0; b) 5.

5. Substitute the symbol "*" with a digit in the number 123*7*, so that the number is divisible by:
a) 2; b) 5; c) 2 and 5; d) 3; e) 2 and 3; f) 4.

6. Substitute some of the digits with zero so that 123 + 456 + 789 + 1011 equals to 2185. What would be the sum if you removed the digits instead of substituting them with zeros?

7. Substitute the sign "*" with suitable digits so the equations are true:
a) $** 5 = (**)^2$; b) $** 1 = 17 \times 1*$; c) $1024 = 2^*$

8. ♠ Restore the missing digits in this addition:

```
      4 *
+   * * 2
  -------
  * * 0 1
```

9. ♠ Restore the missing digits in this multiplication:

```
     * * * *
×        * 2
  ----------
   1 8 * 4 8
 7 4 9 9 *
 -----------
 * * * 6 6 *
```

10. ♠ Find the values of the letters, each of which stands for a particular but different digit.

```
  F O R T Y              H O C U S
+     T E N          +   P O C U S
      T E N              ---------
  ---------            P R E S T O
  S I X T Y
```

11. Several digits "8" are written and some "+" signs are inserted to get the sum 1000. Figure out how it is done.

Worksheet 7

30. 03 1995

PRIME AND COMPOSITE NUMBERS

1. Which of the numbers 17, 21, 29, 101, 127 and 1001 are prime?

2. Write down the first 15 prime numbers.

3. Bertran, a famous mathematician, stated: let n be a natural number greater than 2. There is at least one prime number between n and 2n.

Take some specific examples and check Bertan's statement.

4. Write the following non-prime numbers only as a product of prime numbers (*prime decomposition*):
a) 9;
b) 42;
c) 91;
d) 196;
d) 1001.

5. State the number of factors of:
a) $16 = 2^4$.
b) 32;
c) 125;
d) 32×125;
e) $2^5 \times 5^3 \times 7^9$;
f) $5^n \times 7^m$;

6. A number has a prime decomposition of $2^3 \times 3^2$.
a) Write down its factors.
b) How many factors are there?

7. Write down a number which has:
a) 4 factors;
b) 7 factors;
c) 20 factors.

8. Substitute the symbol "*" with a digit so that the number 13** has only two different prime divisors.

1995

1. Write the fractions $^1/_6$, $^1/_{12}$, $^1/_{20}$ 1, as a difference between two fractions.

2. If n is a positive integer and n is not equal to 0, show that

$$\frac{1}{n(n+1)} = \frac{1}{n} - \frac{1}{n+1}.$$

3. Calculate the value of

a) $1 + \frac{1}{2 \times 3} + \frac{1}{3 \times 4} + \frac{1}{4 \times 5}$,

b) $\frac{1}{20} + \frac{1}{30} + \frac{1}{42} + \frac{1}{56} + \frac{1}{72} + \frac{1}{90} + \frac{1}{110}$.

4. Continue the sequence so, that the sum of all the fractions is equal to 0.750.

$$\frac{1}{2} + \frac{1}{6} + \frac{1}{12} + \dots .$$

5. If n is a positive integer show that

$$\frac{1}{n(n+100)} = \frac{1}{100}\left(\frac{1}{n} - \frac{1}{n+100}\right)$$

6. Substitute the symbol "*" with a digit so that the calculations are true:

a) $\frac{1}{? \times 102} = \frac{1}{100}\left(\frac{1}{*} - \frac{1}{102}\right)$

b) $\frac{1}{3 \times (* + 100)} = \frac{1}{100}\left(\frac{1}{3} - \frac{1}{* + 100}\right)$

Worksheet 9

6. 04. 1995

1. Which of the following numbers is not a prime: 31, 41, 71, 91, 101 ?
A) 31 B) 41 C) 71 D) 91 E) 101.

2. Is 107 a prime number?
Answer ....

3. Write the first 21 prime numbers (2; 3; 5;...). The middle number is:
A) 13 B) 23 C) 37 D) 31 E) 29.

4. The prime decomposition of 120 is:
A) $2^3 \times 3 \times 5$ B) $4 \times 2 \times 3 \times 5$ C) $2 \times 2 \times 3 \times 5 \times 7$ D) $2 \times 3 \times 5$ E) $2 \times 2 \times 2 \times 3 \times 5 \times 7$

5. The prime decomposition of 300 is:
Answer ....

6. The factors of 2^3 are:
A) 1, 2 and 3 B) 1, 2, 4 and 6 C) $2^0, 2, 2^2, 2^3$ D) 2, 4, 8 E) 2^0, 2, 8

7. State the factors of 2^6:
Answer ....

8. Which of the numbers do not have three factors:
A) 25 B) 9 C) 49 D) 121 E) 32

9. A number has a prime decomposition of $2^3 \times 3^5$. How many factors does this number have?
A) 14 B) 15 C) 24 D) 20 E) 8

10. A number has a prime decomposition of $2^3 \times 3^5 \times 7^2 \times 13$. How many factors does this number have?
Answer ....

*11. A certain number has exactly eight factors, of which 49 and 55 are two. The number is:
A) $7^2 \times 5 \times 11$ B) $5 \times 7 \times 11$ C) $5^2 \times 7 \times 11$ D) $5 \times 7 \times 11^2$

*12. Find the smallest positive integer which has amongst its factors 2, 3, 15 and 20.
Answer ....

*13. Take any three-digit number. Write it down twice to make a six digit number. This number will always have amongst its factors:
A) 7, 11, and 17 B) 7, 11 and 13 C) 5, 7, and 11 D) 5, 17, 19 E) 3, 7, 11

14. Which of the following numbers does not have three factors:
A) 13^2 B) 17^2 C) 11^2 D) 14^2 E) 97^2

Worksheet 9 B

6. 04. 1995

1. Which of the following numbers is not prime: 31, 41, 71, 91, 101 ?
A) 31 B) 41 C) 71 D) 91 E) 101.

1*. Write 5 different numbers amongst which only one is a prime.
Answer ....

2. Is 107 a prime number?
Answer ....

3. Write the first 21 prime numbers (2; 3; 5...). The middle number is:
A) 13 B) 23 C) 37 D) 31 E) 29

4. The prime decomposition of 120 is:
A) $2^3 \times 3 \times 5$ B) $4 \times 2 \times 3 \times 5$ C) $2 \times 2 \times 3 \times 5 \times 7$ D) $2 \times 3 \times 5$ E) $2 \times 2 \times 2 \times 3 \times 5 \times 7$

5. The prime decomposition of 300 is:
Answer ....

6. The factors of 2^3 are:
A) 1, 2 and 3 B) 1, 2, 4 and 6 C) $2^0, 2, 2^2, 2^3$ D) 2, 4, 8 E) $2^0, 2, 8$

6*. Write a number which has exactly 4 factors.
Answer ....

7. State the factors of 2^6.
Answer ....

8. Which of the numbers does not have three factors:
A) 25 B) 9 C) 49 D) 121 E) 32

8*. Write a number which has three factors.
Answer ....

9. A number has a prime decomposition of $2^3 \times 3^5$. How many factors does this number have?
A) 14 B) 15 C) 24 D) 20 E) 8

10. A number has a prime decomposition of $2^3 \times 3^5 \times 7^2 \times 13$. How many factors does this number have?
Answer ....

10*. Write a number which has 12 factors.
Answer ....

11. A certain number has exactly eight factors, of which 49 and 55 are two. The number is:
A) $7^2 \times 5 \times 11$ B) $5 \times 7 \times 11$ C) $5^2 \times 7 \times 11$ D) $5 \times 7 \times 11^2$

12. Find the smallest positive integer which has amongst its factors 2, 3, 15 and 20.
Answer............................

13. Take any three-digit number. Write it down twice to make a six digit number. This number will always have amongst its factors:
A) 7, 11, and 17 B) 7, 11 and 13 C) 5, 7, and 11 D) 5, 17, 19 E) 3, 7, 11

14. Find which number does not have three factors:
A) 13^2 B) 17^2 C) 11^2 D) 14^2 E) 97^2

Worksheet 10

13. 04. 1995

1.The factors of 3^5 are:
A) $1, 3, 3^5$ B) $1, 3, 3^2, 3^3, 3^4$ C) $1, 3, 3^2, 3^3, 3^4, 3^5$ D) $3, 3^2, 3^3, 3^4, 3^5$

2. The number 3^6 is divisible by:
A) 1, 3 and 3^6; B) $1, 3, 3^2, 3^3$ and 3^6; C) $1, 3, 3^2, 3^3, 3^4$ and 3^5;
D) $1, 3, 3^2, 3^3, 3^4, 3^5$ and 3^6

3. The Least Common Multiple of 3^4 and 3^6 is:
A) 3^5 B) 3^4 C) 3^6, D) 3^7

4. The Least Common Multiple of 3, 4 and 5 is:
A) 20 B) 70 C) 12 D) 60

5. A two digit number has the property that, if you subtract 3 from it the result is divisible by 3, if you subtract 4 from it the result is divisible by 4, and if you subtract 5 from it the result is divisible by 5. The number is:
A) 60 B) 63 C) 67 D) 120

6. The Least Common Multiple of 2×3^4, 2×3^3 and $2^2 \times 3^6 \times 5$ is:
A) 2×3^5 B) $2^2 \times 3^4 \times 5$ C) $2^2 \times 3^6 \times 5$, D) $2^2 \times 3^4 \times 5^2$

7. The smallest number with the property that division by each of 3, 4 and 5 yields a remainder of 1 is:
A) 60 B) 61 C) 59 D) 100

8. The smallest number which when divided by 3 gives a remainder of 1, when divided by 4 gives a remainder of 2, when divided by 5 gives a reminder of 3 is:
A) 60 B) 62 C) 58 D) 102

9. I am thinking of a number. The Least Common Multiple of my number and 9 is 45. My number could be:
A) only 5 B) only 45 C) only (9 or 45) D)only (5, 15, or 45)

10. Three ships leave Perth for Sydney on the same day. The round-trip takes the first ship 6 days, the second ship 7 days and the third ship 8 days. Assume continuous round trip activities for all three ships. The three ships will leave Perth again together in minimum of:
A) 167 days B) 168 days C) 169 days D) 336 days

11. Two neon signs are turned on at the same time. One blinks every 4 seconds; the other blinks every 6 seconds. Per minute they blink together:
A) 7 times B) 6 times C) 5 times D) 8 times

12*. Write some numbers which have: exactly 3, exactly 5, exactly 7, exactly 11 or exactly 13 factors. What pattern can you draw?

13*. Write some numbers which have: exactly 16, exactly 22 or exactly 36, factors. What pattern do you find?

Worksheet 11B

11. 05. 1995

Formula 1: $a.(b + c) = a(b + c) = a \times b + a \times c = ab + ac$

1. $2(3 + X) = 2 \times 3 + 2 \times X$ — Your examples:
2. $y(3 + 2y) = 3y + 2y^2$
3. $2x(3 + 2x) =$

Formula 2: $a(b - c) = ab - ac$

4. $2(3 - X) = 2 \times 3 - 2 \times X$ — Your examples:
5. $y(3 - 2y) = 3y - 2y^2$
6. $x(1 - 3x) =$
7. $2x(4 - 2x) =$

Formula 3: $(a + b)(a + b) = (a + b)^2 = a^2 + 2ab + b^2$

8. $(3 + b)^2 = 3^2 + 2 \times 3 \times b + b^2$ — Your examples:
9. $(Y + 2a)^2 = y^2 + 2\,y2a + (2a)^2 = y^2 + 4ay + 4a^2$
10. $(x + 1)^2 =$
11. $(3x + 1)^2 =$
12. $(3x + a)^2 =$

Formula 4: $(a - b)(a - b) = (a - b)^2 = a^2 - 2ab + b^2$

13. $(3 - b)^2 = 3^2 - 2 \times 3 \times b + b^2$ — Your examples:
14. $(y - 2a)^2 = y^2 - 2y(2a) + (2a)^2 = y^2 - 4ay + 4a^2$
15. $(x - 1)^2 =$
16. $(3x - 1)^2 =$
17. $(3x - a)^2 =$

$$1$$
$$1 \quad 1$$
$$a \quad b$$
$$1 \quad 2 \quad 1$$
$$a^2 + 2ab + b^2$$
$$1 \quad 3 \quad 3 \quad 1$$
$$a^3 + 3a^2b + 3ab^2 + b^3$$
$$1 \quad 4 \quad 6 \quad 4 \quad 1$$
$$a^4 + 4a^3b + 6a^2b^2 + 4ab^3 + b^4$$

Formula 5: $(a - b)(a + b) = a^2 - b^2$; $\quad (a^2 - b^2)(a^2 + b^2) = a^4 - b^4$

Formula 6: $(a + b)(a^2 - ab + b^2) = a^3 + b^3$; $\quad (a - b)(a^2 + ab + b^2) = a^3 - b^3$

$(x - 1)^2 + 2x = 3(x - 1) + x^2 - 4;$

$x^2 - 2x + 1 + 2x = 3x - 3 + x^2 - 4$

$\underline{x^2} - \underline{2x + 2x} - 3x - \underline{x^2} = -3 - 4 - 1$

$-3x = -8,\ x = {}^{8}/_{3} = 2\,{}^{2}/_{3}$

Solve the equation: $(x + 1)^2 - (x - 2)^2 = (x + 2)(x - 2) - x(x - 1) + 4$

Now write one similar and difficult problem for me to solve:

Worksheet 12

25. 05. 1995

1. The product $3(x + 2)$ equals:
A) $3x$; B) $3x + 2$; C) $3x + 6$; D) $3x + 5$.

2. The product $5a(a + 2)$ equals:
A) $5a^2$; B) $5a^2 + 2$; C) $5a^2 + 5$; D) $5a^2 + 10$.

3. The product $(x - 3)5$ equals:
A) $5x^2$; B) $5x - 3$; C) $x - 15$; D) $5x - 15$.

4. The product $(x + 3)(x - 3)$ equals:
A) $x^2 + 6$; B) $x^2 - 6$; C) $x^2 - 9$; D) $x^2 - 6x + 9$.

5. The product $(x + 3)(x + 3)$ equals:
A) $x^2 + 6$; B) $x^2 + 9$; C) $x^2 + 6x + 9$; D) $x^2 - 6x + 9$.

6. The product $(x - 3)(x - 3)$ equals:
A) $x^2 + 9$; B) $x^2 - 9$; C) $x^2 + 6x + 9$; D) $x^2 - 6x + 9$.

7. The product $(x - 2)(x - 3)$ equals:
A) $x^2 - 5x - 6$; B) $x^2 + 5x - 6$; C) $x^2 - 5x + 6$; D) $x^2 + x - 5$.

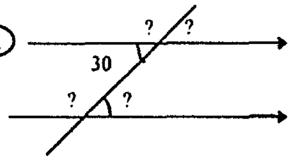
8 The number of the additive terms in the expression $(a + b)(x + y)$ is:
A) 9; B) 4; C) 6; D) 20.

9. The product $(x + 3)(x + 2)$ equals:
A) $x^2 + 5x$; B) $x^2 + 6x$; C) $x^2 + 6$; D) $x^2 + 5x + 6$.

10. The number of the additive terms in the expression $(a + b + c)(x + y)$ is:
A) 9; B) 12; C) 5; D) 6.

11*. The product $(x^2 + 2x + 2)(x - 1)$ equals:
A) $x^3 - 2x - 2$; B) $x^3 + 2x^2 - x - 2$; C) $x^3 + 2x^2 - x - 2$; D) $x^3 + x^2 - 2$.

1



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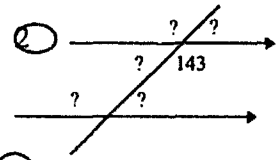

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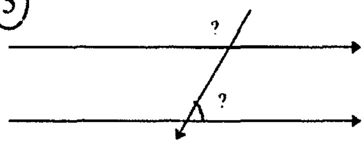

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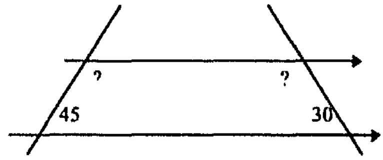

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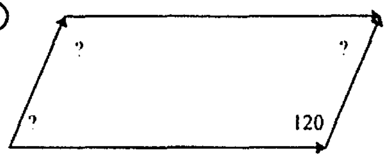

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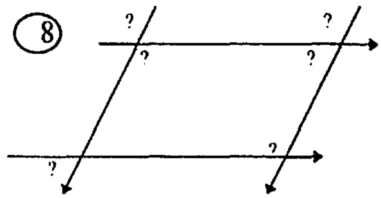

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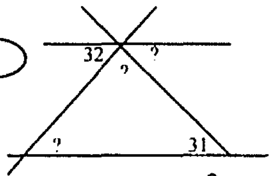

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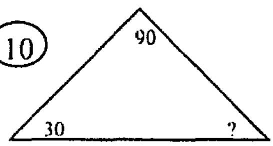

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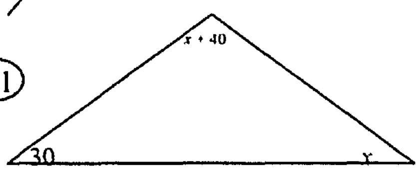

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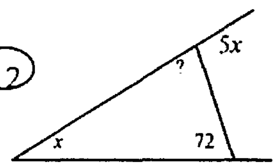

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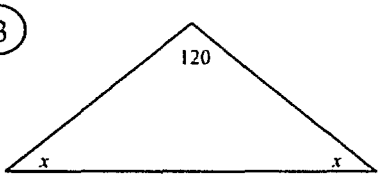

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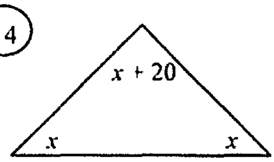
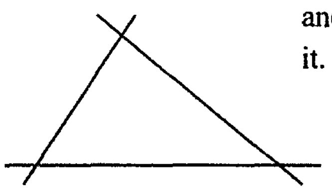


15 Now make your own problem on the basis of:



and ask your friend to solve it.

Worksheet 13

1. 06. 1995

Solve the problem:
1. *Seven* sausages are to be divided *equally* amongst *five* people.
What is the smallest number of pieces of sausage necessary to make this possible?
Solution:

In the above problem, some parts of the problem statement are missing. Finish the problems structure by using suitable wording:

2. sausages are to be divided *equally* amongst people.
What is the smallest number of pieces of sausage necessary to make this possible?

3. sausages are to be divided amongst people.
What is the smallest number of pieces of sausage necessary to make this possible?

4. S*even* sausages are to be divided *equally* amongst *five* people.

5. ....sausages are to be divided ....amongst............people.

8. 06. 1996

This is a problem posed by Norm. How does the problem relate to the "sausage" problem?

There are 30 Alan Bonds, and they have to pay 80 bills. If they share the bills, all the bills, how many total bills will be, assuming that, if 2 Alan Bonds have to share the same bill, then it's counted as 2 bills.

15. 06. 1995

1. Find the 27th term of the arithmetic sequence: 3, 11, 19,...

Answer:

2. Find the 21st term of the arithmetic sequence: 7, 14, 21,...

Answer:

3. Find the 20th term of the arithmetic sequence: 2, 4, 6,...

Answer:

4. Write a formula for the n-th even number.

Answer:

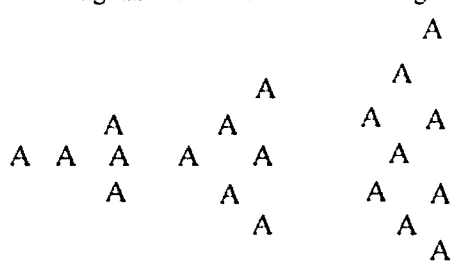
5. Find the sum of the first n natural numbers (1, 2, 3, ...).

Answer:

6. The first term in an arithmetic sequence is 1 The fourth is 10:
a) Find the 15th term of this sequence.

Answer:

7. The numbers 1, 3, 6, 10 are the first four of the triangular numbers. There is a correspondence between the triangular numbers and the following configuration:



1 (1) + 2 (1 + 2) + 3 (1 + 2 + 3) + 4

a) Find the 15th term of this sequence.
b) Find the nth triangular number.

8. ♠ Last night there was a party and the host's doorbell rang 20 times. The first time the doorbell rang only one guest arrived. Each time the doorbell rang after that two more guests arrived than had arrived on previous ring. How many guests arrived at the party?

15. 06. 1995

1. The first term in an arithmetic sequence is 1. The second term is 10.
.... term is:
A) 18; B) 16; C) 9; D) 19.

2. The first term in an arithmetic sequence is 1. The fourth term is 10.
The second term is:
A) 3; B) 4; C) 5; D) 12.

3. The first term in an arithmetic sequence is 1. The fourth is 10.
The ....is:
A) 11; B) 20; C) 22; D) 33.

4. The first term in an arithmetic sequence is 1. The fourth is 10.
The sum of the first twenty terms is:
A) 1 810; B) 2 000; C) 1 730; D) 1 840.

5. Last night there was a party and the host's doorbell rang 10 times. The first time the doorbell rang only one guest arrived. Each time the doorbell rang after that three more guests arrived than had arrived on previous ring.

Ask as many questions as you can. Try to put them in suitable order.

15. 06. 1995

1. The product of $(x + 5)(x - 5)$ is:
A) $x^2 + 25$; B) $x^2 - 25$; C) $x^2 - 10$; D) $x^2 - 10x + 25$.

2. The product of $(x + \boxed{\bigcirc})^2$ is:
A) $x^2 + 25$; B) $x^2 - 25$; C) $x^2 + 10x + 25$; D) $x^2 - 10x + 25$.

3. The product of $(x - \boxed{\bigcirc})^2$ is:
A) $x^2 - 25$; B) $x^2 + 25$; C) $x^2 - 10x + 25$; D) $x^2 + 10x + 25$.

4.* The product of $(x + 5)(x + \boxed{\bigcirc})$ is:
A) $x^2 + 5x$; B) $x^2 + 2x$; C) $x^2 + 5x + 7$; D) $x^2 + 7x + 10$.

5. The Least Common Multiple of 5, 10 and 15 is:
A) 20 ; B) 30; C) 120; D) 60.

6. The smallest number with the property that division by each of 5, 10 and 15 yields a remainder of 1 is:
A) 21 ; B) 31; C) 121; D) 61.

7. Take any two-digit number. Write it down twice to make a four digit number. This number always will have among its factors:
A) 11; B) 101; C) 1001; D)10.

8. Take any three-digit number. Write it down twice to make a six digit number. This number always will have among its factors:
A) 11; B) 101; C) 1001; D)10.

9. How will you finish the problem if you want one of the answers below to be right?
Take any .... Write it down twice to make a .... digit number. This number always will have among its factors:
A) 11; B) 101; C)1001; D)10 001.

22. 06. 1995

1. In an algebra class, the students voted to have a new operation on numbers called "super multiplication."

They defined it by $a*b = a + b + ab$. The "super product" of 2 and 3, i.e 2 * 3, equals:

A) 12 B) 11 C) 5 D) 6

2. If the operation * is defined by $a*b = {}^{1}/_{ab}$ then $4*(3*2)$ equals:

A) ${}^{6}/_{4}$ B) 24 C) $1/_{24}$ D) ${}^{4}/_{6}$

3. For all numbers a, b the operation a*b is defined by

$a*b = ab - a + b$. The solution of the equation $6*x = 15$ equals:

A) $2\,{}^{1}/_{5}$ B) 10 C) 3 D) $2\,{}^{3}/_{15}$

4. For integer numbers a and b we define $a*b = 4a + 3b$.

If $2*x = 68$ then the value of x is

A) 34 B) 20 C) 25 D) 13

LINEAR EQUATIONS

5. Twins Toni and Toby are given the same amount of money. Toni buys 2 apples and has 70 cents left. Toby buys 4 apples and has 20 cents left. What amount of money did each receive?

A) $1.40 B) $1.20 C) $1.60 D) $1.80

6. A bag contains 20 marbles coloured either red, white, blue or green. There is one more red than white, 4 more white than blue and one more blue than green. The number of red marbles is

A) 2 B) 8 C) 10 D) 7

7. ♠ A person's age on his birthday in 1995 is equal to the sum of the digits of the year $19xy$ in which he was born. Therefore x and y satisfy the relation

A) $75 - 10x - y$ B) $95 - x - y$ C) $95 - 11x - 2y$ D) $85 - 11x - 2y$

Worksheet 16B

22. 06. 1995

1. In an algebra class, the students voted to have a new operation on numbers called "super multiplication."

They defined it by $a*b = a + b + ab$. The "super product" of 2 and 3, i.e $2 * 3$, equals:

A) 12 B) 11 C) 5 D) 6

2. If the operation $*$ is defined by $a*b = {}^{1}/_{ab}$ then $4*(3*2)$ equals:

A) ${}^{6}/_{4}$ B) 24 C) $1/_{24}$ D) ${}^{4}/_{6}$

3. For all numbers a, b the operation $a*b$ is defined by

$a*b = ab - a + b$. The solution of the equation $6*x = 15$ equals:

A) $2\,{}^{1}/_{15}$ B) 10 C) 3 D) $2\,{}^{3}/_{15}$

4. For integer numbers a and b we define $a*b$ = ...

....equals:

A) 2 B) 4 C) 6 D) None of them

LINEAR EQUATIONS

5. Twins Toni and Toby are given the same amount of money. Toni buys 2 apples and has 70 cents left. Toby buys 4 apples and has 20 cents left. What amount of money did each receive?

A) $1.40 B) $1.20 C) $ 1.60 D) $1.80

6. A bag contains 20 marbles coloured either red, white, blue or green. There is one more red than white, 4 more white than blue and one more blue than green. The number of red marbles is

A) 2 B) 8 C) 10 D) 7

7*. Make up a problem with the same method of solution as problem 6.

8. A person's age on his birthday in 1995 is equal to the sum of the digits of the year $19xy$ in which he was born. Therefore x and y satisfy the relation

A) $75 - 10x - y$ B) $95 - x - y$ C) $95 - 11x - 2y$ D) $85 - 11x - 2y$

Worksheet 17 A

29. 06.1995.

THE PIGEON-HOLE PRINCIPLE:

Examples:

1. If I have 3 pigeons and 2 pigeon-holes, then at least one hole will contain 2 or more pigeons.

2. If I have 13 pigeons and 6 pigeon-holes, then at least one hole will contain 3 or more pigeons.

Problems:

1. What is the least number of people that must be chosen to be sure that at least 2 have the same first initial?
Answer:

2. What is the least number of people that must be chosen to be sure that at least 2 have the same first initial?
Answer:

3. What is the least number of people that must be chosen to be sure that at least 3 have their birthday on the same day of the week.
Answer:

4. A consumer organiser selects eleven phone numbers from the phone book. Show that at least 2 have the same last digit.

5. What is the least number of phone numbers that must be chosen to be sure that at least 4 have the same last digit?
Answer:

6. A box contains 11 French books, 30 Spanish books, 7 German books, 14 Russian books, 23 English books, and 10 Italian books. How many must I choose to be sure I have 13 books in the same language?
Answer:

7. Show that in any set of 5 different positive integer numbers, at least two of the numbers will have the same remainder when divided by 4.

8. There are 15 people at a party. Some of them shake hands with some of the others. Prove that at least two people have shaken hands the same number of times.

Worksheet 17 B

THE PIGEON-HOLE PRINCIPLE

1. If I have 3 pigeons and 2 pigeon-holes, then one hole will contain 2 or more pigeons.

2. If I have 13 pigeons and 6 pigeon-holes, then at least one hole will contain 3 or more pigeons.

1. What is the least number of people that must be chosen to be sure that at least 2 have the same first initial?
Answer:

2. What is the least number of people that must be chosen to be sure that at least 2 have the same first initial?
Answer:

3. What is the least number of people that must be chosen to be sure that at least 3 have their birthday on the same day of the week.
Answer:

4. A consumer organiser selects eleven phone numbers from the phone book. Show that at least 2 have the same last digit.

5. What is the least number of phone numbers that must be chosen to be sure that at least 4 have the same last digit?
Answer:

6. A box contains 11 French books, 30 Spanish books, 7 German books, 14 Russian books, 23 English books, and 10 Italian books. How many must I choose to be sure I have 13 books in the same language?
Answer:

7. Show that in any set of 5 different positive integer numbers, at least two of the numbers will have the same remainder when divided by 4.

I. Write down a problem based on the Pigeon-hole principle and solve it. How can you increase its difficulty?

8. There are 15 people at a party. Some of them exchange handshakes with some of the others. Prove that at least two people have shaken hands the same number of times.

Identify the main steps in the solution of problem 6 given below and try to rewrite it better:
You need 12 books from every language. There are 6 languages, so $6 \times 12 = 72$. $72 + 1 = 73$. Because the French, German, Italian books don't have 12 then the number is less. Take away the difference between 12 and those languages and it equals 65.

$F = 1$,
$G = 5$,
$\underline{I = 2}$
8

$73 - 8 = 65$.

Worksheet 18

27. 07. 1995:

STRATEGIES FOR SOLVING MULTIPLE-CHOICE QUESTIONS PROBLEMS

Solve the problems below and determine your strategy for finding the right answer.

1. $6(3 - x) - 2(1 - x)$ simplifies to

A) $16 - 8x$ B) $16 + 4x$ C) 16 D) $16 - 4x$ E) $12 - 2x$

2. If $ab = 12$, $bc = 20$, $ac = 15$ and a is positive, then abc equals:

A) 360 B) 3 600 C) 60 D) 36 E) 600

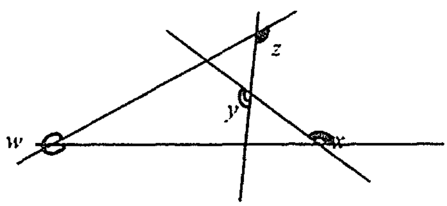
3. A rectangle has perimeter $20cm$ and area $21cm^2$. What are its dimensions, in centimetres?

A) 1 and 20 B) 4 and 4 C) 9 and 2 D) 3 and 7 E) 6 and $3^1/_2$

4. Let a, b, c be distinct integers from one to nine inclusive. The largest possible value of $\frac{a+b+c}{abc}$ is

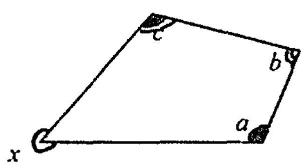
A) 2 B) $^3/_4$ C) $^1/_{21}$ D) 1 E) $^4/_3$

5. Four straight lines intersect as shown.
The value of $x + y + z + w$ is



A) 360 B) 630 C) 450 D) 540 E) 720

6. Angles of size a^0, b^0, c^0, and x are shown.
What is the value of x?



A) $360 - (a + b + c)$ B) $a + c - b$ C) $a + b + c$
D) $360 + b - a - c$ E) $360 + a + c - b$

3. 08. 1995

If one operation can be done in n different ways and if in every case a second operation can be done in m different ways, the two operations can be performed in succession in nxm ways.

1.The menu at a restaurant offers 3 main courses and 4 desserts. How many different two-course meals can be obtained?

A) 12 B) 7 C) 24 D) 30 E) 3

2. The menu at a restaurant offers 5 main courses and 6 desserts. How many different two-course meals can be obtained?

Answer:

3. In how many ways can 3 students stand in a queue?

Answer:

4. In how many ways can 4 students stand in a queue?

Answer:

5. In how many ways can 10 students stand in a queue?

A) 100 000 B) 3828900 C) 3628800 D) 50 E) 1

6. In how many ways can a first and a second prize be awarded in a class of 30?

A) 15 B) 30 C) 10 000 D) 870 E) 900

7. From the set of digits (0, 2, 3, 6, 7, 9):

A) how many different 2 digit integers can be formed?

Answer:

B) how many *even* 2 digit integers can be formed?

Answer:

C) how many 3 digit integers can be formed?

Answer:

D) how many *odd* 3 digit integers can be formed?

Answer:

E) how many *even* 4 digit integers can be formed?

Answer:

(Assume that no digit may be used more than once.)

8. How many diagonals does a 15-sided polygon have?

Answer:

Worksheet 18B

3. 08. 1995

If one operation can be done in n different ways and if in every case a second operation can be done in m different ways, the two operations can be performed in succession in nxm ways.

1.The menu at a restaurant offers 3 main courses and 4 desserts. How many different two-course meals can be obtained?

A) 12 B) 7 C) 24 D) 30 E) 3

2. The menu at a restaurant offers .... main courses and desserts. How many different two-course meals can be obtained?

Answer:

3. In how many ways can 3 students stand in a queue?

Answer:

4. In how many ways can...... students stand in a queue?

Answer:

5. In how many ways can 10 students stand in a queue?

A) 100 000 B) 3828900 C) 3628800 D) 50 E) 1

6. In how many ways can a first and a second prize be awarded in a class of 30?

A) 15 B) 30 C) 10 000 D) 870 E) 900

7. From the set of digits (0, 2, 3, 6, 7, 9):

A) how many different 2 digit integers can be formed?

Answer:

B) how many *even* 2 digit integers can be formed?

Answer:

C) how many .... digit integers can be formed?

Answer:

D) how many .... digit integers can be formed?

Answer:

E) how many .... digit integers can be formed?

Answer:

(Assume that no digit may be used more than once.)

8. How many diagonals does a 15-sided polygon have?

Answer:

10. 08. 1995

IDENTIFYING THE MAIN SOLUTION IDEA

Read the solutions of the problems given below and formulate their main idea:

Problem 1♠: The left-most digit of a six-digit number N is 1. If this digit is removed and then written as a right-most digit, the number thus obtained is three times N. Find N.
Solution:

```
1ABCDE      3×E=*1               2.....
×    3      E=7                  1ABCD7      3×D+2=*7
ABCDE1                ⇒         ×     3          D=5
                                 ABCD71                ⇒

..........12                      2.....
1ABC57      3×C+1=*5            1AB857      3×B+2=*8
×    3      C=8                 ×    3      B=2
ABC571                ⇒         AB8571                ⇒

1A2857      3×A=*2              142857
×    3      A=4                 ×    3
A28571                ⇒         428571
```

Problem 2: Read the solution structure of the problem given below, formulate the main idea and write the solution precisely.

♠ *In each of the ten boxes there is a digit - two of them are shown. When the digits in three successive boxes are added, the total is always 20. What digits are in the other boxes?*

5 □ □ □ □ □ □ □ 7 □

Solution idea:

5 □ □ □ □ □ □ □ 7 □

5 □ □ 5 □ □ □ □ 7 □

5 □ □ 5 □ □ 5 □ 7 □

5 □ □ 5 □ □ 5 □ 7 5

$5 + \square + 7 = 20 \quad \square = 8$

5 □ □ 5 □ □ 5 8 7 5

5 □ □ 5 □ □ 5 8 7 5

Answer:
5 8 7 5 8 7 5 8 7 5

2. Try to pose a problem which could be solved using the same idea.

3. Read the Challenge Problem and suggest ways of changing the problem.

♠ Four friends A, B, C and D are racing together down a flight of stairs. A goes two steps at a time, B-three at a time, C- four at a time and D-5 steps at a time. The only steps which all four tread on are the top one and the bottom one. How many stairs in the flight were stepped on exactly once?

Worksheet 19A

10. 08. 1995

SELECTED AMC 1995 PROBLEMS

1. I have \$46.20 in my pocket in \$2, \$1, 50c, 20c, 10c and 5c coins, and I have an equal number of each coin type. The number is
A) 10 B) 11 C) 12 D) 13 E) 14

2. Mrs Stoyanova counted her class in groups of 4 and there were 2 left over. She then counted in groups of 5 and there was 1 left over. If 15 of her class were girls and she had more girls than boys, the number of boys in her class was
A) 7 B) 8 C) 9 D) 10 E) 11

3. Students in a maths test can score 0, 1, 2 or 3 marks on each of six questions. There is only one way of scoring 18 and six ways of scoring 17. The number of ways a student can score 16 is
A) 6 B) 12 C) 15 D) 21 E) 42

4. ♠ At various times the boss gives her secretary letters to type. The boss puts them in the in-tray one at a time, in order 1, 2, 3, 4, 5, 6 and when time permits between other duties the secretary takes a letter from the top to type. Which of the following could not be the order in which the letters eventually get typed?
A) 1, 2, 3, 4, 5, 6 B) 1, 2, 5, 4, 3 ,6 C) 3, 2, 5, 4, 6, 1 D) 4, 5, 6, 2, 3, 1 E) 6, 5, 4, 3, 2, 1

5. At a school 15 students were absent on Monday, 12 absent on Tuesday and 9 absent on Wednesday. If 22 students were absent at least once during these three day, what is the number of students who could have been absent on all three days?
A) 5 B) 6 C) 7 D) 8 E) 9

18. 08.1995

1. The value of
$1 - 2 + 3 - 4 + 6 + ... - 998 + 999 - 1000 + 1001$ is
A) 500 B) 501 C) -501 D) -1001 E) 100

2. Find the sum of all the two-digit numbers greater than 10 such that the tens digit is one less than the units digit.
A) 476 B) 414 C) 486 D) 404 E) 495

3. A mathematics test consists of 10 questions. Ten points are given for each correct answer and 3 points are deducted for each incorrect answer. If Peter did all questions and scored 61, the number of correct answers he had was
A) 7 B) 5 C) 9 D) 8 E) 6

4. ♠ A table with p rows and q columns is filled with the whole numbers from 1 to pq. They are written in increasing order, along row 1, then row 2, etc. The number 20 is in the third row, 41 is in the fifth row and 103 in the last row. Find $p + q$.
A) 21 B) 22 C) 23 D) 24 E) 25

5. S is the set of numbers from 1 to 100 whose smallest prime factor is 7. How many numbers are in S ?
A) 14 B) 7 C) 4 D) 3 E) 5

6. What would be the third number from the left of the 89th row of the accompanying triangular number pattern?

1
2 3 4
5 6 7 8 9
10 11 12 13 14 15 16

A) 8103 B) 6982 C) 10681 D) 7747 E) 7924

7. $(5x + 2y) - (2x - 5y)$ equals
A) $3x + 3y$ B) $3x - 3y$ C) $3x - 7y$ D) $2x + 3y$ E) $3x + 7y$

8. The manufacturers of a certain pen claim that it can draw a line $1km$ long before it runs dry. If the line it draws is $0.4mm$ wide, then the area, in square meters, that the pen is expected to cover is
A) 4000 B) 400 C) 40 D) 4 E) 0.4

9. ♠ Students in a group dancing class are spaced evenly around a circle and are then numbered consecutively from number 1. Student 20 is directly opposite student 53. How many students are there in the group?
A) 60 B) 62 C) 64 D) 66 E) 68

10. ♠ What is the first time after 4 o'clock that the hands of the clock make an angle of 65^0
A) 4.06 B) 4.07 C) 4.08 D) 4.09 E) 4.10

Worksheet 21

24. 08. 1996

INTRODUCTORY PROBLEMS:

1. I have 2 skirts and 3 blouses. How many different combinations can I wear? (In how many ways can I wear a blouse and a skirt?)

2. If I have 3 skirts and 4 blouses, then in how many ways can I wear a blouse and a skirt?

3. Make up a problem similar to the one above by using the numbers 3 and 5.

4. Make up a problem which can be solved in the same way as problem #1 and has an answer of 12.

5. Generalisation : n and m and n×m combinations.

6. Can you make up examples giving a meaning *n* and *m*?

4. Look at the problem below which is presented "briefly":

2 skirts (ice cream); How many combinations?

3 blouses (lollies) |

4 pairs of shoes (cups) Answer: 120

a) Why is 120 the right answer?

b) What might the meaning of the brackets be?

c) Can you improve the solution structure given below?

5. *There are 4 students* (Ask a meaningful question on the basis of only this information.)

In what wayis the problem given below different from problem 5?

6. *Given:*	*Goal:*
2 boys	How many groups can be made if in every
3 girls;	group we have 1 boy, 1 girl and 1 teacher
4 teachers.	

For the above situation state some meaningful questions:

7. In how many ways can 3 people be chosen out of 10?

$^{10}C_3 = (\ ) = \dfrac{\text{number of groups}}{\text{the times the same group occur}}$

8. Make up a problem from the same type for your friend and explain its solution for him/her.

What is the difference between problem 4*a* and problem 4*b* ? Do they have the same method of solution? Make some changes in the statement which preserve the solution method.

9. Make up a problem similar to Problem 4 which has the same solution method.

Worksheet 22

31. 08. 1995

THE MULTIPLICATION PRINCIPLE

1. How many integers between 1000 and 9999 have all digits different?
Answer:

2. In how many ways can a group of 2 be chosen from 6 people?
Answer:

The Addition Principle

3. How many different groups of 2 or more can be formed from 5 people?
Answer

4. How many one, two or three-digit positive integers are there in base 10?
Answer:

The Restriction Principle

5. In how many ways can 3 boys and 4 girls stand in a queue if the boys insist on occupying the first and the last places?
Answer:

6. How many even 4- digit positive integers are there in base 10?
Answer:

7. How many even 4- digit positive integers are there in base 5?
Answer:

AMC Problems

8♠. Six different Easter eggs are to be shared completely between Greg and Amanda. The eggs are to be shared between them in such a way that no egg is broken, and each gets at least one egg. In how many different ways the eggs can be shared?
Answer:

9♠. Of the numbers from 1 to 1000 inclusive, how many are divisible by 5 or 9 but not both?
A) 311 B) 289 C) 267 D) 200 E) 100

10♠. A table with p rows and q columns is filled with the whole numbers from 1 to pq. They are written in increasing order, along row 1, then row 2, etc. The number 20 is in the third row, 41 is in the fifth row and 103 in the last row. Find $p + q$.
A) 21 B) 22 C) 23 D) 24 E) 25

My AMC Problem is:
(Please write it down).

Worksheet 23

31. 08. 1995

1. Some integers are arranged in the way shown below:

1
2 3 4
5 6 7 8 9
10 11 12...13.. 14 15 16
17 ..25

a) What would be the third number from the left of the 89th row of the accompanying triangular number pattern?

A) 8103 B) 6982 C) 10681 D) 7747 E) 7924

b) State other meaningful questions.

Worksheet 24

7. 09.1995

1. Substitute the symbol "*" with a digit in the number 123*7*, so that the number is divisible by:

a) 2;
b) 5;
c) 2 and 5;
d) 2, 5 and 3.
Find how many solutions there are in each case.

Answer:

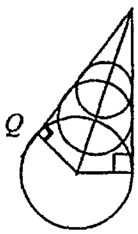
2. How many odd 3-digit positive integers are there in base 10?
Answer:
Solution:

3. Find the last digit of the sum $3^{21}+7^{17}$.
Answer:

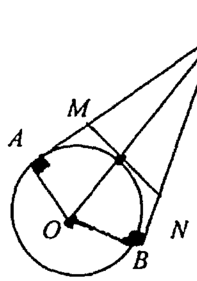
4♠. A table with p rows and q columns is filled with the whole numbers from 1 to pq. They are written in increasing order, along row 1, then row 2, etc. The number 35 is in the 3-rd row, 63 is in the 6-th row and 125 in the last row.

a) Find the values of p and q.
b) In which row will 97 be?
c) How many digits have been used for the numbers in the 10-th row?
Solution:

15. 09.1995



t - tangent
t
$t \perp r$



$AC = BC$

OC- bisector

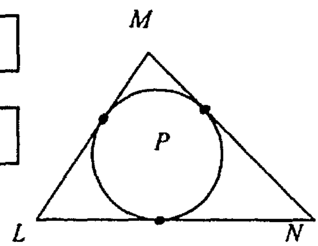
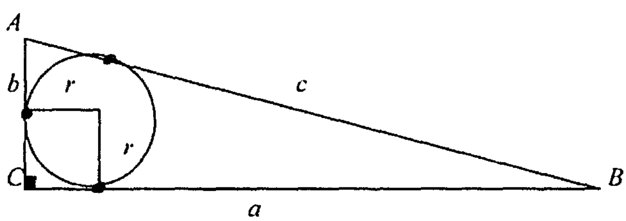


Fig.1 Fig.2 Fig.3

1. Given: ∇ABC: $\angle ACB = 90^0$.



Show that

$a + b = c + 2r$

$(a + b + c) \times r = 2S$

$a^2 + b^2 = c^2$

2. Write the fractions $^1/_{12}$, $^1/_6$, $^1/_{20}$ as a difference between two fractions:

3. If n is a positive integer and n is not equal to 0, show that

$$\frac{1}{n(n+1)} = \frac{1}{n} - \frac{1}{n+1}.$$

4. Calculate the value of

a) $1 + \frac{1}{2 \times 3} + \frac{1}{3 \times 4} + \frac{1}{4 \times 5}$,

b) $\frac{1}{20} + \frac{1}{30} + \frac{1}{42} + \frac{1}{56} + \frac{1}{72} + \frac{1}{90} + \frac{1}{110}$.

21. 09.1995

1. Write the fractions $^1/_{12}$, $^1/_6$, $^1/_{20}$ as a difference between two fractions.

2. If n is a positive integer and n is not equal to 0, show that
$$\frac{1}{n(n+1)} = \frac{1}{n} - \frac{1}{n+1}.$$

3. Calculate the value of

a) $1 + \frac{1}{2 \times 3} + \frac{1}{3 \times 4} + \frac{1}{4 \times 5}$,

b) $\frac{1}{20} + \frac{1}{30} + \frac{1}{42} + \frac{1}{56} + \frac{1}{72} + \frac{1}{90} + \frac{1}{110}$.

4. Carol wants to visit her friend. She remembers that the number of the house she leaves has three digits and it gets 7 times smaller if the middle digit is deleted. What number does the house Carol friend have?

5. After a week of hard calculation Peter figured out 3^{1995}. Then he added up all digits and thus obtained a new number. Next he added up all digits of this new number and obtained another number. He continued doing this. Eventually, he obtained a one-digit number. What was that number?

6. 1999 children are placed along the circumference of a circle. When the years of the ages of any four successive children are added, the total is 44. Find the ages of all these 1999 children.

Worksheet 27

19. 10. 1995

THE MULTIPLICATION PRINCIPLE

1. In how many ways can a group of 3 be chosen from 6 people?
Answer

THE ADDITION PRINCIPLE

2. How many different groups of 3 or more can be formed from 6 people?
Answer

THE RESTRICTION PRINCIPLE

3. In how many ways can 12 boys and 10 girls stay in a line if the girls insist to occupy the middle two places?
Answer:

4♠. Of the numbers from 1 to 2000 inclusive, how many are divisible by 5 or 9 but not both?
Answer

5. How many even numbers are there on the 100-th row

1
2 3 4
5 6 7 8 9
10 11 12...13.. 14 15 16
17 ..25

Answer:

6. Substitute the symbol "*" with a digit in the number 973*1*, so that the number is divisible by:

a) 2, 5 and 3.
b) 5 and 4.

Write down all of the solutions in each case.
Answer:

Worksheet 28

Revision Problems

26. 10.1995

1. The value of $\frac{1}{1995} + \frac{1996 \times 1994}{1995} - 1996$ is:

A) $\frac{1}{1995}$ B) - 1 C) $-\frac{1}{1995}$ D) 1 E) 2

2. If n is a positive integer show that

$$\frac{1}{n(n+100)} = \frac{1}{100}\left(\frac{1}{n} - \frac{1}{n+100}\right)$$

3. Substitute the sign "*" with a digit so that the calculations are true:

a) $\frac{1}{2 \times 102} = \frac{1}{100}\left(\frac{1}{*} - \frac{1}{102}\right)$

b) $\frac{1}{3 \times (* + 100)} = \frac{1}{100}\left(\frac{1}{3} - \frac{1}{* + 100}\right)$

4. Calculate without using a calculator:

$$\frac{1}{1 \times 101} + \frac{1}{2 \times 102} + \frac{1}{3 \times 103} + \ldots + \frac{1}{10 \times 110}.$$

5. The value of 100! is the product of all whole numbers from 1 to 100 inclusive, i.e. 100! = 1 × 2 ×... × 99 × 100. The maximum number of times that 2 will divide into 100! is:
Solution:

9. 11.1995

1. Given the sequence 1, 2, 3, 4, 5, ... N.

a) If $N = 200$, how many digits have been used?

b) Which digit is on the 147-th place?

c) If the last number is 999, how many 3's in total have been used?

d) If the last number is 200 how many prime numbers are there?

e) If the last number is 250 how many numbers of this sequence are divisible by 2, 3, and 4 but are not divisible by 5?

f) Other questions?

2. A pencil and a rubber cost 25 cents. Seven pencils and 4 rubbers cost $1.30.
a) How much should Greg pay for 2 pencils and 2 rubbers?

b) What will be the price of 1 pencil?

c) What will be the price of 1 rubber?

d) How much should Ben pay for 3 pencils and 2 rubbers?

e) Other questions?

Appendix 5. A Sample of Additional Materials Developed for the Study.

SUCCESSIVE NUMBERS

1. Write the missing consecutive number:

a) □, n, $n + 1$, □;

b) □, n, $n + 2$, □;

c) □, $2k$, □, $2k + 4$;

d) □, □, $2k + 1$, $2k + 3$;

2. Find three consecutive numbers such that the sum of the first and the third is 376.

3. Find three consecutive even numbers, such that the sum of the first and the second is 358.

4. What two numbers, neither ending in zero, when multiplied together equal exactly:
a) 10;
b) 100;
c) 10 000;
d) 10^7.

5. Without calculating determine the numbers of the zeros in the product of 5^{17} and 2^{17}.

POWERS. EXTENSION

1. Which digit has to be into the □ in order the equality holds:

a) $1^{2^{3^4}} = 2^{3^4}$;

b) $5^{4^5} = 5$;

c) $(-10)^{100^{200}} = -10$;

d) $(-3)^{100^{200}} = 1$.

2. Explain to your friend why the inequality holds:

a) $(3^4)^5 < 3^{4^5}$;

b) $(-3^5)^4 > (-3)^{5^{40}}$;

c) $(x - 15)^{(x-10)^{(x+20)}}$, if $x = 10$ is less than $(x - 15)^{(x+10)^{(x+30)}}$, if $x = 15$.

EVEN AND ODD NUMBERS

Definition of an *even* number: A positive integer number divisible by 2 is called an even number.

Definition of an *odd* number: A positive integer number which is not even.

1. Which of the following numbers are even:
11, 372, 446, 171, 737, 753, 984, 655, 993 678, 3 720;

2. Write down some even numbers:..

3. Write down six even numbers:..
Can you explain why some of them (at least two) have the same last digits?

4. Write down several examples of even numbers, which have a different last digit: ....

5. Find the missing digit:
a) 326 = 320 + □

b) 4□7 = 450 + 7

c) 46□ = 460 + 8

d) 2□ = 20 + 5

e) 1 6□5 = 1 670 + 5

6. Discuss with your partner why the following statements are true:

a) *Conjecture 1*: A number is divisible by 2 when its last digit is divisible by 2.

b) *Conjecture 2*: A number is divisible by 4 when the number from its last two digit is divisible by 4.

Write down 5 numbers divisible by 4. Use a calculator to check if they are divisible by 4.

c) *Conjecture 3:* A number is divisible by 5 when its last digit is 5 or 0.

Write down 5 numbers divisible by 5. Use a calculator to check if they are divisible by 5.

d) *Conjecture 4*: A number is divisible by 3 when the sum of all its digits is divisible by 3.

Write down 5 numbers divisible by 3. Use a calculator to check if they are divisible by 3.

e) *Conjecture 5:* A number is divisible by 9 when the sum of all its digits is divisible by 9.

Write down 5 numbers divisible by 9. Use a calculator to check if they are divisible by 9.

f) *Conjecture 6:* A number is divisible by 10 when its last digit is 0.

Write down 5 numbers divisible by 10. Use a calculator to check if they are divisible by10.

PRIME AND COMPOSITE NUMBERS

1. Continue the sequence:
2, 3, 5, 7, 11, 13, 17,...

2. Write some examples of numbers which are not prime.

3. Is 127 a prime number? Give arguments.

4. State the factors of 2^3.

5. Give an example of a number which has exactly 7 factors.

6. Write the missing digits $\square^{11}$ or $\square^{11}$, so that both numbers have the same number of factors.

7. Which of the numbers has more factors: 4^{11} or 5^{11}. Give arguments.

8. Write a suitable number in the $\square^{11}$ and $\square^{11}$ so, that both numbers do not have the same number of factors.

9. Write a suitable number in the $\square^{11}$ so, that number has more than 12 factors.

10. Determine the number of the factors of :
a) 6, 14, 15, 21;

b) 20, 26, 45, 78;

11. Write a number which will have the same number of factors as 54.

12. Determine the number of the factors of 240.

13. Which are the factors of:
a) 2^3;

b) 3^7;

c) 4^3;

d) $2^3 \times 3^7$;

e) $2^3 \times 3^7 \times 11^3$;

14. How many factors does the number have:
a) 5^6;

b) 13^7;

c) 4^{12};

d) $12^3 \times 6^7$

e) $2^4 \times 3^6 \times 15^5$;

15. Write down some examples of numbers which have an even numbers of factors. What conclusion can you draw?

THE LEAST COMMON MULTIPLE

Definition: The *least common multiple* of two natural numbers is the smallest natural number which is multiple of both numbers.

1. Find the least common multiple of:
a) 2 and 3;

b) 2 and 6;

c) 2^2 and 2^3;

d) 3^7 and 3^6;

Answer: *LCM* (3^7; 3^6) is 3^7, because $3^7$is the smallest number divisible by 3^6.

To find the least common multiple we choose the highest power of prime occurring in either number and take a product of these numbers.

2. Find the *LCM* of:
a) $2^3 \times 3^4 \times 5$ and $2^4 \times 3^2 \times 7$;

b) $3^4 \times 5^6 \times 11$ and $3^2 \times 5^4 \times 7^2$;

3. Find the least common multiple of:
a) 48 and 72;

b) 16, 12 and 90;

c) 3, 4, 5, 6, 7, 8 and 9.

4. ♠ Betty, David and Rebecca are friends. They love mathematics. Betty goes to maths classes every second day, David-every fourth day, and Rebecca-every sixth. If today they are having a class together, in how many days they will be in the class together again?

Let us make a problem similar to the above, with the same method of solution.

5. What is the smallest number which gives a remainder of 1 if it is divided by 2, 4 and 6?

6. At a parade, the general wanted his solders to go in front of the Queen in lines of equal groups. He tried to make groups of 12, 11, 10, 9, 2, but always one solder was left. At the end they had to go one after another. Find out what the smallest number of the solders could be?

7. Three ships are leaving the Peth harbour for Darwin today. The round trip takes the first ship 4 days, the second - 18 days and the third - 12 days. In how many days the three ships will leave Perth harbour together again? (Assume continuos round trip activities for all three ships.)

Let us make a problem similar to the above, with the same method of solution:

8. Four ships are leaving Perth for Disneyland today. The round trip of the first one takes 36 days, the second - 48 days, the third - 49 and the fourth - 54 days. If the ships are leaving Perth harbour today in how many days they will leave Perth together again?

THE HIGHEST COMMON FACTOR

Definition: A *common factor* of two natural numbers ***a*** and ***b*** is a natural number which is factor of both **a** and **b**.

1. Examples:
a) Let 6 be a common factor of two numbers, than the numbers might be 12 and 30 or 6 and 42.

b) One is common factor of 3 and 5.

c) Three is a common factor of 81 and 123, both numbers are divisible by 3.

2. Write the factors of 2^5 and 2^6. Which is the largest their common factor?

3. Write the factors of 3^4 and 3^7. Which is the largest their common factor?

4. The *prime decompositions* of two numbers are $2^3 \times 3^4 \times 5$ and $2^4 \times 3^2 \times 5^2$. State some of their common factors. Which is the largest their common factor?

Solution:

Obviously, the highest common factor of two numbers will be a number which has as factors powers with bases 2, 3 and 5. The *largest* common factor of 2^3 and 2^4 is 2^3, of 3^4 and 3^2 is 3^2 and of 5^2 and 5 is 5. Then the largest common factor of both numbers will have as factors 2^3, 3^2 and 5. The *smallest number* with this property is $2^3 \times 3^2 \times 5$.

Definition: The *highest common factor* of two natural numbers ***a*** and ***b*** is the *largest* natural number which is factor of both ***a*** and ***b***.

5. Find the highest common factor of:
a) 448 and 240;

b) 3 and 5;

Definition: Two numbers ***a*** and ***b*** are called *relatively prime* if their highest common factor is 1.

Let us take the product of prime decompositions of any two numbers, for example $(2^3 \times 3^4 \times 5)$ and $(2^4 \times 3^2 \times 5^2)$. It is clear that $(2^3 \times 3^4 \times 5)$ and $(2^4 \times 3^2 \times 5^2)$. = $(2^3 \times 3^2 \times 5) \times (2^4 \times 3^4 \times 5^2)$ = HCF × *LCM.*

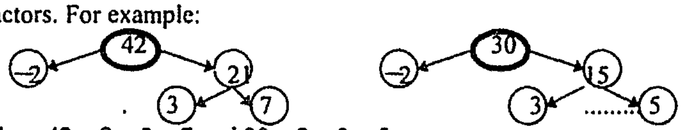
6. Show that $240 \times 448 = LCM(240, 448) \times HCF(240, 448)$.

7. Explain to your partner why always $a \times b = LCM(a, b) \times HCF(a, b)$.

Let us summarise the ways we can find the *Highest Common Factors* of two numbers:

Alternative 1.

We write the prime decompositions of the numbers and *HCF* is the product of all common factors. For example:



Thus, $42 = 2 \times 3 \times 7$ and $30 = 2 \times 3 \times 5$.
Hence $HCF(42, 30) = 2 \times 3 = 6$.
Obviously, this method can be applied the *HCF* of more than two numbers to be found.

Alternative 2.

The second alternative is based on finding common factor of all numbers simultaneously. It is illustrated in the table below.

Solution:

Number 1	Number 2	Common Factor
42	30	2
21	15	3
7	5	1

$HCF(42; 30) = 2 \times 3 \times 1 = 6$.

Alternative 3:

The third alternative is based on the equality: $m \times n = HCF(n; m) \times LCM(n; m)$.

Hence the $HCF(m, n) = \dfrac{m \times n}{LCM(n, m)}$

In this case, $HCF(42; 30) = \dfrac{(42 \times 30)}{(2 \times 3 \times 5 \times 7)} = 6$.

Alternative 4:

Next alternative is based on so called *Euclidean* algorithm (see p. 14 in *Euler booklet*):

$42 = 1 \times 30 + 12;$

$30 = 2 \times 12 + 6$

$12 = 2 \times 6$

$HCF(42; 30) = 6$.

Applications:

1. Mary has a bunch of three types of flowers. From the first type she has 6 flowers, from the second - 4 and from the third - 12. She wants to make several smaller bunches from the same type (they should have the same number of flowers from any type). How many bunches could Mary make? How many flowers from any type will be in a bunch?

2. In the above problem assume that the number of the flowers are respectively:
a) 16; 4 and 12;
b) 16, 7 and 24;
c) 4, 6, 8 and 12.

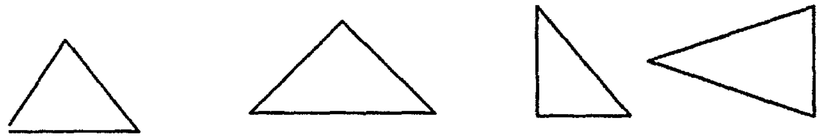
3. Make a problem similar to the one above and suggest a solution idea.

4. Forget about *HCF* and try to solve the problem (practically) without using the concept of *HCF*.

SPECIAL TRIANGLES

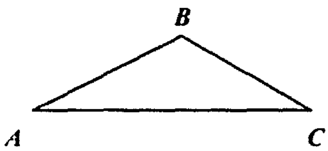
Definition: A triangle is called isosceles if it has two of its sides equal in length.

Examples:



Problem 1. The triangle *ACB* is isosceles (*AB* = *BC*). Show that two of its angles are congruent.
A "short" version of the same problem can be presented as follows:

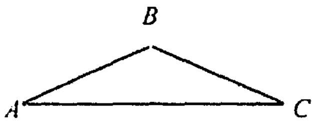
Pr. 1.



Given:
$\triangledown ABC$,
$AB = AC$

Show that:
$\angle A = \angle C$

Problem. 2.

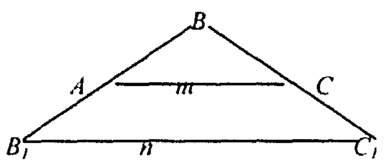


Given:
$\triangledown ABC$.
$\angle A = \angle C$

Show that:
$AB = AC$

Theorem: A triangle is isosceles if and only if two of its angles are congruent.

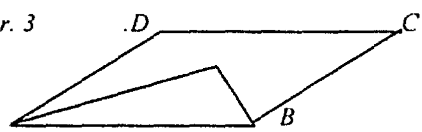
Pr. 2.



Given:
$\triangledown ABC$.
$\angle A = \angle C$
$m // n$

Show that:
$\angle B_1 = \angle C_1$

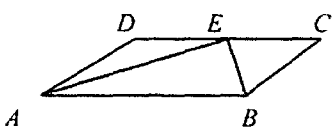
Pr. 3



Given:
ABCD - a parallelogram
AE - bisector of $\angle A$
BE - bisector of $\angle B$

Find out:
$\angle AEB$ = ?

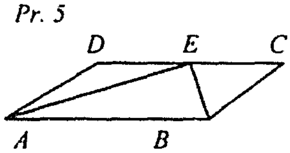
Pr. 4



Given:
ABCD - a parallelogram
BE - bisector of $\angle B$
AE - bisector of $\angle A$
$E \in CD$

Find out:
Which triangles are isosceles ?

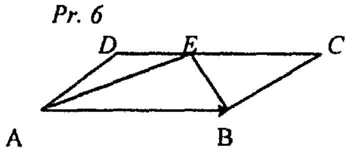
Pr. 5



Given:
ABCD - a parallelogram
BE - bisector of $\angle B$
AE - bisector of $\angle A$
$E \in CD$

Show that:
$DC = 2\,AD$

Inverse problem:
Pr. 6



Given:
ABCD - a parallelogram
$DC = 2\,AD$
E - a midpoint of *AB*

Find out:
$\angle DEC$ = ?

ARITHMETIC SEQUENCES

Problem 1. Try to continue the following sequences of numbers. (The dots indicate that the numbers continue.)

1) 1, 2, 3, 4, . . .

2) 0, 2, 4, 6, . . .

3) 1, 3, 5, 7, . . .

4) 1, 5, 2, 8, . . .

5) 1, 2, 3, 7, 1, 15, . . .

6) 0, 2, 4, 7, 8, . . .

7) 1, 5, 3, 9, 7, . . .

8) 16, 15 14, 13, . . .

9) 2, 4, 8, 16, . . .

New vocabulary:
a term - first, second, . . . - each number in a sequence is called a term. For example, the number 1 is the first term in the first sequence, 2 is the second term, 3 is the third.

Answer the questions:
1. What is the value of the third term in sequence 7) ?

2. What is the value of the fifth term in sequence 8)?

A sequence is called ***arithmetic sequence*** if the difference between any two consecutive terms is always the same. The *difference* is usually denoted by the letter *d*.

3. Which of the sequences in Problem 1are arithmetic sequences?

4. Write down your own examples of sequences which are arithmetic. Give an example of a sequence which is not arithmetic, but there is a pattern between the terms.

5. Write down a formulae for the t_n For the arithmetic sequences in Problem 1.

6. In a sequence $t_1 = 2$, $d = 3$. What is the 50th term?

7. In a sequence the first term is 2, the 45th term is 90. What is the third term equal to?

8. For the sequence below state as many questions as you can:
1, 2, 3, 4, . . . , 1000, . . .

a) Find the 45th term;

b) Find the sum of first 1000 terms;

c) Find the sum of first *n* terms.

d) Find the sum of all numbers between 1 and 1000 which are not divisible by 5.

Solution:

$S_{1000} = 1 + 2 + 3 + ... + 998 + 999 + 1000$

$+ \underline{S_{1000} = 1000 + 999 + 998 + ... + 3 + 2 + 1}$

$2\,S_{1000} = (1 + 1000) + (2 + 999) + ... + (999 + 2) + (1000 + 1)$

$S_{1000} = \frac{1001}{2} \times 1000$

The same idea could be applied for finding the sum of any n terms in an arithmetic sequence.

$S_n = t + (t + d) + ... + (t + (n - 1)d)$

$\underline{S_n = (t + (n - 1)d) + ... + (t + d) + t}$

$2\,S_n = (t + t + (n - 1)d) \times n$

$S_n = \frac{(2t + (n - 1)d)}{2} \times n.$

If we use that $t_n = t + (n - 1)d$, then we get the following formula:

$S_n = \frac{(t + t_n)}{2} \times n.$

9. Find the sum of the first n natural numbers.

10. Find the sum of first n even numbers.

11. A number of apples have been divided between 20 students in this way: the first student got one, the second - two, the third - three and so on. How many apples have been used altogether?

12. A group of students has 26 marbles altogether. The first student has 2 marbles, the second - four, the third - six and so on. How many students are there in the group?

THE PIGEON-HOLE PRINCIPLE

1. In the table below replace the missing words with a synonym:

....	at most 3 people
at least 2	....
not less than 4	....
....	5, 6, 7 or more
not more than 7	....
....	0, 1, 2 or 3 apples
less than 5 plums	....
. ...	more than 4 cars

2. Solve the following problems and underline the words that determine the problem solution.:

A) Peter has at least 5 marbles. How many marbles might he have?

B).Helen has more than 5 apples. How many apples might she have?

C) In my pocket I have not less than 5 coins. How many coins do I have?

3. For the problem situation given choose a meaningful conclusion from those listed below:
There are 10 rabbits into 3 boxes. Then:
A) there is a box with at least two rabbits;

B) there is a box with at least three rabbits;

C) there is a box with not less than 4 rabbits;

D) there is a box with 4 or more rabbits.

E) there is box with 3 or more rabbits.

4. Without adding more information formulate a meaningful conclusion for the problem situations listed below:

A) There are 3 pigeons in 2 pigeon holes. Then..

B) Mrs. Simpson has three children. Then ....

C) In my maths class I have 27 students. Then..

D) This week Carole has been to the library 8 times. Then.......................................

E) There are $k+1$ pigeons into k pigeon-holes. Then ....

The Pigeon-hole Principle: If $k+1$ pigeons go into k pigeon-holes, then at least one pigeon-hole will have one or more pigeons.

In some cases, as those listed below, stronger claims can be formulated. For example, if I distribute 5 apples between my two sons, one will have at least three appels. In the worse case, both could have less than 3 apples, then the maximum number of apples they could have is 4. Because one apple is left, one of them will have at least 3 apples.

5. Formulate a meaningful conclusion for the problem situations listed below:

A) There are 7 pigeons in 2 pigeon-holes. Then..

B) Mrs. Simpson has 5 children. Then ....

C) This week Carole has been to the library 8 times. Then.......................... ....

D) In my maths classes I have 53 students altogether. Then.......................................

E) There are $k \times m + 1$ pigeons into m pigeon-holes. Then ....

6. Finish the problem situations below so that the solution method implies the pigeon-hole principle.

There are 5 pigeons into ....pigeon holes.	Show that there is a pigeon-hole with at least two pigeons
There are .... in my class.	Why at least 2 two students were born on the same day of the week?
There are pigeons into 4 pigeon-holes.	Prove that there is a pigeon-hole with at least two pigeons.
There are ....into .... boxes.	Then there are.....................................
There are	Prove that there are.............................

7. Solve the problem below and write the solution *precisely*:

a) There are 27 students in a class. While doing a keyboard skills test one student made 12 mistakes, while the rest made fewer mistakes. Show that at least 3 students made the same number of mistakes.

Appendix 6. A Sample of Revision Papers Adapted for the Program.

Revision paper 1

THE COMMUTATIVE LAW

There several ***basic laws*** in the **Algebra**.

1. The Commutative Law.

Let a means any number which we will regard as a *first* and b means any number which we regards as the *second* number. Then the commutative law for addition can be written as:

$$a + b = b + a.$$

Its meaning is that the *exchange* of the *terms* in addition *does not change* the *sum.*

For example, according to the commutative law, 3 + 5 = 5 + 3, or if you prefer three-digit numbers, 123 + 456 = 456 + 123.

The commutative law for multiplication can be written as:

$$a \times b = b \times a,$$

and it means that the *exchange* the *factors does not change* the *product.*

2. The Associative Law.

What about if we have more than two numbers? The law which is applied in that case is called by the mathematicians *the associative law.*

In symbols for the addition operation it looks like:

$$(a + b) + c = a + (b + c),$$

and for multiplication operation:

$$(a \times b) \times c = a \times (b \times c).$$

We use the laws mentioned above ***not only*** to add or multiply more than two numbers but also to *curtain* the process of addition or multiplication.

Examples:

Calculate without using a calculator:

a) 13 × 156 × 0 × 3 678 × 12 × 234567;

b) 678 + 1346 + 322 + 654.

3. The Distributive Law.

The law which ***bridges*** addition and multiplication operations is termed as the *distributive law.*

Let us consider the following ever-day-life situation: *Three boys and four girls get 9 apples each.* Then the boys get 3 × 9 = 27 apples, the girls get 4 × 9 = 36 apples. Altogether they get

$$3 \times 9 + 4 \times 9 = 27 + 36 = 63 \text{ (apples)}.$$

The same answer can be calculated in a different way: there are 3 + 4 = 7 children and each of them gets 9 apples, so the total number of apples is (3 + 4) × 9 =

$$(3 + 4).9 = 63.$$

Therefore,

$$(3 + 4).9 = 3.9 + 4.9$$

and in ***general,***

$$(a + b) \times c = a \times c + b \times c$$

or

$$c \times (a + b) = c \times a + c \times b.$$

In the next examples we apply the distributive law for *removing* the brackets. This operation is called: **expansion.** For example, $3(x + 4y) = 3x + 3 \times 4y = 3x + 12y$.

Examples:

$x(y + z) =$	$x(a + b)=$	$x(y + z) =$	$c(a + 2) =$
$c(a + 4) =$	$c(2 + b) =$	$3(a + b) =$	$4(a + x) =$

What will happen if some of the variables are *negative* numbers?

$$a(b - c) = a(b +(- c)) = ab + a(- c) = ab - ac.$$

Let us consider one specific example: $3(x - 6) = 3(x + (- 6)) = 3x - 18$.

Let us present some examples which illustrate some applications of the distributive law:

1. Calculate verbally:

a) 1001 . 30;

b) 1001 . 234.

2. Expand the brackets:

$2(3 - x) =$	$2(3 + x) =$
$3(x - 1) =$	$3(x + 1) =$
$(a - 3)x =$	$(a + 3) =$
$2a(a - 1) =$	$2a(a + 1) =$

What will happen if we have a product of two (or more!) sums? For example, $(a + b)(m + n)$?

4. The distributive Law. Extensions.

Let us assume that the number $(m + n)$ is the sum of the two numbers m and n. We can replace $(m + n)$ with C in the above expression and we will get:

$$(a + b) . C = a . C + b . C, \text{ but } C = m + n, \text{ thus}$$

$$(a + b) .(m + n) = a . (m + n) + b .(m + n).$$

If we apply the distributive law once again, then

$$a(m + n) + b(m + n) = am + an + bm + bn.$$

Examples:

1. *Without expanding the brackets, can you guess how many terms will be after using the distributive law for:*

$(a + b + c + d)(x + y + z)$

2. *Expand the brackets:*

$(x + 3)(x + 2) =$

$(x + 2)(x - 7) =$

$(x + y - 2)(x - 3) =$

THE SQUARE OF A SUM/DIFFERENCE

$$(a+b)^2 = (a+b)(a+b) = a^2 + 2ab + b^2$$
$$(a-b)^2 = (a-b)(a-b) = a^2 - 2ab + b^2$$

Let us go back to the distributive law again and consider the letters inside the brackets are the same. We get

$$(a+b)(a+b) = aa + ab + ba + bb$$

or

$$(a+b)^2 = (a+b)(a+b) = a^2 + 2ab + b^2$$

Example 1: The above formula also could be interpreted geometrically. Let a square with a side $a + b$ is given. Try to cut into two squares which have a sides a and b, and two rectangles with sides a and b.

Example 2: The rule $(a + b)^2 = a^2 + 2ab + b^2$ may be used for generating equalities, for example, such as:

a) $(2 + 17)^2 = 2^2 + 2 . 2 . 17 + 17^2$,
b) $(2 + x)^2 = 2^2 + 2 . 2 . x + x^2$,
c) $(2 + 3x)^2 = 2^2 + 2 . 2 . 3x + (3x)^2 = 2^2 + 12x + 9x^2$.

Example 3: Apply the above rule to:

$(a + 1)(a + 1)=$	$(x + 3)^2=$
$(a + 2)^2=$	$(y + 10)^2=$
$(1 + 2a)^2=$	$(2x + \frac{1}{6}y)^2 =$

These numbers may, of corse, be *negative*. For example, for $a = 3$ and $b = - 5$ we get
$(3 + (- 5))^2 = 2^2 + 2 . 2 . (- 5) + (- 5)^2 = 2^2 - 2 . 2 . 5 + 5^2$.

The same thing could be done for any other numbers, so the general rule is that:
$(a - b)^2 = (a - b)(a - b) = a^2 - 2ab + b^2$

Example 4:

1. *Calculate verbally:* 99^2 and 998^2.

2. *Apply the above rule to*:

$(a - 1)^2=$	$(a - 10)^2=$
$(x - 3)^2=$	$(a - 2)^2=$
$(x - 2a)^2=$	$(3a - 2.5b)^2 =$

THE DIFFERENCE OF SQUARES

$$(a + b).(a - b) = a^2 - b^2$$

Let us multiply $a + b$ and $a - b$:

$$(a + b).(a - b) = a(a - b) + b(a - b) = a^2 - ab + ab - b^2 = a^2 - b^2.$$

So we get the formula:

$$(a + b).(a - b) = a^2 - b^2.$$

Examples:

1. *Calculate verbally:*
a) 101×99;
b) 1999×2001.

2. *Apply the above formula to:*

$(a + 1)(a - 1)=$	$(x + 3)(x - 3)=$
$(a + 10)(a - 10)=$	$(x + 2a)(x - 2a)=$
$(a + 2)(a - 2)=$	$(2x - 3a)(2x + 3a) =$

3. *Write your own examples which are similar to the problems presented in 2).*

Appendix 7. A Sample of Some of the *Challenge Problems* and Hints Given to the Participants in the Program.

A sample of selected *Challenge Problems*

♠ *Challenge Problem 1:* The leftmost digit of a six-digit number *N* is 1. If this digit is removed and then written as a rightmost digit, the number then obtained is three times *N*. Find *N*.

♠ *Challenge Problem 2:* In each of the ten boxes there is a digit - two of them are shown. When the digits in three successive boxes are added, the total is always 20. What digits are in the other boxes?

□ □ □ □ □ □ □ □ □ □

♠ *Challenge Problem 3:* Four friends are racing together down a flight of stairs. *A* goes 2 steps at a time, *B* 3 at a time, *C* 4 at a time and *D* 5 steps at a time. The only steps which all four tread on are the top one and the bottom one. How many stairs in the flight were stepped on exactly once?

♠ *Challenge Problem 4:* Find two 3-digit numbers whose product is 555 555. Show there is only one way to do this.

HINTS FOR UNDERSTANDING THE CHALLENGE PROBLEMS

Challenge Problem 1:
Let us consider a specific example:
If $N = 123\,456$, when the leftmost digit of it is removed, the number 23 456 is obtained. Thus $N = 23\,456 + 100\,000$, or $23\,456 = N - 100\,000$.
When 1 is written as the rightmost digit, 234 561 is obtained, thus
$234561 = 23\,456 \times 10 + 1 = (N - 100\,000) \times 10 + 1$.
Now, you have to compare two numbers and to write a mathematical relationship between them.
Answer: $N = 142\,857$.

Challenge Problem 3:
Notice that the only steps on which all four tread are the top one and the bottom one. The total number of stairs in the flight excluding the top one is equal to . . . Let us number these steps from 1 to . . . The stairs that are stepped on by A only have numbers of the form 2 x *m*, where *m* is divisible by none of the numbers . . .

Challenge Problem 4:
The answer is: 777×715.

Challenge Problem 7:
$Y_1 = (1 + 1)^2 + 2 \times 1$
$Y_2 = (2 + 1)^2 + 2 \times 3$
$Y_3 = (3 + 1)^2 + 4 \times 3$
Can you see the pattern now?

Challenge Problem 8:
Divide 43, 92 and 83 by 7 and look for a pattern in the remainders of:
$43, 43^2, 43^3, 43^4, 43^5, 43^6, \ldots$
$92, 92^2, 92^3, 92^4, 92^5, 92^6, \ldots$
Did you find that the remainders are the same?
What is the remainder of 83:7 ?
Now, if a number is divisible by 7, then the remainder is 0 or 7.

Challenge Problem 9:

Let the number of chairs in each side sections of a row be x. and the number of rows be y. The number of all chairs is 1776 which is (....).x.(.....). So, what are the divisors of 1776?

Challenge Problem 10:

The answer is 52.8 *km/h*.

Let S be the distance between A and B. The time from A to B is . . . If you know the distance between A and B and the time the trip takes. How will you find the average speed?

Challenge Problem 12:

A bunch that would meet all the requirements has to have 4 flowers of one kind and 3 of each of the two others. In how many ways can you choose 4 out of 6 and 3 out of 5 and 3 out of 4?

Challenge Problem 13:

See *Euler Students Notes*, p. 56, Problem 8.

Challenge Problem 14:

Answer the following questions: What time had they been riding till they met for the first time? What distance did they ride together? What distance will they.....

Challenge Problem 15:

Answer the following questions: When the buses will meet for the first time? In how many minutes is the second meeting? In how many minutes is the third meeting, the fourth? . . .Can you see the pattern?

Challenge Problem 16:

Let your number be ***abc***, it is equal to $a \times 100 + b \times 10 + c$, which is $12(a + b + c)$. Now try to simplify

$a \times 100 + b \times 10 + c = 12(a + b + c)$ and solve it. Did you get $2 \times b = 88 \times a - 11 \times c$? What are the divisors of both sides?

The answer is 108.

Good luck!

Appendix 8. A Sample of Selected Structured Notes Taken by an Independent Observer:

THURSDAY 6TH APRIL 1995

FIRST CLASS: 4.00PM - 5.00PM

Introduction: Around 3 minutes long.

Discussed using bases, eg base 10, base 2, etc.

Children involved? Actively involved within 5 minutes, asked series of questions, progressively more difficult, to demonstrate theory of bases.

Almost all children involved here, raising hands.

Students asked to invent their own questions for each other, almost all the students became actively involved.

4.20PM: Students asked who finds the classes difficult?

None said very difficult.

Almost all said a bit difficult.

Norm (near front) said easy.

Revision of previous week's work: Elena explained work on prime numbers, factors etc again (some remembered from last week but most admitted they had forgotten).

Students attentive—many wrote down the example problem and solved it for themselves while Elena was going through it.

When answering questions, the 'louder' students (eg Norm, other boy in front row) answered to the whole class, quieter students only directly to Elena.

Whole class attentive when used the example about food—desserts and main courses, in relation to combinations—many of the children clearly understood the concept after this example.

4.35PM: Left to complete worksheet in pairs.

Worksheets used: titled PRIMES AND COMPOSITES.

Elena moved around and worked with any children who asked questions, and particularly checked on the two pairs of students being tape-recorded/observed: Tom and Daniel (in front row) worked well together, discussing a lot (see tape). Other observed pair (two girls in back row) more quiet/shy and finished worksheet quickly, (probably not much on tape).

Norm pointed out error in a question, long discussion with Elena - see tape.

Some pairs worked well together but some needed encouragement to even talk together! Several ignored their partners and worked on the problems alone.

5.10PM: Class finished.

SECOND CLASS: 5.10PM - 6.10PM

Introduction: Brief, really straight into discussion of last week's work, in detail because it is Sheryl's first week in the class.

Didn't look at 'bases' question as first class had done.

Asked questions often during explanation and because the class was so small, made sure that each student answered some questions—they were all quiet but happy to respond when asked (ie mostly they knew the answers to the questions but were too shy to say them aloud).

Here all students answered directly to Elena, not to the class as a whole.

Concepts were taught by showing patterns, eg 7^{11} has 12 factors, etc, gave many examples and asked for more examples, gradually each student caught on and could give their own example.

5.45PM: Students asked if they found the classes very difficult?

All said no, not really.

5.50PM: Students left to work on question sheets in pairs (Sheryl & Karel, Nicki & Samantha), however they really only worked as individuals.

As before Elena helped students when they asked questions, and also particularly helped Karel (young one).

See tape for discussions between Elena and individual students, especially Karel (Elena spent most of the time helping him).

Confusion about using letters eg students could answer easily a question about 5^{13} but when asked the same question about 5^n they were quite baffled. Elena told them it was just the same thing, just a letter instead of a specific number, but clearly the class was not convinced.

6.15PM: Class finished.

THURSDAY 4TH MAY 1995

FIRST CLASS: 4.00PM - 5.00PM

Introduction: Discussion about difficulty of the program. (No child involvement.)

4.05pm: Brief discussion about prime decomposition using trees:

Children asked to call out two factors (some confusion though as few realised they needed to call out pairs of factors that multiplied to give the appropriate number, eg some may have just said '2 and 10' as two factors of 300).

Children involved? Students asked for answers, involved here-then asked to do some problems on paper, Elena worked with individual children who had problems.

4.15pm: Discussion about LCM, children very involved, eg reading sections from text aloud, Elena asked many questions, and waited until the majority of students had their hands up before asking for responses. Students also involved in creating the problems, Elena asking "What would my question be?" etc.

4.45pm: Students left to work on worksheet: LCM.

4.55pm: Went through answers - asked students for answers and a general consensus on if they agreed.

5.00pm: Class finished.

SECOND CLASS: 5.10PM - 6.10PM

Introduction: Discussion about prime decomposition, using trees, similar to first class.

Children involved: Only by listening—none taking notes, even though several were new to the class.

5.15pm: Given time to work through examples, then discuss solutions. Also looked at text.

Very similar treatment to first class. Then discussed LCM.

5.35pm: Elena made them make up their own questions to do with LCM: eg Nicole (usually very quiet) made up problem to do with LCM of 3, 4 and 5—if there is a girl at every 3rd desk, a cockroach on every 4th desk, beetle on every 5th desk, which is the next desk that will have all three? (See tape, although may not be very clear). Also see tape for Nora's problem—eating every 5, 6 and 7 hours. Students enjoyed the novelty of creating their own problems, and clearly they understood the concepts to be able to create and answer their own problems.

6.00pm: Students left to work on worksheet: LCM. Elena worked with individual students, eg Samantha, Martin (Question 10) - see tape.

6.10pm: Went through some of the answers.

6.15pm: End of class.

THURSDAY 11TH MAY 1995

FIRST CLASS: 4.00PM - 5.00PM

Introduction: Revision of prime factorisation using trees (revision from last week)

Children involved? Immediately, Elena asked many questions as part of the examples shown; asked often "what am I going to do next", etc. Students appeared to understand this revision, and responded to Elena's questions well.

Then gave brief introduction to Highest Common Factor, asked students to help invent examples and questions, some students caught on and able to make up questions, others not too sure.

4.15pm: Looked at worksheet on algebra, asked students if they wanted any of it explained, and if so which parts, students not very vocal (because they understood almost none of it, as it turned out), agreed to explain it all!

4.20pm: Explanations of first sections of algebra, left to do relevant questions in between explanations, Elena helped individual students.

4.30pm: Left to finish remainder of algebra examples.

4.40pm: Went through answers to examples on board, quite quickly. Then looked at geometry side of worksheet, students were asked questions throughout the examples, were able to answer these much better than questions asked about the algebra.

4.50pm: Left to finish the geometry questions.

See tape—some short conversations between Elena and students, eg Simon's explanations, boy next to him—trouble understanding question 10.

4.55pm: Gave some helpful hints to class for solving some of the more difficult problems.

5.00pm: End of class

SECOND CLASS: 5.10PM - 6.10PM

Introduction: Explanation of algebra needed to complete the algebra worksheet—in detail this time, building up from the basics, since the first class had so much trouble understanding.

Children involved: Involved quite quickly, asked to do some examples by themselves, building up slowly in difficulty. Responded much better than the first class as they were eased into the harder algebra problems.

See tape—discussion between Elena, Nicki and Samantha—Nicki and Samantha had been told to go on with worksheet since they were the only Year 9's and had already been taught the material.

5.25pm: Went through the answers to the examples with the main class.

Elena asked for questions to be made up (see tape):

eg Martin made up question, then Nora solved it, then vice versa, then Nora asked if Martin's answer was right. Like last week, the students responded well and appeared to have fun inventing their own questions (a lot of giggling!).

5.40pm: Left to do some of the questions on algebra worksheet.

See tape for more discussion with Nicki and Samantha.

5.50pm: Explanation of geometry—very similar to explanation in first class, asked example questions throughout, class able to answer these quite well.

6.00pm: Left to finish the geometry problems on the sheet.

See tape—discussion with Karel and other children.

6.15pm: End of class.

THURSDAY 18TH MAY 1995

FIRST CLASS: 4.00PM - 5.00PM

Introduction: Revision of indices, from a prepared overhead.

Children involved? Immediately, Elena asked questions from prepared overhead. Initially only 2 or 3 students answered questions (Norm, etc) and Elena tried to wait for more students to try to answer, but there was little response.

4.10pm: Prepared overhead—revision of Highest Common Factor. Here more students involved at first, Elena wrote down all the 'possible' answers that the students called out then asked for consensus on which one was the answer.

Question about bunches of flowers—meaning unclear though, some students involved in discussion about what the question was trying to ask.

Then asked for someone to make up a similar but harder problem, students just provided some different numbers for the same question.

4.20pm: Gregory, some questions about HCF's.

4.25pm: Administration, results from Canberra.

4.30pm: Explanation of hints for Challenge Problems.

4.35pm: Explanation/revision of algebraic factorisation—like last week, but slower explanation. Went through example but didn't wait for class response.

4.40pm: Left to complete "Mathematics Questions - Set 2".
(See tape: discussions with students - with Norm, and later with Elizabeth).
5.05pm: Class finished.

SECOND CLASS: 5.10PM - 6.10PM

Introduction: Students had already begun the 'Mathematics Questions: Set 2' sheet so left to complete it, allowed 20 minutes.

5.30pm: Revision of indices, from prepared overhead. Asked students to make up their own questions and then answer them. Responses asked for to example questions from overhead, students not very confident about answering (like first class where only a couple answered). However has lesson went on the students were able to at least respond when asked.

5.40pm: HCF from prepared overhead. Bunches of flowers problem (made clear now by changing wording)—asked students to describe/explain what the question meant—Nicki explained it well, then Elena asked Carol to repeat the question in her own words, she could do that well too, all students seemed to understand the question and helped give the answers. Then the students helped to create another question, Carol answered it (using HCF theory). Elena asked how they would have solved the question if they didn't know about HCF—students were stumped for a while, but Nicki managed to explain (really made them think about what HCF meant, rather than just knowing it was a HCF problem and solving it without thinking about the meaning).

6.00pm: Admin.—results from Canberra. Then discussion about hints for *Challenge Problems.*

6.15pm: End of class.

THURSDAY 25TH MAY 1995

FIRST CLASS: 4.00PM - 5.00PM

Introduction: Admin. Then answers to worksheet on algebra, which the students had completed while waiting for the class to start.

Children involved? Students not very involved, answers called out by Elena, little response from students. Students were asked which rules had been used to find the answers for these problems, only one or two students (eg Norm) provided the answers.

4.15pm: Explanation of some algebraic rules (distributive/expansion) to help with algebra at school.

Students asked which was the most difficult problem on the algebra worksheet, and which one/s they would like explained in more detail, but there was no response from students.

4.25pm: Explanation of prepared overhead on HCF—4 ways of finding it. Most students still didn't get involved, only Norm.

4.30pm: Revision (from a prepared overhead) of geometry. (finding angles etc.) Most students just watched, occasionally put hands up when questions were asked, but mainly were very unresponsive.

4.45pm: Students left to do some geometry problems from board. Many of the geometry problems used algebraic concepts which needed clarification and explanation, this was given then a few algebraic exampl[illegible] rranging etc) put on board for students to try.

Throughout this [illegible] lesson the students were quiet and unresponsive, for the first three quarters they mainly ha[illegible] listen, not doing anything on paper etc and they became inattentive.

5.00pm: Class finished.

SECOND CLASS: 5.10PM - 6.10PM

Introduction: Admin. then given 10 minutes to complete the algebra worksheet.

Children involved: Immediately by being asked to complete worksheet.

5.25pm: Solutions to algebra questions. Similarly to first class, asked which rules were used to find the answers. But then (different to first class) the students were asked to invent their own questions, similar to the ones they had completed, and have other students answer them; very successful and one girl (Sheryl) who did not understand at first, caught on after many problems had been invented.

5.45pm: Showed prepared overhead—HCF, 4 ways to find HCF. Instead of simply explaining (as did for first class) Elena asked the class "What do you think I have written here, what do I want to tell you, what's it all about?" etc. ie. asked students to explain what was meant by the four different ways, students showed much more interest and understanding than the first class.

5.55pm: Geometry revision from overhead.

6.10pm: Like first class, explanation of rearranging algebraic expressions on board.

6.20pm: End of class.

THURSDAY 8TH JUNE 1995

FIRST CLASS: 4.00PM - 5.00PM

Introduction: Re handout of AMOC competition & showed a student's well-written solution for Challenge Problem 1.

Children involved? Only listening at this stage.

4.05pm: Norm's problem which he has invented. (Based on sausage problem from last week).

There are 30 Alan Bonds. They have to pay off 80 bills. If they share the bills, what is the least amount of total bills? (If 2 Alan Bonds share one bill, it is counted as 2 bills.).

Many suggestions for solution (see tape: Tom, Brad, Hary).

Students seemed to get very involved with this problem, and not so worried about being wrong/right, because the 'teacher' was only another student.

4.15pm: Prepared overhead of arithmetic sequence. (Chapter 5)

Students involved constantly answering if sequences were arithmetic or not, and what the next terms are. Also creating/making up new sequences for example problems.

Left to do some calculations for a few minutes, eg find 20th term in the sequence.

Good discussion input, especially Norm and Hary (see tape).

Asked students to suggest formulae for summing arithmetic terms—some varying suggestions, eg Hary for 1001, use 'pairing' idea for 1 - 1000 then plus 1001 at end.

Students kept involved by asking specific students to calculate results on calculator etc.

Trying to show some real-life applications of arithmetic sequence, asked for suggestions, examples from students (should really only have happened in 2nd class).

5.00pm: Given problem (arithmetic sequence) to solve, most students looked like they had a serious go at working it out.

5.05pm: Class finished.

SECOND CLASS: 5.10PM - 6.10PM

Introduction: Discussed similar problems to sausage problem (from Problems I Like Sharing, last week's handout).

5.15pm: Shared Norm's problem, and asked 'can we solve it in the same way as the sausage problem, can you see the connection?'

Students given a minute to solve the problem. Then asked to invent similar problems, using the same idea in a different situation. Martin just changed the numbers, Nicki came up with a quite different problem altogether (sort of similar to original problem).

5.35pm: Arithmetic sequence, from prepared overhead.

Students asked what kind of questions could be asked about arithmetic sequences—Carol: Find the 21st number; Martin: What number term is the number 120.

5.50pm: Moved on to summing arithmetic sequences, students asked 'What is the problem about?'

Then shown some application-type problems of arithmetic sequences, asked to guess what [illegible] problems are about, etc.

6.20pm: End of class.

THURSDAY 22ND JUNE 1995

FIRST CLASS: 4.00PM - 5.00PM

Introduction: Explanation of special "operations" from handout.

Children involved? (Observer not present until 4.15—chasing a working microphone/recorder).

4.15pm: Students asked "Can you make an example like that?" (this type of question should be restricted to the second class).

4.20pm: "Word problem" about marbles—students asked for suggestions for solution then left to write their solution in a pair on an overhead.

Tom's idea:	S	B	R	E
			0	3
		0	2	5
	0	1	3	6

$1 + 3 + 6 = 10$, $58 - 10 = 48$, ${}^{48}/_{4} = 12$ = Sally

Rene's idea: $x + 1 + 2 + 3 = 58$, and solve. Hary's idea, similar to Rene's.

4.30pm: Hary explained his solution to the class, then Dany's explanation, followed by Blair & Robert's solution—used same method as Dany & Tom but stopped at point when Sally had 1, not when Sally had 0 (as Dany & Tom did). (All three used same ideas but had slightly different methods). Maria's explanation—more of a trial and error method—said ${}^{58}/_{4}=14.5$, guessed 13 = Beth, then checked if it worked; then Lena's explanation (similar to Mary's but more precise). All students (numbers much lower than usual today) were very involved, especially enthusiastic about writing their solutions for the overhead projector.

4.45pm: Elena's explanation of the solution.

4.50pm: Given 10 minutes to complete worksheet called "Problems I Like Sharing" (see tape for conversation between Hary and friend).

5.05pm: Went through solutions to worksheet.

5.15pm: Class finished.

SECOND CLASS: 5.10PM - 6.10PM

Introduction: "Hints"/explanations for questions on worksheet (special operations etc).

Children involved: Students asked for their own examples etc, so involved almost immediately. When discussing 'super-products' got examples from students, eg asked Eddie for a definition of the "Eddie-operation", etc.

5.35pm: Problem about marbles—just put up overhead of diagram of problem and asked students to suggest what the problem might be about. 3 different students offered quite reasonable suggestions:

Eddie: Thought it might be an arithmetic sequence, but understood why not when Elena explained it wasn't.

Nora: Had right idea, guessed it was to do with ages (ie one person 2 years older, one 3 older, etc) rather than marbles.

Samantha: At first she thought she didn't have enough information to solve the problem, but thought that she could try by using x and finding an equation. But couldn't say what the equation would be like.

Then Elena solved the problem on the board algebraically (didn't get students to solve it themselves on overhead as had happened in the first class).

5.45pm: Left to complete worksheet of "Problems I Like Sharing" (slightly different version to first class's handout, included things like "make up a similar problem" etc.). Various students asked to do particular questions for the overhead. See tape for discussion between Nicki & Samantha, who were asked to invent a problem similar to one on the worksheet.

6.10pm: Went through problems created by students— Carol & Nora's, Samanta & Nicki's (about dying dogs different colours), and Karel's. (see overhead copies of them).

6.20pm: End of class.

THURSDAY 29TH JUNE 1995

FIRST CLASS: 4.00PM - 5.00PM

Introduction: Admin. re yesterday's certificate presentation, and Westpac maths competition.

4.05pm: Introduction to Chp 11, The Pigeonhole Principle.

Children involved? First involved when left to read Chp 11 in their texts.

Then asking questions re "pigeons in pigeon holes" from overhead, straightforward questions to students to lead to understanding, 'medium' response from students.

*Then asked students if they could make up their own pigeon hole principle problems!—(such questions should be in second class!)—surprisingly several students had good ideas.

Then continued explanations on overhead, gradually covering more complicated ideas.

Used lots of 'realistic' examples but (mostly) without asking for students to create their own problems.

4.40pm: Went through question (similar to Challenge Problem 13) about students and 'pigeon holes'; discussion about solution with students, good involvement.

4.50pm: Left to do some pigeonhole principle questions from the worksheet in pairs. (See extra tape of student discussion for discussion between Hary and Brad; see class tape for discussions with Brad, Robertino, Ben, Ben and Sarah).

5.00pm: Went through worksheet answers, asking students for answers; Blair showed his solution for Question 6 on overhead.

5.10pm: Class finished.

SECOND CLASS: 5.10PM - 6.10PM

Introduction: Admin re certificate presentations.

Children involved: Immediately after this by being asked what they thought the pigeonhole principle questions/question on overhead were all about.

5.20pm: Discussion led about pigeonhole principle. Constant questioning, eg 'What do you think this might mean' etc, and gradually introducing more complicated examples.

5.40pm: Left to work through pigeonhole principle worksheet with a partner, worksheet slightly different to Class 1, included questions like 'Make a problem similar ...'.

(See extra tape of student discussion—Nicki and Samantha).

6.10pm: Showed Brad's solution to Question 6 and asked students to describe the various steps involved.

6.15pm: Class finished.

THURSDAY 27TH JULY 1995

FIRST CLASS: 4.00PM - 5.00PM

Children involved? Straight away—given 10 minutes to complete the worksheet (with 6 m/c questions), told the lesson would concern "strategies for solving multiple choice questions" (in preparation for Westpac Maths Competition 1/8/95).

4.15pm: Began going through solutions to above questions—asking students for various strategies for solving the questions. Emphasised that for multiple choice format not always necessary to solve problem completely, that this is often too time consuming in a competition situation such as Westpac.

See tape for strategies from:

Qn 2: Valery. Norm, Hary.
Qn 3: Emily, David, Norm.
Qn 4: Hary, Brad, Norm

5.00pm: Class finished.

SECOND CLASS: 5.10PM - 6.10PM

Children involved: Same as first class, given 10 minutes to complete worksheet, etc.

5.20pm: Similar discussion to first class re strategies for questions, but sometimes also asked students to create a similar problem.

See tape for strategies:

Qn 1: Nicki.
Qn 2: Tom
Qn 3: Nora
Qn 4: Samantha, Nicki
Qn 6: Nora

6.05pm: Looked at previous Westpac paper and explained format (eg mark off for incorrect but no penalty for no response).

6.10pm: End of class.

THURSDAY 3RD AUGUST 1995

FIRST CLASS: 4.00PM - 5.00PM

Introduction: Discussion--Tuesday's Westpac maths competition (sharing Easter eggs question)

4.05pm: Introduction to Chapter 10: Counting techniques. Students asked to read over the worksheet for a few minutes.

4.10pm: Further explanation and discussion about questions similar to the those on the worksheet (see tape). Many questions asked of the students along the way; building up to harder examples from easy ones.

4.45pm: Left to work on the worksheet, individually at first and then to compare answers in pairs.

4.55pm: Went over solution to Problem 7.

5.05pm: Class finished.

SECOND CLASS: 5.10PM - 6.10PM

Introduction: Re Chapter 10 Counting techniques: but different to first class— Elena not 'lecturing' but asking "What am I going to ask" (from what the students could guess from the handout) etc as introduction to topic.

Showed similar examples as in first class, increasing in difficulty, but often asked students to make up a similar problem; "Can you tell me a problem which I can solve in the same way," or given some information, "What could the question be?"

5.40pm: Left to finish worksheet, individually, but to compare answers in pairs.

5.50pm: Went through answers to worksheet (NB different copy of worksheet given to each class). For question 4 asked almost everybody for the question they had made up.

6.00pm: Went through question 7 together, asking for suggestions for making up questions for parts (b) to (e).

6.10pm: End of class.

THURSDAY 10TH AUGUST 1995

FIRST CLASS: 4.00PM - 5.00PM

Introduction: Solutions outlined to Challenge Problems 1 and 2. Showed a sample solution (one of the students') to Problem 1.

Children involved? Brighter students (Brad, Hary) answered questions about the solution but other students were hardly involved.

4.15pm: Then left to complete worksheet (counting type questions from Junior Westpac Comp. paper)

See tape for conversation between Hary and Brad.

4.35pm: Went over solutions to worksheet. Asked various students to explain their solutions, almost all students involved. See tape for explanations from following students:

Qn. 1: Robert, Sarah.

Qn. 2: Siobhan, Robert, Vivien, Rene, David.

Qn. 3: Mary.

Qn. 4: David.

Qn. 5: Valery.

5.10pm: Class finished.

SECOND CLASS: 5.10PM - 6.10PM

Introduction: Discussion with Nicki and Samantha about teachers!

5.20pm: Solution to Challenge Problem 1.

5.25pm: Solution to Challenge Problem 2.

In both instances, very little interaction from students.

5.30pm: Given 15 minutes to work through worksheet (counting type problems from Westpac).

See tape for discussions between Nicki & Samantha, and Martin & Eddie.

5.50pm: Went through solutions to worksheet, asking for strategies used to find solutions.

See tape for explanations from following students:

Qn. 1: Ingrid, Martin Nicki, Tom, Nora.

Qn. 2: Martin.

Qn. 3: Ingrid, Karel.

Qn. 4: Tom, Martin.

6.20pm: Went through solutions for questions 5 and 6 quickly (out of time).

6.25pm: End of class.

THURSDAY 17TH AUGUST 1995

FIRST CLASS: 4.00PM - 5.00PM

Introduction: Discussion about classes for next year.

4.05pm: Further explanation of Challenge Problem 1 (to explain about Hary's misunderstanding from last week). Asked other students to help in explanation, ie to explain it to Hary. Asked what was the most important component to remember for solving this type of problem.

4.15pm: Review of solution to Challenge Problem 2. Then showed similar problems and asked if they could still be solved using the same method.

4.25pm: Solution to Challenge Problem 3 (given for the first time). Elena asked students for their varying ideas on how to solve it. (See tape: Mary, Hary, Cathy.)

4.35pm: Solution to Challenge Problem 4.

4.45pm: Left to work on worksheet "Problems I Like Sharing" in pairs—several pairs asked to write their solutions on overheads for particular problems. Elena walked around helping several students (see tape).

5.00pm: Went through solutions to these problems: used overheads created by students to outline solutions, only a little discussion with students (short of time).

5.10pm: Class finished.

SECOND CLASS: 5.10PM - 6.10PM

Introduction: Discussion re classes for next year.

5.20pm: Review of solution to Challenge Problem 2. Said interested in finding out how much students remember about a problem after a week or two. Also asked what was the most important thing to remember after solving it (several suggestions, see tape).

Then asked students to create similar problems, and asked if they could be solved using the same method of solution.

5.30pm: Review of Challenge Problem 1. Pointed out annotations on solution and asked students to guess what they meant, ie what did they have to do with the solution (rather than simply explaining it, as in first class).

5.35pm: Solution to Challenge Problem 3. Asked students what was the most difficult part, or where it would be easy to make a mistake.

Then asked for similar situations, ie problems using a similar idea and method of solution.

5.45pm: Solution to Challenge Problem 4—outline but no discussion with students (short of time).

5.50pm: Left to work in pairs on "Problems I Like Sharing" worksheet; overhead sheets left with pairs to write their solution for particular questions.

6.05pm: Went through solutions to above questions. Asked students to quickly explain what they had written on their overheads. For Question 1 asked what sort of similar problems could be made.

6.15pm: Class finished.

THURSDAY 24TH AUGUST 1995

FIRST CLASS: 4.00PM - 5.00PM

Introduction: Introduction to combinations and permutations.

Children involved? Students asked questions immediately, from prepared overhead of questions and solutions.

Explanation about combining two different things, with questions along the way, gradually increasing in difficulty and asking questions similar to examples shown, eventually to algebraic representation (not emphasised).

4.10pm: Same as above but for combining three different things.

4.15pm: More examples on all types so far.

Then reminder about 'factorial' by showing a familiar problem.

4.30pm: Questions on overhead, students asked to solve them straight away with a partner. (see tape for conversations with individual students.)

4.35pm: Solutions to these questions, asking students for answers and explanations.

4.40pm: Went through two more complicated problems, similar to the ones just completed.

4.45pm: Left to solve questions on 'Counting Techniques' handout, asked to do questions 1, 3, 5 (one of each section). Some students asked to write their solutions on an overhead.

5.00pm: Solutions discussed from students' overheads.

Students given examples of similar problems and asked if they would be solved in the same way.

5.10pm: Class finished.

SECOND CLASS: 5.10PM - 6.10PM

Introduction: Began with same examples as first lesson, but instead asked students 'Can you tell me what this problem might be about?' and also asked students to create new situations/problems with the same numbers.

Then went through other overhead questions, as for first class, but constantly asking students to invent new, similar questions and answer them.

6.00pm: Left to do questions 1, 3, 5 off 'Counting Techniques' handout. (see tape for conversations with Nora, Tom, Martin.)

6.10pm: Brief explanation of 'base,' ie base 10, etc, needed for some questions on handout.

6.15pm: Went through solutions to handout questions. For each, asking 'what if' different parts of the question were changed.

6.25pm: Class finished.

THURSDAY 31ST AUGUST 1995

FIRST CLASS: 4.00PM - 5.00PM

Introduction: Discussion re next year's classes.

4.05pm: Went through solutions to challenge problem re coins, ie choosing coins (combinations). Asked students questions along the way, then asked about the idea of the problem and what was the most difficult part.

4.15pm: Review of counting techniques from a prepared overhead. Discussed multiplication principle, addition principle and restriction principle, asking students revision type questions along the way.

4.25pm: Discussion re challenge problem with rows and columns, (20 in 3rd row, 41 in 5th row, etc). Asked students to help solve, careful to encourage precise reasoning.

4.40pm: Showed problem about numbers arranged in rows in a triangle. Went through some questions, asking students to help, then left students to work in pairs on handout which had similar questions about the triangle of numbers.

Asked some students to write their solutions on overhead slides.

4.55pm: Quickly went over answers, students explaining their solutions from their overheads.

5.00pm: Class finished.

SECOND CLASS: 5.10PM - 6.10PM

Introduction: Discussion re coins as for first class, but showed students the top part of the problem (list of coins) and asked them to make up a suitable question.

Martin actually 'invented' the question that had been discussed in the first class and then this was solved through discussion with the students.

5.20pm: Review of Counting Techniques, as for first class (from overhead) but in more depth as Nicki and Irene have been absent.

Also asked students for examples of questions for each principle, then asked students to solve them.

5.30pm: Challenge problem re rows and columns, as for first class. After solving asked students to make a problem similar.

5.45pm: Discussion re problem with numbers in triangle—showed students triangle and asked them to make up questions about it (similar to those on worksheet).

5.50pm: Students left to work on handout, and asked to write some more questions about the triangle. Some students asked to write their solutions on overhead slides.

6.05pm: Discussed Tournament of the Towns Competition (Curtin Uni).

6.10pm: Solutions to handout, students read off their overheads.

6.25pm: End of class.

THURSDAY 7TH SEPTEMBER 1995

FIRST CLASS: 4.00PM - 5.00PM

Introduction: Students given Westpac Intermediate level paper and answers for Westpac Junior level paper.

Then looked at some solutions on overheads that students had written last week, to check who had written them, and also asked students about the solutions: could they remember or guess the question?

4.10pm: Revision of problem with rows and columns from last week—went through solution again, step by step, using precise reasoning.

4.15pm: Revision of question from first term, what is the last digit of 3 to the power 15, etc.

4.20pm: Asked students to take notes re geometry.

Introduced/revised tangents, diameter, radius, and showed some proofs of 'important theorems in geometry'—no student involvement, just 'lecturing.' Also mentioned Question 24 from Intermediate Westpac paper, (will use some of these theorems to solve).

4.45pm: Students left to solve 2 questions from handout, in pairs.

See tape for some conversations with students.

5.00pm: Went over solutions, students explaining their answers.

5.10pm: Class finished.

SECOND CLASS: 5.10PM - 6.10PM

Introduction: Re Westpac papers/answers as for first class.

5.20pm: Showed solutions from last week that were written by students, as for first class, asked if they could describe the idea of the solutions, and guess the questions.

5.30pm: Reminded re finding last digit of 3 to the power 15, etc. Then asked students to make up and solve similar questions.

5.35pm: Showed students last overhead used in first class geometry 'lecture' and asked students to guess what it could be about. After some discussion re tangents etc went back to first overhead to give definitions and theorems, as for first class, but a little more student involvement.

5.55pm: Students left to work on handout in pairs.

6.05pm: Discussion re bases (since question 2 used terminology of 'base 10'), then solutions for handout problems, given by students.

6.30pm: Class finished.

THURSDAY 14TH SEPTEMBER 1995

FIRST CLASS: 4.00PM - 5.00PM

Introduction: Re Tournament of the Towns Competition (Curtin Uni)

4.10pm: Overview/discussion re 'famous and useful theories in Geometry', worksheet for today and Challenge Problems.

4.15pm: Revision of geometry from last week, students asked some questions.

4.30pm: Gave students geometry problem to do, similar to one they had just been shown—but then went through solution on the board, students not given time to do it themselves.

4.40pm: Another similar geometry problem, students asked to say which 'segments' were equal. Then went over Westpac Question 24 (geometry), to see what more they knew about it now.

4.55pm: Went over one more similar geometry problem, students asked some questions along the way.

5.00pm: Class finished.

SECOND CLASS: 5.10PM - 6.10PM

Introduction: Re Tournament of the Towns.

5.15pm: Revision of geometry from last week—but for this class put up diagram and asked them to explain what they knew, and what sort of questions they could ask about the diagram, etc.

5.35pm: Went through specific problems (some revision), then asked students to describe the main idea of the solution, how they would remember the solution, etc.

5.45pm: Discussed Westpac question 24, how they would solve it now that they knew more geometric theorems.

5.55pm: Asked to choose partners and solve a problem similar to one they have seen, but to show the solution in 2 different ways (some solutions to be written on overhead slides).

6.05pm: Solutions to problem, Elena explained students' solutions for them. But then asked them to explain the main idea of the solution.

6.15pm: End of class.

THURSDAY 21ST SEPTEMBER 1995

FIRST CLASS: 4.00PM - 5.00PM

Introduction: Discussion re hints for Challenge problems and when challenge problems must be submitted by.

4.05pm: Went over hint for Challenge Problem number 10, but didn't ask students many questions. Also showed overhead with similar question to 10 and its solution.

4.20pm: Hint for Challenge Problem 12, asked a student to read out the question, then went through hints on whiteboard, students asked some questions throughout.

4.40pm: Geometry: revision of last few weeks' work by discussion about 3 diagrams on overhead.

4.45pm: Students left to do one question, review of last week's work.

4.55pm: Went through 3rd geometry problem, students asked questions throughout.

5.10pm: Class finished.

SECOND CLASS: 5.10PM - 6.10PM

Introduction: All students given an overhead slide to write down a problem 'similar to their favourite problem' for next week.

5.20pm: Re Challenge problems and hints, etc.

Firstly discussed Challenge Problem 10, students asked many questions throughout.

5.35pm: Similar problem to problem 10 put on overhead, students asked to guess what the problem is about, ie to state the question from the diagram. Students suggested several different ideas. (see tape).

Then students asked to write down their solutions to the 'real' question.

Explanation re solution given, similar to first class but students more involved in helping with answers etc.

5.55pm: Briefly described hints for problems 14, 15, etc.

6.00pm: Review of geometry, showed overhead as for first class. Students given a problem to solve, and left to work on it for 5 minutes, then went through solution.

6.15pm: Asked students to create similar questions.

6.25pm: Class finished.

THURSDAY 28TH SEPTEMBER 1995

FIRST CLASS: 4.00PM - 5.00PM

Introduction: Admin. re competition forms, etc.

4.05pm: Review of geometry—summary of recent lessons, 'lecturing' from whiteboard.

4.15pm: Looked at handout—did problem 1 all together on whiteboard. Elena asked students questions throughout.

4.25pm: Looked at question 3 on the handout, went over on board, then students left to write answer to 3(b).

4.35pm: Went through solution briefly, students asked to help with answers.

Then looked at question 4, students asked to help with solution.

5.00pm: Briefly looked at question 5, students asked questions again.

5.05pm: Admin. re classes next year.

5.10pm: Class finished.

SECOND CLASS: 5.10PM - 6.10PM

Introduction: Began with problems students have made (a problem similar to their favourite problem).

First, Nicki's problem, asked Nicole to explain and other students to suggest how to solve it.

Then asked students to make a similar problem.

5.25pm: Karel's problem: didn't solve, just asked Karel to explain it briefly.

5.30pm: Discussion re problems 1 to 3 on handout, but Elena asked students to guess what she would write next rather than just 'lecturing.'

When problem solved, asked students to describe the main idea of the problem.

5.40pm: Students given time to solve a problem similar to 3(b) (after making this problem themselves).

5.50pm: Went through solution on board, asking students to help.

5.55pm: Question 4, asked students to explain what it was about, then to help solve it.

6.10pm: Asked to make up a problem similar to question 4.

6.15pm: Admin. re next year's classes.

6.25pm: Class finished.

THURSDAY 21ST SEPTEMBER 1995

FIRST CLASS: 4.00PM - 5.00PM

4.00pm: Students left to work on 'test'—Set 1 and 2.

4.40pm: Students asked to look at other page of problems, and hand in Set 1 and 2.

4.45pm: Admin. re classes continuing, presentation etc.

Then briefly went over answers to some of the Set 1 and 2 questions (in depth explanation of Qn 7.)

5.00pm: Discussed solutions to 3rd page of problems.

5.10pm: Class finished.

SECOND CLASS: 5.10PM - 6.10PM

5.15pm: Admin. re classes continuing, etc. Left to do 'test'—Set 1 and 2.

5.55pm: Collected test; discussion re next week's classes and next year's classes.

6.00pm: Discussion about third page of questions, then discussion of Qn 7 from the test.

6.15pm: Class finished.

THURSDAY 2ND NOVEMBER 1995

COMBINED CLASS: 4.00PM - 5.00PM

Introduction: 'Game' to introduce some algebra (ie pick a number, double it, add 4, etc, and get back to original number).

Children involved? Children immediately involved as all had to join in game.
Karel invented his own similar, but more complicated problem and told class. (Although it turned out to be incorrect).

4.10pm: Discussion re percentages—if 15% of a group of people have black hair, and 20% of the remaining people have blonde hair—what questions can be asked about this? Many different types of questions suggested.

4.20pm: Eventually students were steered towards problem similar to '15% discount followed by 20% discount' and left to solve a problem like this.

4.25pm: Went through solution to above problem.

4.40pm: From prepared overheads—'Principle of Inclusion-Exclusion'—ie Venn diagram type problems (using students studying French and German). Students left to work on one, using whatever method they knew or could guess.

4.45pm: Solutions to above—some used Venn diagram, others just words.

4.55pm: Showed formula for working out above problem, then discussed adapting it for three languages instead of two.

5.10pm: Solution to above.

5.15pm: Class finished.

THURSDAY 9TH NOVEMBER 1995

COMBINED CLASS: 4.00PM - 5.00PM

Introduction: Admin re classes for next year.

4.05pm: Revision question—Venn diagram type problem, similar to last week. Showed question on overhead and asked students several questions along the way.

4.15pm: Used similar problem but used context of staircase challenge problem, asked students, "What kind of question do you expect from me?"

Then rephrased question to be about numbers divisible by 2, 3, etc, instead of steps (same problem, different context).

Students left to solve whichever problem they chose.

Then went through solutions (Sarah, Samantha)

4.35pm: Left to solve a problem of their choice off worksheet.

4.50pm: Solutions to problems on worksheet—Question 1, Sarah, Question 2, Brad.

5.00pm: Explanation re simultaneous equations (for Year 8's who haven't yet learnt this at school).

5.10pm: Class finished.

THURSDAY 16TH NOVEMBER 1995

COMBINED CLASS: 4.00PM - 5.00PM

Introduction: Discussion re problems from *Cross-Section* magazine.

4.05pm: Discussion about equations—wrote notes on overhead and encouraged students to make their own notes.

Began by showing a list of different types of equations and asking students for their solutions, then discussed general ways to solve linear equations.

4.25pm: Solved some more complicated linear equations. by asking different students for each step. Then asked Martin to invent a similar one, and given some time to solve it.

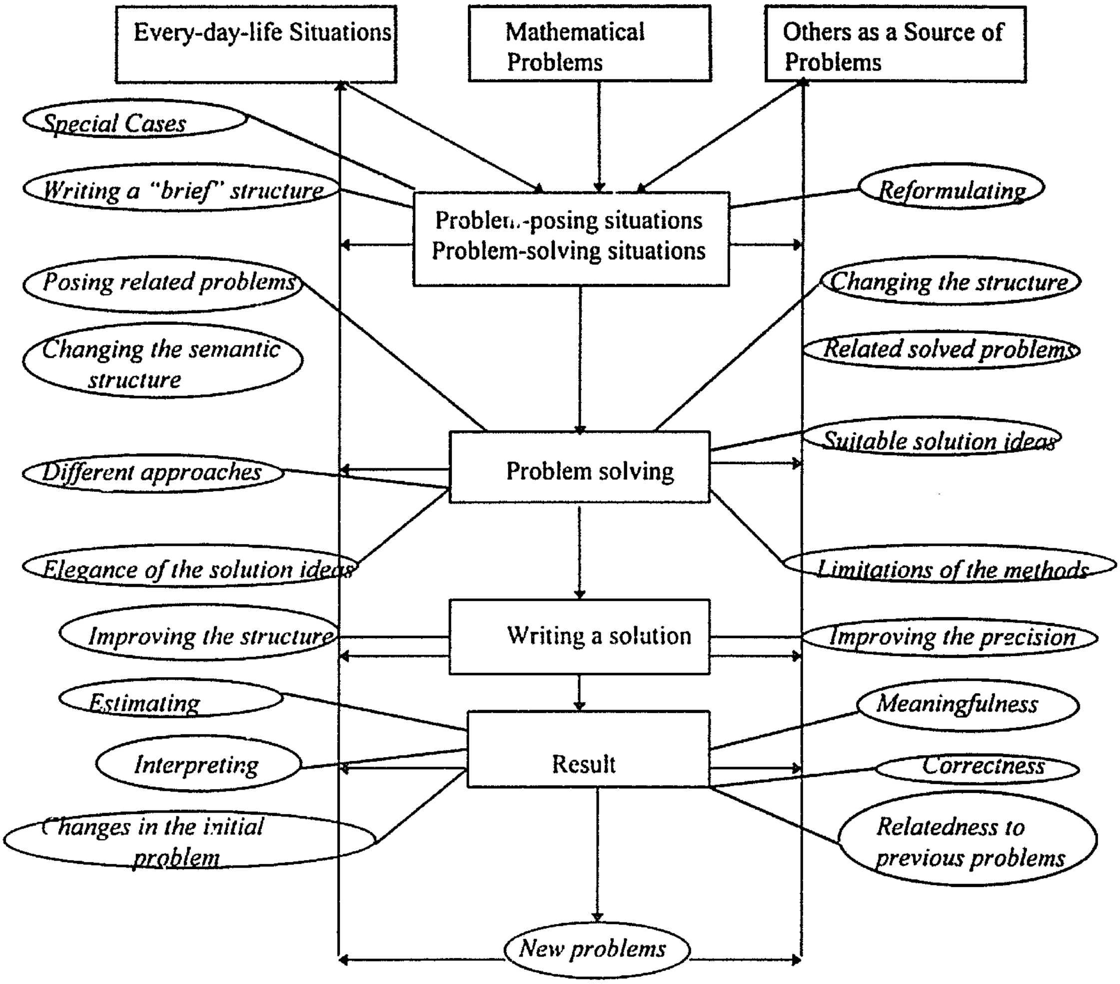
4.35pm: Went through solution to Martin's problem. Then talked about equations like x+y=3, and graphed it.

4.45pm: Looked at simultaneous equations, notes on overhead Students left to solve one by themselves.

4.55pm: Went through solution to above. Then discussed question 15 from the handout (arranging numbers so sums are 15).

5.05pm: Class finished.

Appendix 9. **The Refined Model Describing the Possible Models of Interactions Between Problem Posing and Problem Solving.**



Appendix 10. Classification of Problem-posing Situations.

Problem-posing categories: **Problem-posing situations:**

Free

Posing problems which were found to be interesting;
Posing problems about a particular topic;
Posing problems for a mathematics competition;
Posing problems on every-day-life contexts;
Posing problems from data;
Posing problems with given answers;
Posing problems written to be solved by the teacher;
Posing problems which were found to be difficult;
Posing problems which involved a use of a specific mathematical concept(s);
Posing problems which involved a use of a specific mathematical method;
Situations based on posing problems which involved an use of a specifics solution method, etc.

Semi-structured

Problem posing situations based on a specific problem structure:

Problem posing based on a problem structure with an unstated *Goal*;
Problem posing based on a problem structure with missing elements in a combination of the*Given*, the *Obstacles* and the *Goal*;
Problem posing based on a problem structure with surplus information:
- Situations with surplus information in the *Given*,
- Situations with surplus information in the *Obstacles*.
- Situations with surplus information in a combination of the *Given* and *the Obstacles*;

Posing problems on the basis of different interpretations of a mathematical concept;
Posing problems which have more than one solution, etc.

Problem posing situations based on a specific solution structure:

Problem posing which involves the use of a specific mathematical method within a given problem structure, etc.

Structured

Problem-posing situations based on a specific problem:

Posing problems by varying the mathematical vocabulary of a problem;
Problem posing by presenting a specific problem in students' own words without changing the nature of the problem;
Posing problems by varying the semantic structure of a problem;
Posing multiple goal statements on the basis of a well-structured problem;
Posing problem chains-problem series, problem fields and problem cycles;
Posing problems which are variations of a given problem;
Presenting a problem statement "briefly", etc.

Problem-posing situations based on a specific solution:

Formulating the main solution idea;
Restating a problem on the basis of its solution;
Posing problems with unrealistic solutions;
Problem posing established on the basis of a problem with several solution approaches;
Posing sets of problems which might have a common solution approach;
Posing sets of problems which resemble a given problem but have different solution approaches, etc.