Vol. 29 2003 No. 4

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A CONTROL DESIGN FOR LINEAR TIME-DELAY SYSTEMS

In this paper, we propose a control design for single-input linear systems with a known delay in the state. A systematic construction of the controller gain is given such that the system is asymptotically stable in the sense of Krasovskii. An academic example is used to show the performance of the controller via simulation.

1. INTRODUCTION

In this paper, we consider the problem of control design of single-input systems with known delay in the state of the following form

$$\dot{x}(t) = Ax(t-\tau) + Bu(t); \quad t \geq 0 \tag{1}$$

where $x \in R^n$, $u \in R$ and

$$A = \begin{pmatrix} 0 & 1 & 0 & \dots & 0 \\ 0 & 0 & 1 & \ddots & \vdots \\ \vdots & \vdots & \ddots & \ddots & 0 \\ 0 & 0 & \dots & 0 & 1 \\ a_1 & a_2 & \dots & \dots & a_n \end{pmatrix}, \quad B = \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ 1 \end{pmatrix}.$$

The time delay τ is assumed to be known and the initial condition $x(t) = g(t)$, $-\tau \leq t \leq 0$ where $g(t)$ is a continuous function on the interval $[-\tau, 0]$. It should be first noticed that any single-input system described by

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$$\dot{z}(t) = Fz(t-\tau) + Gu(t) \tag{2}$$

where the pair (F, G) is controllable can be transformed into system (1) by a change of variables (see, e.g., [13]). Consequently, the control design for the class of systems described by (1) is not too restrictive.

We are primarily concerned with the stabilisation of the state vector $x(t) \in R^n$ to the origin by using a control law of the form

$$u(x(t)) = Lx(t-\tau). \tag{3}$$

Under such a control law, the closed-loop system is given by

$$\dot{x}(t) = (A + BL)\, x(t-\tau). \tag{4}$$

The main difficulty in the controller design (3) lies in the choice of gain L such that the system (4) is asymptotically stable in some sense.

By taking the Laplace transform of system (4), the characteristic equation is given by

$$p(s) = \det[sI - (A+BL)e^{-s\tau}]. \tag{5}$$

The characteristic polynomial (5) is said to be asymptotically stable if all of the zeros of $p(s) = 0$ are situated in the left half of the complex plane. The presence of the exponential term $e^{-s\tau}$ means that there are an infinite number of zeros, and therefore it is difficult to determine explicitly all the roots of the equation. In effect, in proper terms, the characteristic polynomial (5) is called a quasi-polynomial. It is because of the above difficulty of roots location that many approaches to control design for time delay systems have been adopted such as spectral decomposition theory, finite spectrum assignment technique and delay independent approaches (see references herein). As a result, several less restrictive definitions of asymptotic stability have been derived in the literature such as γ-stability (see, e.g., [9]) and asymptotic stability in the sense of Krasovskii. For control and observer design purposes the asymptotic stability in the sense of Krasovskii is most commonly used and is defined as follows (see [11], [12]):

DEFINITION 1

The characteristic equation (5) is said to be asymptotically stable in the sense of Krasovskii if all the solutions of $p(s) = 0$ are situated in the semi-plane $\{s\text{: } Re(s) \leq -\alpha;\ \alpha > 0\}$.

The shaded region of Fig. 1 depicts the Krasovskii stability region. The constant α is referred to as the stability margin of the characteristic equation. It is shown in (see [11], [12]) that a sufficient condition for asymptotic stability of (4) is that its associated characteristic equation is stable in the sense of Krasovskii. In a recent paper [1], a new definition of asymptotic stability called the (α, r)-stability was introduced and is defined as follows:

DEFINITION 2

The characteristic equation (5) is said to be (α, r)-stable if all the solutions of $p(s) = 0$ are situated in the semi-plane $\{s: Re(s) \leq -\alpha;\ \alpha > 0\} \cap \{s: |s| \geq r > 0\}$.

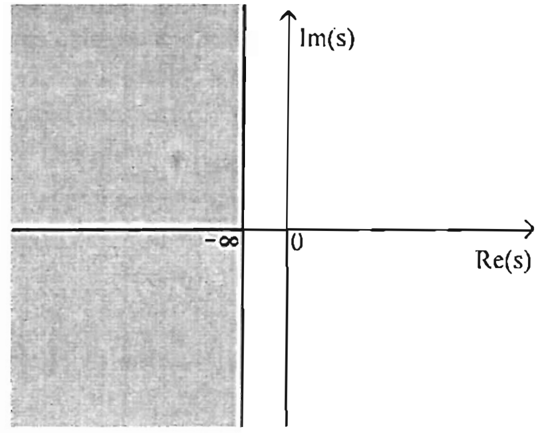

Fig. 1. Krasovskii stability region

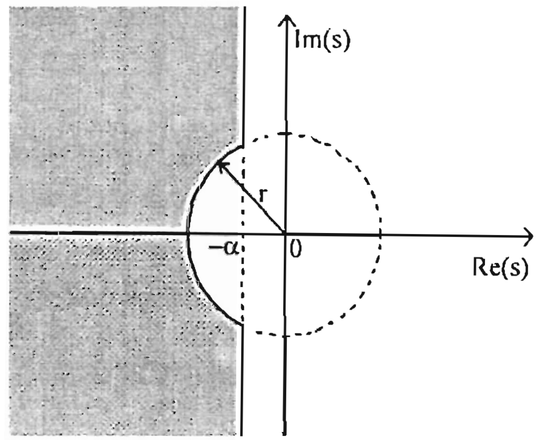

Fig. 2. (α, r)-stability region

The shaded region of Fig. 2 depicts the (α, r)-stability region. We shall say that a linear system is (α, r)-stable if its corresponding characteristic equation is stable in the sense of Definition 2. It can easily be seen that, in general, (α, r)-stability is stronger than Krasovskii stability. More precisely, if a system is (α, r)-stable, then it is stable in the Krasovskii sense for similar values of α; since the (α, r)-stability region is included in the Krasovskii stability region.

In this paper, we shall give a constructive and systematic design of a gain L for the controller (3) such that the characteristic equation of its closed-loop dynamics is asymptotically stable in the sense of Definition 2. This will automatically guarantee that the closed-loop is stable in the Krasovskii sense and which, in turn, will guarantee that the system is asymptotically stable. An academic example is treated to show the procedure of the controller design. Some simulations are also carried out to show the performance of the controller. Finally, some conclusions are drawn.

2. MAIN RESULT

Consider the control law given by equation (3) and define L as follows:

$$L(\theta, \alpha) = -L_0 - L_1(\theta, \alpha)$$

where

$$L_0 = [a_1 \ \ a_2 \ \ \dots \ \ a_n]; \tag{6}$$

$$L_1(\theta, \alpha) = [\theta^n e^{-n\alpha\tau} C_n^n \ \ \dots \ \ \theta^2 e^{-2\alpha\tau} C_n^2 \ \ \theta e^{-\alpha\tau} C_n^1],$$

with $\theta, \alpha > 0$ and $C_n^p = \dfrac{n!}{p!(n-p)!}$. Note that L_0 is the last row of A.

More specifically, consider the control law

$$u(x(t)) = -L_0 x(t-\tau) - L_1(\theta, \alpha)\, x(t-\tau). \tag{7}$$

We can now state the following:

THEOREM 1

Let $r_0 > 0$ be an arbitrary positive constant. Then, for all $\theta \in [0, r_0]$ the origin of the closed-loop system

$$\dot{x}(t) = Ax(t-\tau) + Bu(x(t)) \tag{8}$$

where $u(x)$ is described by (7), is (α, r_0)-stable.

Proof: First notice that the matrix A can be decomposed as $A = A_0 + BL_0$ where

$$A_0 = \begin{pmatrix} 0 & 1 & 0 & \dots & 0 \\ 0 & 0 & 1 & \ddots & \vdots \\ \vdots & \vdots & \ddots & \ddots & 0 \\ \vdots & 0 & \dots & 0 & 1 \\ 0 & 0 & \dots & \dots & 0 \end{pmatrix}.$$

By applying the control law (7) to system (1), we obtain the following closed-loop system

$$\dot{x}(t) = Ax(t-\tau) - BL_0 x(t-\tau) - BL_1 x(t-\tau).$$

Since $A = A_0 + BL_0$, we have

$$\dot{x}(t) = A_0 x(t-\tau) + BL_0(t-\tau) - BL_0 x(t-\tau) - BL_1 x(t-\tau) = (A_0 - BL_1)x(t-\tau). \quad (9)$$

By taking the Laplace transform of (9) (with zero initial conditions), we obtain the following characteristic equation

$$p(s) = \det(sI - (A_0 - BL_1)e^{-\tau s}) = \det M(s)$$

where

$$M(s) = sI - (A_0 - BL_1)e^{-\tau s}.$$

Now, define the diagonal matrix Δ_s as

$$\Delta_s = \begin{bmatrix} e^{n\tau s} & 0 & \dots & 0 \\ 0 & e^{(n-1)\tau s} & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ 0 & \dots & 0 & e^{\tau s} \end{bmatrix}.$$

Then,

$$\begin{aligned} \Delta_s M(s)\,\Delta_s^{-1} &= \Delta_s[sI - (A_0 - BL_1)\,e^{-\tau s}]\,\Delta_s^{-1} \\ &= s\Delta_s I \Delta_s^{-1} - \Delta_s(A_0 - BL_1)\,e^{-\tau s}\Delta_s^{-1} \\ &= sI - (\Delta_s A_0 \Delta_s^{-1} - \Delta_s BL_1 \Delta_s^{-1})\,e^{-\tau s}. \end{aligned}$$

It can be checked that

$$\Delta_s A_0 \Delta_s^{-1} = e^{\tau s} A_0 \quad \text{and} \quad \Delta_s B = Be^{\tau s}$$

so that

$$\Delta_s M(s)\Delta_s^{-1} = sI - (A_0 - BL_1\Delta_s^{-1}).$$

On the other hand,

$$\begin{aligned} p(s) &= \det M(s) \\ &= \det(\Delta_s)\det M(s)\det(\Delta_s^{-1}) \\ &= \det(\Delta_s M(s)\Delta_s^{-1}) \\ &= \det[sI - (A_0 - BL_1\Delta_s^{-1})]. \end{aligned}$$

Now, due to the special structure of the matrix $A_0 - BL_1\Delta_s^{-1}$, it can be verified that

$$p(s) = s^n + C_n^1 s^{n-1}[\theta e^{-(s+\alpha)\tau}] + C_n^2 s^{n-2}[\theta^2 e^{-2(s+\alpha)\tau}] + \ldots + [\theta^n e^{-n(s+\alpha)\tau}]$$
$$= (s + \theta e^{-(s+\alpha)\tau})^n . \tag{10}$$

Hence, the solutions of the equation

$$p(s) = (s + \theta e^{-(s+\alpha)\tau})^n = 0 \tag{11}$$

satisfy the following equality

$$s = -\theta e^{-(s+\alpha)\tau} = -\theta e^{-\alpha\tau} e^{-s\tau} \tag{12}$$

or equivalently

$$s e^{s\tau} = -\theta e^{-\alpha\tau} . \tag{13}$$

Note that $s = 0$ is not a solution of (12) or (13) since $\theta > 0$.

Now, consider a solution s_0 of (11) with a magnitude $r > 0$; i.e., $p(s_0) = 0$. In other words, let

$$s_0 = \mu + j\omega \quad \text{with} \quad |s_0| = \sqrt{\mu^2 + \omega^2} = r > 0 . \tag{14}$$

Then, replacing (14) in (13), we get

$$\begin{aligned} s e^{s\tau} &= e^{(\mu\tau + j\omega\tau)}(\mu + j\omega) \\ &= e^{\mu\tau} e^{j\omega\tau}(\mu + j\omega) \\ &= e^{\mu\tau}(\mu \cos\omega\tau - \omega \sin\omega\tau) + j e^{\mu\tau}(\omega \cos\omega\tau + \mu \sin\omega\tau) \\ &= -\theta e^{-\alpha\tau} . \end{aligned}$$

This implies that

$$e^{\mu\tau}(\mu \cos\omega\tau - \omega \sin\omega\tau) = -\theta e^{-\alpha\tau} \tag{15}$$

and

$$e^{\mu\tau}(\omega \cos\omega\tau + \mu \sin\omega\tau) = 0 . \tag{16}$$

From equation (16), we see that

$$\begin{aligned} \omega \cos\omega\tau + \mu \sin\omega\tau &= \sqrt{\mu^2 + \omega^2}\, \sin(\omega\tau + \varphi) \\ &= |s_0| \sin(\omega\tau + \varphi) \\ &= r \sin(\omega\tau + \varphi) = 0 \end{aligned}$$

where $\varphi = \tan^{-1}\left(\dfrac{\omega}{\mu}\right) = \arg(s_0)$.

Hence, the solution $s_0 = \mu + j\omega$ of (11) has real and imaginary parts such that

$$\omega\tau + \tan^{-1}\left(\frac{\omega}{\mu}\right) = k\pi \tag{17}$$

where k is an integer.

On the other hand,

$$\mu\cos\omega\tau - \omega\sin\omega\tau = \sqrt{\mu^2+\omega^2}\cos(\omega\tau+\varphi).$$

Now, from (17), we can see that $\cos(\omega\tau+\varphi) = (-1^k)$; that is

$$e^{\mu\tau}(\mu\cos\omega\tau - \omega\sin\omega\tau) = e^{\mu\tau}r(-1^k).$$

Consequently, from (15), we have

$$e^{\mu\tau}r(-1^k) = -\theta e^{-\alpha\tau}.$$

Now since θ, $r > 0$, the integer k must be odd (to match the signs) for the previous equality to be satisfied.

Therefore, the real part μ of s_0 satisfies

$$e^{\mu\tau}r = \theta e^{-\alpha\tau}.$$

In other words,

$$\mu = -\alpha + \frac{1}{\tau}\ln\left(\frac{\theta}{r}\right).$$

From this we see that the real part of s_0 depends on the choice of θ. Now, let r_0 be an arbitrary positive number and consider all solutions $s = \mu + j\omega$ of (11) with a magnitude $r \geq r_0 > 0$.

Then,

$$\mu = -\alpha + \frac{1}{\tau}\ln\left(\frac{\theta}{r}\right) \leq -\alpha + \frac{1}{\tau}\ln\left(\frac{\theta}{r_0}\right).$$

If we choose $\theta \leq r_0$, then $\ln\left(\dfrac{\theta}{r_0}\right) \leq 0$ and

$$\mu \leq -\alpha + \frac{1}{\tau}\ln\left(\frac{\theta}{r_0}\right) \leq -\alpha.$$

This means that if $0 < \theta \leq r_0$, then all complex numbers s such that $|s| = r > r_0$ which satisfy the equation $se^{s\tau} = -\theta e^{-\alpha\tau}$ are given by $s = \mu + j\omega$ where

$$\mu = -\alpha + \frac{1}{\tau}\ln\left(\frac{\theta}{r}\right) \leq -\alpha$$

and

$$\omega\tau + \tan^{-1}\left(\frac{\omega}{\mu}\right) = k\pi, \quad \text{where} \quad k \in Z \quad \text{is odd.}$$

As a result, the system (9) is (α, r_0)-stable. Consequently, the controlled system is also Krasovskii stable with stability margin α; hence asymptotically stable.

This completes the proof of Theorem 1.

2.1. REMARK

1. Note that in the non-delayed case ($\tau = 0$) all the poles of the closed-loop system (8) will be located at $-\theta$ in the left-half complex plane when controller (7) is applied with $\tau = 0$.

2. The above control design technique can be extended to the following class of systems

$$\dot{x}(t) = Ax(t-\tau) + \tilde{A}x(t) + Bv(t); \quad t \geq 0 \tag{18}$$

where $x \in R^n$, $v \in R$ and $\tilde{A}$ is of the special form $\tilde{A} = BK$ for some vector $K \in R^n$. Indeed, if we apply the preliminary control

$$v(t) = -Kx(t) + u(t) \tag{19}$$

to system (18), we obtain

$$\dot{x}(t) = Ax(t-\tau) + Bu(t); \quad t \geq 0$$

which is of the form described by system (1). Hence, the complete stabilising feedback for system (18) is given by

$$v(x(t)) = -Kx(t) + u(x(t))$$

where $u(x(t))$ is given by (7).

3. EXAMPLE

In this section, we shall illustrate the previous design methodology through an academic example. Consider the following 2nd-order single-input system.

$$\dot{x}(t) = Ax(t-0.2) + Bu(t) \tag{20}$$

where

$$x(t) = \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix}, \quad A = \begin{bmatrix} 0 & 1 \\ 1 & 2 \end{bmatrix} \quad \text{and} \quad B = \begin{bmatrix} 0 \\ 1 \end{bmatrix}.$$

The above system is of the form (1) with a time lay $\tau = 0.2$ s. Consequently, by applying the previous design methodology, the control law that will stabilise the system at the origin is given by

$$u(x(t)) = -L_0 x(t-\tau) - L_1(\theta, \alpha, \tau)\, x(t-\tau)$$

where

$$L_0 = [1 \quad 2]$$

and

$$L_1(\theta, \alpha, \tau) = [\theta^2 e^{-2\alpha\tau} \quad 2\theta e^{-\alpha\tau}] = [\theta^2 e^{-0.4\alpha} \quad 2\theta e^{-0.2\alpha}]$$

with $\alpha, \theta > 0$. Hence,

$$u(x(t)) = -(1+\theta^2 e^{-0.4\alpha})x_1(t-0.2) - 2(1+\theta e^{-0.2\alpha})x_2(t-0.2).$$

The closed-loop system is given by

$$\dot{x}(t) = \bar{A}x(t-0.2)$$

where $\bar{A} = \begin{bmatrix} 0 & 1 \\ -\theta^2 e^{-0.4\alpha} & -2\theta e^{-0.2\alpha} \end{bmatrix}$.

Simulation

Several sets of simulations were carried out to show the behaviour of the closed-loop system. Here we have chosen $\alpha = 0.1$ and $r_0 = 4$ so that $0 < \theta \le 4$. Figures 3 and 4 show the profile of x_1 and x_2 respectively, when $\theta = 1$. It can be seen that the state variables converge to the origin as expected. Figures 5 and 6 show the same profile when $\theta = 3$. Here it can be seen that the convergence is much quicker.

In general, the higher the value of θ (which implies larger values of r_0) the quicker the convergence. This is because the poles are pushed further to the left in the left-half complex plane. However, as one would expect in such a case a larger control effort is needed to achieve this. This is confirmed in Fig. 7.

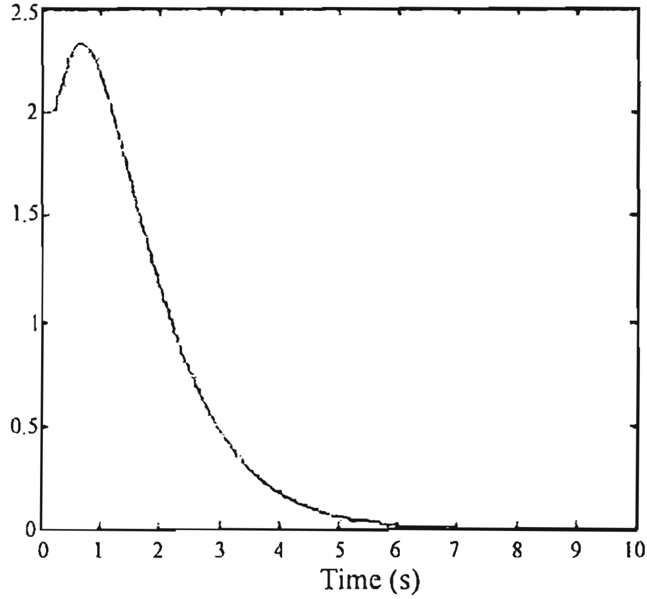

Fig. 3. Profile of x_1 when $\theta = 1$

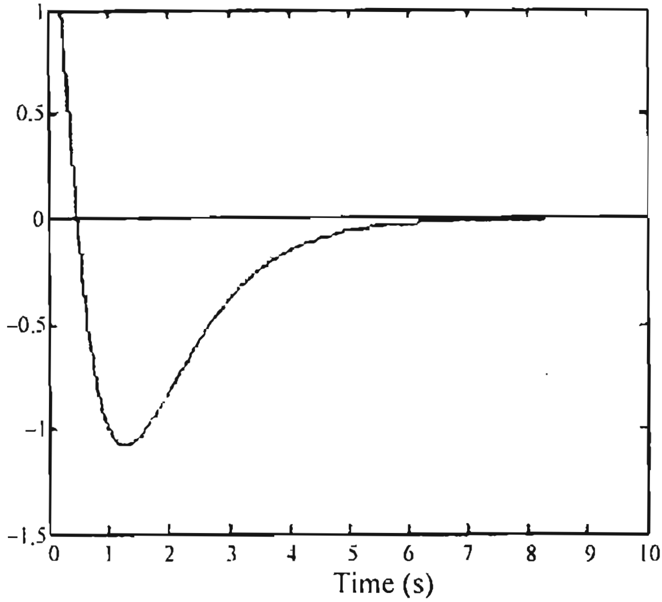

Fig. 4. Profile of x_2 when $\theta = 1$

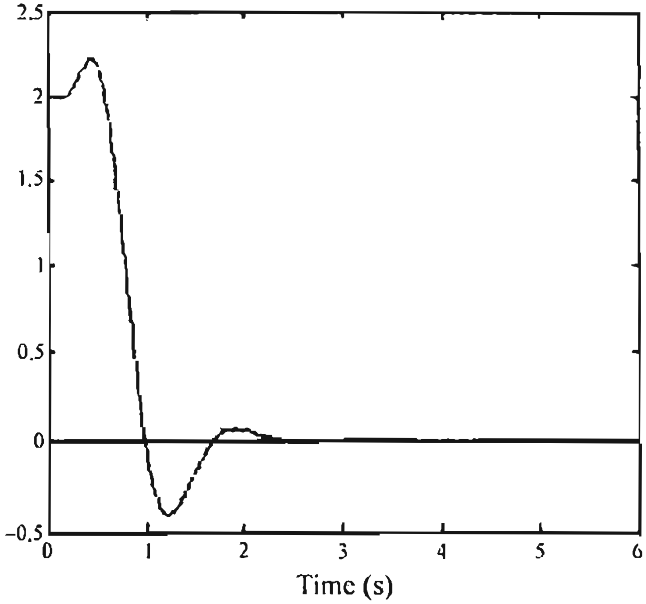

Fig. 5. Profile of x_1 when $\theta = 3$

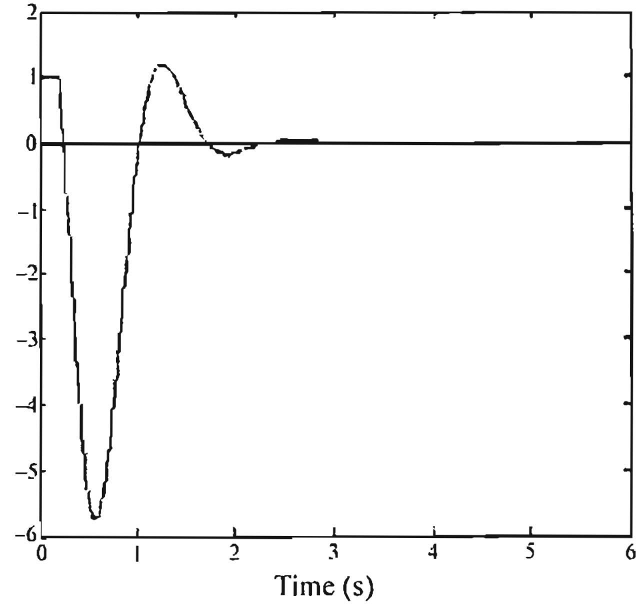

Fig. 6. Profile of x_2 when $\theta = 3$

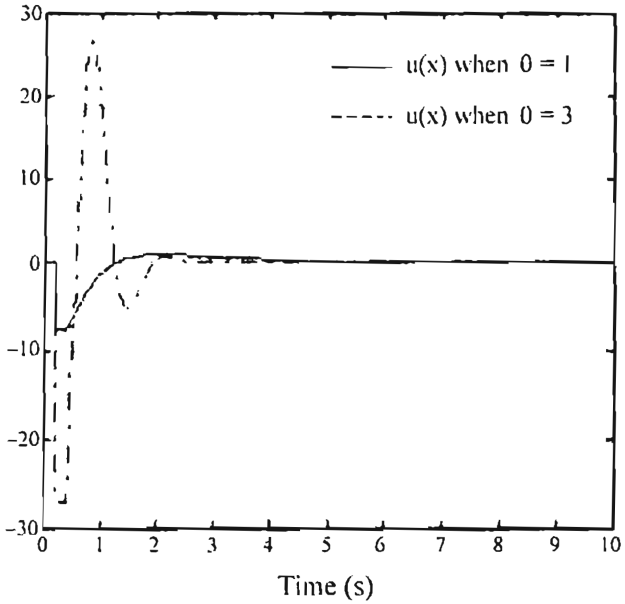

Fig. 7. The control $u(x)$ when $\theta = 1$ and $\theta = 3$

In addition, large values of θ yield large transient peaks. Therefore there is a trade off between the desired magnitude of r_0 and the amplitude of acceptable transient peaks and control effort.

4. CONCLUSIONS

In this paper, we have proposed a control design for a class of single-input linear systems with known state delay. The design is based on the specific choice of controller gain such that the closed-loop system is asymptotically stable in the sense of (α, r)-stability. A simulation example has demonstrated the good convergence performance of the controller when the controller gain is chosen appropriately.

ACKNOWLEDGEMENT

This research is funded and supported by the Engineering and Physical Sciences Research Council (EPSRC).

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Received March 15, 2004