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A NEW PROCESS FOUNDATION FOR THE APPLIED TOPOS

60th birthday tribute to the pioneering work of Martin Hyland and Peter Johnstone in the tradition of Cambridge mathematics

Michael Heather & Nick Rossiter
School of Computing, Engineering and Information Sciences, Northumbria University, NE1 8ST, UK
michael.heather@cantab.net nick.rossiter@unn.ac.uk,
WWW home page: `http://computing.unn.ac.uk/staff/CGNR1/`

EXTENDED ABSTRACT

The world is in turmoil for want of sound reasoning. Economics and the environment are but two of many areas of human endeavour badly betrayed through a failed combination of physical and information science and the rule of law. Logic is the fabric of pure mathematics as the foundation of applied mathematics on which all science is based from the physical through biological and medical to the social sciences. However the symbolic logic of today seems of scarce more use than the syllogisms of Aristotle as observed by Francis Bacon nearly 400 years ago:

> The logic now in use serves rather to fix and give stability to the errors which have their foundation in commonly received notions than to help the search after truth. So it does more harm than good [*Novum Organon* Aphorism XII, 1620].

The well known phrase of 'new foundations' was used by Quine for new axioms of set theory but it echoes a long tradition at Cambridge for foundation principles from Bacon's call for a new version of Aristotle's organon, through Newton's *Principia* to the *Principia Mathematica* of Whitehead and Russell. However these are but conspicuous nodes in a continuum of mathematical advancement at Cambridge. The separate painstaking work of Hyland and Johnstone on the in-depth study of the topos should surely be seen in that tradition when its significance is fully recognised. The historical development of the argument is summarised in the table below.

The world's present problems lie in in the logic of open interactive global systems --- this is the structure of the topos. In the light of current problems thrown up by the real world, the outcome of the work of the last thirty years shows clearly the need for new foundations. The topos so far has been constructed in a category theory based in a set theory of closed world axioms. Martin Hyland and Peter Johnstone have faithfully expounded the topos in the context of the twentieth century mathematics of proof theory on the one hand with the 'effective topos' and within model theory on the other with sheaves, relations, allegories, sketches as well as in extensions of set theory like topology, homotopy, cohomology and differential geometry.

An example of a topos structure is the tranching in the securatization of subprime mortgages. Based on first order models these were considered safer because of the internal spread of risk:

> As do the physicists who promote nuclear energy, those bankers who promote securitisation and the originate-and-distribute model stress that these financial innovations can benefit society. Indeed, some of the bankers who do the promoting are trained physicists and the models that they use to make their case borrow from science some very advanced mathematics. [Securitisation and the Originate and Distribute Model: Does it have a future? Speech by Thomas Huertas, Director, Banking Sector, FSA Euro 50 Group, London 21 April 2008]

Unfortunately this 'very advanced mathematics' is not advanced enough. For it needs the internal structure of a preorder with a subobject classifier that can be either the initial object (false) or the terminal object (true) according to the extraneous ambient context. This is just an example of the well known 'Schrödinger cat' in quantum mechanics. Another example is the cause of climate cange which is likewise undecidable at the first order level of statistical modelling.

Another important area calling for more rigorous category theory is information technologies and communications including the development of the quantum computer where the current theories of the different interpretations of quantum mechanics are first order models and need the higher order construction of the topos. There are very many examples of problems with failed information systems. Whatever the design criteria and the project management issues there is always the underlying conceptual problem of the mapping of the real world on to the von Neumann architecture of current computers. The architypal situation is to be found in Codd's relational database model where there is always a need to loose information in the process of normalization of data. It is not usually possible to know in advance the status of this lost information. The problem arises directly out of the theoretical basis of set theory.

Another pervasive example to be found over a wide field of applications is that of turbulence This has a long history of study by Cambridge applied mathematicians following the work of G I Taylor in the middle of the last century. However all attempts to produce an analytical solution to the problem have failed. Chaos theory is a first order 'make do' that again can lead to even dangerous consequences when risk analysis is needed in hazardous environments.

Despite the later work on the axioms of set theory throughout the twentieth century, nevertheless the classical foundation for category theory still rests ultimately on Whitehead and Russell's *principia mathematica*. The best candidate to provide the new foundations which accords with the naturaleness of (and as defined in) category theory seems to be the process philosophy of Whitehead's later period. There metaphysics replaces the model and assumptions can be discarded to give a reliable theory needed for the applied mathematics of the 21st century.

MATHEMATICAL DEVELOPMENT	SIGNIFICANCE	CONSEQUENCES	CATEGORY THEORY
Port Royal Logic (1662)	different possible logical views/meanings	closed world assumption (CWA) only holds locally	Exact/coexact adjointness
Frege's predicate logic (1879)	meaning as arithmetic	Not universal because of CWA	co-limits
Whitehead & Russell's (1910) *principia mathematica*	axioms of set theory give rise to paradox	*pm* repudiated explicitly by BR implicitly by ANW by conduct	category of sets a model restricted by the axioms of set theory
Gödel's doctoral thesis (1930)	first order (boolean) semantics complete	20th century science holds to first order Query eg methods for economic/climate models, LHC, etc	validates current CT development to first order
Gödel's metamathematical 5th theorem (1930)	predicates realizable as coded by number	logic of mathematics is not the logic of physics	justifies number in CT by assumption
Gödel's 6th theorem (1930)	axiomatic systems of number/sets are undecidable	Hilbert's programme/ *Entscheidungsproblem* undecidable	CT needs higher order validation
Church conjecture (1932)	effective calculability of number	scientific method relying on measurement only valid to first order	The 'effective topos' is a model
Church-Turing Thesis (1936)	effective computability of number	computer with von Neumann architecture is a calculator not a logic machine	CT in computer science restricted to categorification
Church-Turing-Deutsch Principle (1985)	effective quantum computability in parallel	quantum computer needs full quantum theory beyond first order model	CT in quantum mechanics restricted to categorification
Whitehead's *Process & Reality* (1929)	Universe as non-staionary process	Process is metaphysics while set theory is a model	• arrow as process; • physical Universe as a topos; • CT as metaphysics instead of categorification of sets