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On the Hamiltonian Circuits and Hamiltonian Decomposition of $\vec{C}_{n_1}\times\vec{C}_{n_2}\times\cdots\times\vec{C}_{n_k}$

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Abstract Let $n_1\leqslant n_2\leqslant\cdots\leqslant n_k$ be some positive integers. $D=\vec{C}_{n_1}\times\vec{C}_{n_2}\times\cdots\times\vec{C}_{n_k}$ is the cartesian product of directed circuits. In this paper, we prove that D has hamiltonian circuits if $n_i|n_k(1\leqslant i\leqslant k-1)$. When $n_1=n_2=\cdots=n_k$, we confirm that D has $\lfloor k/2\rfloor$ arc disjoint hamiltonian circuits. As a by-product, we deduce that $\Gamma\times\Gamma$ is an hamiltonian digraph if Γ is an hamiltonian digraph.

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Let G be an abelian group with operation "+" and zero element 0. For every subset $S\subset G$, the Cayley digraph $C(G,S)$ is defined as follows. All the elements of G are the vertices of $C(G,S)$ and, for each $s\in S$, $(u, u+s)$ is an arc of $C(G,S)$ which will be called the s-arc. An arc sequence M on S is a listing of some elements of S like that $M=[a_1,a_2,\cdots a_n]$, where $a_i(1\leqslant i\leqslant n)$ are some elements of S which are not necessarily different. Define $S_l(M)=\sum_{i=1}^{l}a_i(1\leqslant l\leqslant n)$. For convenience, we set $S_0(M)=0$ and $S(M)=S_{|M|}$. If $S_{l_1}(M)\neq S_{l_2}(M)$ for $l_1\neq l_2$, then M corresponds a directed path $P(M)=(0,S_1(M),S_2(M),\cdots,S_n(M))$ traversing from 0 by the arcs $a_1,a_2,\cdots,an$ successively. We call M an hamiltonian sequence on S if $P(M)$ is a directed hamiltonian circuit of $C(G,S)$. Thus by definition, $C(G,S)$ has an hamiltonian circuit if and only if there exists an hamiltonian sequence on S. If there are t arcs appearing consecutively in M, we abbrevate them by $t*a$. Similarly, $t*M$ denotes the sequence of t consecutive copies of M. For example, $M=[2*a_1,a_2]=[a_1,a_1,a_2]$ and $2*M=[M,M]=[a_1,a_1,a_2,a_1,a_1,a_2]$, and so on. Denoted by $\langle S(M)\rangle$ the subgroup generated by $S(M)$ and $o(S(M))$ is the order of $S(M)$ in G.

In this work, we state our results in the following way.

Ⅰ. Let $n_1<n_2<\cdots<n_k$ be some positive integers and $\vec{C}_{n_i}$ be the directed circuit. If $n_i|n_k$(1

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$\leqslant i<k)$. then $\vec{C}_{n_1}\times\vec{C}_{n_2}\times\cdots\times\vec{C}_{n_k}$ has a hamiltonian circuit.

Ⅰ. If $n=n_1=n_2=\cdots=n_k$, then $\vec{C}_n\times\vec{C}_n\times\cdots\times\vec{C}_n=\vec{C}_n^k$ contains $\lfloor k/2\rfloor$ arc disjoint hamiltonian circuits.

Ⅲ. Let Γ be a digraph with a hamiltonian circuit. Then the cartesian product $\Gamma\times\Gamma\times\cdots\times\Gamma=\Gamma^k$ contains $\lfloor k/2\rfloor$ arc disjoint hamiltonian circuits. In particular, $\Gamma\times\Gamma$ is an hamiltonian digraph.

First we introduce a lemma which takes another form in [6]. Here we give it a little generalization.

Lemma 1 Let G be an abelian group and $M=[a_1,a_2,\cdots,a_n]$ the arc sequenceon S. $o(S(M))*M$ is an hamiltonian sequence of $C(G,S)$ if and only if $o(S(M))n=|G|$ and $S_i(M)-S_j(M)\notin\langle S(M)\rangle$ for $1\leqslant i\neq j\leqslant n$.

Proof. Set $r=o(S(M))$. For the sufficency, we need to show that $P(M)=(S_0(M), S_1(M),\cdots,S_{rn-1}(M), S_{rn}(M))$ is an hamiltonian circuit. Since $rn=|G|$ and $S_{rn}(r*M)=rS(M)=0$, it suffices to check that $S_i(r*M)$ and $S_j(r*M)$ are distinct for $0\leqslant i\neq j\leqslant rn$. To the contrary, suppose that $S_i(r*M)=S_j(r*M)(i\neq j)$. Let $i=mn+p$ and $j=m'n+p'$, where $0\leqslant p,p'<n$. Then by definition, $S_{mn+p}(r*M)=S_{m'n+p'}(r*M)\Rightarrow mS(M)+S_p(M)=m'S(M)+S_{p'}(M)\Rightarrow S_p(M)-S_{p'}(M)\in\langle S(M)\rangle\Rightarrow p=p'$. This implies $m=m'$, and so $i=j$. A contradiction. The necessity is obvious by the above disccussion.

Let $Z_{n_i}=\langle a_i\rangle$ be a cyclic group of order n_i, where $i=1,2,\cdots,k$. Set $G=Z_{n_1}\times Z_{n_2}\times\cdots\times Z_{n_k}$. Since $\overline{Z}_{n_i}=\{(0,\cdots,a_i,\cdots,0)\}|a_i\in Z_{n_i}\}$ is a subgroup of G isomorphic to Z_{n_i}, we can regard $\overline{Z}_{n_i}$ as Z_{n_i}. Thus $S=\{a_1,a_2,\cdots,a_k\}$ is a generating subset of G. It is easy to see that $C(G,S)$ is just the cartesian product $D=\vec{C}_{n_1}\times\vec{C}_{n_2}\times\cdots\times\vec{C}_{n_k}$. Let M be the arc sequence on S and $\underline{M}$ be the arc sequence obtained from M by deleting the last element of M. For instance, let $M=[2*a_1,a_2,a_3]$, then $\underline{M}=[2*a_1,a_2]$.

Now we define inductively the arc sequence $H_1,H_2,\cdots,H_{k-1}$ as follows. $H_1=[(n_1-1)*a_1,a_2],H_2=[(n_2-1)*H_1,\underline{H_1},a_3],H_3=[(n_3-1)*H_2,\underline{H_2},a_4],\cdots,H_{k-1}=[(n_{k-1})*H_{k-2},\underline{H_{k-2}},a_k]$. It is not difficult to see that

Claim 1 $H_i(1\leqslant i<k)$ is an arc sequence on S which contains precisely $n_1n_2\cdots n_i$ arcs and there is only one a_i lying on the last position of H_i.

Claim 2 $S(H_i)=n_iS(H_{i-1})-a_i+a_{i+1}$, and so $S(H_i)=n_i\cdots n_2(n_1-1)a_1+n_i\cdots n_3(n_2-1)a_2+\cdots+(n_i-1)a_i+a_{i+1}$, where $i=1,2,\cdots,k-1$.

Claim 3 Let $n_1\leqslant n_2\leqslant\cdots\leqslant n_k$ and $n_i|n_k$ for $i<k$. Then $o(S(H_{k-1}))=n_k$.

Theorem 1 Under the assumption of Claim 3 n_k*H_{k-1} is an hamiltonian sequence on S, and so $C(G,S)$ has the directed hamiltonian circuit $P(n_k*H_{k-1})$.

Proof By Claim 1 and Claim 3, H_{k-1} contains exactly $n_1n_2\cdots n_{k-1}$ arcs and $o(S(H_{k-1}))n_1n_2\cdots n_{k-1}=n_1n_2\cdots n_k=|G|$. Thus according to Lemma 1, we need only to check that $S_i(H_{k-1})-S_j(H_{k-1})\notin\langle S(H_{k-1})\rangle$ for $0\leqslant i\neq j<|G|$. We will do this by induction on the index l of $H_l(1\leqslant l\leqslant k-1)$.

It is obvious by definition that $S_i(H_1)-S_j(H_1)\notin\langle S(H_{k-1})\rangle$ for $0\leqslant i\neq j<n_1$. Suppose

we have established that $S_i(H_l) - S_j(H_l) \notin \langle S(H_{k-1})\rangle$ for $l < k-1$ and $0 \leqslant i \neq j < n_1 n_2 \cdots n_l$. To the contrary, assume that there exist $0 \leqslant i \neq j < n_1 n_2 \cdots n_{k-1}$ such that $S_{i_0}(H_{k-1}) - S_{j_0}(H_{k-1}) = mS(H_{k-1})$, where $0 \leqslant m < n_k$.

By Claim 2, the right side contains a term ma_k. But the left side contains no such term. Hence $m = 0$, and so $S_{i_0}(H_{k-1}) = S_{j_0}(H_{k-1})$. Let $i_0 = pn_1 n_2 \cdots n_{k-2} + r$ and $j_0 = p' n_1 n_2 \cdots n_{k-2} + r'$, where $0 \leqslant r, r' < n_1 n_2 \cdots n_{k-2}$ and $0 \leqslant p, p' < n_{k-1}$. We have $pS(H_{k-2}) + S_r(H_{k-2}) = p'S(H_{k-2}) + S_{r'}(H_{k-2}) \Rightarrow (p-p')S(H_{k-2}) + S_r(H_{k-2}) - S_{r'}(H_{k-2}) = 0$. Set

$$u = (p-p')S(H_{k-2}) - (p-p')a_{k-1} + S_r(H_{k-2}) - S_{r'}(H_{k-2}) = -(p-p')a_{k-1}.$$

Then $u \in (Z_{n_1} \times \cdots \times Z_{n_{k-2}}) \cap Z_{n_{k-1}}$. This implies that $p - p' = 0 (\mathrm{mod}\, n_{k-1})$, and so $p = p'$. Thus $S_r(H_{k-2}) = S_{r'}(H_{k-2})$. By induction hypothesis, $r = r'$. But then it contradicts with $i_0 \neq j_0$. So we complete our proof.

Corollary 1 Let $n_1 \leqslant n_2 \leqslant \cdots \leqslant n_k$ be some integers with $n_i | n_k (1 \leqslant i < k)$ and $D = \vec{C}_{n_1} \times \vec{C}_{n_2} \times \cdots \times \vec{C}_{n_k}$. Then D has a directed hamiltonian circuit.

Example 1 Let $G = Z_2 \times Z_3 \times Z_6$ be the cartesian product of the cyclic groups. Set $a_1 = (1,0,0)$, $a_2 = (0,1,0)$ and $a_3 = (0,0,1)$, where $o(a_1) = 2$, $o(a_2) = 3$ and $o(a_3) = 6$. Take $H_2 = [a_1, a_2, a_1, a_2, a_1, a_3]$. Then $6 * H_2$ is an arc hamiltonian sequence which conrresponds to a directed hamiltonian circuit in the cartesian product of $\vec{C}_2 \times \vec{C}_3 \times \vec{C}_6$.

In this section, we further consider the problem of the hamiltonian decomposition of $D = \vec{C}_n \times \vec{C}_n \times \cdots \times \vec{C}_n$. Let $Z_n = \langle a \rangle$ be the cyclic group of order n and $G = Z_n \times Z_n \times \cdots \times Z_n \times = Z_n^k$. We again regard $a_1, a_2, \cdots, a_k$ as $(a,0,\cdots,0), (0,a,\cdots,0), \cdots, (0,\cdots,0,a)$, respectively. Set $S = \{a_1, a_2, \cdots, a_k\}$. Then S generates G.

Let $\mathfrak{R}_k$ be the symmentric group on $\{1,2,\cdots,k\}$. Then for each $\pi \in \mathfrak{R}_k$, π derives an automorphism of G like that $\pi: j_1 a_1 + j_2 a_2 + \cdots + j_k a_k \to j_1 a_{\pi(1)} + j_2 a_{\pi(2)} + \cdots + j_k a_{\pi(k)}$. Denote by Aut G and Aut$C(G,S)$ the automorphism groups of G and $C(G,S)$ respectively.

Lemma 2 $\mathfrak{R}_k$ is the subgroup of Aut$G$, and so is the subgroup of Aut$C(G,S)$.

According to the above lemma, for each π of $\mathfrak{R}_k$, π maps an arc sequence to another. For example, set $H_1 = [n * a_1, a_2]$. Then $\pi(H_1) = [n * a_{\pi(1)}, a_{\pi(2)}]$. Commonly, if M is an arc sequence on S, Then $\pi(M)$ is the sequence got from M by replacing each a_i in M with $a_{\pi(i)}$. By Theorem 1 and Lemma 2, we have

Lemma 3 For every $\pi \in \mathfrak{R}_k$, $n * \pi(H_{k-1})$ is an hamiltonian sequence of $C(G,S)$.

Let u and u' be any two integers with $0 \leqslant u, u' < n^k$. Set $u = t_{k-1}n^k + t_{k-2}n^{k-2} + \cdots + t_1 n + t_0$ and $u' = t'_{k-1}n^{k-1} + t'_{k-2}n^{k-2} + \cdots + t'_1 n + t'_0$, where $0 \leqslant t_i, t'_i < n$ and $0 \leqslant i < k$.

Lemma 4 Suppose $S_u(n * H_{k-1}) = S_u'(n * \pi(H_{k-1}))$ for $\pi \in \mathfrak{R}_k$. Then $t_0 = t'_0$.

Proof By definition, we have

$$S_u(n * H_{k-1}) = t_{k-1}S(H_{k-1}) + t_{k-2}S(H_{k-2}) + \cdots + t_1 S(H_1) + S_{t_0}(H_1). \tag{1}$$

$$S'_u(n * \pi(H_{k-1})) = t'_{k-1}S(\pi(H_{k-1})) + t'_{k-2}S(\pi(H_{k-2})) + \cdots + t'_1 S(\pi(H_1)) + S_{t'_0}(\pi(H_1)). \tag{2}$$

By Claim 2, we have

$$S(H_i) = n^{i-1}(n-1)a_1 + n^{i-2}(n-1)a_2 + \cdots + (n-1)a_i + a_{i+1} = (n-1)a_i + a_{i+1}$$

$$S(\pi(H_i)) = n^{i-1}(n-1)a_{\pi(1)} + n^{i-2}(n-1)a_{\pi(2)} + \cdots + (n-1)a_{\pi(i)} + a_{\pi(i+1)}$$
$$= (n-1)a_{\pi(i)} + a_{\pi(i+1)},$$

where $i = 1,2,\cdots,k-1$. Return to (1) and (2), we have

$$S_u(n * H_{k-1}) = t_{k-1}((n-1)a_{k-1} + a_k) + t_{k-1}((n-1)a_{k-2} + a_{k-1}) + t_{k-3}((n-1)a_{k-3} + a_{k-2}) + \cdots + t_1((n-1)a_1 + a_2) + t_0 a_1$$
$$= t_{k-1}a_k + (t_{k-2} - t_{k-1})a_{k-1} + (t_{k-3} - t_{k-2})a_{k-2} + \cdots + (t_1 - t_2)a_2 + (t_0 - t_1)a_1, \quad (3)$$
$$S_{u'}(n * \pi(H_{k-1})) = t'_{k-1}a_{\pi(k)} + (t'_{k-2} - t'_{k-1})a_{\pi(k-1)} + (t'_{k-3} - t'_{k-2})a_{\pi(k-2)} + \cdots + (t'1 - t'_2)a_{\pi(2)} + (t'_0 - t'_1)a_{\pi(1)}. \quad (4)$$

Let c and c' be the sum of the coefficients of all terms in (3)and (4),respectively. Then $c = t_0$ and $c' = t'_0$. Because of G being the cartesian product of k $Z_n's$, for each term of a_i in (3) there is precisely one term of $a_{\pi(\pi^{-1}(i))}$ in (4) such that they are equal to each other, and so the difference of their coefficients can be devisable by n. Thus $n \mid c - c' = t_0 - t'_0$. This gives $t_0 = t'_0$.

Lemma 5 If there sxists $\pi \in \Re_k$ such that $\pi(a_1) \neq a_1$ and $\pi(\{a_2,a_3\}) \cap \{a_2,a_3\} = \varnothing$, then $P(n * H_{k-1})$ and $P(n * \pi(H_{k-1}))$ are two arc disjoint hamiltonian circuits.

Proof For convennience, we also use $[s_1, s_2, \cdots, s_{n^k}]$ and $[s'_1, s'_2, \cdots, s'_{n^k}]$ to present the arc sequence $n * H_{k-1}$ and $n * \pi(H_{k-1})$, respectively. Our aim is to prove that $s_{u+1} \neq s'_{u+1}$ whenever $S_u(n * H_{k-1}) = S_u(n * \pi(H_{k-1}))$.

By Lemma 4, we have $t_0 = t'_0$. Two cases will be considered bellow.

Case 1 $0 \leqslant t_0 = t'_0 < n-1$. In this case, according to(1),(2)and the definition of H_{k-1}, have $s_{u+1} = a_1$ and $s'_{u'+1} = a_{\pi(1)}$. It immediately follows our consequence by the assumption of π.

Case 2 $t_0 = t'_0 = n-1$. In this case, for the similar reason as above, we have

$$s_{u+1} = \begin{cases} a_2, & \text{if } t_0 = n-1 \text{ and } t_1 < n-1, \\ a_3, & \text{if } t_0 = n-1 \text{ and } t_1 = n-1, \end{cases} \qquad s'_{u'+1} = \begin{cases} a_{\pi(2)}, & \text{if } t_0 = n-1 \text{ and } t'_1 < n-1, \\ a_{\pi(3)}, & \text{if } t_0 = n-1 \text{ and } t'_1 = n-1, \end{cases}$$

It is clear that $s_{u+1} \neq s'_{u'+1}$ since $\pi(\{a_2,a_3\}) \cap \{a_2,a_3\} = \varnothing$. Thus we finish our proof.

For every $i \in \{1,2,\cdots,k\}$, define $\pi_j: i \to i + 2j(\mathrm{mod}\,k)$, where $0 \leqslant j < k$. Clearly $\pi_j \in \Re_k$. Then by Lemma 3, $P(n * \pi_j(h_{k-1}))$ is an hamiltonian circuit. If j is confined to $1 \leqslant j < \lfloor k/2 \rfloor$, one can simply see that $\pi_j(a_1) \neq a_1$ and $\pi(\{a_2,a_3\}) \cap \{a_2,a_3\} = \varnothing$.

Theorem 2 Let $G = Z_n \times Z_n \times \cdots \times Z_n = Z_n^k (Z_n = \langle a \rangle)$ and $S = \{a_i = (0,\cdots,a,\cdots,0) \mid i = 1,2,\cdots k\}$. Then $C(G,S)$ contains $\lfloor k/2 \rfloor$ arc disjoint hamiltonian circuits.

Proof We need to prove that $\{P(n * \pi_j(H_{k-1})) \mid j = 0,1,\cdots,\lfloor k/2 \rfloor - 1\}$ are arc disjoint.

By Lemma 2, $P(n * \pi_j)(H_{k-1}))$ and $P(n * \pi_{j'}(H_{k-1}))$ are arc disjoint if and only if $P(n * H_{k-1})$ and $P(n * \pi_l(H_{k-1}))$ are arc disjoint, where $l = j - j'$ if $j > j'$ or $l = j' - j$ otherwise. Since the action of π_l on $\{a_1, a_2, a_3\}$ is $\{a_{1+2l}, a_{2+2l}, a_{3+2l}, \}$ and $1 \leqslant l < \lfloor k/2 \rfloor$, π_l satisfies the condition of Lemma 5. Hence $P(n * H_{k-1})$ and $P(n * \pi_l(H_{k-1}))$ are indeed arc disjoint. It readily follows our result.

Corollary 2 $\vec{C}_n \times \vec{C}_n \times \cdots \times \vec{C}_n = \vec{C}_n^k$ contains $\lfloor k/2 \rfloor$ arc disjoint hamiltoinan circuits.

Let Γ be a digraph with hamiltonian circuit C_n. Then the cartesian product $C_n \times C_n$ is the subgraph of Γ. The following result is a by-product of Corollary 1 and Corollary2.

Theorem 3 Let Γ be a digraph with an hamiltonin circuit. Then the cartesian product $\Gamma \times \Gamma \times \cdots \times \Gamma = \Gamma^k$ contains $\lfloor k/2 \rfloor$ arc disjoint hamiltonian circuits. In particular, $\Gamma \times \Gamma$ is an hamiltonian digraph.

In fact, the condition of the above Theorem can be extended to the different digraphs with the same number of vertices as well.

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关于直积 $\vec{C}_{n_1} \times \vec{C}_{n_2} \times \cdots \times \vec{C}_{n_k}$ 的哈密顿圈及哈密顿分解

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摘 要

设 $n_1 \leqslant n_2 \leqslant \cdots \leqslant n_k$ 是正整数，$D = \vec{C}_{n_1} \times \vec{C}_{n_2} \times \cdots \times \vec{C}_{n_k}$ 是有向圈的直积. 在本文中，我们证明了如果 $n_i | n_k (1 \leqslant i \leqslant k-1)$，则 D 含有哈密根图. 当 $n_1 = n_2 = \cdots = n_k$ 时，我们进一步得到 D 含有 $\lfloor k/2 \rfloor$ 个弧不交的哈密顿圈. 作为副产品，我们推出当 Γ 是哈密顿有向图时 $\Gamma \times \Gamma$ 也是哈密顿有向图.

关键词:Cayley 有向图;哈密顿圈;哈密顿分解