第 46 卷 第 4 期 2013 年 12 月
数 学 研 究
Journal of Mathematical Study
Vol. 46 No. 4 Dec. 2013

Avalanche Sizes of the Abelian Sandpile Model on Unicyclic Graphs

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Abstract We first give a definition of a maximal avalanche sequence for a configuration of the Abelian sandpile model and characterize some avalanche properties. Applying these properties to unicyclic graphs, we determine their number of topplings on each vertex in principal avalanches and avalanche polynomials of the Abelian sandpile model, which generalizes R. Cori's results on cycles.

Key words Sandpile model; Recurrent configuration; Avalanche sizes

CLC number O 157.5 **Document code** A

1 Introduction

The concept of self-organized criticality was introduced by Bak, Tang and Wiesenfeld in 1987[1]. The dynamics of self-organized critical systems which give rise to the robust power law correlations seen in the non-equilibrium steady states in nature must not involve any fine-tuning of parameters. The systems under their natural evolution are driven to a critical state which shows long range spatio-temporal fluctuations similar to those in equilibrium critical phenomena[2,3]. This mechanism has been invoked to describe a large variety of physical systems such as forest-fires, earthquakes, punctuated equilibrium in biology, stock-market fluctuations etc[2−4].

Received date: 2013-10-11.
Foundation item: The work was supported by NSFC (11171279, 11101358), the NSF of Fujian (2012D140), the Scientific Research Fund of Fujian Provincial Education Department (JA12208, JA12266), the Scientific Research Foundation of Guangxi Education Committee (200103YB069) and the Doctor Foundation of Minnan Normal University (201008).

Sandpile model (SM) is the paradigm of self-organized criticality in physics[1−4]. It is a discrete model defined on a lattice and possesses a cellular automaton type of dynamics. A general analysis of the original sandpile model was undertaken by Dhar[5]. He showed that the general sandpile model features an Abelian group (hence refer to this model as the Abelian sandpile model) and characterized its critical state. It had been considered by many combinatorialists as a game on a graph called the chip firing game or the dollar game[6−11].

The Abelian sandpile model can be described informally as the dynamics on a connected graph G with a special vertex q, called the sink. We assume that a pile of particles is placed on each vertex different from the sink q in the graph G at the beginning of the dynamics. The evolution rule in the dynamics consists of selecting a vertex with at least as many particles as its degree and passing a particle from the vertex to each neighboring vertex. We call this a vertex toppling. In the dynamics, the sink does not topples. This model with the assignments of particles on G and the evolution rule is written for $ASM(G, q)$.

Before we continue, we need some definitions. Let $G = (V, E)$ be a simple connected graph with n+1 vertices:

$$V(G) = \{q, v_1, \cdots, v_n\}.$$

The order of G is denoted by $|G|$. A configuration of the $ASM(G, q)$ is a non-negative integer vector

$$s = (s(v_1), s(v_2), \cdots, s(v_n)),$$

where the non-negative integer $s(v_i)$ is considered as the number of particles placed on the vertex v_i. In particular, $\epsilon_{v_i} = (\epsilon_{v_i}(v_1), \epsilon_{v_i}(v_2), \cdots, \epsilon_{v_i}(v_n))$ with

$$\epsilon_{v_i}(v) = \begin{cases} 1 & \text{if } v = v_i \\ 0 & \text{if } v \in V(G)/\{v_i, q\} \end{cases}$$

is a configuration of the $ASM(G, q)$. Note that the configurations of the $ASM(G, q)$ depend on the sink q. A vertex v_i is stable in s if $s(v_i) < d_G(v_i)$ (the degree of v_i in G). A configuration s is said to be stable when all the vertices different from q are stable. The Laplacian matrix $L = (l_{uw})$ of G is the $(n + 1) \times (n + 1)$ matrix whose rows and columns are indexed by the vertices $q, v_1, \cdots, v_n$, entries $l_{uu} = d_G(u)$, and

$-l_{uw}(u \neq w)$ is the number of edges joining vertex u and vertex w. The $n \times n$ matrix $Q = (q_{ij})$ obtained from the Laplacian matrix L by deleting its row q and column q is called the toppling matrix. The time evolution of a configuration of the $ASM(G, q)$ is defined by the following rules:

1. taking a configuration
2. adding a particle on a randomly chosen vertex that different from q
3. performing topplings until a new stable configuration is obtained.

If there exists some unstable vertex v_i, then it topples. The configuration is updated according to the rule :

$$s(v_j) \rightarrow s(v_j) - q_{ij}, \quad j = 1, \cdots, n.$$

If there remains or appears another unstable vertex v_i, then it also topples. The process stops when all vertices different from q are stable. The sequence of topplings in the process is an avalanche, the size of an avalanche is the number of topplings performed. Let $\mathcal{S}$ be a non-empty finite sequence of (not necessarily distinct) vertices of G except for q. The sequence $\mathcal{S}$ is legal for a configuration s if and only if starting with s, the vertices can topple in the order of $\mathcal{S}$. Since the graph G is connected, for any initial configuration s, after a finite sequence of topplings, we can obtain a stable configuration t. This means that the sink collects all particles getting out of the dynamics.

A configuration is recurrent in the $ASM(G, q)$ if it is a stable configuration which is met infinitely often when performing the above operations 2 and 3. Note that not all stable configurations are recurrent. Dhar[12] and Creutz[13] showed that the set of recurrent configurations of $ASM(G, q)$ has a group structure (the Abelian group) with a natural addition. Let R be the set of all recurrent configurations of the $ASM(G, q)$ with $V(G) = \{q, v_1, \cdots, v_n\}$. Suppose $s \in R$ and an avalanche starts with the configuration $s + \epsilon_{v_i}$ created from s and $\epsilon_{v_i} = (\epsilon_{v_i}(v_1), \epsilon_{v_i}(v_2), \cdots, \epsilon_{v_i}(v_n))$ by the normal vector addition, then we call it a principal avalanche. The size of the avalanche for configuration $s + \epsilon_{v_i}$ is denoted by $\alpha(s, v_i)$. Associate a polynomial encoding the sizes of principal avalanches to the $ASM(G, q)$ and call it an avalanche polynomial which is given by

$$A_G(x) = \sum_{i=1}^{n} \sum_{s \in R} x^{\alpha(s, v_i)} = \sum \alpha_k x^k,$$

where α_k is the number of principal avalanches of size k.

For the Abelian sandpile model, one of the main concerns of scientists is the distribution of sizes of the principal avalanches. Experimental results show that, in physics, the distributions seem to have a power law tail[2,3]. But for general graphs the story seems different. To determine the exact size of an avalanche is difficult. So up to now, along this line, the result is very little. In paper [14], Dhar and Majumdar obtained the total number of topplings in an avalanche of the Abelian sandpile model on the Bethe lattice (∞-dimensions Abelian sandpile model). A well recognized fact about the Abelian sandpile model on the Bethe lattice is that its vertices are very near to the sink. Thus they only considered the size of avalanches caused by adding a particle at a vertex very far from the sink. For finite dimensional Abelian sandpile model, Cori, Dartois and Rossin[15] considered the trees, the cycles, the complete graphs and the lollypop graphs.

In this paper, we are interested in the distribution of sizes of the principal avalanches of the finite dimensional Abelian sandpile model on some graphs. We take a somewhat different perspective, viewing the Abelian sandpile model as some kind of dynamic process. In this context, we first give a definition of a maximal avalanche sequence for a configuration of the Abelian sandpile model and characterize some avalanche properties. Applying those properties to the unicyclic graphs with the sink being on the unique cycle, we also determine the number of topplings on each vertex in the principal avalanches and the avalanche polynomials of the Abelian sandpile model, which generalizes the results on cycles in [15].

2 Lemmas

Suppose $\mathcal{S}$ and $\mathcal{S}'$ are two non-empty finite vertex sequences of a graph G (the vertices in the sequence are not necessarily distinct). Write $\mathcal{S}\backslash\mathcal{S}'$ for the sequence obtained from $\mathcal{S}$ by deleting every vertex in $\mathcal{S}'$, also define the notation $(\mathcal{S},\mathcal{S}')$ as the concatenation of the sequences.

Lemma 1[9] If $\mathcal{S}$ and $\mathcal{S}'$ are two legal sequences for a configuration t of the $ASM(G,q)$, then the sequence $(\mathcal{S},\mathcal{S}'\backslash\mathcal{S})$ is also a legal sequence for the configuration t.

In order to look at the Abelian sandpile model as some kind of dynamic process, we define a special legal sequence. Let t be a starting unstable configuration of the $ASM(G,q)$. A maximal vertex sequence for the configuration t is a legal sequence $\mathcal{S}=v_1,\cdots,v_k$ such that every vertex in G appears at most once and the length k of $\mathcal{S}$ is as large as possible.

Lemma 2 Let t be an unstable configuration of the $ASM(G,q)$, $\mathcal{S}$ and $\mathcal{S}'$ be two maximal vertex sequences for the configuration t. Then a vertex in G does appear in both $\mathcal{S}$ and $\mathcal{S}'$ or in neither $\mathcal{S}$ nor $\mathcal{S}'$, and $\mathcal{S}$, $\mathcal{S}'$ lead to the same configuration.

Proof Since $\mathcal{S}$ and $\mathcal{S}'$ are two maximal vertex sequences for the configuration t, they are legal sequences and the sequences $(\mathcal{S},\mathcal{S}'\backslash\mathcal{S})$, $(\mathcal{S}',\mathcal{S}\backslash\mathcal{S}')$ are legal for the configuration t by Lemma 1. If there is a vertex v in the sequence $\mathcal{S}$ and not in $\mathcal{S}'$, then the sequence $\mathcal{S}\backslash\mathcal{S}'$ is a non-empty sequence. Thus the length of the sequence $(\mathcal{S}',\mathcal{S}\backslash\mathcal{S}')$ is larger than the length of the sequence $\mathcal{S}'$, which contradicts to the maximality. Hence $\mathcal{S}$ and $\mathcal{S}'$ have the same vertices, and they lead to the same configuration. The proof is completed.

The following property plays the principal role in the dynamics of the Abelian sandpile model.

Lemma 3[5,6] The stable configuration after an avalanche depends only on the starting configuration of the avalanche, and does not depend on the possible choice of the order of topplings during the avalanche.

Suppose that t_0 is an unstable configuration of the $ASM(G,q)$. We now describe a maximal avalanche sequence for the configuration t_0. During an avalanche starting with configuration t_0, if the configuration t_i $(i\geq 0)$ is defined, then the configuration t_{i+1} is defined as the configuration obtained by toppling a maximal vertex sequence from configuration t_i (unless t_i is stable) and t_{i+1} is determined uniquely by Lemma 2. In view of Lemma 3, we may choose a possible order of topplings during the avalanche. Thus suppose that S_j $(j\geq 1)$ is one maximal vertex sequence for the configuration t_{j-1} and t_k $(k\geq 1)$ is stable, then the toppling sequence $S_1,\cdots,S_k$ is called a maximal avalanche sequence for the configuration t_0. An example of one maximal avalanche sequence is shown in Figure 1. For the configuration $t_0=(5,4,3,1)$ of the $ASM(K_5,q)$, there are four maximal vertex sequences $v_1v_2v_3v_4$, $v_1v_3v_2v_4$, $v_2v_1v_3v_4$ and $v_2v_3v_1v_4$. Each of them can be chosen as $\mathcal{S}_1$ and leads to the same configuration $t_1=(4,3,2,0)$. $\mathcal{S}_2=v_1v_2v_3$ is the only maximal vertex sequence for the configuration t_1 and leads to the stable configuration $t_2 = (2, 1, 0, 3)$. Therefore the maximal avalanche sequences for the configuration t_0 are $v_1v_2v_3v_4v_1v_2v_3$, $v_1v_3v_2v_4v_1v_2v_3$, $v_2v_1v_3v_4v_1v_2v_3$ and $v_2v_3v_1v_4v_1v_2v_3$.

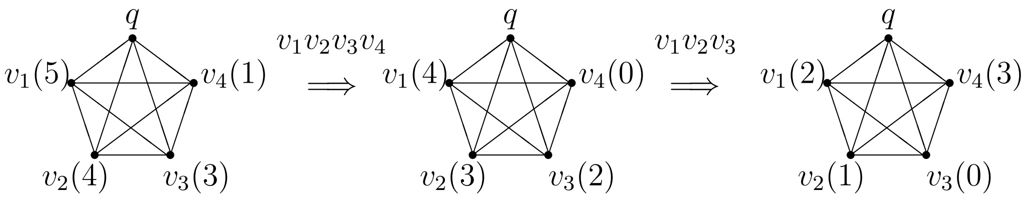

Figure 1

Lemma 4[9] The number of recurrent configurations of the $ASM(G, q)$ is equal to the number of spanning trees of G.

Let T be a tree with the sink q. Then by Lemma 4, the $ASM(T, q)$ has only one recurrent configuration s_T. It is pointed out in [15] that

$$s_T(u) = d_T(u) - 1, \quad u \in V(T) \setminus \{q\}. \tag{1}$$

We consider the tree T in such a way that the sink q is the root of T. We write T_u for the subtree rooted in u. For the recurrent configuration s_T, if u is a son of the root q of T, then adding a particle to vertex u gives a sequence of topplings of all the vertices of T_u, and the principal avalanche ends there. If u is not a son of q, suppose that $v(\neq q)$ is the father of u. Then v gets one particle after this first sequence of topplings of all the vertices of T_u and a new sequence of topplings can be performed. Thus we have

$$\alpha(s_T, u) = \begin{cases} |T_u| & \text{if the father of } u \text{ is } q; \\ \alpha(s_T, v) + |T_u| & \text{otherwise.} \end{cases} \tag{2}$$

For the $ASM(G, q)$, denote by $N_G(S)$ the neighborhood of $S \subset V(G)$, that is, if $w \in N_G(S)$, then $w \notin S$ and there is a vertex $u \in S$ such that $wu \in E(G)$. If $S \subset V(G)$, then $G - S$=$G[V \setminus S]$ is the subgraph of G obtained by deleting the vertices in S and all edges incident to them. Similarly, if $E' \subset E(G)$, then $G - E'$=$(V(G), E(G) \setminus E')$. Suppose that G contains an induced subgraph T being a tree not including the vertex q and there exist $x, y \in V(G)$ such that $\{x, y\}$=$N_G(V(T))$, $N_G(\{x\}) \cap V(T) = \{u\}$ and $N_G(\{y\}) \cap V(T) = \{v\}$. We write $u = v_1v_2 \cdots v_l = v$

for the unique path from u to v in this tree T, $T_1 \cup \cdots \cup T_l$ for the graph obtained from this tree T by deleting each edge of the unique path, where T_i is a subtree of T and $V(T_i) \cap V(T) = \{v_i\}$, $1 \le i \le l$ (see Figure 2).

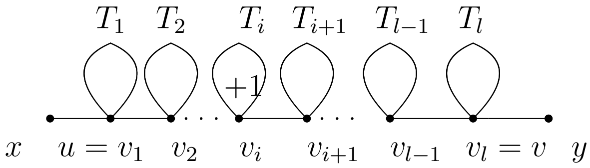

Figure 2

Lemma 5 Suppose that G contains an induced subgraph T being a tree not including the sink q, there exists $x, y, u, v \in V(G)$ such that $\{x, y\} = N_G(V(T))$, $|N_G(\{x\})| \ge 2$, $|N_G(\{y\})| \ge 2$, $N_G(\{x\}) \cap V(T) = \{u\}$, $N_G(\{y\}) \cap V(T) = \{v\}$ and a unique (u, v)-path $v_1v_2\ldots v_l$ in T shown in Figure 2. If there is a stable configuration t of the $ASM(G, q)$ with $t(v) = d_G(v) - 1$ for each vertex $v \in V(T)$ and $t(v) \le d_G(v) - 2$ for $v \in \{x, y\} \backslash \{q\}$, then

(i) for the configuration $t + \epsilon_{v^*}$, if $v^* \in V(T_i)$ $(1 \le i \le l)$, then there is a maximal vertex sequence $\mathcal{S}$=$\mathcal{T}_i$,$\mathcal{T}_{i-1}, \ldots, \mathcal{T}_1$,$\mathcal{T}_{i+1}, \ldots$,$\mathcal{T}_l$, where $\mathcal{T}_k$ consists of all the vertices in $V(T_k)(k = 1, 2, \ldots l)$, which results in the configuration t' with

$$t'(v) - \epsilon_{v^*}(v) = \begin{cases} t(v) + 1 & \text{if } v = x \ne q, \text{ or } v = y \ne q, \\ t(v) - 1 & \text{if } v = v_1 \text{ or } v = v_l\ , \\ t(v) & \text{otherwise.} \end{cases}$$

Especially, $\mathcal{S}$ is also a maximal avalanche sequence if $v* = v_1$ or $v* = v_l$.

(ii) for the configuration $t + \epsilon_{v_i}$ $(1 \le i \le l)$, there is a maximal avalanche sequence $\mathcal{S}_1$, $\mathcal{S}_2$, $\cdots$, $\mathcal{S}_k$ $(k = min\{i, l - i + 1\})$ such that $\mathcal{S}_{j+1}$ is a subsequence of $\mathcal{S}_j$ and $|\mathcal{S}_j|$, the length of sequence $\mathcal{S}_j$, is equal to $|\mathcal{S}_{j+1}| + |T_j| + |T_{l-j+1}|$ $(1 \le j \le k - 1)$;

(iii) $\alpha(t, v) = \alpha(s_{T_i}, v) + \sum_{j=1}^{l} a_j |T_j|$ for $v \in V(T_i)$ $(1 \le i \le l)$, where s_{T_i} is the recurrent configuration of $ASM(T_i, v_i)$ and $a_j = min\{i, j, l - i + 1, l - j + 1\}$.

Proof (i) Since G is connected and $t(v) = d_G(v) - 1$ for $v \in V(T)$. For v^* in $V(T_i)$ and the configuration $t + \epsilon_{v^*}$, there is a legal sequence $\mathcal{T}_i$ which consists of all the vertices in T_i. After the legal sequence $\mathcal{T}_i$, each neighbor of v_i in $V(G) \setminus V(T_i)$ gets a particle. If they are different from x and y, then v_{i-1} and v_{i+1} get a particle respectively and T_{i-1} and T_{i+1} can topple. From this argument and Lemma 3, we can obtain a legal sequence $\mathcal{T}_i, \mathcal{T}_{i-1}, \cdots, \mathcal{T}_1, \mathcal{T}_{i+1}, \cdots, \mathcal{T}_l$ for configuration $t+\epsilon_{v^*}$, where $\mathcal{T}_i$ consists of all the vertices in $V(T_i)(i=1,2,\ldots l)$. This results in a new configuration t' with $t'(x)=t(x)+1$ if $x\neq q$, $t'(y)=t(y)+1$ if $y\neq q$, $t'(v_1)=t(v_1)-1$, $t'(v_l)=t(v_l)-1$, $t'(v_j)=t(v_j)+1$ and $t'(z)=t(z)$ for $z\in V(G)\setminus(\{x,y,v_1,v_l,v_j,\}\cup\{q\})$. The legal sequence is also a maximal vertex sequence for configuration $t+\epsilon_{v^*}$ since each vertex in $V(G)\backslash V(T)$ can not topple for the configuration t'. If $v*=v_1$ or $v*=v_l$, then the configuration t' is stable and the sequence $\mathcal{S}$ is also a maximal avalanche sequence.

(ii) By Lemma 5(i), for the configuration $t+\epsilon_{v_i}, i\in\{1,2,\ldots,l\}$, there is a maximal vertex sequence $\mathcal{S}_1$=$\mathcal{T}_i\mathcal{T}_{i-1},\ldots,\mathcal{T}_1\mathcal{T}_{i+1},\ldots,\mathcal{T}_l$, which results in the configuration t_1' with

$$t_1'(v)-\epsilon_{v_i}(v)=\begin{cases} t(v)+1 & \text{if } v=x \text{ and } x\neq q, \text{ or } v=y \text{ and } y\neq q,\\ t(v)-1 & \text{if } v=v_1 \text{ or } v=v_l,\\ t(v) & \text{otherwise,}\end{cases}$$

where $\mathcal{T}_j$ consists of all the vertices in $V(T_j)(j=1,2,\ldots l)$. If $k=min\{i,l-i+1\}=1$, that is $i=1$ or $i=l$, then the configuration t_1' is stable and the sequence $\mathcal{S}_1$ =$\mathcal{T}_1\mathcal{T}_2,\ldots,\mathcal{T}_l$ or $\mathcal{T}_l\mathcal{T}_{l-1},\ldots,\mathcal{T}_1$ is also a maximal avalanche sequence. If $k>1$, then set $t_1=t_1'-\epsilon_{v_i}$ and t_1 is a stable configuration with $t_1(v_1)=t(v_1)-1=d_G(v_1)-2$, $t_1(v_l)=t(v_l)-1=d_G(v_l)-2$, $t_1(v)=t(v)=d_G(v)-1$ for $v\in V(T_2)\cup V(T_3)\cup\cdots\cup V(T_{l-1})$. Note that $T'=T-(T_1\cup T_l\cup\{v_1v_2,v_{l-1}v_l\})$ is also a tree of G and $N(V(T'))\cap V(G)=\{v_1,v_l\}$. By application of Lemma 5(i) again, we can obtain the maximal vertex sequence $\mathcal{S}_2$=$\mathcal{T}_i\mathcal{T}_{i-1},\ldots,\mathcal{T}_2\mathcal{T}_{i+1}\mathcal{T}_{i+2},\ldots,\mathcal{T}_{l-1}$ for the configuration $t_1'=t_1+\epsilon_{v_i}$ $(i\in\{2,3,\ldots,l-1\})$ and reach the configuration t_2'. Clearly, $|\mathcal{S}_1|=|\mathcal{S}_2|+|T_1|+|T_l|$. If $k=2$, that is $i=2$ or $i=l-1$, then t_2' is stable and $\mathcal{S}_1\mathcal{S}_2$ is a maximal avalanche sequence for $t+\epsilon_{v_i}$. If $k>2$, then $i\neq 2,l-1$. Repeating the above process until $k=min\{i,l-i+1\}$, the assertion holds.

(iii) Let s_{T_i} be the configuration of t restricted in $T_i-\{v_i\}$, $1\leq i\leq l$. Then s_{T_i} with $s_{T_i}(v)=d_G(v)-1$ for $v\in V(T_i)\setminus\{v_i\}$ is the unique recurrent configuration of $ASM(T_i,v_i)$ by equation (1). By Lemma 3, an avalanche for a configuration does not depend on the possible choice of the order of topplings. For the configuration $t' = t + \epsilon_v$ obtained from t by adding a particle on $v \in V(T_i)$, we first topple the vertices in $T_i - \{v_i\}$ until they are stable. Hence, after toppling $\alpha(s_{T_i}, v)$ vertices for the configuration t' by equation (2), the configuration $t + \epsilon_{v_i}$ is reached. Especially, for $v = v_i$, $\alpha(s_{T_i}, v_i) = 0$. Therefore,

$$\alpha(t, v) = \alpha(s_{T_i}, v) + \alpha(t, v_i).$$

By Lemma 5(ii), $\alpha(t, v_i) = |\mathcal{S}_1| + |\mathcal{S}_2| + \cdots + |\mathcal{S}_k|$, $k = min\{i, l-i+1\}$ and $|\mathcal{S}_j| = |\mathcal{S}_{j+1}| + |T_j| + |T_{l-j+1}|$ $(1 \le j \le k-1)$. Noting that $|\mathcal{S}_1| = |T| = |T_1| + |T_2| + \cdots + |T_l|$, we have $\alpha(t, v_i) = \sum_{j=1}^{l} a_j |T_j|$, where $a_j = min\{i, j, l-i+1, l-j+1\}$. The proof is completed.

Note that if $x = y$ in Lemma 5 and the other conditions are satisfied, then the results hold.

3 Avalanche Polynomials for Unicyclic Graphs

Whether a configuration is recurrent can be tested by the well-known burning algorithm as follows.

Lemma 6[5] Let $\varphi(v)$ be the number of edges from v to the sink in G. The configuration s is recurrent if and only if it is stable and adding $\varphi(v)$ particles at each vertex causes every vertex to topple exactly once. Moreover, at the end of the topplings we arrive at the initial configuration s.

Let $C_{l+1} = qv_1v_2 \cdots v_lq$ be a cycle of length $l+1$ and G be any unicyclic graph created from C_{l+1} by attaching $l+1$ trees $T_q, T_1, \cdots, T_l$ to $q, v_1, \cdots, v_l$, respectively. For the unicyclic graph G, if $|T_q| \ge 2$, then we can consider the construction of the unicyclic graph in such a way that it consists of merging the sinks of two graphs, say a tree and a unicyclic graph with $|T_q| = 1$. Thus the avalanche polynomial of the unicyclic graph is translated into the avalanche polynomial of the unicyclic graph with $|T_q| = 1$ plus l times the avalanche polynomial of the tree. So we just consider the case of $|T_q| = 1$ in G.

Theorem 1 Suppose G is a unicyclic graph created from $C_{l+1} = qv_1v_2 \cdots v_lq$ by attaching l trees $T_1, \cdots, T_l$ to $v_1, \cdots, v_l$, respectively. Then the set of recurrent configurations on the $ASM(G, q)$ is $R = \{s_0, s_1, \cdots, s_l\}$, where

$$s_0(v) = d_G(v) - 1 \text{ for } v \in V(G)/\{q\},$$

$$s_i(v) = \begin{cases} d_G(v_i) - 2 & \text{if } v = v_i \\ d_G(v) - 1 & \text{if } v \in V(G)/\{v_i, q\} \end{cases}, \ i = 1, 2 \ldots, l.$$

Proof It is easy to obtain that s_i $(0 \le i \le l)$ is stable. Moreover, the number of spanning trees of G is $l + 1 = |R|$ which is equal to the number of recurrent configurations on the $ASM(G, q)$ by Lemma 4. Therefore by Lemma 6, it is sufficient to prove that each vertex topples only once, that is, we can go from the configuration $s_i + \epsilon_{v_1} + \epsilon_{v_l}$ $(0 \le i \le l)$ on G to the stable configuration s_i. Let us apply Lemma 5(i) with $x = y = q$ and $t = s_0$. Then $s_0 + \epsilon_{v_1}$ reaches the stable configuration s_l and $s_0 + \epsilon_{v_l}$ reaches the stable configuration s_1. It is clear that $s_1 + \epsilon_{v_1} = s_l + \epsilon_{v_l} = s_0$. Hence the configurations $s_0 + \epsilon_{v_1} + \epsilon_{v_l}$, $s_1 + \epsilon_{v_1} + \epsilon_{v_l}$ and $s_l + \epsilon_{v_1} + \epsilon_{v_l}$ arrive at the configurations s_0, s_1 and s_l, respectively. Similarly, let us apply Lemma 5(i) with $\{q, v_i\} = \{x, y\}$ and $t = s_i$. Then the configuration $s_i + \epsilon_{v_1}$ $(1 < i < l)$ on G leads to the configuration s_{i-1}, and the configuration $s_i + \epsilon_{v_l}$ $(1 \le i < l)$ on G reaches the configuration s_{i+1}. Thus the configuration $s_i + \epsilon_{v_1} + \epsilon_{v_l}$ $(1 < i < l)$ reaches the configuration s_i. Therefore $R = \{s_0, s_1, \cdots, s_l\}$ is the set of recurrent configurations on the $ASM(G, q)$.

Suppose that s is a recurrent configuration of the $ASM(G, q)$ with $V(G) = \{q, v_1, \cdots, v_{|G|-1}\}$. The principal avalanche matrix for the recurrent configuration s is a matrix $A_{|G|-1}(s)$ in which its rows and columns are indexed by the vertices of $V(G) \setminus \{q\}$ such that the v_iv_j−entry of $A_{|G|-1}(s)$ is equal to the number of topplings performed on v_j when adding a particle at v_i for the recurrent configuration s on the $ASM(G, q)$ and reaching a stable configuration.

By Theorem 1, all recurrent configurations on the $ASM(C_{n+1}, q)$ are $s_0 = (1, \cdots, 1) \in \mathbb{Z}^n$ and $s_k = (1, \cdots, 1, 0, 1, \cdots, 1) \in \mathbb{Z}^n$, where 0 is in position k, $1 \le k \le n$. Denote the matrix $A_n(s_i) = A_n^i$, $i = 0, 1, \cdots, n$. By Lemma 5(ii), if let $x = y = q$, $t = s_0$ on G and set $|T_i| = 1 (i = 1, 2, \ldots, n)$, then for configuration $s_0 + \epsilon_{v_i}$ $(1 \le i \le n)$, there is a maximal avalanche sequence $\mathcal{S}_1, \cdots, \mathcal{S}_k$ with

$$\mathcal{S}_1 = v_i, v_{i-1}, \cdots, v_1, v_{i+1}, v_{i+2}, \cdots, v_n,$$

$$\mathcal{S}_2 = v_i, v_{i-1}, \cdots, v_2, v_{i+1}, v_{i+2}, \cdots, v_{n-1},$$

$$\vdots$$

$$\mathcal{S}_k = v_i, v_{i-1}, \cdots, v_k, v_{i+1}, v_{i+2}, \cdots, v_{n-k+1},$$

$k = min\{i, l-i+1\}$, $|\mathcal{S}_j|=|\mathcal{S}_{j+1}|+2$ $(1 \le j \le k-1)$. Thus it is not difficult to obtain that the principal avalanche matrix for s_0 on the $ASM(C_{n+1}, q)$ is $A_n^0 = (b_{ij}^0)_{n\times n}$, where $b_{ij}^0 = min\{i, j, n-i+1, n-j+1\}$; that is,

$$A_n^0 = (b_{ij}^0)_{n\times n} = \begin{pmatrix} 1 & 1 & 1 & \dots & 1 & 1 & 1 \\ 1 & 2 & 2 & \dots & 2 & 2 & 1 \\ 1 & 2 & 3 & \dots & 3 & 2 & 1 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ 1 & 2 & 3 & \dots & 3 & 2 & 1 \\ 1 & 2 & 2 & \dots & 2 & 2 & 1 \\ 1 & 1 & 1 & \dots & 1 & 1 & 1 \end{pmatrix}.$$

For the recurrent configuration $s_k = (1, \cdots, 1, 0, 1, \cdots, 1) \in \mathbb{Z}^n$, where 0 is in position $k(1 \le k \le n)$, we have $s_k(v_k) = d_{C_{n+1}}(v_k) - 2$ and $s_k(v_j) = d_{C_{n+1}}(v_j) - 1$ for $j \ne k$. By Lemma 5(ii), let paths $v_1v_2\dots v_{k-1}$ and $v_{k+1}v_{k+2}\dots v_n$ in C_{n+1}instead of the tree T in G respectively. We can obtain the principal avalanche matrix

$$A_n^k = (b_{ij}^k)_{n\times n} = \begin{pmatrix} A_{k-1}^0 & O_{(k-1)1} & O_{k(n-k)} \\ O_{1(k-1)} & 0 & O_{1(n-k)} \\ O_{(n-k)k} & O_{(n-k)1} & A_{n-k}^0 \end{pmatrix},$$

where O_{lh} is the $l \times h$ matrix each entry of which is zero, $1 \le l, h \le n-1$.

Theorem 2 Suppose G is a unicyclic graph created from $C_{l+1} = qv_1v_2\cdots v_lq$ by attaching l trees $T_1, \cdots, T_l$ to $v_1, \cdots, v_l$ respectively and $R = \{s_0, s_1, \cdots, s_l\}$ is the set of recurrent configurations on the $ASM(G, q)$ as in Theorem 1. Then

$$A_G(x) = \sum_{v\in V(G)/\{q\}} \sum_{0\le i\le l} x^{\alpha(s_i,v)},$$

where

$$\alpha(s_i, v) = \sum_{r=1}^{l} b_{jr}^i |T_r| + \alpha(s_{T_j}, v),\ v \in V(T_j),\ 1 \le j \le l,$$

b^i_{jr} is the (j,r) entry of A^i_l, $0 \le i \le l$, and s_{T_j} is the unique recurrent configuration of the $ASM(T_j, v_j)$.

Proof By Lemma 5(ii), let $x = y = q$ and $t = s_0$. Then for configuration $s_0 + \epsilon_{v_i}$ $(1 \le i \le l)$ on the $ASM(G,q)$, there is a maximal avalanche sequence $\mathcal{S}_1$, $\cdots$, $\mathcal{S}_k$ with

$$\mathcal{S}_1 = \mathcal{T}_i, \mathcal{T}_{i-1}, \cdots, \mathcal{T}_1, \mathcal{T}_{i+1}, \mathcal{T}_{i+2}, \cdots, \mathcal{T}_l,$$

$$\mathcal{S}_2 = \mathcal{T}_i, \mathcal{T}_{i-1}, \cdots, \mathcal{T}_2, \mathcal{T}_{i+1}, \mathcal{T}_{i+2}, \cdots, \mathcal{T}_{l-1},$$

$$\vdots$$

$$\mathcal{S}_k = \mathcal{T}_i, \mathcal{T}_{i-1}, \cdots, \mathcal{T}_k, \mathcal{T}_{i+1}, \mathcal{T}_{i+2}, \cdots, \mathcal{T}_{l-k+1},$$

$k = min\{i, l-i+1\}$, $|\mathcal{S}_j| = |\mathcal{S}_{j+1}| + |T_j| + |T_{l-j+1}|$ $(1 \le j \le k-1)$, where $\mathcal{T}_j$ consists of all the vertices in $V(T_j)$ $(j = 1, 2, \ldots l)$. Furthermore, by Lemma 5(iii), for $v \in V(T_j)$ $(1 \le j \le l)$, $\alpha(s_0, v) = \sum_{r=1}^{l} b^0_{jr} |T_r| + \alpha(s_{T_j}, v)$, where $b^0_{jr} = min\{j, r, l-j+1, l-r+1\}$ which is the entry (j,r) of A^0_l and s_{T_j} is the unique recurrent configuration of the $ASM(T_j, v_j)$. For the recurrent configuration s_i, $i \in \{1, 2, \ldots, l\}$, by Theorem 1,

$$s_i(v) = \begin{cases} d_G(v_i) - 2 & \text{if } v = v_i\ , \\ d_G(v) - 1 & \text{if } v \in V(G)/\{v_i, q\}, \end{cases}$$

we consider the sizes of principal avalanches on vertices $v \in V(T_j)$, $1 \le j \le l$. Since $s_i(v_i) = d_G(v_i) - 2$, by Lemma 5(iii), set $x = q$ and $y = v_i$. Then for $v \in V(T_j)$, $1 \le j \le i-1$, the vertices in $V(T_j)$ $(i \le j \le l)$ in G do not topple, and we have $\alpha(s_i, v) = \sum_{r=1}^{i-1} b^0_{jr} |T_r| + \alpha(s_{T_j}, v)$, where b^0_{jr} is the (j,r) entry of A^0_{i-1} and s_{T_j} is the unique recurrent configuration of the $ASM(T_j, v_j)$. Similarly, for $i+1 \le j \le l$, we also have $\alpha(s_i, v) = \sum_{r=1}^{l-i} b^0_{(j-i)r} |T_r| + \alpha(s_{T_j}, v)$, where $b^0_{(j-i)r}$ is the $((j-i), r)$ entry of A^0_{l-i}. For $j = i$, $\alpha(s_i, v) = \alpha(s_{T_i}, v)$ since $s_i(v_i) = d_G(v_i) - 2$, especially, for $v = v_i$, $\alpha(s_{T_i}, v_i) = 0$. Hence

$$\alpha(s_i, v) = \sum_{r=1}^{l} b^i_{jr} |T_r| + \alpha(s_{T_j}, v),\ v \in V(T_j),$$

b^i_{jr} is the (j,r) entry of A^i_l, $0 \le i \le l$, and s_{T_j} is the unique recurrent configuration of the $ASM(T_j, v_j)$. Therefore the avalanche polynomial of the $ASM(G,q)$ on the unicyclic graph G is

$$A_G(x) = \sum_{v\in V(G)/\{q\}} \sum_{s\in R} x^{\alpha(s,v)} = \sum_{v\in V(G)/\{q\}} \sum_{0\le i\le l} x^{\alpha(s_i,v)},$$

where

$$\alpha(s_i,v) = \sum_{r=1}^{l} b_{jr}^{i}|T_r| + \alpha(s_{T_j},v),\ v\in V(T_j),\ 1\le j\le l,$$

b_{jr}^{i} is the (j,r) entry of A_l^i, $0\le i\le l$, and s_{T_j} is the unique recurrent configuration of the $ASM(T_j,v_j)$.

In fact, if G in Theorem 2 is a cycle, then we have an immediate consequence due to R. Cori in [15].

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单圈图上Abelian沙堆模型的雪崩大小

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摘 要 研究图上Abelian沙堆模型问题. 首先给出关于沙堆模型常返构型的极大雪崩序列, 然后刻画了一些雪崩性质. 基于这些性质, 我们确定了单圈图的基本雪崩中每个顶点的topplings数及它的雪崩多项式, 推广了R. Cori的结果.

关键词 沙堆模型; 常返构型; 雪崩大小