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On Blow-up Rate of Solution of Semilinear Parabolic Equations System

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Abstract In this paper we carry on the literature [5], to study the Blow-up rete of solution for some semilinear parabolic equation in the cases of the single point Blow-up near the Blow-up point.

Key Words Blow-up time, Blow-up rate, Compact subset

Introduction

In [5] We had discussed Blow-up phenomenon of solution for some semilinear parabolic equations system. Now we shall study the Blow-up rate of solution in the cases of the single point Blow-up t→T where T Blow-up time.

We Consider the following problem

$$\begin{cases} u_t - \Delta u = uf_1(v) & \\ v_t - \Delta v = vf_2(u) & (x.t) \in \Omega \times (o,T) \equiv Q_T \quad (1) \\ u(x,t) = 0, v(x,t) = 0 & (x,t) \in \partial\Omega \times [0,T) \quad (2) \\ u(x,o) = \varphi(x), v(x,t) = \psi(x) & X \in \Omega \quad (3) \end{cases}$$

Where Ω is a bounded domain R^n with appropriate Smooth bouundary $\partial\Omega$.

And f1, f2, φ, ψ satifies that

$$\begin{aligned} &\varphi(x) \geqslant 0,\ \psi(x) \geqslant 0, \varphi \in c^1 \quad \psi \in c^1.\ in\Omega \\ &f_i(s) \geqslant 0,\ f'_i(s) > 0,\ f''_i(s) > 0,\ for\ s > 0,\ i = 1,2 \end{aligned} \quad (4)$$

In order to discuss the Blow-up rate of soluution of eguation system, We give the definition of Blow-up point and Blow-up time as follows.

Definition If a unigue solution $\{u,v\}$ of (1)-(3) exists on $\Omega\times[0,\sigma]$, Set $T=\operatorname{Sup}\{\sigma|(u,v)$exist in $[0,\sigma)\}$. And either

$$\lim_{t\to T^-}\max u(x,t) = \infty \ or \ \lim_{t\to T^-}\max v(x,t) = \infty \quad (5)$$

or Both of (5) holds. Then we call the Blow-up to the solution and time T is called Blow-up time of the solution.

Lemma O In problem (1)～(3), set $\Omega=B_R=\{X|X\in R^n, |X|\leqslant R\}$ $r=|X|$, if (4) holds and $\varphi_r<0, \psi_r<0$. And assume that There exist two positive functions F_1, F_2 such that

$$\begin{cases} f'_1(v) - 2\varepsilon F'_1(v) \geqslant 0, \quad f_1(v) - 2\varepsilon F_1(v) \geqslant 0 \\ v[f'_1(v)F_2(u) - F'_1(v)f_2(u)] - 2\varepsilon[F_1^2(v) + F'_1(v)F_2(u)] \geqslant 0 \\ f'_2(u) - 2\varepsilon F'_2(u) \geqslant 0, \quad f_2(u) - 2\varepsilon F_2(u) \geqslant 0 \\ u[f'_2(u)F_1(v) - F'_2(u)f_1(v)] - 2\varepsilon[F_2^2(u) + F'_2(u)F_1(v)] \geqslant 0 \end{cases} \tag{6}$$

hold, then the point r=0 is the only Blow-up point By the theorem 2.5 of [5], We can obtain that proof of lemma.

The estimate of Blow-up rate

As above, We assume that $\Omega = B_R$ and the point r=0 is unigue Blow-up point for the solution of the problem (1)~(3) then{u,v} take positive maximum is r=0, thus we get $\Delta u(0,t) \leqslant 0, \Delta v(0,t) \leqslant 0$, when $t < T$. Set

$$U(t) = \max_{\bar{\Omega}} u(x,t) = u(0,t) \qquad V(t) = \max_{\bar{\Omega}} v(x,t) = v(0,t)$$

then we have

$$\begin{cases} \dfrac{dU(t)}{dt} \leqslant Uf_1(v) & (7) \\ \dfrac{dV(t)}{dt} \leqslant Vf_2(U) & (8) \\ U(0) = \psi(0), V(0) = \varphi(0) \end{cases}$$

Theorem 1, For the problem (1)~(3), Suppose that (4) holds, Assume that $\varphi_r < 0, \psi_r < 0$. $f_1(s) = s^{1+\alpha}, f_2(s) = s^{1-\beta}$ and the Blow-up of solutions of the problem(1)~(3)occurs in T1, then we have

$$\begin{aligned} U(t) &\leqslant c_1(T_1 - t)^{\frac{1}{1-\beta}} \\ V(t) &\leqslant c_2(T_1 - t)^{\frac{1}{1-\alpha}} \end{aligned} \tag{9}$$

where $\alpha > 0$, $\beta > 0$

Proof. Consider thet the follwing equations system

$$\begin{cases} \dfrac{d\bar{U}}{dt} = (\bar{U} + u)(\bar{V} + \lambda)^{1+\alpha} & (10) \\ \dfrac{d\bar{V}}{dt} = (\bar{V} + \lambda)(\bar{U} + \mu)^{1+\beta} & (11) \\ \bar{U}(0) = \varphi(0) - \mu, \quad \bar{V}(0) = \psi(0) - \lambda \end{cases}$$

by (10)~(11),we can get

$$(\bar{V} + \lambda)^{\alpha} d\bar{V} = (\bar{U} + \mu)^{\beta} d\bar{U} \tag{12}$$

Integrating (12) with respest to t from 0 to t, we get

$$\frac{(\bar{V} + \lambda)^{1+\alpha}}{1+\alpha}\Big|_0^t = \frac{(\bar{U} + \mu)^{1-\beta}}{1+\beta}\Big|_0^t$$

namely

$$\frac{[\bar{V}(t) + \lambda]^{1-\alpha} - [\psi(0)]^{1+\alpha}}{1+\alpha} = \frac{[\bar{U}(t) + \mu]^{1+\beta} - [\varphi(0)]^{1+\beta}}{1+\beta} \tag{13}$$

If

$$\frac{[\psi(0)]^{1+\alpha}}{1+\alpha} \leqslant \frac{[\varphi(0)]^{1-\beta}}{1+\beta} \tag{14}$$

then

$$[\overline{V}(t)+\lambda]^{1+\alpha}\leqslant\frac{1+\alpha}{1+\beta}[\overline{U}(t)+\mu]^{1+\beta} \tag{15}$$

By the first equuation of (10), we get

$$\frac{d\overline{U}}{dt}\leqslant\frac{1+\alpha}{1+\beta}(\overline{U}+\mu)^{2+\beta} \tag{16}$$

integrsting (16) With respect to t from o to t ,we have

$$\begin{aligned}\frac{1}{[\overline{U}(t)+\mu]^{1+\beta}} &\geqslant \frac{1}{[\psi(0)]^{1+\beta}}-(1+\alpha)t\\ &=(1+\alpha)[\frac{1}{(1+\alpha)[\varphi(0)]^{(1+\beta)}}-t]\\ &\triangleq(1+\alpha)[T_1-t]\end{aligned} \tag{17}$$

It follows that

$$\overline{U}(t)+\mu\leqslant c_1(T_1-t)^{-\frac{1}{1+\beta}},\quad c_1=(1+\alpha)^{-\frac{1}{1+\beta}} \tag{18}$$

By the second equation of (10) and (18) we obtain that

$$\frac{d\overline{V}}{dt}\leqslant(\overline{V}+\lambda)\frac{1}{(1+\alpha)(T_1-t)} \tag{19}$$

Integrating (19) with respect to from o to t, we get

$$(\overline{V}+\lambda)^{1+\alpha}\leqslant\frac{c}{T_1-t},\qquad c=T_1[\psi(0)]^{1+\alpha}$$

thus we conclude that

$$\overline{V}+\lambda\leqslant c_2(T_1-t)^{-\frac{1}{1+\alpha}},\qquad c_2=T_1{}^{1+\alpha}[\psi(0)] \tag{20}$$

We Compare the Soluuution $\{U(t),V(t)\}$ of the problem (7)～(8) with the soluution $\{\overline{U}(t),\overline{V}(t)\}$ of the problem (10)～(11), easily prove that.

$$U(t)\leqslant\overline{U}(t),\qquad V(t)\leqslant\overline{V}(t)$$

thus we hhave

$$U(t)\leqslant c_1(T_1-t)^{-\frac{1}{1+\beta}},\qquad V(t)\leqslant c_2(T_1-t)^{-\frac{1}{1+\alpha}} \tag{21}$$

The following, We wish to obtain an estimates of the solution $\{u,v\}$ near $r=0$ for the non-symmetric domain.

Lemma 2 For the problem (1)～(3), We suppose that the condition (4) hold, and $\Delta\varphi+\varphi f_1(\psi)\geqslant0,\Delta\psi+\psi f_2(\varphi)\geqslant0$, But non-identidy with zero. then the solution of the problem satisfy

$$u_t>0,\quad v_t>0,\qquad (x,t)\in Q_T \tag{22}$$

proof. differenting the equation(1) with respect to t, we get

$$\begin{cases}u_{tt}-\Delta u_t=u_tf_1(v)+uf'_1(v)v_t\\ V_{tt}-\Delta v_t=v_tf_2(u)+vf'_2(u)u_t\\ u_t(x,0)=\Delta\varphi+\varphi f_1(\psi)\geqslant0\\ v_t(x,0)=\Delta\psi+\psi f_2(\varphi)\geqslant0\\ u_t(x,0)=0,\quad v_t(x,t)=0\end{cases} \begin{matrix}(23)\\ \\(24)\end{matrix}$$

because the right side of equation (23) and initial boundary condition are non-negative. Hence by the lemma 1.1 of [5] we have

$$u_t(x,t) \geqslant 0, \quad v_t(x,,t) \geqslant 0.$$

Further from $\Delta\varphi+\varphi f_1(\psi)\not\equiv 0$, and $\Delta\psi+\psi f_2(\varphi)\not\equiv 0$, it follws that

$$u_t(x,t) > 0 \text{ and } v_t(x,t) > 0$$

this completes proof.

Lemma 3 For the problem (1) ~ (3), Suppose that (4) holds, and the Blow-up point set S is a compact subset of Ω, and $\Delta\varphi+\varphi f_1(\psi)>0$, $,\psi+\psi f_2(\varphi)>0$. and there exist positive functions F_1, F_2 such that (6) (7) hold, Then there exist $\delta>0$ such that

$$u_t \geqslant \delta u F_1(v),\ vt \geqslant \delta v F_2(u),\ (x,t) \in \Omega^\eta \times (\eta,T) \tag{25}$$

proof. We introduce fuunction

$$J_1 = u_t - \delta u F_1(v)$$
$$J_2 = v_t - \delta v F_2(u)$$

Passing to the compunction we get

$$\begin{aligned} J_{1t}-J_{1rr}-\frac{(n-1)}{r}J_{1r} &= (u_t-u_{rr}-\frac{(n-1)}{r}u_r)_t-\delta u_1 F'_1(v)(v_t-v_{rr}-\frac{(n-1)}{r}v_r) \\ &\quad -\delta F_1(v)(u_t-u_{rr}-\frac{(n-1)}{r}u_r)+2\delta F_1(v)u_r v_r+\delta u F''_1(v)v_r^2 \\ &= f_1(v)J_1+uf'_1(v)J_2+\delta uv[f'_1(v)F_2(u)-F'_1(v)f_2(u)] \\ &\quad +2\delta F_1(v)u_r v_r+\delta u F''_1(v)v_r^2. \end{aligned}$$

By (6) we can get

$$J_{1t} - J_{1rr} - \frac{(n-1)}{r}J_{1r} - f_1(v)J_1 \geqslant uf'_1(v)J_2$$

Similarly we get that

$$J_{2t} - J_{2rr} - \frac{(n-1)}{r}J_{2r} - f_2(u)J_2 \geqslant vf'_2(u)J_1$$

Since Blow-up point set is a compact sudset of Ω, thus when η is small enough, $uF_1(v)$ and $vF_2(u)$ are bounded. and by $\Delta\varphi+\varphi f_1(\psi)>0$ and $\Delta\psi+\psi f_2(\varphi)>0$ and

$$u_t > 0, v_t > 0 \qquad in\ Q_T$$

Then we have

$$u_t \geqslant c > 0, v_t \geqslant c > 0 \qquad \text{on parobolic boundary of } \Omega^\eta \times (\eta,T)$$

thus when δ small enough, we have $J1>0, J2>0$ on the parabolic boundary of $\Omega^\eta \times (\eta,T)$. from the [5] lemma 1.1 again. it follows that

$$J_1 \geqslant 0,\ J_2 \geqslant 0\ in\ \Omega^\eta \times (\eta,T)$$

namely

$$u_t \geqslant \delta u F_1(v), v_t \geqslant \delta v F_2(u)\ in\ \Omega^\eta \times (\eta,T)$$

The proof is complete.

Where $\Omega^\eta = \{x \mid x \in \Omega,\ dist(x,,\partial\Omega) > \eta\}$

For the $F1$, $F2$ in lemma 3, we introduce the function

$$G(u,v)=\frac{1}{(v+\lambda)}\int_u^\infty\frac{ds}{F_2(s)}+\frac{1}{(u+\mu)}\int_v^\infty\frac{ds}{F_1(s)}\qquad \lambda,\mu>0$$

where

$$\int_u^\infty\frac{ds}{F_2(s)}<\infty,\qquad \int_v^\infty\frac{ds}{F_1(s)}<\infty$$

thus we have

$$-[G(u,v)]_t=\frac{u_t}{(v+\lambda)F_2(u)}+\frac{v_t}{(u+\mu)F_1(v)}\geqslant 2\delta$$

so that, by integration with respect to t from t to T. we get

$$G(u(x,t),v(x,t))-G(u(x,T),v(x,T))\geqslant 2\delta(T-t)$$

thus have

$$G(u(x,t),v(x,t))\geqslant 2\delta(T-t)$$

we choose that

$$F_1(v)=(v+\mu)^{1+\alpha},F_2(u)=(u+\mu)^{1+\beta},1>\alpha,\beta>0$$

thus we have

$$\frac{1}{(v+\lambda)(u+\mu)^\beta}+\frac{1}{(u+\mu)(v+\lambda)^\alpha}\geqslant 2\delta(T-t)$$

In view of estimate when $t\to T$, we can assume that u, $v>1$. thus we have

$$\frac{1}{v+\lambda}+\frac{1}{(v+\lambda)^\alpha}\geqslant 2\delta(T-t)$$

$$\frac{2}{(v+\lambda m)^\alpha}\geqslant 2\delta(T-t)$$

$$v\leqslant\delta^{-\frac{1}{\alpha}}(T-t)^{-\frac{1}{\alpha}}$$

It is similar to that above mentioned, we can obtain

$$u\leqslant\delta^{-\frac{1}{\beta}}(T-t)^{-\frac{1}{\beta}}$$

Using above lemma, we further follow that the estimate of the Blow-up rate.

We consider following the initial value Problem of equations system

$$\begin{cases}\dfrac{dU_1}{dt}=\delta(u_1+\mu)F_1(v_1+\lambda)\\ \dfrac{dv_1}{dt}=\delta(v_1+\lambda)F_2(U_1+\mu)\qquad 0<t<T\\ U_1(0)=\psi(0)-\mu\\ V_1(0)=\psi(0)-\lambda\end{cases}\tag{27}$$

assume the solution of (27) $\{U_1V_1\}$, we again compare the solution $\{U_1,V_1\}$ with the solution $\{u,v\}$ of the problem

$$\begin{cases}u_t\geqslant\delta uF_1(v)\\ v_t\geqslant\delta vF_2(u)\\ u(0)=\varphi(0)\\ v(0)=\psi(0)\end{cases}\tag{28}$$

We easily follow that

$$U(x,t) \geqslant U_1 + \mu, \qquad v(x,t) \geqslant V_1 + \lambda$$

Now we consider $f_1(s)=s^{1+\alpha}$, $f_2(s)=S^{1+\beta}$, $F_1(s)=s^{1+\alpha_1}$, $F_2(s)=s^{1+\beta_1}$ and $\alpha_1, < \alpha, \beta_1 < \beta$. In view of (27). We get

$$(U_1 + \mu)^{\beta_1} \frac{\mathrm{d}u_1}{\mathrm{d}t} = (v_1 + \lambda)^{\alpha_1} \frac{\mathrm{d}v_1}{\mathrm{d}t}$$

by integration from o to t, we get

$$\frac{1}{\beta_1 + 1}(U_1 + \mu)^{1+\beta_1} - \frac{1}{1+\beta_1}[\varphi(0) + \mu]^{1+\beta_1} = \frac{1}{1+\alpha_1}(V_1 + \lambda)^{1+\alpha_1} - \frac{1}{1+\alpha_1}[\psi(0) + \lambda]^{1+\alpha_1}$$

(1)if

$$\frac{1}{1+\beta_1}[\varphi(0) + \mu]^{1+\beta_1} \geqslant \frac{1}{1+\alpha_1}[\psi(0) + \lambda]^{1+\alpha_1}$$

then

$$(V_1 + \lambda)^{1+\alpha_1} \leqslant \frac{1+\alpha_1}{1+\beta_1}(U_1 + \mu)^{1+\beta_1}$$

thus

$$\frac{dU_1}{dt} \leqslant \delta\left(\frac{1+\alpha_1}{1+\beta_1}\right)(U_1 + \mu)^{2+\beta_1}$$

by integration form t to T, we have

$$U_1 + \mu \geqslant \left[\frac{1}{\delta(1+\alpha_1)}\right]^{\frac{1}{1+\beta_1}}(T-t)^{\frac{-1}{1+\beta_1}} \triangleq C_1(T-t)^{-\frac{1}{1+\beta_1}} \tag{29}$$

Uusing (27)(29) We get

$$\frac{dv_1}{dt} \geqslant (v_1 + \lambda)\frac{1}{\delta(1+\alpha_1)}(T-t)^{-1}$$

by again integration we have

$$(V_1 + \lambda)^{\delta(1+\alpha_1)} \geqslant \frac{c}{T-t}$$

Let t=0 we have $c=T[\psi(0)]^{\delta(1+\alpha_1)}$, it follw that

$$V_1 + \lambda \geqslant c_2(T-t)^{\frac{-1}{\delta(1+\alpha_1)}}, \qquad c_2 = \psi(0)T^{\frac{1}{\delta(1+\alpha_1)}}$$

thus we have

$$\begin{aligned} U(t) \geqslant u(x,t) \geqslant U_1 + \mu \geqslant c_1(T-t)^{-\frac{1}{1+\beta_1}},\ c_1 = \left[\frac{1}{\delta(1+\alpha_1)}\right]^{\frac{1}{1+\beta_1}} \\ V(t) \geqslant v(x,t) \geqslant V_1 + \lambda \geqslant c_2(T-t)^{-\frac{1}{\delta(1+\alpha_1)}},\ c_2 = \psi(0)T^{\frac{1}{\delta(1+\alpha_1)}} \end{aligned} \tag{30}$$

(2) If

$$\frac{1}{1+\beta_1}[\varphi(0) + \mu]^{1+\beta_1} \leqslant \frac{1}{1+\alpha_1}\{\psi(0) + \lambda]^{1+\alpha_1}$$

then

$$(U_1 + \mu)^{1+\beta_1} \leqslant \frac{1+\beta_1}{1+\alpha_1}(V_1 + \lambda)^{1+\alpha_1}$$

thus we have

$$\frac{dv_1}{dt} \leqslant \delta\left(\frac{1+\beta_1}{1+\alpha_1}\right)(v_1+\lambda)^{2+\alpha_1}$$

by integration from t to T, We get

$$V_1+\lambda \geqslant \left[\frac{1}{\delta(1+\beta_1)}\right]^{\frac{1}{1+\alpha_1}}(T-t)^{-\frac{1}{1+\alpha_1}} \triangleq c_3(T-t)^{-\frac{1}{1+\alpha_1}}$$

and from (27) we have that

$$U_1+\mu \geqslant c_4(T-t)^{-\frac{1}{\delta(1+\beta_1)}}, c_4=\varphi(0)T^{\frac{1}{\delta(1+\beta_1)}}$$

thus we get

$$U(t) \geqslant u(x,t) \geqslant U_1+\mu \geqslant c_4(T-t)^{-\frac{1}{\delta(1+\beta_1)}}$$
$$V(t) \geqslant v(x,t) \geqslant V_1+\lambda \geqslant c_3(T-t)^{-\frac{1}{1+\alpha_1}} \qquad (31)$$

where (30)(31) hold for 0<t<T.

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关于半线性抛物型方程组解的短破率

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摘 要 本文将对某一类半线性抛物型方程组在解的单点爆破情况下,估计当点邻近短爆点时,解的爆破率.

关键词 爆破时刻,爆破率,紧子集