Lower and Upper Orientable Strong Radius and Diameter of Cartesian Product of Paths*

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Abstract: For two vertices u and v in a strong digraph D, the strong distance $sd(u,v)$ between u and v is the minimum size (the number of arcs) of a strong sub-digraph of D containing u and v. For a vertex v of D, the strong eccentricity $se(v)$ is the strong distance between v and a vertex farthest from v. The strong radius $srad(D)$ (resp. strong diameter $sdiam(D)$) is the minimum (resp. maximum) strong eccentricity among the vertices of D. The lower (resp. upper) orientable strong radius $srad(G)$ (resp. $SRAD(G)$) of a graph G is the minimum (resp. maximum) strong radius over all strong orientations of G. The lower (resp. upper) orientable strong diameter $sdiam(G)$ (resp. $SDIAM(G)$) of a graph G is the minimum (resp. maximum) strong diameter over all strong orientations of G. In this paper, we determine the lower orientable strong radius and strong diameter of Cartesian product of paths, and give bounds on the upper orientable strong radius and a conjecture of the upper orientable strong diameter of Cartesian product of paths.

Key words: Strong distance; Lower orientable strong radius and strong diameter; upper orientable strong radius and strong diameter

CLC number: O157.5 **Document Code:** A **Article ID:** 1000-2839(2009)01-0033-05

路和路的笛卡尔积的最小和最大定向强半径和强直径

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摘 要： 强有向图D中任意两个点u,v的强距离$sd(u,v)$ 定义为D中包含u和v的最小有向强子图D_{uv}的大小(弧的数目). D中一点u 的强离心率$se(u)$定义为u到其他顶点的强距离的最大值. 强有向图D的强半径$srad(D)$（相应的强直径$sdiam(D)$）定义为D中所有顶点强离心率的最小值（相应的最大值）. 无向图G的最小定向强半径$srad(G)$（相应的最大定向强半径$SRAD(G)$）定义为D中所有强定向的强半径的最小值（相应的最大值）. 无向图G的最小定向强直径$sdiam(G)$（相应的最大定向强直径$SDIAM(G)$）定义为D中所有强定向的强直径的最小值（相应的最大值）. 本文确定了路和路的笛卡尔积的最小定向强半径$srad(P_m\times P_n)$和强直径的值$sdiam(P_m\times P_n)$，给出了最大定向强半径$SRAD(P_m\times P_n)$的界并提出关于最大定向强直径$SDIAM(P_m\times P_n)$的一个猜想.

关键词： 强距离; 最小定向强半径和强直径; 最大定向强半径和强直径

1 Introduction

In [2], Chartrand *et al.* defined the strong distance $sd(u,v)$ between two vertices u and v in a strong digraph D as the minimum size (the number of arcs) of a strong sub-digraph of D containing u and v. A (u,v)-geodesic is a strong sub-digraph of D of size $sd(u,v)$ containing u and v. Here we consider only strong oriented graphs of simple graphs. Clearly, if $u \neq v$ then $sd(u,v) \geq 3$. And $sd(u,v) = 3$ if and only if u and

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v belong to a directed 3-cycle in D. Fig. 1 shows a strong digraph with $sd(w,v)=3$, $sd(u,w)=5$ and $sd(u,x)=6$.

The strong eccentricity $se(v)$ of a vertex v in a strong digraph D is

$$se(v)=\max\{sd(v,x)|\ x\in V(D)\}.$$

The strong radius $srad(D)$ of D is

$$srad(D)=\min\{se(v)|\ v\in V(D)\},$$

while the strong diameter $sdiam(D)$ of D is

$$sdiam(D)=\max\{se(v)|\ v\in V(D)\}.$$

The strong radius and strong diameter of a strong digraph satisfy the following inequality.

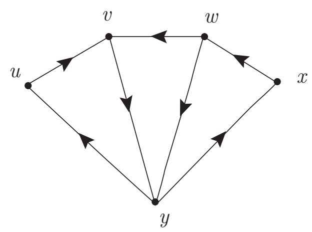

Fig 1. Strong distance in a strong digraph

Theorem 1[2] For every strong digraph D,

$$srad(D)\le sdiam(D)\le 2srad(D).$$

In [2], Chartrand *et al*. showed that, for any integers r, d with $3\le r\le d\le 2r$, there existed a strong oriented graph D such that $srad(D)=r$ and $sdiam(D)=d$; and gave an upper bound on strong diameter of a strong oriented graph D.

Theorem 2[2] If D is a strong oriented graph of order $n\ge 3$, then

$$sdiam(D)\le\lfloor\frac{5(n-1)}{3}\rfloor.$$

In [5], Dankelmann *et al*. observed the structure of a (u,v)-geodesic for any $u,v\in V(D)$, where D is a strong digraph, and established the following result.

Theorem 3[5] Let D be a strong digraph. For $u,v\in V(D)$, let D_{uv} be a (u,v)-geodesic. Then $D_{uv}=P\cup Q$, where P and Q are a directed (u,v)-path and a directed (v,u)-path, respectively, in D_{uv}. There exist directed cycles $C_1,\ C_2,\ \ldots,\ C_k\subset D_{uv}$ such that

(i) $u\in V(C_1)$, $v\in V(C_k)$;

(ii) $\bigcup_{i=1}^k C_i=D_{uv}$;

(iii) each C_i contains at least one arc that is in P but not in Q, and at least one arc that is in Q but not in P;

(iv) $C_i\cap C_{i+1}$ is a directed path for $i=1,\ 2,\ \ldots,k-1$;

(v) $V(C_i)\cap V(C_j)=\emptyset$ for $1\le i<j-1\le k-1$.

In [5], Dankelmann *et al*. presented the upper bounds on strong diameter of D in terms of order n, directed girth $g\ge 2$, and strong connectivity κ as $sdiam(D)\le\lfloor\frac{(n-1)(g+2)}{g}\rfloor$ and $sdiam(D)\le\frac{5}{3}(1+\frac{n-2}{\kappa})$. They also gave an upper bound on the strong radius of a strong oriented graph D.

Theorem 4[5] For any strong oriented graph D of order n, $srad(D)\le n$, and this bound is sharp.

In [7], for a connected graph G, Lai *et al*. defined the lower orientable strong radius $srad(G)$ of G as

$$srad(G)=\min\{srad(D)|\ D\ is\ a\ strong\ orientation\ of\ G\},$$

while the upper orientable strong radius $SRAD(G)$ of G is

$$SRAD(G)=\max\{srad(D)|\ D\ is\ a\ strong\ orientation\ of\ G\},$$

they also defined the lower orientable strong diameter $sdiam(G)$ of G as

$$sdiam(G)=\min\{sdiam(D)|\ D\ is\ a\ strong\ orientation\ of\ G\},$$

while the upper orientable strong diameter $SDIAM(G)$ of G is

$$SDIAM(G)=\max\{sdiam(D)|\ D\ is\ a\ strong\ orientation\ of\ G\}.$$

Some known results of strong distance, strong radius and strong diameter of graphs are due to [3-5, 8-9].

In this paper, we investigate strong orientations of Cartesian product $P_m\times P_n$ of two paths P_m and P_n. $P_m\times P_n$ might denote a map of streets in a city. A strong orientation of $P_m\times P_n$ with lower strong diameter could be used to improve traffic jam of a city. We determine the lower orientable strong radius and strong diameter of Cartesian product of paths, and give bounds on the upper orientable strong radius, and a conjecture of the upper orientable strong diameter of Cartesian product of paths.

2 The lower orientable strong radius and diameter of Cartesian product of paths

In this section, we consider the lower orientable strong radius and diameter of Cartesian product of paths.

The Cartesian product $G=G_1\times G_2$ of two vertex-disjiont graphs G_1 and G_2 has vertex set $V(G)=V(G_1)\times V(G_2)$ and two vertices (u_1,u_2) and (v_1,v_2) of G are adjacent if and only if either $u_1=v_1$ and $u_2v_2\in E(G_2)$ or $u_2=v_2$ and $u_1v_1\in E(G_1)$.

For the Cartesian product $P_m\times P_n$ of paths P_m and P_n, with $m\ge 2$ and $n\ge 2$, let $V(P_m\times P_n)=\{(i,j)|\ 1\le i\le m,\ 1\le j\le n\}$, for $m\ge 2$ and $n\ge 2$. Thus, two vertices (i,j) and (i',j') are adjacent in $P_m\times P_n$ if and only if $i=i'$ and $|j-j'|=1$ or $j=j'$ and $|i-i'|=1$.

Proposition 1 Let D be a strong orientation of a graph G. Then $sd_D(u,v)\ge 2d_G(u,v)$ for any $u,v\in V(D)$, where $d_G(u,v)$ denotes the length of a shortest (u,v)-path in G.

Proof For any $u,v\in V(D)$, let D_{uv} be a (u,v)-geodesic. By Theorem 3, there exist directed cycles $C_1,C_2,\ldots,C_k\subset D_{uv}$ such that$D_{uv}=\bigcup_{i=1}^{k}C_i$ satisfying (i)-(v) in Theorem 3. Furthermore, $D_{uv}=P\cup Q$, where P is a directed (u,v)-path and Q is a directed (v,u)-path. Let $P_i=C_i\cap C_{i+1}$ be the directed path for $1\le i\le k-1$, a_i and b_i be the first and last vertex of P_i, respectively. Let

$$\widetilde{P}=\begin{cases} P(u,a_1)\cup Q(a_1,b_2)\cup P(b_2,a_3)\cup\cdots\cup Q(a_{k-1},v), & \text{if } k \text{ is even};\\ P(u,a_1)\cup Q(a_1,b_2)\cup P(b_2,a_3)\cup\cdots\cup P(b_{k-1},v), & \text{if } k \text{ is odd}.\end{cases}$$

and

$$\widetilde{Q}=\begin{cases} Q(u,b_1)\cup P(b_1,a_2)\cup Q(a_2,b_3)\cup\cdots\cup P(b_{k-1},v), & \text{if } k \text{ is even};\\ Q(u,b_1)\cup P(b_1,a_2)\cup P(a_2,b_3)\cup\cdots\cup Q(a_{k-1},v), & \text{if } k \text{ is odd}.\end{cases}$$

where $P(x,y)$ (resp. $Q(x,y)$) denotes the undirected sub-path of P (resp. Q) with end vertices x and y. Clearly, $\widetilde{P}$ and $\widetilde{Q}$ are two edge-disjoint undirected path connecting u and v. So $sd_D(u,v)=|A(D_{uv})|\ge|E(\widetilde{P})|+|E(\widetilde{Q})|\ge 2d_G(u,v)$.

Theorem 5 Let $2 \le m \le n$. Then $srad(P_m \times P_n) = 2(\lfloor \frac{m}{2} \rfloor + \lfloor \frac{n}{2} \rfloor)$.

Proof For any strong orientation D of $P_m \times P_n$ and $u \in V(D)$, let v be a vertex farthest from u, then, $d(u,v) \ge \lfloor \frac{m}{2} \rfloor + \lfloor \frac{n}{2} \rfloor$, and $sd_D(u,v) \ge 2d(u,v) \ge 2(\lfloor \frac{m}{2} \rfloor + \lfloor \frac{n}{2} \rfloor)$ by Proposition 5. Hence, $se(u) \ge 2(\lfloor \frac{m}{2} \rfloor + \lfloor \frac{n}{2} \rfloor)$. Therefore, $srad(D) \ge 2(\lfloor \frac{m}{2} \rfloor + \lfloor \frac{n}{2} \rfloor)$, which implies that $srad(P_m \times P_n) \ge 2(\lfloor \frac{m}{2} \rfloor + \lfloor \frac{n}{2} \rfloor)$.

Now it is enough to give a strong orientation D of $P_m \times P_n$ such that $srad(D) = 2(\lfloor \frac{m}{2} \rfloor + \lfloor \frac{n}{2} \rfloor)$.

Let $u = (\lceil \frac{m}{2} \rceil, \lceil \frac{n}{2} \rceil) \in V(P_m \times P_n)$, D be a strong orientation of $P_m \times P_n$ such that $V(D) = V(P_m \times P_n)$ and arc set

$$A(D) = \{((i,j),(i,j+1)) | 1 \le i \le m \text{ and } i \ne \lceil \frac{m}{2} \rceil,\ 1 \le j \le \lceil \frac{n}{2} \rceil - 1\}$$

$$\cup \{((i,j),(i,j-1)) | 1 \le i \le m \text{ and } i \ne \lceil \frac{m}{2} \rceil,\ \lceil \frac{n}{2} \rceil + 1 \le j \le n\}$$

$$\cup \{((i,j),(i-1,j)) | 1 \le j \le n \text{ and } j \ne \lceil \frac{n}{2} \rceil,\ 2 \le i \le \lceil \frac{m}{2} \rceil\}$$

$$\cup \{((i,j),(i+1,j)) | 1 \le j \le n \text{ and } j \ne \lceil \frac{n}{2} \rceil,\ \lceil \frac{m}{2} \rceil \le i \le m-1\}$$

$$\cup \{((\lceil \frac{m}{2} \rceil, j),(\lceil \frac{m}{2} \rceil, j+1)) | \lceil \frac{n}{2} \rceil \le j \le n-1\} \cup \{((\lceil \frac{m}{2} \rceil, j),(\lceil \frac{m}{2} \rceil, j-1)) | 2 \le j \le \lceil \frac{n}{2} \rceil\}$$

$$\cup \{((i, \lceil \frac{n}{2} \rceil),(i-1, \lceil \frac{n}{2} \rceil)) | \lceil \frac{m}{2} \rceil + 1 \le i \le m\} \cup \{((i, \lceil \frac{n}{2} \rceil),(i+1, \lceil \frac{n}{2} \rceil)) | 1 \le i \le \lceil \frac{m}{2} \rceil - 1\}.$$

Clearly, the orientation D is strong and, for any $v \in V(D) - u$, v and u are contained in a directed cycle of length at most $2(\lfloor \frac{m}{2} \rfloor + \lfloor \frac{n}{2} \rfloor)$. So $se(u) = 2(\lfloor \frac{m}{2} \rfloor + \lfloor \frac{n}{2} \rfloor)$. Moreover, for any $v \in V(D) - u$, there is a vertex w such that $d_G(v,w) \ge (\lfloor \frac{m}{2} \rfloor + \lfloor \frac{n}{2} \rfloor)$, and so $se(v) \ge sd(v,w) \ge 2d_G(v,w) \ge 2(\lfloor \frac{m}{2} \rfloor + \lfloor \frac{n}{2} \rfloor)$. Thus $se(u) = min\{se(v) | v \in V(D)\}$, and $srad(D) = 2(\lfloor \frac{m}{2} \rfloor + \lfloor \frac{n}{2} \rfloor)$. Hence, $srad(P_m \times P_n) = 2(\lfloor \frac{m}{2} \rfloor + \lfloor \frac{n}{2} \rfloor)$. The proof is completed.

For any strong orientation D of $P_m \times P_n$, $sd_D((1,1),(m,n)) \ge 2d_G((1,1),(m,n)) = 2(m+n-2)$. So $sdiam(D) \ge 2(m+n-2)$, that is, $sdiam(P_m \times P_n) \ge 2(m+n-2)$, where $2 \le m \le n$. In the following, we will prove $sdiam(P_m \times P_n) = 2(m+n-2)$.

Theorem 6 Let $2 \le m \le n$. Then $sdiam(P_m \times P_n) = 2(m+n-2)$.

Proof From the above analysis, we only need to give a strong orientation D of $P_m \times P_n$ such that $sdiam(D) = 2(m+n-2)$.

Case 1. $m = 2$ and $n \ge 2$. Let $A(D) = \{((1,i),(1,i+1)) | 1 \le i \le n-1\} \cup \{((2,i),(2,i-1)),((1,i),(2,i)) | 2 \le i \le n\} \cup \{((2,1),(1,1))\}$, then D is a strong orientation of $P_2 \times P_n$ with $sdiam(D) = 2n$.

Case 2. $m = n = 3$. Let $A(D) = \{((1,1),(1,2)),((1,2),(1,3)),((1,3),(2,3)),$
$((2,3),(3,3)),((3,3),(3,2)),((3,2),(3,1)),((3,1),(2,1)),((2,1),(1,1)),((1,2),(2,2)),$
$((3,2),(2,2)),((2,2),(2,3)),((2,2),(2,1))\}$, then D is a strong orientation of $P_3 \times P_3$ with $sdiam(D) = 8$.

Case 3. $m \ge 3$ and $n \ge 4$. Let $A(D) =$

$$\{((i,j),(i,j+1)) | 1 \le i \le \lfloor \frac{m}{2} \rfloor,\ 1 \le j \le n-1\} \cup \{((i,j),(i+1,j)) | 1 \le i \le m-1, \lfloor \frac{n}{2} \rfloor + 1 \le j \le n\}$$

$$\cup \{((i,j),(i-1,j)) | 2 \le i \le m,\ 1 \le j \le \lfloor \frac{n}{2} \rfloor\} \cup \{((i,j),(i,j-1)) | \lfloor \frac{m}{2} \rfloor + 1 \le i \le m,\ 2 \le j \le n\}.$$

It is easy to verify that $sd((1,1),(m,n)) = 2(m+n-2) = max\{sd(u,v) | u,v \in V(D)\}$. So $sdiam(D) \le 2(m+n-2)$. On the other hand, by the above analysis, we have $sdiam(D) \ge 2(m+n-2)$. Therefore, $sdiam(D) = 2(m+n-2)$.

The proof is completed.

3 The upper orientable strong radius and diameter of Cartesian product of paths

In this section, we give a conjecture of the upper orientable strong diameter and the bounds on the upper orientable strong radius of Cartesian product of paths.

Lemma 1 Let $2\le m\le n$. Then there exists a strong orientation D of $P_m\times P_n$ such that $sdiam(D)=mn+n-2$.

Proof Let P be the Hamiltonian path of $P_m\times P_n$ starting from $x=(1,1)$ such that for any vertices (i,j) and (i',j'), if $j<j'$, then, on P, (i,j) precedes (i',j'). If n is odd (resp. even) then the terminating vertex y of P is equal to (m,n) (resp. $(1,n)$). Let D be a strong orientation of $P_m\times P_n$ such that the path P is a shortest directed (x,y)-path, and let Q be a shortest directed (y,x)-path. Then $A(Q)\setminus A(P)$ contains exactly $n-1$ arcs, and $P\cup Q$ is a (x,y)-geodesic with $sd(x,y)=mn+n-2$, which contains all vertices of D. Consequently, $sdiam(D)=mn+n-2$.

By Lemma 1, we have the following.

Theorem 7 Let $2\le m\le n$. Then $SDIAM(P_m\times P_n)\ge mn+n-2$.

In addition, we have the following conjecture.

Conjecture 1 Let $2\le m\le n$, then $SDIAM(P_m\times P_n)=mn+n-2$.

Obviously, the conjecture holds for $m=2$.

Now, by Theorem 1, Theorem 4 and Theorem 9, we can give the bounds on the upper orientable strong radius.

Theorem 8 Let $2\le m\le n$. Then $SRAD(P_2\times P_2)=4$, and $\lceil\frac{mn+n-2}{2}\rceil\le SRAD(P_m\times P_n)\le mn$ for $(m,n)\ne(2,2)$.

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