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The Finite Groups With the Same Degree and the Prime Number of Non-linear Irreducible Characters

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Abstract: In this paper, we classify all the finite groups which have prime number of non-linear irreducible characters and whose degrees for these non-linear irreducible characters are equal.

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In this paper we only consider finite groups G. We let $\mathrm{Irr}(G)$ denote all complex irreducible characters of G. Let $\mathrm{cd}(G)$ denote the set of degrees of χ when χ ranges over $\mathrm{Irr}(G)$. There are many results to classify finite groups G based on $\mathrm{cd}(G)$. For example, for a given set $S=\{1,a,b\}$ with some conditions on $a, b\in \mathbf{N}$, the integer set, Thomas Noritzsch[10] studies the finite groups satisfying $|\mathrm{cd}(G)|=S$. Some other related results can be found[6].

Let $\mathrm{Irr}_1(G)$ denote all non-linear irreducible characters of a group G. In 1992, Y. Berkovich[1] et al. classified all finite groups with distinct degrees for distinct non-linear characters of G. In 1996, Y. Berkovich[4] further classified all finite solvable groups with only two nonlinear irreducible characters having equal degree.

In the case of $|\mathrm{Irr}_1(G)|=1$ or 2 or 3, it is almost well-known to classify these finite groups by the published articles[8,4,2]. Among these results[4,2] to classify all finite groups of $|\mathrm{Irr}_1(G)|=3$, one important case is $|\mathrm{cd}(G)|=2$. In this paper, we consider more general cases and suppose $\mathrm{Irr}_1(G)$ has prime elements. We classify all these finite groups in two cases: (1) G is nilpotent; (2) G is non-nilpotent. The group in the first case still be unknown in a lot of cases. For the second case, that is to say, G is not nilpotent and $|\mathrm{cd(G)}|=2$, there is a result to describe this type of group in [9] and T. Noritzsch[10]. We will give a more delicate classification based on their results.

In this paper, Our notations are most from Isaacs's book[5]. We use G' to denote the commutator subgroup of G. $Z(G)$ denotes the center of the group G. $F(G)$ and $\Phi(G)$ denotes the Fitting group of G and Frattini subgroup of G, respectively. Usually p denotes a prime

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number. For $\lambda \in \mathrm{Irr}(H)$ and $H \le G$, we denote the induction of λ to G by λ^G.

1 Lemmas and Definitions

For the convenience of the reader, we collect some well-known facts as our lemmas in this section.

The first result is about finite group G with only one non-linear irreducible character.

Lemma 1.1[8] G has only one nonlinear irreducible character if and only if one of the following assertions holds:

(1) G is an extra-special 2-group.

(2) G is a Frobenius group with an abelian Frobenius kernel N and an abelian Frobenius complement H of $|H| = |N| - 1$.

Lemma 1.2[9, 10] Let G be a non-nilpotent group and $\mathrm{cd}(G) = \{1, m\}$, then

(1) $F(G)$ is the only abelian subgroup of G of index m;

(2) If a Sylow p-subgroup of G is non-abelian, then $|G : F(G)| = p$;

(3) If all Sylow subgroups of G are abelian, then $F(G) = G' \times Z(G)$, $G/Z(G)$ is a Frobenius group with Frobenius kernel $F(G)/Z(G)$.

Definition 1.3[10] A non-abelian p-group G is called semiextra-special if for all maximal subgroups N of $Z(G)$, the factor group G/N is extra-special.

Definition 1.4[7] A group G is called a Camina group if G' is a proper normal subgroup and $\{[g, x] : x \in G\} \supseteq G'$ for all $g \in G - G'$.

Lemma 1.5[10] Let G be a non-abelian p-group, then the following two statements are equivalent:

(1) $Z(G) = G'$ and $\mathrm{cd}(G) = \{1, |G : Z(G)|^{\frac{1}{2}}\}$;

(2) G is semiextra-special.

Lemma 1.6[7] If G is a finite Camina p-group, then $G_4 = 1$.

2 Main Theorems

Our main theorems about classification of all finite groups satisfying our conditions will be proved in this section.

The first case is nilpotent group G.

Theorem 2.1 Let G be nilpotent and $|\mathrm{Irr}_1(G)| = q, q$ a prime number. Then $|\mathrm{cd}(G)| = 2$ if and only if one of the following assertions holds:

(1) G is an extra-special 3-group;

(2) $|G| = 2^{2m+2}, |Z(G)| = 4, |G'| = 2$;

(3) $D(16)$, $Q(16)$, $SD(16)$;

(4) G is a semiextra-special 2-group with $|G| = 2^{2m+t}$ and $|G'| - 1 = 2^t - 1 = q$ is a prime number;

(5) $G = P \times A$, P is an extra-special 2-group, $|A| = q, q > 2$.

Proof If G is nilpotent with $|\mathrm{Irr}_1(G)| = 2$ and $|\mathrm{cd}(G)| = 2$, then it is one of groups in (1), (2) by [2].

If G is nilpotent with $|\mathrm{Irr}_1(G)| = q$, where $2 < q$ is a prime number.

First, if G is p-group, by Burnside theorem[8, Question 3.16], if $|G|$ is odd, then the number of nonlinear irreducible characters which have same degrees is even, this is contradiction with $q \neq 2$. We get $p = 2$. Assume that $|G| = 2^n$, $|G : G'| = 2^k$, $\mathrm{cd}(G) = \{1, 2^m\}$, then $2^n = 2^k + q \times 2^{2m}$, so $2^{n-2m} = 2^{k-2m} + q$. we get $k = 2m$ and $4 \mid (q+1)$. By [3] we have that G is a Camina 2-group and of class 2 or of maximal class.

(a) The nilpotency class of $G : \mathrm{cl}(G) = 2$.

That is $G' \leq Z(G)$. According to $2^{2m} \leq |G : Z(G)|$, we have

$$|Z(G)| \leq 2^{n-2m} = |G'|,$$

so $Z(G) = G'$ and $\mathrm{cd}(G) = \{1, |G : Z(G)|^{\frac{1}{2}}\}$. By Lemma 1.5 we get the group (4).

(b) $|G| = 2^n$, $\mathrm{cl}(G) = n-1$.

$G_n = 1$, then $|G'| = 2^{n-1}$ or 2^{n-2}. Since $|2^{n-2m}| = |G'|$, we get $|G'| = 2^{n-2}$ and $\mathrm{cd}(G) = \{1, 2\}$. So $n \geq 4$. But by Lemma 1.6, $G_4 = 1$. Then $n = 4$. It is one of the groups in (3) by [2].

Second, if G is nilpotent, G is not 2-group, then there exists $P \in \mathrm{Sylow}_p(G)$, P non-abelian, p is a prime factor of $|G|$. Assume $G = P \times A$. $|A| \neq 1$. By ([8], Theorem 4.21)

$$\mathrm{Irr}(G) = \{\theta \times \alpha | \theta \in \mathrm{Irr}(P), \alpha \in \mathrm{Irr}(A)\},$$

then

$$|\mathrm{Irr}_1(G)| = q = |P : P'||\mathrm{Irr}_1(A)| + |A : A'||\mathrm{Irr}_1(P)| + |\mathrm{Irr}_1(A)||\mathrm{Irr}_1(P)|.$$

Since $|\mathrm{cd}(G)| = 2$, P nonabelian, then $|\mathrm{cd}(P)| = 2$, $|\mathrm{cd}(A)| = 1$. So $q = |A||\mathrm{Irr}_1(P)|$, then $|A| = q$ and $|\mathrm{Irr}_1(P)| = 1$, by Lemma 1.1 we have the result (5).

Obviousely, all groups G in (1)–(5) are satisfying our conditions.

Remark 2.2 By Theorem 2.1 above, If G is a p-group and satisfies the conditions in Theorem 2.1, then G is a 2-group.

Now we consider another case: G is non-nilpotent, $|\mathrm{Irr}_1(G)| = q$(q is a prime number) and $|\mathrm{cd}(G)| = 2$.

Theorem 2.3 Let G be non-nilpotent and $|\mathrm{Irr}_1(G)| = q$, where q is a prime number. Then $|\mathrm{cd}(G)| = 2$ if and only if one of the following assertions holds:

(1) $F(G)$ is abelian, $|G : F(G)| = |F(G) : G'| = p$, p is a prime divisior of $|G'|$, $|G'| - 1$ is a prime number.

(2) G is a semidirect product of G' with A. G' is an abelian p-group and A is cyclic, $|A| = q(|G'| - 1)$(q is a prime number). $G/Z(G)$ is a Frobenius group with Frobenius kernel isomorphic to G' and an a cyclic Frobenius complement of order $|G'| - 1$.

(3) $G = N_1 \times N_2$, where N_1 is Frobenius group with an elementary abelian kernel N and a cyclic complement H of $|H| = |N| - 1$. $|N_2| = q$, where q is a prime number.

(4) G is Frobenius group with an elementary abelian kernel N and a cyclic complement H of $q|H| = |N| - 1$,where q is a prime number.

Proof If G is type (1), by $|G| = |G : G'| + \sum_{\chi \in \mathrm{Irr}_1(G)} \chi(1)^2$, we have $|\mathrm{Irr}_1(G)| = |G'| - 1 = q$, $|\mathrm{cd}(G)| = 2$.

If G is type (2), we have $|Z(G)| = q$. Suppose $\chi \in \mathrm{Irr}_1(G/Z(G))$, then $\chi(1) = |G'| - 1$. Assume that $[\chi_{G'}, \lambda] \neq 0$, $\lambda \in \mathrm{Irr}(G')$. Then we have that $\lambda \neq I_{G'}$. $G/G' \cong A$ is cyclic.

Now we want to prove $\lambda^G = \sum_{i=1}^{s}\chi_i$, $\chi_1 = \chi$, $\chi_i \in \mathrm{Irr}_1(G)$, where all χ_i have the same degree $\chi(1) = |G'| - 1, i = 1, 2, \cdots, s$. Let $T = I_G(\lambda)$. Since $T = G'(T \cap A)$ is a semidirect of G' with $T \cap A, \lambda$ can be extended to a $\mu \in \mathrm{Irr}(T)$. Then by T/G' is cyclic, we have

$$\lambda^T = \sum_{\alpha_i \in \mathrm{Irr}(T/G')} \mu\alpha_i,$$

where $\mu\alpha_i$ run over $\mathrm{Irr}(T|\lambda)$ when α_i run over $\mathrm{Irr}(T/G')$. Thus by Clifford Theorem[8, Theorem 6.11], we have

$$\lambda^G = \sum_{\alpha_i \in \mathrm{Irr}(T/G')} (\mu\alpha_i)^G.$$

Let $\chi_i = (\mu\alpha_i)^G$, then they will run over all $\mathrm{Irr}(G|\lambda)$ when α_i run over $\mathrm{Irr}(T/G')$. We can see that $\chi_i(1) = |G/T|$ is the same number for all χ_i. In particular, suppose $\chi = \chi_1$, then $\chi_i(1) = \chi_1(1) = |G'| - 1$.

Thus $|G : G'| = s(|G'| - 1)$, so $s = q$ and $\mathrm{Irr}_1(G)$ has at least q non-linear irreducible characters of degree $|G'| - 1$. From $|G| = |G : G'| + \sum_{\chi \in \mathrm{Irr}_1(G)} \chi(1)^2$, it follows that $|\mathrm{Irr}_1(G)| = q, \mathrm{cd}(G) = \{1, |G'| - 1\}$.

If G is type (3), by Lemma 1.1, N_1 has only one non-linear irreducible character. For $|N_2| = q$, then $|\mathrm{Irr}_1(G)| = q$, and $|\mathrm{cd}(G)| = 2$.

If G is type (4), since G/N is abelian, then $(I_N)^G = \sum_{\Psi \in \mathrm{Lin}(G/N)} \Psi$. For any $\chi \in \mathrm{Irr}_1(G)$, we have $N \nleq \ker\chi$, so we can see that $\mathrm{Irr}_1(G) = \mathrm{Irr}(G) - \mathrm{Irr}(G/N)$, by the properties of Frobenius group. By [8, Theorem 6.34], we know the map $\lambda \mapsto \lambda^G$ is a map from $\mathrm{Irr}(N) - \{I_N\}$ onto $\mathrm{Irr}(G) - \mathrm{Irr}(G/N)$. For any $\lambda, \mu \in \mathrm{Irr}(N), \lambda^G = \mu^G$ if and only if $\lambda = \mu^k, k \in G$. So $|\mathrm{Irr}_1(G)|$ is the number of the orbits of the conjugacy action of G on $\mathrm{Irr}(N) - \{I_N\}$.

Since N is abelian, then $\mathrm{cd}(G) = \{1, |G : N|\} = \{1, |H|\}, q|H| = |N| - 1$, so the conjugacy action of G on $\mathrm{Irr}(N) - \{I_N\}$ has q orbits, then $|\mathrm{Irr}_1(G)| = q$.

Conversely, G is non-nilpotent and $\mathrm{cd}(G) = \{1, m\}$. We get $|F(G)| = |G'| \times |Z(G)|$, and $G/F(G)$ is a cyclic group of order m. Thus G is one of the following two groups by [9]:

(a) $F(G)$ is abelian, $|G : F(G)| = m = p$, p is a prime divisior of $|G'|$.

If $|\mathrm{Irr}_1(G)| = q$(a prime number), from $|G| = |G : G'| + \sum_{\chi \in \mathrm{Irr}_1(G)} \chi(1)^2$ and $|F(G)| = |G'| \times |Z(G)|$ it follows that $|Z(G)|(|G'| - 1) = pq$. Since G is non-nilpotent, then $G' \nleq \Phi(G)$. Therefore there exists a maximal subgroup A, we have $G' \nleq A$, then $G = G'A$. If $|G'| = 2$, we obtain $A \trianglelefteq G$ and A abelian, then G is abelian, a contradiction. So $|Z(G)| = p, |G'| = q + 1$ by $p \mid |G'|$. We get group (1).

(b) $G' \cap Z(G) = 1$ and $\bar{G} = G/Z(G)$ is a Frobenius group with Frobenius kernel $(\bar{G})' = G' \times Z(G)/Z(G)$ and a cyclic complement $\bar{A} = A/Z(G)$ of order $|G : G' \times Z(G)| = m$. So $\mathrm{cd}(\bar{G}) = \mathrm{cd}(G) = \{1, m\}$. If $|\mathrm{Irr}_1(G)| = q$(a prime number), then $|\mathrm{Irr}_1(\bar{G})| = i, i = 1, 2, \cdots, q$. Considering the conjugacy action of $\bar{G}$ on $\mathrm{Irr}((\bar{G})') - \{I_{(\bar{G})'}\}$, by the same arguments as above, we get $im = |(\bar{G})'| - 1 = |G'| - 1$.

(i) $|\mathrm{Irr}_1(\bar{G})| = 1$.

Then $\mathrm{cd}(G) = \{1, |G'| - 1\}$. By $|G| = |G : G'| + \sum_{\chi \in \mathrm{Irr}_1(G)} \chi(1)^2$, it follows that $|G : G'|(|G'| - 1) = q(|G'| - 1)^2$, so $|A|(|G'| - 1) = q(|G'| - 1)^2$. Since $|A| = |Z(G)||\bar{A}| = |Z(G)|(|G'| - 1)$, then $|Z(G)| = q$.

In order to get the structure of G, we discuss as follows according to A being cyclic or not.

If A is cyclic, then we have type (2).

If A is not cyclic, we suppose tht $A = \langle a, Z(G)\rangle$. If $\langle a\rangle \bigcap Z(G) \neq 1$, then $Z(G) \leq \langle a\rangle$, so A is cyclic, a contradiction. Therefore $A = \langle a\rangle \times Z(G)$. We consider $N_1 = G'\langle a\rangle$, for any $h \in C_{N_1}(g), \forall g \in G', g \neq 1$, then $hZ(G) \in C_{G/Z(G)}(gZ(G)) = (G' \times Z(G))/Z(G)$, so $h \in G' \times Z(G)$. Assume $h = g_1a_1 = g_2a_2$, where $g_1, g_2 \in G'$, $a_1 \in Z(G)$, $a_2 \in \langle a\rangle$. We get $g_2^{-1}g_1 = a_2a_1^{-1} \in G' \cap A = 1$, then $a_1 = a_2 \in Z(G) \cap \langle a\rangle = 1$, therefore $h \in G'$. By [7, Question 7.1], N_1 is a Frobenius group with G' as Frobenius kernel and $\langle a\rangle$ as Frobenius complement. We get type (3).

(ii) When $q > 2$, considering $|\mathrm{Irr}_1(\bar{G})| = j$, $j = 2, 3, \cdots, q-1$, so that $\mathrm{cd}(G) = \{1, \frac{|G'|-1}{j}\}$, as the same calculation as that in (i), we have $|Z(G)| = \frac{q}{j}$, a contradiction.

(iii) $|\mathrm{Irr}_1(\bar{G})| = q$.

As the same calculation as that in (i), we have $|Z(G)| = 1$, type (4) follows.

With the two theorems above, the classification of all finite groups satisfying our conditions is completed.

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非线性不可约特征标维数相等且为素数个的有限群

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摘要：对于具有素数个非线性不可约特征标且它们的维数相等的有限群，我们给出一个分类.

关键词：特征标；维数；导群