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Boundedness for Commutators of Multilinear Fractional Integral Operators

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Abstract: By some estimates for the sharp maximal functions, the $L^p(\mathbb{R}^n)$-boundedness for a class of commutators for multilinear fractional integral operators with $\mathrm{BMO}(\mathbb{R}^n)$ functions are obtained.

Key words: multilinear operator; fractional integral; commutator; sharp maximal function; fractional maximal operator; $\mathrm{BMO}(\mathbb{R}^n)$

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0 Introduction

In recent years, the study of multilinear operators has been attracting many researchers. Many results obtained parallel the linear theory of classical operators but new interesting phenomena have also been observed. For details one can see [2, 4–6, 9] et al. Simultaneously commutators of multilinear operators continue to attract much attention, many basic properties can be found in the article[12,14,10]. In this paper, we consider the L^p-boundedness for certain commutators of multilinear fractional integral operators.

Let $\mathbb{R}^n$, $n \geq 2$, be the n-dimensional Euclidian space, Kenig and Stein[9] studied the multilinear operators of fractional integral type, see also [4, 7]. For the following multilinear fractional integrals

$$I_{\alpha,m}(f_1, \cdots, f_m)(x) = \int_{(\mathbb{R}^n)^m} \frac{f_1(x-y_1)f_2(x-y_2)\cdots f_m(x-y_m)}{|(y_1, y_2, \cdots, y_m)|^{mn-\alpha}} dy_1 \cdots dy_m, \tag{0.1}$$

Kenig and Stein[9] obtained the following result.

Theorem A Let $m \in \mathbb{N}$, $\frac{1}{s} = \frac{1}{r_1} + \cdots + \frac{1}{r_m} - \frac{\alpha}{n} > 0$ with $0 < \alpha < mn$, $1 \leq r_j \leq \infty$. Then,

(a) If each $r_j > 1$,

$$\|I_{\alpha,m}(f_1, \cdots, f_m)\|_{L^s(\mathbb{R}^n)} \leq C \prod_{j=1}^{m} \|f_j\|_{L^{r_j}(\mathbb{R}^n)}. \tag{0.2}$$

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(b) If $r_j = 1$ for some j,

$$\|I_{\alpha,m}(f_1, \cdots, f_m)\|_{L^{s,\infty}(\mathbb{R}^n)} \leq C \prod_{j=1}^{m} \|f_j\|_{L^{r_j}(\mathbb{R}^n)}. \tag{0.3}$$

Obviously, $I_{\alpha,1}$ is the classical Riesz potential operator I_α, which has many well-known results (see [1, 3, 9] etc.). The corresponding commutators of I_α were also studied extensively (see [8, 3, 11] etc.). A natural problem is to consider the commutators of multilinear fractional integral operators. Here, motivated by [14], we will consider the following commutator of multilinear fractional integral operators $I_{\alpha,2}$:

$$\begin{aligned}[b_1, b_2, I_{\alpha,2}](f_1, f_2)(x) &= b_1(x)b_2(x)I_{\alpha,2}(f_1, f_2)(x) - b_1(x)I_{\alpha,2}(f_1, b_2f_2)(x) \\ &\quad - b_2(x)I_{\alpha,2}(b_1f_1, f_2)(x) + I_{\alpha,2}(b_1f_1, b_2f_2)(x),\end{aligned} \tag{0.4}$$

where $I_{\alpha,2}$ is as in (0.1) for $m = 2$, b_1 and b_2 are locally integrable function in $\mathbb{R}^n$. Our main result can be formulated as follows.

Theorem 1 Let $b_i \in \mathrm{BMO}(\mathbb{R}^n)$, $i = 1, 2$, $0 < \alpha < 2n$. Then $[b_1, b_2, I_{\alpha,2}]$ is a bounded operator from $L^{q_1}(\mathbb{R}^n) \times L^{q_2}(\mathbb{R}^n)$ to $L^q(\mathbb{R}^n)$ with $1 < q_1, q_2 < \infty$ and $\frac{1}{q} = \frac{1}{q_1} + \frac{1}{q_2} - \frac{\alpha}{n}$.

We would remark that our some ideas used in the proof of Theorem 1 are taken from [13, 14]. Throughout this paper, we always use the letter C to denote a positive constant that may vary at each occurrence but is independent of the main parameters.

1 Proof of Theorem 1

Before proving our results, let us recall some relevant definitions and notation. For a locally integrable function f, we denote by Mf the standard Hardy-Littlewood maximal function, and for $\delta > 0$ we denote by M_δ the operator

$$M_\delta f = [M(|f|^\delta)]^{\frac{1}{\delta}}.$$

Let $M^\sharp f$ be the Fefferman-Stein sharp function of f defined by

$$M^\sharp f(x) = \sup_{Q \ni x} \frac{1}{|Q|} \int_Q |f(y) - m_Q(f)| dy,$$

where the supremum is taken over all cubes Q with sides parallel to the coordinate axes, and $m_Q(f)$ is the mean value of f on the cube Q. Similarly as above we define $M_\delta^\sharp f$ by

$$M_\delta^\sharp f = [M^\sharp(|f|^\delta)]^{\frac{1}{\delta}}.$$

Also, for $0 < \alpha < n$, let $M^{(\alpha)}$ be the fractional maximal operator defined by

$$M^{(\alpha)} f(x) = \sup_{Q \ni x} |Q|^{(\frac{\alpha}{n})-1} \int_Q |f(y)| dy.$$

In addition, as usual, a function $A: [0, \infty) \to [0, \infty)$ is a Young function if it is continuous, convex, and increasing satisfying $A(0) = 0$ and $A(t) \to \infty$ as $t \to \infty$. The complementary Young function associated to A is defined by

$$\bar{A}(s) = \sup_{0 \leq t < \infty} [st - A(t)], \quad 0 \leq s < \infty.$$

Given a Young function A and a cube Q, the mean Luxemburg norm of a measurable function f on Q is defined by

$$\|f\|_{A,Q} = \inf\left\{\lambda > 0:\ \frac{1}{|Q|}\int_Q A\left(\frac{|f(x)|}{\lambda}\right)dx \le 1\right\}.$$

Associated to the average $\|f\|_{A,Q}$ and $0 < \alpha < n$, we define a fractional maximal operator $M_{A,\alpha}$ by

$$M_{A,\alpha}f(x) = \sup_{Q\ni x}|Q|^{\frac{\alpha}{n}}\|f\|_{A,Q}.$$

The main example that we are going to be using is $A(t) = t(1+\log^+ t)$. The complementary Young function is given by $\bar{A}(t) \approx e^t$. We then write

$$\|f\|_{L\log L,Q} = \|f\|_{A,Q} \quad \text{and} \quad \|f\|_{\exp L,Q} = \|f\|_{\bar{A},Q}.$$

We also need to use the generalized Hölder inequality

$$\frac{1}{|Q|}\int_Q |f(y)g(y)|dy \le \|f\|_{A,Q}\|g\|_{\bar{A},Q}, \tag{1.1}$$

and the following fact[3]

$$M_{L\log L,\alpha}f(x) \approx M^{(\alpha)}Mf(x). \tag{1.2}$$

Lemma 1.1[4,3] Let $b \in \mathrm{BMO}(\mathbb{R}^n)$, $0 < p < \infty$, there exists a finite positive constant C such that

$$|Q|^{-1}\int_Q |b(x) - m_Q(b)|^p dx \le C\|b\|_*^p, \tag{1.3}$$

and

$$\|b - m_Q(b)\|_{\exp L,Q} \le C\|b\|_*, \tag{1.4}$$

where $m_Q(b) = |Q|^{-1}\int_Q b(x)dx$, $\|b\|_*$ denotes the BMO norm of b.

Lemma 1.2[4] Let $1 \le p_0 < \infty$ and $0 < \delta < 1$. Then for any p with $p_0 \le p < \infty$ there exists a positive constant C such that for all functions f with $Mf \in L^{p_0}(\mathbb{R}^n)$, we have

$$\|M_\delta(f)\|_{L^p(\mathbb{R}^n)} \le C\|M_\delta^\sharp(f)\|_{L^p(\mathbb{R}^n)}.$$

Lemma 1.3 Let $\tau > 1$, $b_1, b_2 \in L^\infty(\mathbb{R}^n)$ and $0 < \delta < \frac{n}{2n-\alpha}$ where $\alpha = \alpha_1 + \alpha_2$ for some $\alpha_1, \alpha_2 \in (0,n)$, then there exists a constant $C > 0$ such that for all $f_1,\ f_2 \in L_c^\infty(\mathbb{R}^n)$ and $x \in \mathbb{R}^n$

$$\begin{aligned} M_\delta^\sharp\left([b_1, b_2, I_{\alpha,2}](f_1, f_2)\right)(x) \lesssim\ & [\|b_1\|_*\|b_2\|_* M_\tau(I_{\alpha,2}(f_1,f_2))(x) \\ & + \|b_1\|_* M_\tau([b_2, I_{\alpha,2}](f_1,f_2))(x) \\ & + \|b_2\|_* M_\tau([b_1, I_{\alpha,2}](f_1,f_2))(x) \\ & + \|b_1\|_*\|b_2\|_* M_{L\log L,\alpha_1}f_1(x) M_{L\log L,\alpha_2}f_2(x)], \end{aligned} \tag{1.5}$$

and

$$\begin{aligned} M_\delta^\sharp\left([b_1,\ I_{\alpha,2}](f_1,\ f_2)\right)(x) \lesssim\ & \|b_1\|_*\, [M_\tau(I_{\alpha,2}(f_1,\ f_2))(x) \\ & + M_{L\log L,\,\alpha_1}(f_1)(x) M_{L\log L,\,\alpha_2}(f_2)(x)], \end{aligned} \tag{1.6}$$

$$M_\delta^\sharp([b_2, I_{\alpha,2}](f_1, f_2))(x) \lesssim \|b_2\|_* [M_\tau(I_{\alpha,2}(f_1, f_2))(x) + M_{L\log L, \alpha_1}(f_1)(x) M_{L\log L, \alpha_2}(f_2)(x)], \tag{1.7}$$

where

$$[b_1, I_{\alpha,2}](f_1, f_2)(x) = b_1(x) I_{\alpha,2}(f_1, f_2)(x) - I_{\alpha,2}(b_1 f_1, f_2)(x),$$

$$[b_2, I_{\alpha,2}](f_1, f_2)(x) = b_2(x) I_{\alpha,2}(f_1, f_2)(x) - I_{\alpha,2}(f_1, b_2 f_2)(x).$$

Proof By the definition of $M_\delta^\sharp$, to prove (1.5), it suffices to prove that for any $x \in \mathbb{R}^n$ and a cube Q containing x,

$$\begin{aligned}\left(\frac{1}{|Q|}\int_Q |[b_1, b_2, I_{\alpha,2}](f)(z) - h_Q|^\delta dz\right)^{\frac{1}{\delta}} &\lesssim (\|b_1\|_* \|b_2\|_* M_\tau(I_{\alpha,2}(f_1, f_2))(x) \\ &+ \|b_1\|_* M_\tau([b_2, I_{\alpha,2}](f_1, f_2))(x) + \|b_2\|_* M_\tau([b_1, I_{\alpha,2}](f_1, f_2))(x) \\ &+ \|b_1\|_* \|b_2\|_* M_{L\log L, \frac{\alpha}{2}} f_1(x) M_{L\log L, \frac{\alpha}{2}} f_2(x)),\end{aligned}$$

where $h_Q = m_Q(I_{\alpha,2}((m_{2Q}(b_1) - b_1) f_1 \chi_{\mathbb{R}^n \setminus 2Q}, (m_{2Q}(b_2) - b_2) f_2 \chi_{\mathbb{R}^n \setminus 2Q}))$. Obviously

$$\begin{aligned}|[b_1, b_2, I_{\alpha,2}](f)(z) - h_Q| &\le |(b_1(z) - m_{2Q}(b_1))(b_2(z) - m_{2Q}(b_2)) I_{\alpha,2}(f_1, f_2)(z)| \\ &+ |(b_1(z) - m_{2Q}(b_1)) I_{\alpha,2}(f_1, (b_2(z) - b_2) f_2)(z)| \\ &+ |(b_2(z) - m_{2Q}(b_2)) I_{\alpha,2}(b_1(z) - b_2) f_1 - f_2)(z)| \\ &+ |I_{\alpha,2}((b_1 - m_Q(b)_1) f_1, (b_1 - m_Q(b)_1) f_2)(z) - h_Q| \\ &:= \mathrm{I}(z) + \mathrm{II}(z) + \mathrm{III}(z) + \mathrm{IV}(z).\end{aligned}$$

We obtain

$$M_\delta^\sharp [b_1, b_2, I_{\alpha,2}](f_1, f_2)(x) \lesssim [\mathrm{I} + \mathrm{II} + \mathrm{III} + \mathrm{IV}],$$

where $\mathrm{I} = (|Q|^{-1} \int_Q \mathrm{I}(z)^\delta dz)^{\frac{1}{\delta}}$, and II, III, IV are defined as the same.

To estimate I, by Hölder's inequality and (1.3) we obtain

$$\begin{aligned}\mathrm{I} &= \left(\frac{1}{|Q|}\int_Q \mathrm{I}(z)^\delta dz\right)^{\frac{1}{\delta}} \\ &\lesssim \left(\frac{1}{|Q|}\int_Q |b_1(z) - m_{2Q}(b_1)|^{\tau_1} dz\right)^{\frac{1}{\tau_1}} \\ &\times \left(\frac{1}{|Q|}\int_Q |b_2(z) - m_{2Q}(b_2)|^{\tau_2} dz\right)^{\frac{1}{\tau_2}} \\ &\times \left(\frac{1}{|Q|}\int_Q |I_{\alpha,2}(f_1, f_2)|^\tau dz\right)^{\frac{1}{\tau}} \\ &\lesssim \|b_1\|_* \|b_2\|_* M_\tau(I_{\alpha,2}(f_1, f_2))(x),\end{aligned}$$

where $\tau_1 > 1$, $\tau_2 > 1$ and $\frac{1}{\tau} + \frac{1}{\tau_1} + \frac{1}{\tau_2} = \frac{1}{\delta}$.

Next we estimate term II by

$$\begin{aligned}
\mathrm{II} &= \left(\frac{1}{|Q|}\int_Q \mathrm{II}(z)^\delta dz\right)^{\frac{1}{\delta}} \\
&\lesssim \left(\frac{1}{|Q|}\int_Q |b_1(z)-m_{2Q}(b_1)|^s dz\right)^{\frac{1}{s}} \\
&\quad \times \left(\frac{1}{|Q|}\int_Q |[b_2, I_{\alpha,2}](f_1,f_2)|^\tau dz\right)^{\frac{1}{\tau}} \\
&\lesssim \|b_1\|_* M_\tau(b_2, I_{\alpha,2}(f_1,f_2))(x),
\end{aligned}$$

where $s>1$ and $\frac{1}{s}+\frac{1}{\tau}=\frac{1}{\delta}$.

Similarly, we have

$$\mathrm{III} \lesssim \|b_2\|_* M_\tau(b_1, I_{\alpha,2}(f_1,f_2))(x).$$

Finally, we estimate IV. Set $f_j^0=f_j\chi_{2Q}$ and $f_j=f_j^0+f_j^\infty$ for $j=1,2$. Then

$$\begin{aligned}
\mathrm{IV}(z) &\le |I_{\alpha,2}((b_1-m_{2Q}(b_1))f_1^0,(b_2-m_{2Q}(b_2))f_2^0)(z)| \\
&\quad +|I_{\alpha,2}((b_1-m_{2Q}(b_1))f_1^0,(b_2-m_{2Q}(b_2))f_2^\infty)(z)| \\
&\quad +|I_{\alpha,2}((b_1-m_{2Q}(b_1))f_1^\infty,(b_2-m_{2Q}(b_2))f_2^0)(z)| \\
&\quad +|I_{\alpha,2}((b_1-m_{2Q}(b_1))f_1^\infty,(b_2-m_{2Q}(b_2))f_2^\infty)(z)-h_Q| \\
&= \mathrm{IV}_1(z)+\mathrm{IV}_2(z)+\mathrm{IV}_3(z)+\mathrm{IV}_4(z),
\end{aligned}$$

and so we have $\mathrm{IV}\lesssim\sum_{j=1}^4 \mathrm{IV}_j$, where

$$\mathrm{IV}_j=\left(\frac{1}{|Q|}\int_Q \mathrm{IV}_j(z)^\delta dz\right)^{\frac{1}{\delta}},\quad j=1,2,3,4.$$

To estimate IV_1, by (0.3), Kolmogorov's inequality with $p=1$, $q=\frac{n}{2n-\alpha}$, the generalized Hölder's inequality (1.1) and (1.4), we have

$$\begin{aligned}
\mathrm{IV}_1 &= \left(\frac{1}{|Q|}\int_Q \mathrm{IV}_1(z)^\delta dz\right)^{\frac{1}{\delta}} \\
&\lesssim |Q|^{\frac{\alpha}{n}-2}\|I_{\alpha,2}((b_1-m_{2Q}(b_1))f_1^0,(b_2-m_{2Q}(b_2))f_2^0)\|_{L^{\frac{n}{2n-\alpha},\infty}(\mathbb{R}^n)} \\
&\lesssim \prod_{j=1}^2 |Q|^{\frac{\alpha_j}{n}-1}\int_{2Q}|(b_j-m_{2Q}(b_j))f_j(y_j)|dy_j \\
&\lesssim \prod_{j=1}^2 |Q|^{\frac{\alpha_j}{n}}\|b_j-m_{2Q}(b_j)\|_{\exp L,2Q}\|f_j\|_{L\log L,2Q} \\
&\lesssim \prod_{j=1}^2 \|b_j\|_* M_{L\log L,\alpha_j}f_j(x).
\end{aligned}$$

For term IV_2,

$$\begin{aligned}
\mathrm{IV}_2 &= \left(\frac{1}{|Q|}\int_Q \mathrm{IV}_2(z)^\delta dz\right)^{\frac{1}{\delta}} \\
&\lesssim \frac{1}{|Q|}\int_Q\int_{\mathbb{R}^n\setminus 2Q}\int_{2Q}\frac{|(b_1-m_{2Q}(b_1))f_1^0||(b_2(y_2)-m_{2Q}(b_2))f_2^\infty(y_2)|}{|(z-y_1,z-y_2)|^{2n-\alpha}}dy_1dy_2dz
\end{aligned}$$

$$\lesssim \frac{1}{|Q|}\int_Q\int_{2Q}|(b_1(y)-m_{2Q}(b_1))f_1^0(y_1)|dy_1\int_{\mathbb{R}^n\setminus 2Q}\frac{|(b_2(y_2)-m_{2Q}(b_2))f_2^\infty(y_2)|}{|z-y_2|^{2n-\alpha}}dy_2dz$$
$$\lesssim \|b_1-m_{2Q}(b_1)\|_{\exp L,2Q}\|f_1\|_{L\log L,2Q}\int_Q\int_{\mathbb{R}^n\setminus 2Q}\frac{|(b_2(y_2)-m_{2Q}(b_2))f_2(y_2)|}{|z-y_2|^{2n-\alpha}}dy_2dz$$
$$\lesssim \|b_1\|_*M_{L\log L,\alpha_1}f_1(x)|Q|^{1-\frac{\alpha_1}{n}}\sum_{k=1}^\infty\int_{2^{k+1}Q\setminus 2^kQ}\frac{|(b_2(y_2)-m_{2Q}(b_2))f_2(y_2)|}{(2^k-1)^{2n-\alpha}l(Q)^{2n-\alpha}}dy_2$$
$$\lesssim \|b_1\|_*M_{L\log L,\alpha_1}f_1(x)\Bigg(\sum_{k=1}^\infty 2^{k(\alpha-2n)}|Q|^{\frac{\alpha_2}{n}-1}\int_{2^{k+1}Q}|(b_2(y_2)-m_{2^{k+1}Q}(b_2))f_2(y_2)|dy_2$$
$$+\sum_{k=1}^\infty 2^{k(\alpha-2n)}|m_{2^{k+1}Q}(b_2)-m_{2Q}(b_2)||Q|^{\frac{\alpha_2}{n}-1}\int_{2^{k+1}Q}|f_2(y_2)|dy_2\Bigg)$$
$$\lesssim \|b_1\|_*M_{L\log L,\alpha_1}f_1(x)\Bigg(\sum_{k=1}^\infty 2^{k(\alpha_1-n)}|2^{k+1}Q|^{\frac{\alpha_2}{n}}\|b_2-m_{2^{k+1}Q}(b_2)\|_{\exp L,2^{k+1}Q}$$
$$\times\|f_2\|_{L\log L,2^{k+1}Q}+\sum_{k=1}^\infty 2^{k(\alpha_1-n)}\|b_2\|_*M_{\alpha_2}f_2(x)\Bigg)$$
$$\lesssim \|b_1\|_*M_{L\log L,\alpha_1}f_1(x)\left(\|b_2\|_*M_{L\log L,\alpha_2}f_2(x)+\|b_2\|_*M_{\alpha_2}f_2(x)\right)$$
$$\lesssim \prod_{j=1}^2\|b_j\|_*M_{L\log L,\alpha_j}f_j(x).$$

Here we used the fact

$$|m_{2^{k+1}Q}(b_2)-m_{2Q}(b_2)|\le Ck\|b_2\|_*$$

and

$$M_{\alpha_2}f_2(x)\le M_{L\log L,\alpha_2}f_2(x)\quad(\text{see [3]}).$$

Similarly,

$$\mathrm{IV}_3\lesssim\prod_{j=1}^2\|b_j\|_*M_{L\log L,\alpha_j}f_j(x).$$

For term IV_4, we have

$$\mathrm{IV}_4\le\frac{1}{|Q|}\int_Q|I_{\alpha,2}((b_1-m_{2Q}(b_1))f_1^\infty,(b_2-m_{2Q}(b_2))f_2^\infty)(z)$$
$$-I_{\alpha,2}((b_1-m_{2Q}(b_1))f_1^\infty,(b_2-m_{2Q}(b_2))f_2^\infty)(x_0)|\,dz$$
$$\le\frac{1}{|Q|}\int_Q\int_{\mathbb{R}^n\setminus 2Q}\int_{\mathbb{R}^n\setminus 2Q}\left|\frac{1}{|(z-y_1,z-y_2)|^{2n-\alpha}}-\frac{1}{|(x_0-y_1,x_0-y_2)|^{2n-\alpha}}\right|$$
$$\times\left|\prod_{j=1}^2(b_j(y_i)-m_{2Q}(b_j))f_j^\infty(y_j)\right|dy_1dy_2dz$$
$$\lesssim\frac{1}{|Q|}\int_Q\prod_{j=1}^2\int_{\mathbb{R}^n\setminus 2Q}\frac{|x_0-z|^{\frac12}}{|x_0-y_j|^{n-\frac{\alpha}{2}+\frac12}}|b_j(y_i)-m_{2Q}(b_j))f_j(y_j)|dy_jdz$$

$$\lesssim \prod_{j=1}^{2}\sum_{k=1}^{\infty} 2^{-\frac{k}{2}} \left|2^{k+1}l(Q)\right|^{-n+\alpha_j} \int_{2^{k+1}Q} |b_j(y_j) - m_{2Q}(b_j)||f_j(y_j)|dy_j$$

$$\lesssim \prod_{j=1}^{2}\sum_{k=1}^{\infty} 2^{-\frac{k}{2}} \left|2^{k+1}l(Q)\right|^{-n+\alpha_j} \left(\int_{2^{k+1}Q} \left|b_j(y_j) - m_{2^{k+1}Q}(b_j)\right| |f_j(y_j)|dy_j \right.$$

$$\left. +|m_{2^{k+1}Q}(b_j) - m_{2Q}(b_j)| \int_{2^{k+1}Q} |f_j(y_j)|dy_j \right)$$

$$\lesssim \prod_{j=1}^{2}(\|b_j\|_* M_{L\log L,\alpha_j} f_j(x) + \|b_j\|_* M_{\alpha_j} f_j(x))$$

$$\lesssim \prod_{j=1}^{2} \|b_j\|_* M_{L\log L,\alpha_j} f_j(x).$$

This finishes the proof of (1.5). Similarly, we can prove (1.6) and (1.7) and we omit the details, which completes the proof of Lemma 1.3.

Proof of Theorem 1 By the Lebesgue differentiation theorem, we have that

$$|f(x)| \le Mf(x), \quad \text{a.e. } x \in \mathbb{R}^n.$$

Suppose $b_1,\ b_2 \in L^\infty(\mathbb{R}^n)$, by $f_1,\ f_2 \in L_c^\infty(\mathbb{R}^n)$ together with Lemma 1.2 and Lemma 1.3, we obtain

$$\begin{aligned}
\|[b_1,b_2,I_{\alpha,2}](f_1,f_2)\|_{L^q(\mathbb{R}^n)} &\le \|M_\delta([b_1,b_2,I_{\alpha,2}](f_1,f_2))\|_{L^q(\mathbb{R}^n)}\\
&\le \|M_\delta^\sharp([b_1,b_2,I_{\alpha,2}](f_1,f_2))\|_{L^q(\mathbb{R}^n)}\\
&\lesssim \big(\|b_1\|_*\|b_2\|_*\|M_\tau(I_{\alpha,2}(f_1,f_2))\|_{L^q(\mathbb{R}^n)}\\
&\quad + \|b_1\|_*\|M_\tau([b_2,I_{\alpha,2}](f_1,f_2))\|_{L^q(\mathbb{R}^n)}\\
&\quad + \|b_2\|_*\|M_\tau([b_1,I_{\alpha,2}](f_1,f_2))\|_{L^q(\mathbb{R}^n)}\\
&\quad +\|b_1\|_*\|b_2\|_*\|M_{L\log L,\alpha_1}f_1(x)M_{L\log L,\alpha_2}f_2\|_{L^q(\mathbb{R}^n)}\big)\\
&\lesssim \|b_1\|_*\|b_2\|_*\big(\|M_\tau(I_{\alpha,2}(f_1,f_2))\|_{L^q(\mathbb{R}^n)}\\
&\quad +\|M_{L\log L,\alpha_1}f_1 M_{L\log L,\alpha_2}f_2\|_{L^q(\mathbb{R}^n)}\big)\\
&\lesssim \|b_1\|_*\|b_2\|_*\big(\|I_{\alpha,2}(f_1,f_2)\|_{L^q(\mathbb{R}^n)}\\
&\quad +\|M_{L\log L,\alpha_1}f_1\|_{L^{p_1}(\mathbb{R}^n)}\|M_{L\log L,\alpha_2}f_2\|_{L^{p_2}(\mathbb{R}^n)}\big)\\
&\lesssim \|b_1\|_*\|b_2\|_*\|f_1\|_{L^{q_1}(\mathbb{R}^n)}\|f_2\|_{L^{q_2}(\mathbb{R}^n)},
\end{aligned}$$

where $\frac{1}{p_j} = \frac{1}{q_j} - \frac{\alpha_j}{n}$, $j = 1, 2$. Here, in the last inequality we have used (0.2), (1.2) and the $(L^{q_j}(\mathbb{R}^n),\ L^{p_j}(\mathbb{R}^n))$-boundedness of $M^{(\alpha_j)}$ and $L^{q_j}(\mathbb{R}^n)$-boundedness of M, $j = 1, 2$.

By the same arguments as in [13, p. 686–687], we easily derive the inequality

$$[b_1,b_2,I_{\alpha,2}](f_1,f_2)\|_{L^q(\mathbb{R}^n)} \lesssim \|b_1\|_*\|b_2\|_*\|f_1\|_{L^{q_1}(\mathbb{R}^n)}\|f_2\|_{L^{q_2}(\mathbb{R}^n)}$$

holds for $b_1,\ b_2$ belong to $\mathrm{BMO}(\mathbb{R}^n)$ and $f_1,\ f_2 \in L_c^\infty(\mathbb{R}^n)$. Since $L_c^\infty(\mathbb{R}^n)$ is dense in $L^q(\mathbb{R}^n)$, $1 < q < \infty$, we can extend $[b_1,b_2,I_{\alpha,2}]$ to the whole $L^{q_1}(\mathbb{R}^n) \times L^{q_2}(\mathbb{R}^n)$. This finishes the proof of Theorem 1.

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多线性分数次积分算子交换子的有界性

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摘要：本文利用 sharp 极大函数的估计，证明了一类由多线性分数次积分算子和 BMO($\mathbb{R}^n$) 函数生成的交换子的 $L^p(\mathbb{R}^n)$ 有界性.

关键词：多线性算子；分数次积分；交换子；sharp 极大函数；分数次极大算子；BMO($\mathbb{R}^n$)