数学研究
Journal of Mathematical Study

Vol. 27 No. 2
Dec. 1994

A NEW METHOD FOR AGGREGATION OF PREFERENCE

Gao Hongzhen

(Departmennt of Statistic, Xiamen University, Xiamen 361005)

Abstract In this paper we shall study some paradoxes aggrgation of preference. We review several methods for group decision which has been developed so for. Morever, we shall introduce a new method which allows us to eval those paradoxes.

Key Words Aggregation of preference, Group decision, ε-majority

1 INTRODUCTION

The problem of how "best" to aggregate individual choice into social (or group) preference has attracted many mathematicians, economists and sociologists. This problem can formulated as follows: Given the preference ranking of m alternatives by the members of a group of n individuals, define "fair" methods for aggregating this set of individual ranking into a single ranking for the group, However, many investigations in group decision do not to construct a complete group preference oder, and attempt only to specify alternatives which from some natural viewpoint are "best". A good idea of these developments may be found in work of Fishburn[2], Mirkin[3], Sen[4], and Woodall[5]. In this paper we shall focus our attention on following problem: There are m alternatives and a group consist of n individuals.

Given an integer r ($1<r<m$), each individual place r alternatives by order of preference of alternatives the first choice being marked 1, the second 2, and so on, on the ballot papers. For expositional simplicity it will be assumed throughout that individual indifference between distinct alternatives does not arise. We should like to have a "rational "method that will select r of the m alternatives and rank them. We expect such a method have "majority property": if the majority (or great majorty) place a alternative in k−th ranking, than the group decision should be the same。 Unfortunately, many methods, such as the utilitarianism utility fucton [4], the Borda rule[4], and the method of linear weighted sum [3], have not this property, Thus, some paradoxes may arise, when we use these mothods for aggregation of preference. In section 2 we shall cite instance to illustrate this. Theorem 1 and its corollary show that in order to avoid these paradoxes a new method is needed. In section 3 we shall of-

?1994-2014 China Academic Journal Electronic Publishing House. All rights reserved. http://www.cnki.

fer a new method to aggragate preference, and show that our method possesses the "ε-majority property".

2 SOME PARADOXES

We now introduce the following notation to be useful in what follows. Let

$$A = \{a_1, a_2, \cdots, a_m\}$$

be the set consisted of alternatives $a_1, \cdots, a_m$, and

$$Z_{r+1} = \{1, 2, \cdots, r, r+1\}$$

be a set consisted of r+1 positive integer numbers. Let R^m be a linear space of dimension m, and R be a real number set. Let

$$B = [b_{ij}]$$

be an m by r matrix, its ij-entry is the number of individual who have the alternative i in j-th place in their preference orders. The notation b_i is used to denote the i-the row vector of this matrix. So the problem reduces to: Given an m by r matrix of numbers b_{ij}, to choose r row (alternatives) and rank them.

Definition 1. A ranking function of a group is a mapping from A into Z:

$$F: A \to Z_{r+1}$$

which defined as follows:

$$F(a_i) = \begin{cases} k, & \textit{if the group decision had alternative } a_i \textit{ in k-th place} \\ r+1, & \textit{if thegroup decision didn't choose alternative } a_i. \end{cases} \tag{1}$$

Definition 2. Let σ be a permutation on Z_m, and u be a mapping from R^m into R,

$$u: R^m \to R$$

such that

$$u(b_{\sigma(1)}) \geqslant u(b_{\sigma(2)}) \geqslant \cdots \geqslant u(b_{\sigma(m-1)}) \geqslant u(b_{\sigma(m)})$$

IF

$$F(a_{\sigma(k)}) = \begin{cases} k, & k \leqslant r, \\ r+1, & r < k \end{cases} \tag{2}$$

where F is a ranking function of a group, then u is called a aggregation function of the group.

Thus, a method for aggregation can be determined by a aggregation function. For this reason, the methods, which is mentioned above. can be expressed by following function, respectively.

(1) The aggregation function of the utilitarian approach may be given by

$$u_1(b_i) = \sum_{j=1}^{r} b_{ij} \tag{3}$$

(2) The Borda rule can be seen as based on attaching a number to any alternative equal to the sum of its ranks in each individual's ballot paper. For example, in the case of r=3, n=4, if a_1 is first in one individual's ordering and second in the other three individuals then the "Borda count" for a_1 is 3+2+2+2=9. So the aggregation function of Borda rule can be written as

?1994-2014 China Academic Journal Electronic Publishing House. All rights reserved. http://www.cnki.

$$u_2(b_i) = \sum_{j=1}^{r}(r+1-j)b_{ij} \tag{4}$$

(3)The method of linear weighted sum depends on the weight $w_j(j=1,2,\cdots,r)$. Its aggregation function may given by

$$u_3(b_i) = \sum_{j=1}^{r} w_j b_{ij}, \tag{5}$$

where $w_1,\cdots,w_r$ are constant weight.

We now consider following example: $n=100, r=3$,

$$B = \begin{bmatrix} 60 & 0 & 0 \\ 0 & 10 & 51 \\ 0 & 40 & 19 \\ 30 & 10 & 1 \\ 10 & 30 & 10 \\ 0 & 10 & 20 \end{bmatrix}$$

By the formula(3) we have

$$u_1(a_2) \geqslant u_1(a_1) \geqslant u_1(a_3) \geqslant u_1(a_4) \geqslant u_1(a_5) \geqslant u_1(a_6).$$

Hence, $F(a_2)=1, F(a_1)=2, F(a_3)=3$, and group decision shall place alternative a_2 in first ordering, alternative a_1 in second, and alternative a3 in third. There are some papradoxes:

1. The great majority (in this case is 60%) of individuals have the alternative a_1 in first place, but group decision have it second.

2. None have the alternative a_2 in first place, but group decision have it first.

3. If three individuals, who had the individual a_2 in third place in their original preference, changed their mind and substituted the alternatives a_4 for a_2, then group desion, on contrary, have the alternative a_2 in third place. Conversely we can construct an example: when the number of individual who have the individual a_2 in k-th place increased, the a_2 lost his k-th place in group ranking.

Even if we substitute formula(4) or (5) for (3), we still unable to overcome these difficulties.

Sice the majority property is a strict requirment, we now introduce the concept of "ε-najority property".

Definition 3. Let $0.5<\varepsilon<1$. An aggregation functon u is said to have ε-majority property if $b_{ik} \geqslant \varepsilon n$ imples $i=\sigma(k)$ for each i and k.

It's obviously that if an aggregation functon u have ε_1-majority property, then u have ε_2-majority property when $\varepsilon_1 \leqslant \varepsilon_2$.

The above example show aggregation functon u_1 does not possess 0.6-majority property. We now shall investigate the property of aggregation function u_2 and u_3.

Theorem 1. Let $0.5<\varepsilon<1$. the aggregation function u_3 does not possess ε-majority prop-

?1994-2014 China Academic Journal Electronic Publishing House. All rights reserved. http://www.cnki.

erty, no matter how chosen the weight

$$w_1, w_2, \cdots, w_r \quad (w_i > 0, \sum_{i=1}^{r} w_i = 1) \quad are.$$

Proof. Let e_k, $(k=1,2,\cdots,r)$ be an r-dimensional vector, which has all componets zero except for k-th component $e_k=1$, and let $\tilde{b}_k=\varepsilon n e_k$.

To prove this theorem, it is sufficient to show at least one of following r statements (st. (1), ⋯, st. (r)) is true:

st. (1): There is at lest one i such that $u_3(b_i)>u_3(\tilde{b}_1)$.

st. (2): There are at lest two i such that $u_3(b_i)>u_3(\tilde{b}_2)$.

……

st. (r): There are at lest r subscripts i such that $u_3(b_i)>u_3(\tilde{b}_r)$.

suppose that on one of st. (1), st. (2), ⋯, st. $(r-1)$ is true. Then we shall show st. (r) is true.

Indeed, if st. (1) was not true, then for all i, we have

$$u_3(b_i) \leqslant u_3(\tilde{b}_1).$$

This inequality is the same as

$$\sum_{j=1}^{r} w_j b_{ij} \leqslant w_1 \varepsilon_n. \tag{6}$$

We take $b_1=[(1-\varepsilon)n, \frac{\varepsilon n}{r-1}, \frac{\varepsilon n}{r-1}\cdots, \frac{\varepsilon n}{r-1}]$. It's evidently that as long as n is great enough all components of b_i are integer. Substitution of b_i into (6) yield

$$(1-\varepsilon)nw_1 + \frac{\varepsilon n}{r-1}(1-w_1) \leqslant w_1 \varepsilon n.$$

So we have

$$w_1 \geqslant \frac{1}{1+\beta} \tag{7}$$

where $\beta=(2-1/\varepsilon)(r-1)$.

Choose $b_i=ne_k(k=2,\cdots,r)$,

then form(6) we can find

$$w_1 > w_k, (k=2,\cdots,r). \tag{8}$$

Therefore, if st. (2) was not true, then for every $b_i(b_i \neq \tilde{b}_1)$ we have

$$u_3(b_i) \leqslant u_3(b_2),$$

since

$$u_3(\tilde{b}_1) > u_3(\tilde{b}_2).$$

Now choose $b_i=[\frac{\varepsilon n}{r-1}, (1-\varepsilon)n, \frac{\varepsilon n}{r-1}, \cdots, \frac{\varepsilon n}{r-1}]$

we can also find

$$w_2 \geqslant \frac{1}{1+\beta} \tag{9}$$

and $w_2 \geqslant w_k, (k=3,\cdots r)$. (10)

?1994-2014 China Academic Journal Electronic Publishing House. All rights reserved. http://www.cnki.

Similary, if st. (3), ⋯, st. (r−1) were not true, we can also find

$$w_k \geqslant \frac{1}{1+\beta} \quad and \quad w_k \geqslant w_{k+1}(k=3,4,\cdots,r-1). \tag{11}$$

It follows that

$$W_r = 1 - \sum_{k=1}^{r} w_k \leqslant \frac{2-r+\beta}{1+\beta}.$$

Upon choosing $b_i = ne_i$, $(i=1,2,\cdots,r-1)$, this leads to

$$u_3(b_i) \geqslant u_3(\bar{b}_r), (i=1,2,\cdots,r-1). \tag{12}$$

Let

$$b_r = [\frac{n}{r-1}, \frac{n}{r-1}, \frac{n}{r-1}, 0]$$

from (7)−(11) we can establish the inequality

$$u_3(b_r) \geqslant \frac{n}{1+\beta}. \tag{13}$$

when $\varepsilon < 1$, we obtain

$$u_3(b_r) > \frac{2-r+\beta}{1+\beta}\varepsilon n \geqslant u_3(\bar{b}_r). \tag{14}$$

From (12), (14) we find there are r subscript i such that $u_3(b_i) > u_3(\bar{b}_r)$, and the proof of the theorem is complete.

Corollary. The aggregation function of Borda rule does not possess ε−majority property.

Proof. Let

$$u_4(b_i) = \sum_{k=1}^{r} \frac{2}{r(r+1)}(r-k+1)b_{ik}.$$

It can be seen that u_4 does not possess ε−majority property by theorem 1. But we have

$$u_2(b_i) = \frac{r(r+1)}{2}u_4(b_i).$$

This completes the proof.

3 A NEW METHOD

The theorem 1 and corollary show that: according to u_2 or u_3, even though 99.9% individuals had alternative a_i in k−th place, the group decision does not always the same. In this section we shall suggest a new nethod. For expositional simplicity, we only consider the case: r=3, the result may be generalized. With each b_i we associate an orderd 3-tuple (k_1, k_2, k_3), determined by following equations: $k_j(b_i) = k_{j1}(b_{i1}) + k_{j2}(b_{i2}) + k_{j3}(b_{i3})$, $(j=1,2,3)$,

Where

$$k_{11}(x) = 10x/n,$$

$$k_{12}(x) = \begin{cases} [-\frac{1}{2}(\frac{x}{n} - 0.1)^2 + 1/2]\frac{10x}{n}, & 0 \leqslant x \leqslant 0.2n \\ \frac{10}{3}(0.2 - \frac{x}{n}), & 0.2n < x \leqslant n \end{cases}$$

?1994-2014 China Academic Journal Electronic Publishing House. All rights reserved. http://www.cnki.

$$k_{13}(x)=\begin{cases}[-\frac{1}{3}(\frac{x}{n}-0.1)^2+1/2]\frac{10x}{n}, & 0\leqslant x\leqslant 0.2n\\ \frac{10}{3}(0.2-\frac{x}{n}), & 0.2n<x\leqslant n\end{cases}$$

$$k_{21}(x)=k_{12}(x),\quad k_{31}(x)=k_{13}(x)$$
$$k_{22}(x)=k_{11}(x),\quad k_{32}(x)=k_{12}(x)$$
$$k_{23}(x)=k_{12}(x),\quad k_{33}(x)=k_{11}(x)$$

In these equations, $k_j(b_i)$ will be called a "j-scale of an alternative i" (or, for simplicity, ij-scale).

The "strength", that group decision have alternative i in j place, can be charecterized by the ij-scale. For example, $k_1(b_i)$ consist of three tems, the $k_{11}(b_{i1})$ increases along with the increase of b_{i1}; the $k_{12}(b_{12})$ depends on b_{i2}, which is the number of individual who have alternative i in second place. If $b_{i2}\leqslant\lambda n$ (λ is a suitable critical parameter, here $\lambda=0.2$), then $k_{12}(b_{i2})\geqslant 0$; if $b_{i2}>\lambda n$, then $k_{12}(b_{i2})<0$, because many individuals have alternative i in second place, so that i1-scale should be decrease. The $k_{13}(b_{i3})$ is very similar to the $k_{12}(b_{i2})$. Therefore, b_i can be classified according to the ij-scale. Let

$$k_s(b_i)=\max\{k_1(b_i),k_2(b_i),k_3(b_i)\},$$

and $k_s(b_i*)=\max\limits_{i\in B_s}\{k_s(b_i)\}$, where

$$B_s=\{i\,|\,k_s(b_i)\}=\max\{k_1(b_i),k_2(b_i),k_3(b_i)\}.\ (s=1,2,3).$$

We can choose such a subscript i, and have the alternative i s-th place. Thus the aggregation function of this method can be written as

$$u_5(b_i)=\begin{cases}4-s, & k_s(b_i*)=k_s(b_i)\\ 0, & otherwise.\end{cases}$$

Theorem 2. The aggregation function u_5 possess 0.6-majority property.

Proof. Not lose generality, suppose that $b_{k1}\geqslant 0.6n$ thon $b_{k2}+b_{k3}\leqslant 0.4n$. So at most one of b_{k2} and b_{k3} is big than $0.2n$. Suppose $b_{k2}>0.2n$, then $b_{k3}<0.2n$, so $k_{13}(b_{k3})\geqslant 0$, $k_{12}(b_{k2})>-23$. If $b_{k2}\leqslant 0.2n$, and $b_{k3}\leqslant 0.2$, we have $k_{13}(b_{k3})\geqslant 0$, and $k_{12}(b_{k2})\geqslant 0$. But $k_{11}(b_{k1})\geqslant 6$, when $b_{k1}\geqslant 0.6n$. Therefore $k_1(b_k)\geqslant\frac{16}{3}$.

It's easily check that

$k_2(b_k)<k_1(b_k)$, $k_3(b_k)>k_1(b_k)$. So $k\in B_1$. But for each $j\in B_1$, we find $b_{j1}\leqslant 0.4n$, and

$$k_1(b_j)\leqslant\frac{10}{4}+\frac{1}{2}+\frac{1}{2}<k_1(b_k).$$

We now have

$$k_1(b_k)=\max_{j\in B_1}\{k_1(b_j)\}$$

The proof of theorem is complete.

(下转第 11 页)

?1994-2014 China Academic Journal Electronic Publishing House. All rights reserved. http://www.cnki.

仿射空间 R^{n+1} 中的非退化二次函数的水平曲面

郝　建　华
（山西师范大学数学系，　临汾　041004）

摘　要　本文利用 Hessian 度量和横截向量场，给出了非退化二次函数的水平曲面的一个充要条件.

关键词　水平曲面，　Hessian 度量，横截向量场

（上接第 6 页）

References

1　Arrow K J. Social Choice and Individual Value, New York, 1961
2　Fishburn P C. Condorect Social Choice function, SIAM J. Appl. Math. 1987, 33:, 469-489
3　Mirkin B G. Group Choice, ed. by P. C. Fishburn. Washington, DC; Winson. 1979
4　Sen A. Social Choice Theory, in: "Handbook of Mathematical Economics", Ed. by Arrow and Intriligator, Vol. Ⅲ, North-Holland, New York, 1986, 1073～1160
5　Woodall D R. An Impossibility Theorem for Electoral Systems, Discrete math. 1987, 66: 209～211

集计偏好的一种新方法

高　鸿　桢
（厦门大学计统系，　厦门 361005）

摘　要　本文研究在集计个体偏好中产生的若干悖论，而通常群体决策中有可能产生此类悖论，进而提出一种可避免产生悖论的新集计方法.

关键词　偏好集计，群体决策，ε-多数决

?1994-2014 China Academic Journal Electronic Publishing House. All rights reserved.　http://www.cnki.