THEOREMS OF INTERVAL FUZZY SET AND ITS OPERATION RULES

Wu Shunxiang, Cao Da

(Department of Automation, Xiamen University, Xiamen, 361005, P.R. China)

Abstract: Although the concept of interval fuzzy set and its properties have been defined, its three theorems and their effectiveness are not proved. Therefore, the knowledge presentation and its operation rules of interval fuzzy set are studied firstly, and then the cut set of interval fuzzy set is proposed. Moreover, the decomposition theorem, the representation theorem and the extension theorem of interval fuzzy set are presented. Finally, examples are given to demonstrate that the classical fuzzy set is a special case of interval fuzzy set and interval fuzzy set is an effective expansion of the classical fuzzy set.

Key words: interval fuzzy set; decomposition theorem; representation theorem; extension theorem

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INTRODUCTION

In 1965, Professor Zadeh proposed the basic concept of fuzzy set[1]. The cut set is very useful and important in the research of fuzzy set theory, which is applied in fuzzy topology[2-3], fuzzy algebras[4-5], fuzzy measure and analysis[6-9], fuzzy optimization and decision[10-11], fuzzy reasoning[12-13], fuzzy logic[14] and related domains. The cut set is the bridge to link fuzzy set and classical set. Therefore, decomposition theorem, representation theorem and extension theorem can be constructed based on cut set. In the extension of fuzzy set theory, such as, type-2 fuzzy set[15], type-L fuzzy set[16], intuitionistic fuzzy set[17], interval-valued fuzzy set[18-20] and interval-valued intuitionistic fuzzy set[21], the three corresponding theorems have been constructed, and play an important role in fuzzy theory.

Because of the incompletion of information, the data which are used to describe the characteristics of things are often some interval numbers. When the boundaries between such things are fuzzy, the classical fuzzy method is not appropriate. Therefore, it is necessary to build up a more precise fuzzy set. The combination of interval number and fuzzy set concept is a more effective way. Thus, the concept of interval fuzzy set and its operations are proposed[22].

This paper introduces the basic concept of interval fuzzy set, and presents the cut set. Then, the three theorems are constructed and proved. Finally, the corresponding examples are given to demonstrate the feasibility of the three theorems.

1 BASIC CONCEPT OF INTERVAL FUZZY SET

Definition 1 Suppose that U is the universe, μ_A the arbitrary mapping from U to closed interval $[0, 1]$, as

$$\mu_A: U \to [0, 1],\ u \to \mu_A(u)$$

where A is called the fuzzy set based on U, μ_A the membership function of fuzzy set A, $\mu_A(u)$ the

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E-mail: sxwu@xmu.edu.cn

membership degree of u in fuzzy set A. The membership degree is also denoted as $A(u)$. The set of all fuzzy sets based on universe U is denoted as $F(U)$.

Definition 2 Suppose that U is the universe of fuzzy set A and B, $\forall u \in U$, if $A(u) \leq B(u)$, it means that fuzzy set B contains A, which is denoted as $A \subseteq B$.

If $A \subseteq B$ and $B \subseteq A$, it means that A is equal to B, which is denoted as $A = B$.

If $A \subseteq B$ and $A \neq B$, then, it means that B truly contains A, which is denoted as $A \subset B$.

Definition 3 Suppose that U is the universe of interval fuzzy set X, if there exist classical fuzzy sets P and Q, which satisfy $P \subseteq X \subseteq Q$, namely, $X = \{S \mid S \in F(U) \wedge P \subseteq S \subseteq Q\}$. And for any fuzzy sets P' and Q', which satisfy $P' \subseteq X \subseteq Q'$ and $P' \subseteq P \subseteq X \subseteq Q \subseteq Q'$, X is called the interval fuzzy set with P as its lower bound and Q as its upper bound, which is denoted as $P \dot{\subseteq} X \dot{\subseteq} Q$ or $X\Big|_P^Q$. The set of all interval fuzzy sets based on universe U is denoted as $IF(U)$.

In the interval fuzzy set $X\Big|_P^Q$, P the largest fuzzy set contained in X and Q the smallest fuzzy set containing X. Apparently, for interval fuzzy set X, $\varphi \subseteq X \subseteq U$. Specially, in the interval fuzzy set $X\Big|_P^Q$, if $P = Q$, X can be changed to classical fuzzy set.

2 OPERATION OF INTERVAL FUZZY SET

Definition 4 Suppose that U is the universe of interval fuzzy sets $X_1\Big|_{P_1}^{Q_1}$ and $X_2\Big|_{P_2}^{Q_2}$, then, $X_1 \cdot X_2$ is called the algebra product of interval fuzzy sets X_1 and X_2, namely, $(X_1 \cdot X_2)(u) = \left\{u \in U \,\Big|\, X_1(u) \cdot X_2(u)\right\}$.

Theorem 1 Suppose that U is the universe of interval fuzzy set $X_1\Big|_{P_1}^{Q_1}$ and $X_2\Big|_{P_2}^{Q_2}$, then, $X_1 \cdot X_2 = (X_1 \cdot X_2)\Big|_{(P_1 \cdot P_2)}^{(Q_1 \cdot Q_2)}$.

Defioition 5 Suppose that U is the universe of interval fuzzy sets $X_1\Big|_{P_1}^{Q_1}$ and $X_2\Big|_{P_2}^{Q_2}$, then, $X_1 + X_2$ is called the algebraic sum of interval fuzzy sets X_1 and X_2, namely, $(X_1 + X_2)(u) = \left\{u \in U \,\Big|\, X_1(u) + X_2(u)\right\}$.

Theorem 2 Suppose that U is the universe of interval fuzzy sets $X_1\Big|_{P_1}^{Q_1}$ and $X_2\Big|_{P_2}^{Q_2}$, then, $X_1 + X_2 = (X_1 + X_2)\Big|_{(P_1 + P_2)}^{(Q_1 + Q_2)}$.

Definition 6 Suppose that U is the universe of interval fuzzy sets $X_1\Big|_{P_1}^{Q_1}$ and $X_2\Big|_{P_2}^{Q_2}$, then, $X_1 \cup X_2$ is called the union set of interval fuzzy sets X_1 and X_2, namely, $\left(X_1 \cup X_2\right)(u) = \left\{u \in U \,\Big|\, X_1(u) \vee X_2(u)\right\}$.

Theorem 3 Suppose that U is the universe of interval fuzzy sets $X_1\Big|_{P_1}^{Q_1}$ and $X_2\Big|_{P_2}^{Q_2}$, then, $X_1 \cup X_2 = \left(X_1 \cup X_2\right)\Big|_{\left(P_1 \cup P_2\right)}^{\left(Q_1 \cup Q_2\right)}$.

Definition 7 Suppose that U is the universe of interval fuzzy sets $X_1\Big|_{P_1}^{Q_1}$ and $X_2\Big|_{P_2}^{Q_2}$, then, $X_1 \cap X_2$ is called the intersection set of interval fuzzy sets X_1 and X_2, namely, $\left(X_1 \cap X_2\right)(u) = \left\{u \in U \,\Big|\, X_1(u) \wedge X_2(u)\right\}$.

Theorem 4 Suppose that U is the universe of interval fuzzy sets $X_1\Big|_{P_1}^{Q_1}$ and $X_2\Big|_{P_2}^{Q_2}$, then, $X_1 \cap X_2 = \left(X_1 \cap X_2\right)\Big|_{\left(P_1 \cap P_2\right)}^{\left(Q_1 \cap Q_2\right)}$.

Definition 8 Suppose that U is the universe of interval fuzzy set $X\Big|_P^Q$, then, X^c is called the complementary set of interval fuzzy set X, namely, $X^c = \left\{u \in U \,\Big|\, 1 - X(u)\right\} = \left\{S \,\Big|\, (S \subseteq U) \wedge (Q^c \subseteq S \subseteq P^c)\right\} = X\Big|_{Q^c}^{P^c}$.

Definition 9 Suppose that U is the universe of fuzzy set A and B, if $B \subseteq A$, then, $A - B$ is called the algebraic difference of fuzzy sets A and B, namely, $(A - B)(u) = \left\{u \in U \,\Big|\, A(u)\right.$

$- B(u)\Big\}$.

Theorem 5 Suppose that U is the universe of interval fuzzy set $X\Big|_P^Q$, then

(1) $\left(X\Big|_P^Q\right)^c = X\Big|_{Q^c}^{Q^c+(Q-P)}$.

(2) $\left(X\Big|_P^Q\right)^c = Q^c + X\Big|_{\varphi}^{Q-P}$.

Theorem 6 Suppose that U is the universe of interval fuzzy sets $X_1\Big|_{P_1}^{Q_1}$ and $X_2\Big|_{P_2}^{Q_2}$, if $|Q_1 \cap Q_2| = 0$, then $|X_1 \cap X_2| = 0$.

Theorem 7 Suppose that U is the universe of interval fuzzy sets $X_1\Big|_{P_1}^{Q_1}$ and $X_2\Big|_{P_2}^{Q_2}$, if $Q_1 \subseteq Q_2$, then

(1) $\left(X_1 \cup X_2\right)\Big|_{P_1 \cup P_2}^{Q_2}$.

(2) $\left(X_1 \cap X_2\right)\Big|_{P_1 \cap P_2}^{Q_1}$.

Theorem 8 Suppose that U is the universe of interval fuzzy sets $X_1\Big|_{P_1}^{Q_1}$ and $X_2\Big|_{P_2}^{Q_2}$, if $P_2 \subseteq P_1$ and $Q_1 \subseteq Q_2$, then

(1) $\left(X_1 \cup X_2\right)\Big|_{P_1}^{Q_2}$.

(2) $\left(X_1 \cap X_2\right)\Big|_{P_2}^{Q_1}$.

Definition 10 Suppose that U is the universe of interval fuzzy sets $X_1\Big|_{P_1}^{Q_1}$ and $X_2\Big|_{P_2}^{Q_2}$, if $P_1 = P_2$ and $Q_1 = Q_2$, then interval fuzzy sets X_1, X_2 are equal, which is denoted as $X_1 = X_2$.

Definition 11 Suppose that U is the universe of interval fuzzy sets $X_1\Big|_{P_1}^{Q_1}$ and $X_2\Big|_{P_2}^{Q_2}$, so

(1) if $P_2 \subseteq P_1 \subseteq Q_1 \subseteq Q_2$, the interval fuzzy set X_2 fuzzily contains X_1, which is denoted as $X_1 \subseteq X_2$.

(2) if $P_2 \subset P_1 \subseteq Q_1 \subset Q_2$, the interval fuzzy set X_2 fuzzily and truly contains X_1, which is denoted as $X_1 \subset X_2$.

Theorem 9 Suppose that U is the universe of interval fuzzy sets X, Y and Z, then

(1) Idempotent law

$X \cup X = X$, $X \cap X = X$

(2) Commutative law

$X \cup Y = Y \cup X$, $X \cap Y = Y \cap X$

(3) Associative law

$\left(X \cup Y\right) \cup Z = X \cup \left(Y \cup Z\right)$, $\left(X \cap Y\right) \cap Z = X \cap \left(Y \cap Z\right)$

(4) Distributive law

$\left(X \cup Y\right) \cap Z = \left(X \cap Z\right) \cup \left(Y \cap Z\right)$, $\left(X \cap Y\right) \cup Z = \left(X \cup Z\right) \cap \left(Y \cup Z\right)$

(5) Absorption law

$\left(X \cup Y\right) \cap X = X$, $\left(X \cap Y\right) \cup X = X$

(6) Identity law

$X \cup U = U$, $X \cap U = X$, $X \cup \varphi = X$, $X \cap \varphi = \varphi$

(7) Double complementarity law

$(X^c)^c = X$

(8) Duality law

$\left(X \cup Y\right)^c = X^c \cap Y^c$, $\left(X \cap Y\right)^c = X^c \cup Y^c$

3 PROOF OF THREE THEOREMS

3.1 Decomposition theorem

Definition 12 Suppose that U is the universe of interval fuzzy set $X\Big|_P^Q$, and $0 \leq \lambda \leq 1$, then

(1) $X_\lambda = \left\{u \in U \,\Big|\, X(u)\Big|_{P(u)\geq\lambda}^{Q(u)\geq\lambda}\right\} = X_\lambda\Big|_{P_\lambda}^{Q_\lambda}$ is called the λ cut set of interval fuzzy set X, and λ the level, where Q_λ is the λ upper cut set of X and P_λ the λ lower cut set of X.

(2) $X_\lambda = \left\{u \in U \,\Big|\, X(u)\Big|_{P(u)>\lambda}^{Q(u)>\lambda}\right\} = X_\lambda\Big|_{P_\lambda}^{Q_\lambda}$ is called the λ strong cut set of interval fuzzy set X, where Q_λ is the λ upper strong cut set and P_λ the λ lower strong cut set.

Specially, the cut set of interval fuzzy set X is interval grey set.

Suppose that $\lambda_i \in [0, 1]$, $i \in I$, where I is the universe of i, then $\bigvee_{i\in I} \lambda_i = \sup_{i\in I}\{\lambda_i\}$ and $\bigwedge_{i\in I} \lambda_i = \inf_{i\in I}\{\lambda_i\}$.

Property 1 Suppose that U is the universe of interval fuzzy sets X, Y and $X_i (i \in I)$, then

(1) $\left(X \cup Y\right)_\lambda = X_\lambda \cup Y_\lambda$, $\left(X \cap Y\right)_\lambda = X_\lambda \cap Y_\lambda$, $\left(X \cup Y\right)_\lambda = X_\lambda \cup Y_\lambda$, $\left(X \cap Y\right)_\lambda = X_\lambda \cap Y_\lambda$.

(2) $\left(\bigcup_{i\in I} X_i\right)_\lambda = \bigcup_{i\in I} (X_i)_\lambda$, $\left(\bigcap_{i\in I} X_i\right)_\lambda = \bigcap_{i\in I} (X_i)_\lambda$, $\left(\bigcup_{i\in I} X_i\right)_\lambda = \bigcup_{i\in I} (X_i)_\lambda$, $\left(\bigcap_{i\in I} X_i\right)_\lambda = \bigcap_{i\in I} (X_i)_\lambda$.

(3) $\bigcap_{i\in I} X_{\lambda_i} = X_{(\bigvee_{i\in I}\lambda_i)}$, $\bigcup_{i\in I} X_{\lambda_i} = X_{(\bigwedge_{i\in I}\lambda_i)}$.

(4) $X_0 = U$, $X_1 = \Phi$.

(5) $X\subseteq Y \Rightarrow \lambda\cdot X\subseteq \lambda\cdot Y$.

(6) $(X^c)_{\underset{\cdot}{\lambda}} = (X_{1-\lambda})^c$, $(X^c)_\lambda = (X_{\underset{\cdot}{1-\lambda}})^c$.

The property is the extension of classical fuzzy set and interval grey set.

Definition 13 Suppose that U is the universe of interval fuzzy set $X\Big|_P^Q$, $0\le\lambda\le 1$, then, λX_λ is called the scalar product between λ and X_λ. It is interval fuzzy set, and its membership function is as follows

$$\lambda X_\lambda(u) = \lambda\wedge X_\lambda(u)\Big|_{\lambda\wedge P_{\lambda(u)}}^{\lambda\wedge Q_{\lambda(u)}} = \min\left\{\lambda, X_\lambda(u)\Big|_{\min\{\lambda, P_\lambda(u)\}}^{\min\{\lambda, Q_\lambda(u)\}}\right\}$$

$$\min\{\lambda, R_\lambda(u)\} = \begin{cases}\lambda, u\in R_\lambda\\ 0, u\notin R_\lambda\end{cases}$$

where R_λ represents X_λ, Q_λ or P_λ.

Theorem 10 Decomposition Theorem 1

Suppose that U is the universe of interval fuzzy set $X\Big|_P^Q$, and X_λ the cut set of interval fuzzy set X, $0\le\lambda\le1$, then

$$X = \bigcup_{0\le\lambda\le1}\lambda X_\lambda = \bigcup_{0\le\lambda\le1}\lambda X_\lambda\Big|_{\bigcup_{0\le\lambda\le1}\lambda P_\lambda}^{\bigcup_{0\le\lambda\le1}\lambda Q_\lambda}$$

Proof It is equivalent to demonstrate that $\forall u\in U$, $X(u) = \left(\bigcup_{0\le\lambda\le1}\lambda X_\lambda\right)(u)$, and

$$\left(\bigcup_{0\le\lambda\le1}\lambda X_\lambda\right)(u) = \left(\bigcup_{0\le\lambda\le1}\lambda X_\lambda\right)(u)\Bigg|_{\left\{\bigcup_{0\le\lambda\le1}\lambda P_\lambda\right\}(u)}^{\left\{\bigcup_{0\le\lambda\le1}\lambda Q_\lambda\right\}(u)} =$$

$$\bigvee_{0\le\lambda\le1}(\lambda X_\lambda)(u)\Bigg|_{\bigvee_{0\le\lambda\le1}(\lambda P_\lambda)(u)}^{\bigvee_{0\le\lambda\le1}(\lambda Q_\lambda)(u)} =$$

$$\left\{\underbrace{\left[\bigvee_{0\le\lambda\le X(u)}\lambda\wedge X\lambda(u)\right]}_{1} \vee \underbrace{\left[\bigvee_{X(u)\le\lambda\le1}\lambda\wedge X\lambda(u)\right]}_{2}\right\}\left|\begin{matrix}\left\{\underbrace{\left[\bigvee_{0\le\lambda\le Q(u)}\lambda\wedge Q_\lambda(u)\right]}_{1}\vee\underbrace{\left[\bigvee_{Q(u)\le\lambda\le1}\lambda\wedge Q_\lambda(u)\right]}_{2}\right\}\\ \left\{\underbrace{\left[\bigvee_{0\le\lambda\le P(u)}\lambda\wedge P_\lambda(u)\right]}_{1}\vee\underbrace{\left[\bigvee_{P(u)\le\lambda\le1}\lambda\wedge P_\lambda(u)\right]}_{2}\right\}\end{matrix}\right.$$

According to the first item, because $R(u)\ge\lambda$, $u\in R_\lambda$, i. e, $R_\lambda(u) = 1$, then $\lambda\wedge R_\lambda(u) = \lambda$. However, according to the second item, because $\lambda > R(u)$, $u\notin R_\lambda$, that is to say, $R_\lambda(u) = 0$, then, $\lambda\wedge R_\lambda(u) = 0$.

From the above analysis, we have

$$\left(\bigcup_{0\le\lambda\le1}\lambda X_\lambda\right)(u) =$$

$$\left\{\left[\bigvee_{0\le\lambda\le X(u)}\lambda\right]\vee\left[\bigvee_{X(u)\le\lambda\le1}0\right]\right\}\left|\begin{matrix}\left\{\left[\bigvee_{0\le\lambda\le Q(u)}\lambda\right]\vee\left[\bigvee_{Q(u)\le\lambda\le1}0\right]\right\}\\ \left\{\left[\bigvee_{0\le\lambda\le P(u)}\lambda\right]\vee\left[\bigvee_{P(u)\le\lambda\le1}0\right]\right\}\end{matrix}\right.$$

$$= \bigvee_{0\le\lambda\le X(u)}\lambda\Bigg|_{\bigvee_{0\le\lambda\le P(u)}\lambda}^{\bigvee_{0\le\lambda\le Q(u)}\lambda} = X(u)\Big|_{P(u)}^{Q(u)} = X(u)$$

The decomposition theorem can construct interval fuzzy set with interval grey set, and it is the method for linking interval fuzzy set and interval grey set.

Theorem 11 Decomposition Theorem 2

Suppose that U is the universe of interval fuzzy set $X\Big|_P^Q$, and $X_{\underset{\cdot}{\lambda}}$ the strong cut set of interval fuzzy set X, $0\le\lambda\le1$, then

$$X = \bigcup_{0\le\lambda\le1}\lambda X_{\underset{\cdot}{\lambda}} = \bigcup_{0\le\lambda\le1}\lambda X_{\underset{\cdot}{\lambda}}\Bigg|_{\bigcup_{0\le\lambda\le1}\lambda P_{\underset{\cdot}{\lambda}}}^{\bigcup_{0\le\lambda\le1}\lambda Q_{\underset{\cdot}{\lambda}}}$$

Proof It is similar to prove Theorem 1.

Theorem 12 Decomposition Theorem 3

Suppose that U is the universe of interval fuzzy set $X\Big|_P^Q$, if a pair of set-valued mappings exist, we have H_Q: $[0, 1]\to P(U)$, $\lambda\to H_Q(\lambda)$, H_P: $[0, 1]\to P(U)$, $\lambda\to H_P(\lambda)$. $\forall\lambda\in[0, 1]$, $Q_{\underset{\cdot}{\lambda}}\subseteq H_Q(\lambda)\subseteq Q_\lambda$ and $P_{\underset{\cdot}{\lambda}}\subseteq H_P(\lambda)\subseteq P_\lambda$. Then

(1) $X = X\Bigg|_{\bigcup_{0\le\lambda\le1}\lambda H_P(\lambda)}^{\bigcup_{0\le\lambda\le1}\lambda H_Q(\lambda)}$.

(2) $\lambda_1 < \lambda_2$, $H_Q(\lambda_1)\supseteq H_Q(\lambda_2)$ and $H_P(\lambda_1)\supseteq H_P(\lambda_2)$.

(3) $X_\lambda = X_\lambda\Bigg|_{\bigcap_{\alpha<\lambda} H_P(\alpha)}^{\bigcap_{\alpha<\lambda} H_Q(\alpha)}$, $\alpha\in(0, 1]$ and

$$X_{\underset{\cdot}{\lambda}} = X_{\underset{\cdot}{\lambda}}\Bigg|_{\bigcup_{\alpha>\lambda} H_P(\alpha)}^{\bigcup_{\alpha>\lambda} H_Q(\alpha)},\ \alpha\in[0, 1).$$

Proof

(1) $Q_{\underset{\cdot}{\lambda}}\subseteq H_Q(\lambda)\subseteq Q_\lambda\Rightarrow\lambda\cdot Q_{\underset{\cdot}{\lambda}}\subseteq\lambda\cdot H_Q(\lambda)\subseteq\lambda\cdot Q_\lambda$, then

$$Q = \bigcup_{\lambda\in[0,1)}\lambda\cdot Q_{\underset{\cdot}{\lambda}}\subseteq\bigcup_{\lambda\in[0,1]}\lambda\cdot H_Q(\lambda)\subseteq\bigcup_{\lambda\in[0,1]}\lambda\cdot$$

$Q_\lambda = Q\Rightarrow Q = \bigcup_{\lambda\in[0,1]}\lambda\cdot H(\lambda)$ and $P_{\underset{\cdot}{\lambda}}\subseteq H_P(\lambda)\subseteq P_\lambda\Rightarrow\lambda\cdot P_{\underset{\cdot}{\lambda}}\subseteq\lambda\cdot H_P(\lambda)\subseteq\lambda\cdot P_\lambda$, then

$$P = \bigcup_{\lambda\in[0,1)}\lambda\cdot P_{\underset{\cdot}{\lambda}}\subseteq\bigcup_{\lambda\in[0,1]}\lambda\cdot H_P(\lambda)\subseteq\bigcup_{\lambda\in[0,1]}\lambda\cdot$$

$P_\lambda = P\Rightarrow P = \bigcup_{\lambda\in[0,1]}\lambda\cdot H(\lambda)$, then

$$X = X\Bigg|_{\bigcup_{0\le\lambda\le1}\lambda H_P(\lambda)}^{\bigcup_{0\le\lambda\le1}\lambda H_Q(\lambda)}.$$

(2) $\lambda_1<\lambda_2\Rightarrow H_Q(\lambda_1)\supseteq Q_{\lambda_1}\supseteq Q_{\lambda_2}\supseteq H_Q(\lambda_2)$, i. e., $H_Q(\lambda_1)\supseteq H_Q(\lambda_2)$. And $H_P(\lambda_1)\supseteq P_{\lambda_1}\supseteq P_{\lambda_2}\supseteq H_P(\lambda_2)$, i. e, $H_P(\lambda_1)\supseteq H_P(\lambda_2)$.

(3) $\forall\alpha<\lambda$, $\lambda\in(0, 1]$, then

$H_Q(\alpha) \supseteq Q_\alpha \supseteq Q_\lambda \Rightarrow \bigcap_{\alpha<\lambda} H_Q(\alpha) \supseteq Q_\lambda, \lambda \in (0, 1]$, and $\bigcap_{\alpha<\lambda} H_Q(\alpha) \subseteq \bigcap_{\alpha<\lambda} Q_\alpha = Q_{(\vee_{\alpha<\lambda})} = Q_\lambda,\ \lambda \in (0, 1]$.

Then, $Q_\lambda = \bigcap_{\alpha<\lambda} H_Q(\alpha)$, it is similar to prove $P_\lambda = \bigcap_{\alpha<\lambda} H_P(\alpha)$. Therefore, $X_\lambda = X_\lambda \Big|^{\bigcap_{\alpha<\lambda} H_Q(\alpha)}_{\bigcap_{\alpha<\lambda} H_P(\alpha)}$, $\alpha \in (0, 1], \forall \alpha > \lambda, \lambda \in [0, 1)$, then

$Q_{\underset{\cdot}{\lambda}} \supseteq Q_\alpha \supseteq H_Q(\alpha) \Rightarrow \bigcup_{\alpha>\lambda} H_Q(\alpha) \subseteq Q_{\underset{\cdot}{\lambda}}, \lambda \in [0, 1)$, and $\bigcup_{\alpha>\lambda} H_Q(\alpha) \subseteq \bigcup_{\alpha>\lambda} Q_\alpha = Q_{(\wedge_{\alpha>\lambda} \underset{\cdot}{\lambda})} = Q_{\underset{\cdot}{\lambda}},\ \lambda \in [0, 1)$.

Then, $Q_{\underset{\cdot}{\lambda}} = \bigcup_{\alpha>\lambda} H_Q(\alpha)$, it is similar to prove $P_{\underset{\cdot}{\lambda}} = \bigcup_{\alpha>\lambda} H_P(\alpha)$.

Therefore, $X_{\underset{\cdot}{\lambda}} = X_{\underset{\cdot}{\lambda}} \Big|^{\bigcup_{\alpha>\lambda} H_Q(\alpha)}_{\bigcup_{\alpha>\lambda} H_P(\alpha)}$, $\lambda \in [0, 1)$.

According to decomposition Theorem 3, in interval fuzzy set $X \Big|^Q_P$, taking the upper bound Q for example, Q can be constructed by not only cut set Q_λ or $Q_{\underset{\cdot}{\lambda}}$, but also general set cluster $H_Q(\lambda), \lambda \in [0, 1]$. Namely, $H_Q(\lambda)$ may be Q_λ or $Q_{\underset{\cdot}{\lambda}}$, even the set between them. Because of the flexibility of $H_Q(\lambda)$, $H_Q(\lambda)$ is applied in real life widely.

3.2 Representation theorem

Definition 14 Suppose that set-valued mapping $H: [0, 1] \to P(U)$, $\lambda \to H(\lambda)$, it meets

$\forall \lambda_1, \lambda_2 \in [0, 1], \forall \lambda_1 < \lambda_2 \Rightarrow H(\lambda_2) \subseteq H(\lambda_1)$

Then, H is the nested set based on U, $N(U)$ the set of all nested sets based on U.

Definition 15 Suppose that $H, H_t \in N(U)$, $t \in T$, the operations of $\cup$, $\cap$ and c in $N(U)$ are defined as follows

$\left(\bigcup_{t\in T} H_t\right)(\lambda) \in \bigcup_{t\in T} H_t(\lambda)$, $\left(\bigcap_{t\in T} H_t\right)(\lambda) \in \bigcap_{t\in T} H_t(\lambda)$ $H^c(\lambda) = (H(1-\lambda))^c$, $\lambda \in [0, 1]$

The above three operations are called union, intersection, complement of nested sets, respectively.

According to decomposition Theorem 3, any interval fuzzy set $X \Big|^Q_P$ has a pair of set-valued mapping H_Q and H_P, i.e., interval fuzzy set $X \Big|^Q_P$ has nested set group (H_Q, H_P), where, H_Q, $H_P \in N(U)$. $N'(U)$ is the set of all nested set groups based on U.

Theorem 13 Representation Theorem 1

Suppose that H_Q, $H_P \in N(U)$, and

$$Q = \bigcup_{\lambda\in[0,1]} \lambda \cdot H_Q(\lambda),\ P = \bigcup_{\lambda\in[0,1]} \lambda \cdot H_P(\lambda)$$

Then

$$Q, P \in F(U), \text{ i.e., } X \Big|^Q_P = X \Big|^{\bigcup_{\lambda\in[0,1]} \lambda\cdot H_Q(\lambda)}_{\bigcup_{\lambda\in[0,1]} \lambda\cdot H_P(\lambda)} \in$$

$IF(U)$, and $\forall \lambda \in [0, 1]$, we have

(1) $\lambda \neq 0 \Rightarrow X_\lambda \Big|^{Q_\lambda}_{P_\lambda} = X_\lambda \Big|^{\bigcap_{\lambda>\alpha} H_Q(\alpha)}_{\bigcap_{\lambda>\alpha} H_P(\alpha)}$.

(2) $\lambda \neq 1 \Rightarrow X_{\underset{\cdot}{\lambda}} \Big|^{Q_{\underset{\cdot}{\lambda}}}_{P_{\underset{\cdot}{\lambda}}} = X_{\underset{\cdot}{\lambda}} \Big|^{\bigcup_{\lambda<\alpha} H_Q(\alpha)}_{\bigcup_{\lambda<\alpha} H_P(\alpha)}$.

Proof $\forall \lambda \in [0, 1]$, because $H_Q(\lambda)$ and $H_P(\lambda) \in P(U)$, $\lambda \cdot H_Q(\lambda) \in F(U)$, $\lambda \cdot H_P(\lambda) \in F(U)$, then, $X \Big|^Q_P = X \Big|^{\bigcup_{\lambda\in[0,1]} \lambda\cdot H_Q(\lambda)}_{\bigcup_{\lambda\in[0,1]} \lambda\cdot H_P(\lambda)}$.

According to the decomposition Theorem 3, if it meets condition $Q_{\underset{\cdot}{\lambda}} \subseteq H_Q(\lambda) \subseteq Q_\lambda$ and $P_{\underset{\cdot}{\lambda}} \subseteq H_P(\lambda) \subseteq P_\lambda$, the above (1) and (2) hold. It is equivalent to demonstrate the condition.

$\forall \lambda \in [0, 1]$, if $\lambda \neq 1$, $x \in Q_{\underset{\cdot}{\lambda}} \Rightarrow Q(x) > \lambda \Rightarrow \left((\bigvee_{\alpha\in[0,1]} \alpha \cdot H_Q(\alpha))(x)\right) > \lambda \Rightarrow \left(\bigvee_{\alpha\in[0,1]} (\alpha \wedge H_Q(\alpha)(x))\right) > \lambda \Rightarrow \left(\exists \alpha_0 \in [0, 1]\right)(\alpha_0 \wedge H_Q(\alpha_0)(x) > \lambda) \Rightarrow \alpha_0 > \lambda,\ H_Q(\alpha_0)(x) = 1 \Rightarrow x \in H_Q(\alpha_0) \subseteq H_Q(\lambda)$.

If $\lambda = 1$ and $Q_{\underset{\cdot}{\lambda}} = \varnothing$, $x \in H_Q(\lambda) \Rightarrow H_Q(\lambda)(x) = 1 \Rightarrow \left(\bigvee_{\alpha\in[0,1]} \alpha\ H_Q(\alpha)(x)\right) \geq \lambda \wedge H_Q(\lambda)(x) = \lambda \Rightarrow Q(x) \geq \lambda \Rightarrow x \in Q_\lambda$

Then, $\forall \lambda \in [0, 1]$, $Q_{\underset{\cdot}{\lambda}} \subseteq H_Q(\lambda) \subseteq Q_\lambda$. It is similar to demonstrate condition $P_{\underset{\cdot}{\lambda}} \subseteq H_P(\lambda) \subseteq P_\lambda$.

Theorem 14 Representation Theorem 2

If there is the mapping $\varphi: N'(U) \to IF(U)$, then

$\forall X \Big|^Q_P \in IF(U)\ \exists (H_Q, H_P) \in N'(U)$

$H_Q \to \varphi(H_Q) = \bigcup_{\lambda\in[0,1]} \lambda \cdot H_Q(\lambda)$

$H_Q \to \varphi(H_Q) = \bigcup_{\lambda\in[0,1]} \lambda \cdot H_Q(\lambda)$

Then $<$ is the homomorphism surjection from $\left(N(U), \cup, \cap, c\right)$ to $\left(IF(U), \cup, \cap, c\right)$, and $P\ K \le [0, 1]$, we have

(1) $(<(H_Q))_{\underset{\cdot}{K}}\ H_Q(K)\ A\ (<(H_Q))_K$ and $(<(H_P))_{\underset{\cdot}{K}}\ H_P(K)\ A\ (<(H_P))_K$

(2) $(<(H_Q))_K = \bigcap\limits_{A > K} H_Q(A)$ and $(<(H_P))_K = \bigcap\limits_{A > K} H_P(A)$, $P\ K \le (0, 1]$.

(3) $(<(H_Q))_{\underset{\cdot}{K}} = \bigcup\limits_{A > K} H_Q(A)$ and $(<(H_P))_{\underset{\cdot}{K}} = \bigcup\limits_{A > K} H_P(A)$, $P\ K \le [0, 1)$.

Proof The above (1) - (3) can be proved based on representation Theorem 1 directly, and then this paper proves that $<$ is homomorphism surjection.

$<$ maintains the operations "$\cup$" and "$\cap$".

In view of (3), and $H_t \in N(U)\ (t \in T)$, $P\ K \in [0, 1)$, then

$$\left(<\left(\bigcup_{t\in T}(H_Q)_t\right)\right)_{\underset{\cdot}{K}} = \bigcup_{A > K}\left(\bigcup_{t\in T}(H_Q)_t\right)(A) = \bigcup_{A > K}\left(\bigcup_{t\in T}(H_Q)_t(A)\right) = \bigcup_{t\in T}\left(\bigcup_{A > K}(H_Q)_t(A)\right) = \bigcup_{t\in T}\left(<((H_Q)_t)\right)_{\underset{\cdot}{K}} = \left(\bigcup_{t\in T}<((H_Q)_t)\right)_K$$

Apparently, when $K = 1$, the above formula holds. According to the decomposition theorem, we have

$$<\left(\bigcup_{t\in T}(H_Q)_t\right) = \bigcup_{t\in T}<\left((H_Q)_t\right)$$

It is similar to demonstrate that

$$<\left(\bigcup_{t\in T}(H_P)_t\right) = \bigcup_{t\in T}<\left((H_P)_t\right)$$

Then, it is also similar to demonstrate "$\cap$", where

$$<\left(\bigcap_{t\in T}(H_Q)_t\right) = \bigcap_{t\in T}<((H_Q)_t)$$

$$<\left(\bigcap_{t\in T}(H_P)_t\right) = \bigcap_{t\in T}<((H_P)_t)$$

$<$ maintains the operation c.

$$P\ K \in [0, 1), (<(H_Q))_K = \bigcap_{A > K} H^c_{\widehat{Q}}(A) = \bigcap_{A > K}(H_Q(1 - A))^c = \left(\bigcup_{1-A > 1-K} H_Q(1 - A)\right)^c = \left(\bigcup_{A > 1-K} H_Q(A)\right)^c = ((<(H_Q))_{1-K})^c = ((<(H_Q))^c)_K$$

It is similar to demonstrate that

$$(<(H_P))_K = ((<(H_P))^c)_K$$

Therefore, $<$ is homomorphism surjection.

3.3 Extension theorem

Theorem 15 Extension theorem

Suppose that there is mapping $f: U_1 \to U_2$, then two mappings can be induced, which are denoted as f and f^{-1}, respectively.

$$f: IF(U_1) \to IF(U_2), X\Big|_P^Q \to f\left(X\Big|_P^Q\right)$$

$$f^{-1}: IF(U_2) \to IF(U_1), Y\Big|_{P'}^{Q'} \to f^{-1}\left(Y\Big|_{P'}^{Q'}\right)$$

where

$$f(Q)(u_2) = \begin{cases} \bigvee\limits_{f(u_1) = u_2} Q(u_1) & f(u_2) \ne U \\ 0 & f(u_2) = U \end{cases}$$

$$f(P)(u_2) = \begin{cases} \bigvee\limits_{f(u_1) = u_2} P(u_1) & f(u_2) \ne U \\ 0 & f(u_2) = U \end{cases}$$

Apparently

$$f(Q)(u_2) \ge f(P)(u_2)$$

$$f^{-1}(Q')(u_1) = Q'(f(u_1))$$

$$f^{-1}(P')(u_1) = P'(f(u_1))$$

Then, $f(X)$ is called the image of X based on f, and $f^{-1}(Y)$ the inverse image of Y based on f.

4 EXAMPLES

4.1 Examp e 1

In order to illustrate the decomposition Theorem 1 of interval fuzzy set, the example 1 is given.

Suppose that

$$X\Big|_P^Q = X\Big|_{0.2/x_1 + 0.4/x_2 + 0.4/x_3}^{0.3/x_1 + 0.5/x_2 + 0.6/x_3 + 0.4/x_4 + 0.3/x_5} \quad 0 \le K \le 1$$

Then

$$X_{0.6} = X_{0.6}\Big|_U^{\{x_3\}} = X_{0.6}\Big|_U^{1/x_3}$$

$$X_{0.5} = X_{0.5}\Big|_U^{\{x_2, x_3\}} = X_{0.5}\Big|_U^{1/x_2 + 1/x_3}$$

$$X_{0.4} = X_{0.4}\Big|_{\{x_2, x_3\}}^{\{x_2, x_3, x_4\}} = X_{0.4}\Big|_{1/x_2 + 1/x_3}^{1/x_2 + 1/x_3 + 1/x_4}$$

$$X_{0.3} = X_{0.3}\Big|_{\{x_2, x_3\}}^{\{x_1, x_2, x_3, x_4, x_5\}} = X_{0.3}\Big|_{1/x_2 + 1/x_3}^{1/x_1 + 1/x_2 + 1/x_3 + 1/x_4 + 1/x_5}$$

$$X_{0.2} = X_{0.2}\Big|_{\{x_1, x_2, x_3\}}^{\{x_1, x_2, x_3, x_4, x_5\}} =$$

$$X_{0.2}\Big|^{1/x_1+1/x_2+1/x_3+1/x_4+1/x_5}_{1/x_1+1/x_2+1/x_3}$$

$$0.6X_{0.6}=X_1=X_{0.6}\Big|^{0.6/x_3}_{\mathrm{U}}$$

$$0.5X_{0.5}=X_2=0.5X_{0.5}\Big|^{0.5/x_2+0.5/x_3}_{\mathrm{U}}$$

$$0.4X_{0.4}=X_3=0.4X_{0.4}\Big|^{0.4/x_2+0.4/x_3+0.4/x_4}_{0.4/x_2+0.4/x_3}$$

$$0.3X_{0.3}=X_4=0.3X_{0.3}\Big|^{0.3/x_1+0.3/x_2+0.3/x_3+0.3/x_4+0.3/x_5}_{0.3/x_2+0.3/x_3}$$

$$0.2X_{0.2}=X_5=0.2X_{0.2}\Big|^{0.2/x_1+0.2/x_2+0.2/x_3+0.2/x_4+0.2/x_5}_{0.2/x_1+0.2/x_2+0.2/x_3}$$

According to the decomposition theorem, we have

$$X=\bigcup_{0\le K\le 1} K X K= X_1+X_2+X_3+X_4+X_5=$$

$$X\left|\begin{matrix}\left(0.6/x_3\right)\cup\left(0.5/x_2+0.5/x_3\right)\cup\left(0.4/x_2+0.4/x_3+0.4/x_4\right)\cup\\ \left(0.3/x_1+0.3/x_2+0.3/x_3+0.3/x_4+0.3/x_5\right)\cup\\ \left(0.2/x_1+0.2/x_2+0.2/x_3+0.2/x_4+0.2/x_5\right)\\ \\ \mathrm{U}\cup\mathrm{U}\cup\left(0.4/x_2+0.4/x_3\right)\cup\left(0.3/x_2+0.3/x_3\right)\cup\\ \left(0.2/x_1+0.2/x_2+0.2/x_3\right)\end{matrix}\right.=$$

$$X\left|\begin{matrix}0.2\vee 0.3/x_1+0.2\vee 0.3\vee 0.4\vee 0.5/x_2+\\ 0.2\vee 0.3\vee 0.5\vee 0.4\vee 0.6/x_3+\\ 0.2\vee 0.3\vee 0.4/x_4+0.2\vee 0.3/x_5\\ \\ 0.2/x_1+0.2\vee 0.3\vee 0.4/x_2+\\ 0.2\vee 0.3\vee 0.4/x_3\end{matrix}\right.=$$

$$X\Big|^{0.3/x_1+0.5/x_2+0.6/x_3+0.4/x_4+0.3/x_5}_{0.2/x_1+0.4/x_2+0.4/x_3}$$

It can be seen from the example 1 that by using of the decomposition theorem, the original study of fuzzy object can be decomposed into a series of corresponding classical issues to deal with.

4.2 Example 2

In order to illustrate the decomposition Theorem 2 of interval fuzzy set, the example 2 is given.

Suppose that $U=\{x_1, x_2, x_3, x_4, x_5\}$ is the universe of interval fuzzy set $X\Big|^{Q}_{P}$, and the strong cut set of interval fuzzy set X is

$$XK=\begin{cases} XK\Big|^{\{x_1,x_2,x_3,x_4,x_5\}}_{\{x_1,x_2,x_3,x_4\}} & 0\le K\le 0.2\\ XK\Big|^{\{x_1,x_2,x_3,x_4\}}_{\{x_1,x_3,x_4\}} & 0.2< K\le 0.5\\ XK\Big|^{\{x_1,x_3,x_4\}}_{\{x_1,x_4\}} & 0.5< K\le 0.7\\ XK\Big|^{\{x_1,x_4\}}_{\mathrm{U}} & 0.7< K\le 0.9\\ XK\Big|^{\{x_4\}}_{\mathrm{U}} & 0.9< K\le 1.0\end{cases}$$

Then it can be changed to

$$XK=\begin{cases} XK\Big|^{\{1,1,1,1,1\}}_{\{1,1,1,1,0\}} & 0\le K\le 0.2\\ XK\Big|^{\{1,1,1,10\}}_{\{1,0,1,1,0\}} & 0.2< K\le 0.5\\ XK\Big|^{\{1,0,1,1,0\}}_{\{1,0,0,1,0\}} & 0.5< K\le 0.7\\ XK\Big|^{\{1,0,0,1,0\}}_{\{0,0,0,0,0\}} & 0.7< K\le 0.9\\ XK\Big|^{\{0,0,0,1,0\}}_{\{0,0,0,0,0\}} & 0.9< K\le 1.0\end{cases}$$

$$X = X\Big|^{Q}_{P} = X\Big|^{0.9/x_1+0.5/x_2+0.7/x_3+1.0/x_4+0.2/x_5}_{0.7/x_1+0.2/x_2+0.5/x_3+0.2/x_4}$$

It can be seen from the example 2 that the decomposition theorem provides the possibility of using interval grey set to construct fuzzy set, and establishes the contact between fuzzy set and grey set.

4.3 Example 3

In order to illustrate the extension theorem of interval fuzzy set, the example 3 is given.

Suppose that $U_1=\{x_1, x_2, x_3, x_4, x_5\}$, $U_2=\{a, b, c, d\}$, and

$$f(x)=\begin{cases} a & x\in\{x_1,x_2\}\\ b & x\in\{x_3,x_4\}\\ c & x=x_5\end{cases}$$

$$X\Big|^{0.3/x_1+0.5/x_2+0.6/x_3+0.4/x_4+0.3/x_5}_{0.2/x_1+0.4/x_2+0.4/x_3}$$

Solution According to the extension theorem, $Y=f(X)\in IF(U_2)$, and because $f^{-1}(a)\ne\mathrm{U}$, then

$$f(X)(a)=\bigvee_{f(u_1)=a} X(u_1)\Big|^{\bigvee_{f(u_1)=a}Q(u_1)}_{\bigvee_{f(u_1)=a}P(u_1)}=$$

$$\bigvee_{f(u_1)=a} X(u_1)\Big|^{Q(x_1)\vee Q(x_2)}_{P(x_1)\vee P(x_2)}=$$

$$\bigvee_{f(u_1)=a} X(u_1)\Big|^{0.3\vee 0.5}_{0.2\vee 0.4}=\bigvee_{f(u_1)=a} X(u_1)\Big|^{0.5}_{0.4}$$

Similarly

$$f(X)(b)=\bigvee_{f(u_1)=a} X(u_1)\Big|^{0.6\vee 0.4}_{0.4\vee 0}=\bigvee_{f(u_1)=a} X(u_1)\Big|^{0.6}_{0.4}$$

$$f(X)(c)=\bigvee_{f(u_1)=a} X(u_1)\Big|^{0.3}_{0}$$

Because $f^{-1}(d)=\mathrm{U}$, $f(X)(d)=0$, therefore, $Y=f(X)=Y\Big|^{0.5/a+0.6/b+0.3/c}_{0.4/a+0.4/b}$.

Because P $u_1\in U_1$, $f^{-1}(Y)(u_1)=Y(f(u_1))$, we have

$$f^{-1}(Y)(x_1)=Y(a)\Big|^{Q(a)}_{P(a)}=Y\Big|^{0.5}_{0.4}$$

$$f^{-1}(Y)(x_2) = Y(a)\Big|_{P(a)}^{Q(a)} = Y\Big|_{0.4}^{0.5}$$

$$f^{-1}(Y)(x_3) = Y(b)\Big|_{P(b)}^{Q(b)} = Y\Big|_{0.4}^{0.6}$$

$$f^{-1}(Y)(x_4) = Y(b)\Big|_{P(b)}^{Q(b)} = Y\Big|_{0.4}^{0.6}$$

$$f^{-1}(Y)(x_5) = Y(c)\Big|_{P(c)}^{Q(c)} = Y\Big|_{0}^{0.3}$$

Therefore

$$f^{-1}(Y) = f^{-1}(Y)\Big|_{0.4/x_1+0.4/x_2+0.4/x_3+0.4/x_4}^{0.5/x_1+0.5/x_2+0.6/x_3+0.6/x_4+0.3/x_5}$$

It can be seen from the example 3 that by using the extension theorem, the classical method can be promoted to make the result more close to reality.

5 CONCLUSION

This paper introduces the concept of interval fuzzy set, and then describes the operation of interval fuzzy set. The decomposition theorem, the representation theorem and the extension theorem are constructed and proved. Examples are given to explain the three theorems. Through the proof and examples, it is illustrated that the classical fuzzy set is a special case of interval fuzzy set, and the interval fuzzy set is the effective extension of the classical fuzzy set. Based on the interval fuzzy set, the next research will be further application in fuzzy clustering, fuzzy classification, and fuzzy pattern recognition and so on.

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(厦门大学自动化系，厦门，361005，中国)

：在区间模糊集概念及其性质的基础上，针对区间模糊集的3个理论尚未得到有效证明的现状，首先对区间模糊集的知识表示及其运算法则进行了研究，然后提出了基于区间模糊集的截集概念。并在此基础之上进一步研究了基于区间模糊集的分解定理、表现定理和扩展定理，并通过实例说明了传统模糊集是区间模糊集的一个特例，区间模糊集是传统模糊集的有效扩展。

关键词：区间模糊集；分解定理；表现定理；扩展定理
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