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博 士 学 位 论 文

分数阶对流-扩散方程的基本解和数值方法

Fundamental Solutions and Numerical Methods of the Fractional Advection-Dispersion Equations

沈淑君

指导教师姓名：刘 发 旺 教授
专 业 名 称：计 算 数 学
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摘 要

分数阶微分方程的特点是含有非整数阶导数，能非常有效的描述各种各样的物质的记忆和遗传性质，在物理，数学，机械工程，生物，电子工程，控制理论和金融等领域发挥越来越重要的作用。各种分数阶模型与无秩序的动力系统有着紧密的联系。物理学中的反常扩散最初是从随机游走模型中发展得来的。分数阶对流-扩散方程是模拟各种反常扩散现象的有力工具。分数阶对流-扩散方程是分数阶动力方程的一部分，方程中可以含有空间和时间的分数阶导数算子。本文分别讨论时间、空间、空间-时间的分数阶对流-扩散方程。文中所涉及的空间分数阶导数均为Riesz空间分数阶导数，它含有双侧的Riemann-Liouville分数阶导数。Riesz空间分数阶导数的一个显著优点是适用于高维空间。

本文主要由下面几个部分组成。

首先，引言部分介绍了分数阶微积分的发展历史和已有的一些重要成果。然后介绍分数阶微积分的一些预备知识，给出了分数阶微积分一些基本定义和性质。

其次，第二章从时间分数阶扩散方程出发，提出一显式守恒差分近似，进行稳定性与收敛性分析。将得到的结果推广到时间分数阶对流-扩散方程，对时间分数阶对流-扩散方程的显式守恒差分近似，用数学归纳法进行稳定性与收敛性分析，并用质点的随机游走来解释。实践证明，随机游走是解释许多自然科学学科中随机过程的强有力的模型。

第三章考虑Riesz空间分数阶对流-扩散方程。这一章包含三部分内容。首先考虑初值问题。利用Laplace和Fourier变换得到Riesz空间分数阶对流-扩散方程初值问题的基本解。用格林函数表示基本解，并对其进行概率解释。再利用Riemann-Liouville分数阶导数与Grünwald-Letnikov分数阶导数之间的等价关系，构造一显式有限差分近似，这一离散格式可以解释为一个随机游走模型，并且收敛于稳定的概率分布。第二部分考虑初边值问题，基于Riesz空间分数阶导数可以表示为拉普拉斯算子幂次方这一特点，借助于矩阵转换技巧与分数阶行方法求此方程的数值解，利用特征函数的性质与Laplace变换相结合求出其新的

解析解，再对这两种解进行比较。最后，进一步讨论了初边值问题的有限差分近似，构造显式和隐式两种差分近似，并进行了误差分析。

第四章中考虑Riesz空间-时间分数阶对流-扩散方程。首先考虑初值问题。利用Laplace和Fourier变换得到Riesz空间-时间分数阶对流-扩散方程初值问题的基本解。用格林函数表示此基本解，对其进行概率解释。利用Riemann-Liouville分数阶导数与Grünwald-Letnikov分数阶导数之间的等价关系，构造一显式有限差分近似，这一离散格式可以解释为一个随机游走模型。然后讨论初边值问题。构造显式和隐式两种有限差分近似，进行了误差分析。由于分数阶导数的非局部性结构，使得计算分数阶微分方程的数值方法需要比整数阶花费更多的计算时间和存储要求。因此，在文中最后部分，我们提出了提高计算精度的Richardson外推法和减少计算量的"short-memory"原则，以此来改进我们的数值方法。

在每一章中，均给出数值例子说明所用数值方法的有效性。

关键词：分数阶对流-扩散方程；基本解；数值解；随机游走模型；稳定性；收敛性

Abstract

The characteristic of fractional order differential equation is containing the non-integer order derivative. It can effectively describe the memory and transmissibility of many kinds of material, and plays an increasingly important role in physics, mathematics, mechanical engineering, biology, electrical engineering, control theory, finance and other fields. All kinds of fractional models have close relation with chaotic dynamics. Anomalous diffusion in physics were originally developed from stochastic random walk models. Fractional advection-dispersion equations is powerful tool to simulate all kinds of anomalous diffusion phenomena. They are a subset of fractional kinetic equations that allow fractional derivatives in both the space and time operators. We discuss the time, space, space-time Fractional advection-dispersion equations respectively in this paper. The spatial derivatives discussed in the paper are all Riesz space fractional derivative, which include the left and right Riemann-Liouville fractional derivatives. The notable merit of Riesz space fractional derivative lies in its applicability to higher dimensional space.

This thesis consists of the four chapters.

Introduction presents the developmental history of fractional calculus and some important previous works at first. Then, gives some concerning fractional calculus to prepare the knowledge and present basic definitions and properties of fractional calculus.

In Chapter 2, starting from the time fractional diffusion equation,we present an explicit conservative difference approximation, and give the stability and convergency analysis. Then, we extend the obtained results to the time fractional advection-dispersion equation. For the explicit conservative difference approximation of the time fractional advection-dispersion equation, we analyge the stability and convergency by using mathematical induction, and interpret it as a particle random walk. Random walks have proven to be a useful model in understanding processes across a wide spec-

trum of scientific disciplines.

In Chapter 3, we consider the Riesz space fractional advection-dispersion equation. It has three components. At first, we consider the case of initial value problem. Using the method of the Laplace and Fourier transforms, we obtain the fundamental solution of the equation with initial condition. The fundamental solution is represented by Green function, and can be intergreted the probability interpretation. We construct an explicit finite difference approximation for the equation by using the equivalence relation between Riemann-Liouville fractional derivative and Grünwald-Letnikovmake fractional derivative. The discrete scheme can be interpreted as a discrete random walk model, and the random walk model converges to a stable probability distribution. Secondly, we consider the case of initial-boundary problem. For the Riesz space fractional derivative can be expressed by a fractional power of the Laplacian operator, the numerical solution of our equation can be obtained by recur to matrix transfer technique and fractional method of lines. We also derive the new analytic solution by utilizing the property of eigenfunction and Laplace transform. Furthermore we compare the analytic solution and the numerical solution. Finally, we discuss the finite difference approximations in the case of initial-boundary problem. The explicit and implicit difference approximations are presented and the error analysis is also given.

In Chapter 4, we consider the Riesz space-time fractional advection-dispersion equation. At first, we consider the case of initial value problem. We obtain the fundamental solution by using the method of the Laplace and Fourier transforms. The fundamental solution also be represented by Green function, and also can be proposed the probability interpretation. Using the equivalence relation between Riemann-Liouville fractional derivative and Grünwald-Letnikovmake fractional derivative, an explicit finite difference approximation for the equation is presented. The discrete scheme can be interpreted as a discrete random walk model. Then, the case of initial-boundary problem are discussed. The explicit and implicit finite difference approximations are proposed and the error analysis are also given. The non-local structure of fractional derivatives is one reason, why numerical methods for fractional differential equations are much more costly in computational time and storage requirements that their in-

teger order counterparts. Thus, we propose the Richardson extrapolation which can promote the accuracy and "short-memory" principle which reduce the computational cost finally, these two methods are used to improve our numerical methods.

Some numerical examples are presented in each chapter, which show the efficiency of our numerical methods.

Key words: fractional advection-dispersion equation; fundamental solution; numerical solution; random walk model; stability; convergence

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