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厦门大学

博 士 学 位 论 文

# Navier-Stokes-Poisson 方程组、液晶方程组及相关模型的若干问题

# Navier-Stokes-Poisson Equations、Liquid Crystals Equations and Some Other Related Problems

江 飞

指导教师姓名：谭 忠 教授

专 业 名 称：应 用 数 学

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# 中文摘要

自引力作用下、有粘可压的流体运动模型可用 Navier-Stokes-Poisson 方程组近似描述；如果不考虑粘性，则可用 Euler-Poisson 方程组近似描述。早在十九世纪，对于这两个方程组，特别是后者，引起很多数学工作者的研究，并且得到了很多有意义的结果，其中包括存在性、唯一性、稳定性、稳态解的存在性、多解问题、有固核和无固核的径向解的存在性。本文主要是从数学理论方面对刻画可压等熵自引力并受外力作用下有粘流体的 Navier-Stokes 方程组（简称 N. S. P. 方程组）进行深入研究，并以此为主线也展开对其它相关的流体模型，如液晶模型等的研究。本文的主要内容包括以下四个方面。

1、用 Feireisl 处理 Navier-Stokes 方程组情况的能量方法和标准紧性方法，研究了 N. S. P. 方程组的弱解（关于 3 维空间变量）关于时间的大时间行为。由于引力位势能量的出现，导致 N. S. P. 方程组的总能量可能是负的，为此我们结合引力位势的估计，证明了能量是有下界的。另一方面在做提高密度更高可积性估计时，为了得到引力与密度乘积项能被压强控制，我们利用 Hardy-Littlewood-Sobolev 不等式，得到了能量的有界性估计，即有界吸引集的存在性，然后用标准的紧性收敛方法，证明了闭轨迹的渐近紧性。最后我们利用 $\varrho \in C([0,T],\ L^p(\Omega))$（其中 $1 \le p < \gamma$）推导出引力位势能关于时间的连续性的结果，从而证明了全局吸引子的存在性。

2、证明了 3 维 N. S. P. 方程组，当初始条件是球对称且绝热指数 $\gamma \in (4/3,\ 3/2]$ 时，其 Cauchy 问题存在全局有限（或有界）能量弱解。该结果在球对称情况下推广了 Feireisl 的结果。特别我们还得到：对于 $\gamma = 4/3$ 情况，如果附加“总质量小于某个临界质量”条件，则此结论同样成立。证明的主要想法是先构造有界环区域上的逼近强解，然后借签 S. Jiang 在文献 [1] 中的方法取极限，通过一系列精细的紧性分析方法证明出该极限函数就是问题的弱解。难点主要在于证明能量不等式成立，为此我们进一步利用重化弱解的性质证明逼近密度强解在 $C([0,T],\ L^p(\mathbb{R}^3))$（其中 $1 \le p < \gamma$）中强收敛，从而得到能量不等式。此外为了构造逼近强解，对初始密度进行技巧性处理，以保证在给定的条件下从能量不等式中推导出逼近强解的一致先验估计。

3、我们也用上述紧性处理方法，研究了在 3 维空间中有界区域上向列液晶的近似模型，首先我们用 Galerkin 方法得到带人工粘性项液晶模型的逼近解，然后通过精细估计，得到液晶分子方向函数与密度下界无关的估计，这个关键性估计保证我们可以证明，在初始密度 $\varrho_0 \in L^\gamma(\Omega)$，$3/2 \le \gamma$ 的条件下，全局弱解的存在性，这个结果与 C. Liu 在文献 [2] 中要求初始密度有正下界的结果完全不同。难点主要在于 Galerkin 逼近解及

消粘取极限时紧性的证明，为了得到速度与密度的强紧性，我们采用了 P. L. Lions、C. Liu 和 A. Novotný 关于逼近解的紧性处理方法。值得指出的是，比较 X. G. Liu 和 Z. Y. Zhang 的结果，我们的定理条件不需要方程组中逼近函数项的 Lipschitz 条件。当然这个弱解仍满足能量不等式，及在分部意义下满足微分形式的能量不等式。

4、最后我们指出对于二维的可压正压流的 Navier-Stokes 方程组，其中压强 $p(\varrho) = a\varrho\log^d(\varrho)$（$\varrho$ 充分大，$d > 1$ 和 $a > 0$），在 Orlicz 空间理论体系下，可以得到解集关于有界区域变动的紧性，特别从中推导出了更一般的弱解存在性理论，即区域边界的正则性可放宽成一般有界开集，而不需要其它限制条件。此外，在最后一章附注了其它两个有关流体方程的爆破和弱解存在性结论。

**关键词**：自引力下流体，Navier-Stokes-Poisson 方程组，有界（或有限）能量弱解，大时间行为，液晶。

# Abstract

The motion of compressible, viscous self-gravitating fluids can be expressed by Navier-Stokes-Poisson equations, while this motion without viscosity can be described by Euler-Poisson equations. In the 19th century, this two equations, especially the latter, have catched lots of mathematician's research, and they have gotten various results, including existence, uniqueness, stability, the existence for stationary solutions, and the existence of the symmetric solutions with solid core and without solid core. In this thesis, our work is focused on the mathematical theoretical investigation of the Navier-Stokes system of compressible, viscous, isentropic self-gravitating fluids, which is called N. S. P. equations for short, furthermore we extend the topic to other related models for fluids, for example, liquid crystal model and so on. We organize the paper as follows. These results of this thesis can be brought under four headings.

1. We apply the energy method and standard compactness method, which Feireisl ever used to deal with the Navier-Stokes equations, to study the global behavior of weak solutions of the N. S. P. equations in time in a bounded three-dimension domain-arbitrary forces. Firstly, because the gravitation potential energy may make the total energy negative, we should make use of the fine estimations of gravitation potential energy to prove the lower bound of the total energy. Moreover, in improving the estimate of the density, we use the Hardy-Littlewood-Sobolev inequality to obtain that the product of gravitation and density can be controlled by the pressure, thus we get the bound of energy and prove the existence of bounded absorbing set. Secondly, by the standard compactness method, we obtain asymptotic compactness on closed trajectory. Lastly, we use $\varrho \in C([0,\, T],\, L^p(\Omega))$, where $1 \leq p < \gamma$, to infer that the gravitation potential energy is continuous on time, thus we can prove the existence of global attractor.

2. We prove the global existence of finite energy (or bounded) weak solutions to the Cauchy problem for the N. S. P. equations in $\mathbb{R}^3$ when the Cauchy initial data are radially symmetric. It extends Feireisl's existence theorem to the case $4/3 < \gamma \leq 3/2$ for radially symmetric weak solution. In particular, this conclusion also holds for $\gamma = 4/3$, if the total mass is less than certain critical mass. The main idea of proof is that we firstly construct a family of function sequences of approximate strong solutions on bounded

annulus, then refer to the methods in S. Jiang's paper[1] and take limit, finally by a series of fine compactness analysis, we prove the limit function is the weak solution of the problem . The main difficulty is to prove that the energy inequality holds. To prove this, we use the properties of renormalized continuity equations to prove that the density in approximate strong solutions is strong convergence in $C([0,T],\ L^p(\mathbb{R}^3))$, where $1 \leq p < \gamma$, thus we can get the energy inequality. In addition, to construct the approximate strong solutions, we use technical methods to deal with the initial density to make sure that we can deduce the uniformly estimations on the approximate strong solutions from the energy inequality.

3. We also study a simplified system for the flow of nematic liquid crystals in a bounded domain in the three dimensional space by above compactness method. We firstly construct the approximate solutions of the liquid crystals models with dissipation by Galerkin method, then by careful estimations, we obtain the new estimations with respect to the orientation of the liquid crystal molecules, which is independent of lower bound of the density. This key estimation makes sure that we can prove the global existence of the weak solutions under the condition of the initial density belong to $L^\gamma(\Omega)$ for any $3/2 \leq \gamma$. This result is very different from the C. Liu's results[2], where they need the condition of the positive lower bound of the initial density. The main difficulty is to prove the compactness when we take limit in Galerkin approximate solutions and let dissipation tend to zero. To overcome the difficulty, we use the P. L. Lions, C. Liu and A. Novotný's methods to deal with the compactness problems in order to obtain the strong compactness of velocity and density. It deserves to point out that compared to the X. G. Liu and Z. Y. Zhang's results, we don't need the Lipschitz condition in our condition of theorem, and the weak solutions also satisfy the energy inequality in integral or differential form.

4. Lastly, we point out that for the case of the Navier-Stokes equations of compressible, barotropic flow in two dimensions space, with pressure satisfying $p(\varrho) = a\varrho\log^d(\varrho)$ for large $\varrho$, here $d > 1$ and $a > 0$, under the frame of Orlicz spaces, we can get a compactness result for the solution set of the equations with respect to the variation of the underlying bounded spatial domain. In particular, we conclude a general existence theorem for the system in question with no restrictions on smoothness of the bounded spatial domain. In addition, we remark the conclusions on the blow up and the existence of weak solutions for other two related models on fluids in the last chapter.

**Key words:** self-gravitating fluid, Navier-Stokes-Poisson equations, bounded (or finite) energy weak solution, long-time behavior, liquid crystals.

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