学校编号： 10384　　分类号： ____ 密级： ____
学号： B200223004　　UDC: _____

博　士　学　位　论　文

量子环面上导子李代数的结构和表示

The structure and the representations of the derivation Lie algebras over the quantum torus

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申请学位级别：　博　士　学　位

专　业　名　称：　基　础　数　学

论文提交日期：　2005 年　3 月

论文答辩日期：　2005 年　5 月

学位授予单位：　厦　门　大　学

学位授予日期：　2005 年　　月

答辩委员会主席：_____

评阅人：_____

2005 年　3　月

The structure and the representations of the derivation Lie algebras over the quantum torus

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March, 2005

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二 〇 〇 五 年 三 月

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摘要

众所周知，仿射 Kac-Moody 代数及其表示在数学和物理的许多分支中都有重要应用。仿射 Kac-Moody 代数可看成从一维环面到复数域上有限维单李代数的多项式映射的泛中心扩张，也就是说，它们以单变量的罗朗多项式环为其坐标代数。单变量的罗朗多项式环的导子李代数称为 Witt 代数，而 Witt 代数的泛中心扩张称为 Virasoro 代数。Virasoro 代数的表示在仿射 Kac-Moody 代数的可积模的构造及其结构分析和可积模的分类中扮演着重要的角色（见 [GoO] 和 [GoP]）。同时， Virasoro 代数的 Unitary 表示在 moonshine module 以及顶点算子代数的构造和结构分析中也有许多应用（见 [FLM]）。此外， Virasoro 代数的表示在理论物理的 string 理论中也得到了广泛研究（见 [GSW]），事实上， Virasoro 代数蕴含在任何具有共形不变量的 2 维时空理论中。由此可见，李代数的坐标代数的导子李代数及其中心扩张在李代数的表示的研究中起着重要作用，同时它自身在数学和理论物理中也有许多应用。

作为仿射 Kac-Moody 代数的自然推广，文 [H-KT] 引进了扩张仿射李代数（ EALAs ）的概念。随后，文 [BGK] 和文 [AABGP] 的作者对扩张仿射李代数进行分类时，发现它们不但涉及到多变量的罗朗多项式坐标代数，还涉及到特殊的交错代数、 Jordan 代数及量子环面坐标代数。

量子环面包含了多变量的罗朗多项式环为其特例，同时量子环面的导子李代数还包含了一些特殊的 Jordan 代数的导子李代数为其子代数（见 [T2]）。此外， toroidal 李代数和以量子环面为坐标代数的 EALAs 上的可积模的分类问题可转化为其坐标代数上的导子李代数的模的分类（见 [E8] 、[EJ] 和 [E3] ）。因此，本论文的研究兴趣始终集中在对量子环面的导子李代数及其子代数的结构和表示的研究上。全文分两部分共五章，第一部分研究量子环面上导子李代数的表示，第二部分研究两个变量的量子环面上 Skew 导子李代数的结构；第一部分由三章构成，而第二部分由两章构成。下面我们更加详细地阐述各章的主要内容。

用 $\mathrm{Der}(\mathbf{C_q})$ 表示多变量的量子环面 $\mathbf{C_q}$ 上的导子李代数。在第一章，我们构造了从 gl_d- 模到 $\mathrm{Der}(\mathbf{C_q})$- 模的一族函子 F_g^α，该构造方法包含了沈光宇在 [S1] 中所构造的部分模和 Larsson 在 [L1] 及 Rao 在 [E2] 所构造的模。当量子环面矩阵的所有元素都是单位根时，我们还给出了权空间维数有限的 $\mathrm{Der}(\mathbf{C_q})$- 模的结构的完整刻划。我们证明了与支配整权 ψ 对应的有限维不可约 gl_d- 模 $V(\psi, b)$

在函子 F_g^α 下的像 $F_g^\alpha(V(\psi,b))$, 在 (ψ,b) 不等于 (δ_k,k) 或 $(0,b)$ 时，都是完全可约的。而对 (ψ,b) 等于 (δ_k,k) 或 $(0,b)$ 的情形，我们则给出 $F_g^\alpha(V(\psi,b))$ 的不可约商模。本章的结果包含了 Rao 文 [E2] 中的结果。这些结论已发表在 Journal of Algebra, 275 (2004), 250-274 。

在第二章，我们引进量子环面上 skew 导子李代数 L_q 的概念，并构造了从 sl_d- 模到 L_q- 模的一族函子 F_g^α 。接着，我们研究两个变量的量子环面上的 skew 导子李代数 L_q 的模 $F_g^\alpha(V)$ 。首先，我们给出 $F_g^\alpha(V)$ 的结构的完整刻划；然后证明了下述结论：

对任意有限维 sl_2- 模 V 和 W, 如果 q 是 p 次本原单位根，那么 L_q- 模 $F_{g_1}^\alpha(V)$ 与 $F_{g_2}^\beta(W)$ 同构的充要条件是

$$\alpha-\beta\in\Gamma,\ g_2(\mathbf{s})\equiv f(\alpha-\beta,\mathbf{s})g_1(\mathbf{s}),\ \forall\ \mathbf{s}\in\Gamma$$

而且 sl_2- 模 V 与 W 同构。

由于两个变量的量子环面上的 skew 导子李代数 L_q 包含了 Virasoro-like 代数和它的 q- 类似为其特例，故我们的结果包含了赵开明和张贺春在文 [ZZ] 中的部分结果。这些结果即将发表在《数学进展》（见 [LiT2] ）。

在第三章, 我们解决了 q- 类似 Virasoro-like 代数上中心作用非平凡的 Harish-Chandra 模的分类问题，主要证明了下列几个结论：

(1) q- 类似 Virasoro-like 代数上的任意一个中心作用非平凡的 Harish-Chandra 模都是广义高权模。

(2) 对 q- 类似 Virasoro-like 代数上的任一非平凡广义高权模 V ， V 是 Harish-Chandra 模的充要条件是：存在 $\mathcal{H}_{\mathbf{e_1}}$ 上一个满足定理 3.2.11 条件的线性泛函 ψ 使得 $V\simeq\overline{M}(\mathbf{e_1},\mathbf{e_2},A_{\overline{\psi}})$ 。

(3) 对 q- 类似 Virasoro-like 代数上的任一中心作用非平凡的模 V ， V 是 Harish-Chandra 模的充要条件是：存在 $\mathcal{H}_{\mathbf{e_1}}$ 上一个满足定理 3.2.11 条件的线性泛函 ψ 使得 $V\simeq\overline{M}(\mathbf{e_1},\mathbf{e_2},A_{\overline{\psi}})$ 。

对 Virasoro-like 代数我们也证明了类似的结果，该结果即将发表在 Journal of Pure and Applied Algebra（见 [LiT5] ）.

记两个变量的量子环面上的 skew 导子李代数为 L_q , 并令 $[L_q,L_q]=L_q'$ 。在第四章, 我们给出了李代数 L_q' 的二上同调群 $H^2(L_q',\mathbf{C})$, 并且具体给出了它

的 2-cocyle , 从而得到了 L'_q 的泛中心扩张 $\overline{L'_q}$ 。接着, 我们研究 L'_q 和 $\overline{L'_q}$ 的导子李代数。本章的结果即将发表在 Communication in Algebra (见 [LiT1]).

在第五章, 我们研究李代数 L_q 和 L'_q 的同构映射和自同构群, 并证明了下述结论:

(1) 如果 $\sigma : L'_{q_1} \longrightarrow L'_{q_2}$ 是一个李代数的满同态, 那么 σ 是一个分次同态。

(2) $L_{q_1} \cong L_{q_2}$ 或 $L'_{q_1} \cong L'_{q_2}$ 当且仅当 $q_1 = q_2$ 或者 $q_1 = q_2^{-1}$.

(3) 如果把 L_q 和 L'_q 的自同构群分别记为 $\mathrm{Aut}L_q$ 和 $\mathrm{Aut}L'_q$, 那么

$$\mathrm{Aut}L_q \cong GL_2(\mathbf{Z}) \propto (\mathbf{C}^* \times \mathbf{C}^*) \cong \mathrm{Aut}L'_q.$$

本文第二部分的结果推广了文 [JM1] 和 [JM2] 的结果并包含了上述两篇文章中的结果。

关键词: 量子环面、skew 导子李代数、支配整权、广义高权模、Virasoro-like 代数、 Harish-Chandra 模、自同构群。

Abstract

It is well known that affine Kac-Moody algebras and their representations play important roles in many branches in both mathematics and theoretical physics. Affine Kac-Moody algebras can be viewed as certain central extensions of the Lie algebras of polynomial maps of an one dimensional torus into finite dimensional simple Lie algebras over the field of complex numbers. That is, they used Laurent polynomial ring in one variable as their coordinates. The derivation Lie algebra of Laurent polynomial ring in one variable is called the Witt algebra, and the universal central extension of the Witt algebra is called the Virasoro algebra. The representations of the Virasoro algebra play crucial roles in the construction and the analysis of the structure of the integrable modules of affine Kac-Moody algebras (see [GoO] and [GoP]). Meanwhile, the unitary representations of the Virasoro algebra have many important applications in the construction and the analysis of the structure of the moonshine modules and the vertex operator algebras (see [FLM]). In addition to these, the representations of the Virasoro algebra have been studied extensively in the context of string theories in theoretical physics (see [GSW]). In fact, the Virasoro algebra is relevant in the theory of two dimensional space-time which possesses a comformal invariance. From the above facts, one can see that the derivation Lie algebras of the coordinate algebra of the infinite dimensional Lie algebras and their central extensions have many important applications in the study of the representation theory of Lie algebra. Moreover, they have many applications in both mathematics and theoretical physics.

The extended affine Lie algebras (EALAs for short) were first introduced in the paper [H-KT] as a natural generalization of affine Kac-Moody algebras. One motivation for this work was from quantum gauge theories. Subsequently, in [BGK] and [AABGK], it becomes clear that EALA's allow not only the multiple variables Laurent polynomial rings as their coordinate algebras but also certain alternative and Jordan algebras and the quantum torus as their coordinate algebras.

We notice that the quantum torus contains the Laurent polynomial ring

as its special case, and the derivation Lie algebra of the quantum torus contains the derivation Lie algebra of certain Jordan algebras as its subalgebras (see [T2]). Meanwhile, it is proved that the classifications of the integrable modules over the toroidal Lie algebras and some EALAs can be reduced to the classification of the modules over the derivation Lie algebras of their coordinate algebras (see [E8], [EJ] and [E3]). Therefore, we focus our attentions on the structure and the representations of the derivation Lie algebra of quantum torus in this thesis. The thesis contains two parts. The first part consists of three chapters, and is devoted to the study of the reprtesentations of the derivation Lie algebra over the quantum torus, while the second part consists of two chapters, and is devoted to the study of the structure of the skew derivation Lie algebra over the rank two quantum torus. Now we describe our main contents in more detail.

Let $\mathrm{Der}(\mathbf{C_q})$ be the derivation Lie algebra of the quantum torus $\mathbf{C_q}$. In chapter one, we construct a class of functors F_g^α from gl_d-modules to $\mathrm{Der}(\mathbf{C_q})$-modules, which give a large class of $\mathrm{Der}(\mathbf{C_q})$-modules containing some modules constructed by Guangyu Shen in [S1], and the modules constructed by Larsson in [L1] and Rao in [E2] as special cases. We also give a complete description of the structure of the $\mathrm{Der}(\mathbf{C_q})$-modules when the entries of the quantum torus matrix are roots of unity. We show that for any irreducible gl_d-module $V(\psi, b)$ corresponding to the dominant integral weight ψ, $F_g^\alpha(V(\psi, b))$ is completely reducible $\mathrm{Der}(\mathbf{C_q})$-modules when (ψ, b) does not equal to the fundamental weights (δ_k, k) or $(0, b)$. When (ψ, b) equals to the fundamental weights (δ_k, k) or $(0, b)$, we also give the irreducible quotient modules of $F_g^\alpha(V(\psi, b))$. Our results contain the results obtained in [E2] as special cases. The results of this chapter is published in Journal of Algebra, 275 (2004), 250-274.

In chapter two, we introduce the definition of the skew derivation Lie algebra L_q of the quantum torus $\mathbf{C_q}$ and construct a large class of funtors F_g^α from sl_d-modules to L_q-modules. Then we focus our study on the structure of the modules $F_g^\alpha(V)$ of the skew derivation Lie algebra L_q over the rank

two quantum torus. We describe the structure of the L_q-modules $F_g^\alpha(V)$, and show the following result:

for any finitely dimensional sl_2-module V and W, if q is a p-th root of unity, then L_q-module $F_{g_1}^\alpha(V)$ isomorphic to $F_{g_2}^\beta$ if and only if

$$\alpha-\beta\in\Gamma,\ g_2(\mathbf{s})\equiv f(\alpha-\beta,\mathbf{s})g_1(\mathbf{s}),\ \ \forall\mathbf{s}\in\Gamma$$

and sl_2-module V isomorphic to W.

Since the skew derivation Lie algebra L_q of rank 2 quantum torus contains the Virasoro-like algebra and the q- analog of the Virasoro-like algebra as its special cases, our results contain the results in [ZZ] as spcial cases. The results of this chapter will be published in Advances in Mathematics (Chenese)(see [LiT2]).

In chapter three, we study the classification of the nonzero level Harish-Chandra module over the q-analog of the Virasoro-like algebra. Mainly, we prove the following results.

(1) A nonzero level Harish-Chandra module over the q-analog of the Virasoro-like algebra is a generalized highest weight module.

(2) A nontrivial generalized highest weight module V over the q-analog of the Virasoro-like algebra is a Harish-Chandra module if and only if there exists a linear function ψ over $\mathcal{H}_{\mathbf{e_1}}$ with the property in Theorem 3.2.11 such that $V\simeq\overline{M}(\mathbf{e_1},\mathbf{e_2},A_{\overline{\psi}})$, up to a twist of automorphism of $\widehat{L}$.

(3) A nonzero level module V over the q-analog of the Virasoro-like algebra is a Harish-Chandra module if and only if there exists a linear function ψ over $\mathcal{H}_{\mathbf{e_1}}$ with the property in Theorem 3.2.11 such that $V\simeq\overline{M}(\mathbf{e_1},\mathbf{e_2},A_{\overline{\psi}})$, up to a twist of automorphism of $\widehat{L}$.

We prove similar results for the Virasoro-like algebra, which will be published in Journal of Pure and Applied Algebra (see [LiT5]).

Denote the skew derivation Lie algebra over the rank 2 quantum torus by L_q, and set $[L_q,L_q]=L_q'$. In chapter four , we show that the second cohomology group $H^2(L_q',\mathbf{C})$ of the Lie algebra L_q' is 2-dimensional. Moreover, we obtain the universal central extension $\overline{L_q'}$ of L_q', and the derivation Lie algebra of L_q' and $\overline{L_q'}$.

The results of this chapter will be published in Communication in Algebra (see [LiT1].

In chapter five , we study the isomorphism and the structure of the automorphism group of L_q and L'_q. We prove the following results.

(1) If $\sigma : L'_{q_1} \longrightarrow L'_{q_2}$ is an epimorphism of Lie algebras, then σ is a graded homomorphism.

(2) $L_{q_1} \cong L_{q_2}$ or $L'_{q_1} \cong L'_{q_2}$ if and only if $q_1 = q_2$ or $q_1 = q_2^{-1}$.

(3) Denote the automorphism group of L_q and L'_q by $\mathrm{Aut}L_q$ and $\mathrm{Aut}L'_q$ respectively, then

$$\mathrm{Aut}L_q \cong GL_2(\mathbf{Z}) \propto (\mathbf{C}^* \times \mathbf{C}^*) \cong \mathrm{Aut}L'_q,$$

Our results in part two of this thesis generlize the results of [JM1] and [JM2].

Key words: Quantum torus, skew derivation Lie algebra, Dominant integrable weight, Generalized highest weight module, the Virasoro-like algebra, Harish-Chandra module, Automorphism group.