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Positive Entire Solutions to an Elliptic System

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In a recent paper [1], Ye has demonstrated that the singular nonlinear elliptic equation

$$\Delta u = f(u, |\nabla u|, |x|)/u^{\lambda}, \quad x \in R^2, 0 < \lambda < 1$$

has a positive entire solution under some conditions imposed on f. Problems of finding positive entire solutions to elliptic equations have been studied by many authors since they have certain physical backgrounds and practical applications. For details, see, for example, [1–3] and the references therein.

In this paper, we study an elliptic system of the form

$$\Delta u_k = \frac{f_k(u_1, \cdots, u_m; |\nabla u_1|, \cdots, |\nabla u_m|; |x|)}{g_k(u_1, \cdots, u_m)}, \tag{1}$$

$$x \in R^n, \ n \geq 2, \ k = 1, 2, \cdots, m,$$

with the aim of extending and improving the results in [1]. The following hypotheses are adopted throughout:

(H1) $g \in C(R_+^m; R_+)$, $R_+ := [0, +\infty)$, and there exists a nondecreasing function $G \in C(R_+; R_+)$, with $G(A) > 0$ for some $A > 0$, such that

$$g_k(u) \geq G(\min_{1 \leq j \leq m}\{u_j\}) \qquad \text{for all } u \in R_+^m.$$

(H2) $f \in C(R_+^{2m+1}; R_+)$ and there exists a function $F \in C(R_+^3; R_+)$, which is nondecreasing in the first two arguments, such that for each $k \in \{1, 2, \cdots, m\}$

$$f_k(u, v, t) \leq F\left(\max_{1 \leq j \leq m}\{u_j\}, \max_{1 \leq j \leq m}\{v_j\}, t\right) \qquad \text{for all } (u, v, t) \in R_+^{2m+1}$$

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and for some $B \geq A$

$$\int_0^1 F(2B, B, s)\,ds + \int_1^{+\infty} sF(2B + B\ln s, B/s, s)\,ds \leq BG(B),$$

where A is determined by (H1).

We remark that our conditions allow $g_k(u) \equiv 0$ on a neighbourhood of the origin for some $k \in \{1, \cdots, m\}$ (and hence the system (1) can be singular) and $f(u, v, t) \equiv 0$ for all $(u, v, t) \in R_+^{2m+1}$ (and hence $u(x) := (B, \cdots, B)$ is a positive entire solution to (1) in this case).

Under hypotheses (H1) and (H2), we will use the Schauder fixed point theorem to demonstrate the following existence result.

Theorem 1 *Let* (H1) *and* (H2) *hold. Then the system* (1) *has a radially symmetric solution* $u \in C^2(R^n; R_+^m)$ *satisfying*

$$B \leq u_j(x) \leq \alpha(|x|) \quad \text{and} \quad |\nabla u_j(x)| \leq \beta(|x|), \qquad x \in R^n,\ j = 1, 2, \cdots, m, \tag{2}$$

where

$$\alpha(t) := \begin{cases} 2B, & 0 \leq t \leq 1; \\ 2B + B\ln t, & t \geq 1, \end{cases} \qquad \beta(t) := \begin{cases} B, & 0 \leq t \leq 1; \\ B/t, & t \geq 1. \end{cases} \tag{3}$$

Clearly, our theorem is an extension and improvement of the results in [1].

To prove Theorem 1, we define a mapping $\Phi: D \to D$ by

$$(\Phi y)_k(t) := \begin{cases} 0, & t = 0; \\ \displaystyle\int_0^t \left(\frac{s}{t}\right)^{n-1} \frac{f_k((Jy)(s); y(s); s)}{g_k((Jy)(s))}\,ds, & t > 0, \end{cases} \qquad k = 1, 2, \cdots, m,$$

for any $y \in D$, where

$$D := \{y \in C(R_+; R_+^m);\ y_k(t) \leq \beta(t),\ t \geq 0,\ k = 1, 2, \cdots, m\}$$

and

$$(Jy)_k(t) := B + \int_0^t y_k(s)\,ds, \qquad t \geq 0,\ k = 1, 2, \cdots, m.$$

We claim that (i) $\Phi(D) \subset C^1(R_+; R_+^m)$, (ii) $\Phi(D) \subset D$, (iii) $\Phi(D)$ is sequentially compact in D, and (iv) Φ is continuous on D.

From these claims, we know that Φ is a compact mapping from D into itself and hence has at least one fixed point in D, by the Schauder fixed point theorem. Let $y \in D$ is a fixed point of Φ. Then

$$y_k(t) = (\Phi y)_k(t) = \begin{cases} \displaystyle\int_0^t \left(\frac{s}{t}\right)^{n-1} \frac{f_k((Jy)(s); y(s), s)}{g_k((Jy)(s))}\,ds, & t > 0; \\ 0, & t = 0, \end{cases} \qquad k = 1, 2, \cdots, m,$$

$$w_k(t) := (Jy)_k(t) = B + \int_0^t y_k(s)\,ds, \quad t \geq 0,\ k = 1, 2, \cdots, m,$$

and hence for $k = 1, 2, \cdots, m$

$$\begin{cases} w_k'(t) = y_k(t), \ t \geq 0, \\ w_k''(t) + \dfrac{n-1}{t} w_k'(t) = \dfrac{f_k(w(t); w'(t); t)}{g_k(w(t))}, \ t > 0, \\ w_k(0) = B, \ w_k'(0) = 0, \ k = 1, 2, \cdots, m. \end{cases}$$

Let $u(x) := w(|x|)$. Then it is easy to check that the function $u(x) \in C^2(R^n; R^m_+)$ is a radially symmetric solution to the system (1).

In the sequel, we show claims. From the definition of Φ and (H1) —(H2), we have, for each fixed $y \in D$

$$\lim_{t \downarrow 0} (\Phi y)_k(t) = \lim_{t \downarrow 0} \int_0^t \left(\frac{s}{t}\right)^{n-1} \frac{f_k((Jy)(s); y(s); s)}{g_k((Jy)(s))} \mathrm{d}s = 0 = (\Phi y)_k(0),$$

i.e., $(\Phi y)(t)$ is continuous at $t = 0$;

$$(\Phi y)_k'(0) := \lim_{t \downarrow 0} \frac{(\Phi y)_k(t) - (\Phi y)_k(0)}{t} = \frac{f_k(BE; 0; 0)}{ng_k(BE)}, \qquad k = 1, 2, \cdots, m,$$

where $E := (1, 1, \cdots, 1)$, by the L'Hospital rule;

$$(\Phi y)_k'(t) = \frac{f_k((Jy)(t); y(t); t)}{g_k((Jy)(t))} - \frac{n-1}{t^n} \int_0^t s^{n-1} \frac{f_k((Jy)(s); y(s); s)}{g_k((Jy)(s))} \mathrm{d}s, \ t > 0,$$

$$\lim_{t \downarrow 0} (\Phi y)_k'(t) = \frac{f_k(BE; 0; 0)}{g_k(BE)} - \frac{n-1}{n} \frac{f_k(BE; 0; 0)}{g_k(BE)} = (\Phi y)_k'(0),$$

i.e., $(\Phi y)'(t)$ is continuous at $t = 0$. Claim (i) is thus proved.

Also, we have, for each fixed $y \in D$

$$\begin{aligned} &B \leq (Jy)_k(t) \leq B + \int_0^t \beta(s) \mathrm{d}s \leq \alpha(t), \qquad t \geq 0, \ k = 1, 2, \cdots, m, \\ &0 \leq (\Phi y)_k(t) \leq \int_0^t \left(\frac{s}{t}\right)^{n-1} \frac{F(\alpha(s), \beta(s), s)}{G(B)} \mathrm{d}s, \qquad t > 0, \ k = 1, 2, \cdots, m, \end{aligned} \tag{5}$$

and hence

$$0 \leq (\Phi y)_k(t) \leq \int_0^1 \frac{F(2B, B, s) \mathrm{d}s}{G(B)} \leq B \qquad \text{for } 0 \leq t \leq 1,$$

$$\begin{aligned} 0 \leq (\Phi y)_k(t) &\leq \frac{1}{t} \int_0^1 \frac{F(2B, B, s)}{G(B)} \mathrm{d}s + \int_1^{+\infty} \frac{sF(2B + B\ln s, B/s, s)}{G(B)} \mathrm{d}s \\ &\leq B/t \qquad \text{for all } t \geq 1, \end{aligned}$$

i.e.,

$$0 \leq (\Phi y)_k(t) \leq \beta(t) \qquad \text{for all } t \geq 0, \ k = 1, 2, \cdots, m, \tag{6}$$

which shows that claim (ii) is true.

Next, we prove claim (iii). For any given $\varepsilon > 0$ and each fixed $y \in D$, it follows from (6) that

$$0 \leq (\Phi y_k)(t) \leq \beta(t) = B/t < \varepsilon/2 \qquad \text{for } t \geq N := 2B/\varepsilon + 1$$

and hence for any $h > 0$ and $t \geq N$

$$|(\Phi y)_k(t + h) - (\Phi y)_k(t)| \leq \frac{B}{t + h} + \frac{B}{t} < \varepsilon. \tag{7}$$

From (5), we know that for each fixed $y \in D$

$$|(\Phi y)_k{}'(t)| \leq \frac{F(\alpha(t),\beta(t),t)}{G(B)} + \frac{n-1}{t^n}\int_0^t s^{n-1}\frac{F(\alpha(s),\beta(s),s)}{G(B)}\mathrm{d}s$$

$$\leq \frac{n}{G(B)}\max\{F(\alpha(t),\beta(t),t);\ 0\leq t\leq N+1\} =: M.$$

Let $\delta := \min\{1, \varepsilon/M\}$. Then for any $h\in(0,\delta)$ and $t\in[0,N]$

$$|(\Phi y)_k(t+h) - (\Phi y)_k(t)|$$

$$= \left|\int_t^{t+h}(\Phi y)_k{}'(s)\,\mathrm{d}s\right| \leq Mh < \varepsilon,\qquad k = 1,2,\cdots,m. \tag{8}$$

From (6) —(8), we know that $\Phi(D)$ is a family of functions which are uniformly bounded and equicontinuous on R_+. This shows that claim (iii) is true.

Finally, we prove (iv). For any fixed $y_0\in D$, we can choose a sequence $\{y_j\}\subset D$ converging to y_0 uniformly on R_+. Notice that

$$(\Phi y_j)_k(t) = \begin{cases}\displaystyle\int_0^t\left(\frac{s}{t}\right)\frac{f_k((Jy_j)(s);y_j(s);s)}{g_k((Jy_j)(s))}\mathrm{d}s, & t>0;\\ 0, & t=0\end{cases}$$

for each $k\in\{1,2,\cdots,m\}$ and each $j\in\{0,1,2,\cdots\}$. Since for any fixed $t>0$

$$0\leq \frac{f_k((Jy_j)(s);y_j(s);s)}{g_k((Jy_j)(s))}\leq\frac{F(\alpha(s),\beta(s),s)}{G(B)},\qquad 0\leq s\leq t,$$

$$\lim_{j\to\infty}\frac{f_k((Jy_j)(s);y_j(s);s)}{g_k((Jy_j)(s))} = \frac{f_k((Jy_0)(s);y_0(s);s)}{g_k((Jy_0)(s))},\qquad 0\leq s\leq t,$$

uniformly, we conclude that

$$\lim_{j\to\infty}(\Phi y_j)_k(t) = \lim_{j\to\infty}\int_0^t\left(\frac{s}{t}\right)^{n-1}\frac{f_k((Jy_j)(s);y_j(s);s)}{g_k((Jy_j)(s))}\mathrm{d}s$$

$$= \int_0^t\left(\frac{s}{t}\right)^{n-1}\frac{f_k((Jy_0)(s);y_0(s);s)}{g_k((Jy_0)(s))}\mathrm{d}s$$

$$= (\Phi y_0)(t),\qquad t>0$$

by the Lebesgue dominated convergence theorem. This means that Φ is continuous at $y_0\in D$. Since $y_0\in D$ is arbitrary, Φ is also continuous on D. Thus claim (iv) is proved.

(2) follows from (3), (5), and (6). The proof is complete.

References

[1] Ye Changqing, Existence of positive entire solutions to a class of singular nonlinear elliptic equations, *J. Math. Study*, **31**(4) (1988), 459 —465.

[2] Xu Xingyue, Existence of positive entire solutions to a class of singular nonlinear elliptic equations, *J. Math. (PRC)*, **15**(4) (1995), 421 —428.

[3] Kusano, T. and Usami, H., Positive solutions of a class of second-order semilinear elliptic equations in the plane, *Math. Ann.*, **268**(1984), 255 —264.