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POSITIVE SOLUTIONS TO A SINGULAR SECOND ORDER THREE-POINT BOUNDARY VALUE PROBLEM *

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Abstract: A fixed point theorem is used to study a singular second order three-point boundary value problem. The problem is more general. Combining the method of constructing Green functions with operators defined piecewise, the existence result of positive solutions to a singular second order three-point boundary value problem is established. The nonlinearity can be allowed to change sign.

Key words: positive solution; boundary value problem; existence

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1 Introduction and Main Results

The existence of positive solutions has been established for a nonlinear second order three-point boundary value problem of the form

$$\begin{cases} -y'' = Q(x)f(y) & (0 < x < 1), \\ y(0) = 0, y(1) = \alpha y(\eta) \end{cases} \tag{1}$$

only very recently in [1]. It was assumed there that $0 < \eta < 1$, $0 < \alpha\eta < 1$, $Q(x) \in C([0,1]; \mathbf{R}_+)$, $f(y) \in C(\mathbf{R}_+; \mathbf{R}_+)$, $\mathbf{R}_+ = [0, +\infty)$, and $f(y)$ is superlinear or sublinear at $y = 0$ and $y = +\infty$. And the proof of the result above-mentioned was based upon the following two propositions.

Theorem A[1,2] Let $0 < \eta < 1$, $\alpha\eta \neq 1$ and let $h(x) \in C[0,1]$. Then the linear three-point boundary value problem

$$\begin{cases} -y'' = h(x) & (0 < x < 1), \\ y(0) = 0, y(1) = \alpha y(\eta) \end{cases}$$

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has a unique solution $y(x) \in C^2[0,1]$, which can be expressed by

$$y(x) = -\int_0^x (x-t)h(t)\,dt - \frac{\alpha x}{1-\alpha\eta}\int_0^\eta (\eta-t)h(t)\,dt + \frac{x}{1-\alpha\eta}\int_0^1 (1-t)h(t)\,dt.$$

If $\alpha\eta \in (0,1)$, and $h(x) \le 0$ on $[0,1]$, then $y(x) \ge 0$ on $[0,1]$; further, if $h(x) > 0$ for some $x \in [0,1]$, then $y(x) > 0$ on $(0,1)$.

Theorem B[3] Let E be a Banach space and K be a cone in E. Assume that Ω_1, Ω_2 are open subsets of E with $0 \in \Omega_1$, $\bar{\Omega}_1 \subset \Omega_2$, and let

$$\Phi: K \cap (\bar{\Omega}_2 \setminus \Omega_1) \to K$$

be a completely continuous mapping such that either

i) $\|\Phi y\| \le \|y\|$ $\forall y \in K \cap \partial\Omega_1$ and $\|\Phi y\| \ge \|y\|$ $\forall y \in K \cap \partial\Omega_2$,

ii) $\|\Phi y\| \ge \|y\|$ $\forall y \in K \cap \partial\Omega_1$ and $\|\Phi y\| \le \|y\|$ $\forall y \in K \cap \partial\Omega_2$.

Then the mapping Φ has a fixed point $K \cap (\bar{\Omega}_2 \setminus \Omega_1)$.

In the present paper, we restudy the three-point boundary value problem (1) with the aim of extending and improving the above-mentioned result. The hypothesis we adopt is as follows:

H1) $\eta \in (0,1)$, $\alpha > 0$, $f(y) \in C(\mathbf{R}_+; \mathbf{R})$, and $Q(x) \in L^1_{\mathrm{loc}}(0,1)$ with $Q(x) \ge 0$ a.e. on $(0,1)$ and

$$0 < \int_0^1 (1-x)Q(x)\,dx < +\infty, \qquad \int_0^\eta xQ(x)\,dx < +\infty.$$

There are two points we should emphasize. First, in our problem the function $Q(x)$ is allowed to be singular at $x = 0$ and $x = 1$. For example, the function

$$Q(x) = x^{-a}(1-x)^{-b} \qquad (a, b \in (1,2))$$

satisfies H1). Secondly, our purpose is to deal with not only the case $\alpha\eta \in (0,1)$ but also the case $\alpha\eta \ge 1$. For the latter, Theorem A does not work. For this reason, we need the following two propositions.

Theorem **1** For each given $\rho \ge 0$, initial value problems

$$\begin{cases} w'' = \rho Q(x)w & (0 < x < 1), \\ w(0) = 0, \quad w'(0) = 1, \end{cases}$$

$$\begin{cases} w'' = \rho Q(x)w & (0 < x < \eta), \\ w(0) = 0, \quad w'(\eta) = -1, \end{cases}$$

$$\begin{cases} w'' = \rho Q(x)w & (\eta < x < 1), \\ w(\eta) = 0, \quad w'(\eta) = 1, \end{cases}$$

and

$$\begin{cases} w'' = \rho Q(x)w & (0 < x < 1), \\ w(1) = 0, \quad w'(1) = -1 \end{cases}$$

have solutions respectively $w_1(x) \in AC[0,1] \cap C^1[0,1)$, $w_2(x) \in AC[0,\eta] \cap C^1(0,\eta]$,

$w_3(x) \in AC[\eta,1] \cap C^1(0,1]$ and $w_4(x) \in AC[0,1] \cap C^1(0,1]$, which are all convex on their intervals of existence. Moreover,

$$\begin{vmatrix} w_2(x) & w_1(x) \\ w_2'(x) & w_1'(x) \end{vmatrix} \equiv w_2(0) = w_1(\eta) \qquad \text{on } [0,\eta],$$

$$\begin{vmatrix} w_4(x) & w_3(x) \\ w_4'(x) & w_3'(x) \end{vmatrix} \equiv w_4(\eta) = w_3(1) \qquad \text{on } [\eta,1]$$

and

$$\begin{vmatrix} w_4(x) & w_1(x) \\ w_4'(x) & w_1'(x) \end{vmatrix} \equiv w_4(0) = w_1(1) \qquad \text{on } [0,1].$$

It is indubitable that $w_1(x) = x$, $w_2(x) = \eta - x$, $w_3(x) = x - \eta$, and $w_4(x) = 1 - x$ when $\rho = 0$.

Theorem 2 For each given $\alpha \in \mathbf{R}$, there is a $\rho \geq 0$ such that

$$w_1(1) - \alpha w_1(\eta) > 0. \tag{2}$$

Assume that (2) holds. Then the linear three-point boundary value problem

$$\begin{cases} -y'' + \rho Q(x) y = h(x) \qquad (0 < x < 1), \\ y(0) = 0, \quad y(1) = \alpha y(\eta) \end{cases}$$

has a unique solution

$$y(x) = \begin{cases} \dfrac{w_4(\eta) w_1(\eta)}{w_1(1) - \alpha w_1(\eta)} \left(\displaystyle\int_0^\eta \frac{w_1(t)}{w_1(\eta)} h(t)\,dt + \int_\eta^1 \frac{w_4(t)}{w_4(\eta)} h(t)\,dt \right) \qquad (x = \eta), \\ w_2(x) \displaystyle\int_0^x \frac{w_1(t)}{w_1(\eta)} h(t)\,dt + w_1(x) \int_x^\eta \frac{w_2(t)}{w_1(\eta)} h(t)\,dt + y(\eta) \frac{w_1(x)}{w_1(\eta)} \\ \qquad\qquad (0 \leq x \leq \eta), \\ w_4(x) \displaystyle\int_\eta^x \frac{w_3(t)}{w_4(\eta)} h(t)\,dt + w_3(x) \int_x^1 \frac{w_4(t)}{w_4(\eta)} h(t)\,dt + y(\eta) \frac{w_4(x) + \alpha w_3(x)}{w_4(\eta)} \\ \qquad\qquad (\eta \leq x \leq 1) \end{cases} \tag{3}$$

for any fixed $h(x) \in L^1_{\mathrm{loc}}(0,1)$ with

$$\int_0^\eta w_1(t) \mid h(t) \mid dt + \int_\eta^1 w_4(t) \mid h(t) \mid dt < +\infty.$$

If $\alpha \geq 0$, and $h(x) \geq 0$ a.e. on $(0,1)$, then $y(x) \geq 0$ on $[0,1]$; further, if $\int_0^\eta w_1(x) h(x)\,dx + \int_\eta^1 w_4(x) h(x)\,dx > 0$, then $y(x) > 0$ for all $x \in (0,1]$.

Here a function $y(x)$ is said to be a solution to the three-point boundary value problem (1), if

i) $y(x) \in AC[0,1]$, $y(0) = 0$, $y(1) = \alpha y(\eta)$,

ii) $y'(x) \in AC_{loc}(0,1) \cap L^1(0,1)$, $y''(x) = L^1_{loc}(0,1)$, and

iii) $-y''(x) = Q(x) f(y(x))$ a.e. on $(0,1)$.

If $y(x) > 0$ for all $x \in (0,1]$, then it is called a positive solution to (1).

It is obvious that Theorem 2 is an improvement and extension of Theorem A.

To establish the existence of a positive solution to (1), we further assume that

H2) There exists a $\rho \geq 0$ such that (2) holds and $f^*(y) = f(y) + \rho y$ is nonnegative on $\mathbf{R}_+$, and

H3) One of the following two conditions is fulfilled

$$\limsup_{y \to 0} \frac{f^*(y)}{y} \leq \beta \text{ and } \limsup_{y \to +\infty} \frac{f^*(y)}{y} \geq \eta, \tag{4}$$

$$\limsup_{y \to 0} \frac{f^*(y)}{y} \geq \gamma \text{ and } \limsup_{y \to +\infty} \frac{f^*(y)}{y} \leq \beta. \tag{5}$$

Here β and γ are both constants satisfying

$$\beta M \left(\int_0^\eta w_1(x) Q(x) \mathrm{d}x + \int_\eta^1 w_4(x) Q(x) \mathrm{d}x \right) < 1, \tag{6}$$

$$\frac{\gamma \sigma w_1(\eta)}{w_1(1) - \alpha w_1(\eta)} \int_\eta^1 w_4(x) Q(x) \mathrm{d}x > 1, \tag{7}$$

$$M = 1 + \frac{\max\{w_4(\eta), w_1(\eta)\}}{w_1(1) - \alpha w_1(\eta)} \max_{\eta \leq x \leq 1} \left\{ \frac{w_4(x) + \alpha w_3(x)}{w_4(\eta)} \right\}, \tag{8}$$

$$\sigma = \frac{\min_{\eta \leq x \leq 1} \left\{ \frac{w_4(x) + \alpha w_3(x)}{w_4(\eta)} \right\}}{\left\{ \frac{w_1(1) - \alpha w_1(\eta)}{\max\{w_4(\eta), w_1(\eta)\}} \right\} + \max_{\eta \leq x \leq 1} \left\{ \frac{w_4(x) + \alpha w_3(x)}{w_4(\eta)} \right\}} < 1. \tag{9}$$

It is clear that H2) allows $f(y)$ to change sign when $\rho > 0$.

Applying Theorem 2 and Theorem B, we can prove the existence results below-mentioned.

Theorem 3 Let H1) - H3) be fulfilled. Then the three-point boundary value problem (1) has a positive solution.

Theorem 4 Let $0 < \eta < 1$, $0 < \alpha\eta < 1$, $f(y) \in C(\mathbf{R}_+; \mathbf{R}_+)$ and $Q(x) \in L^1_{loc}(0,1)$ with $Q(x) \geq 0$ a.e. on $(0,1)$ and

$$0 < \int_0^1 (1 - x) Q(x) \mathrm{d}x < +\infty, \quad \int_0^\eta x Q(\mathrm{d}x) < +\infty.$$

Then the three-point boundary value problem (1) has a positive solution, provided that one of the following two conditions holds

i) $\lim_{y \to 0} \frac{f(y)}{y} = 0$ and $\lim_{y \to +\infty} \frac{f(y)}{y} = +\infty$,

ii) $\lim_{y \to 0} \frac{f(y)}{y} = +\infty$ and $\lim_{y \to +\infty} \frac{f(y)}{y} = 0$.

Being a consequence of Theorem 3, Theorem 4 improves and extends the result in [3].

It must be pointed out that the condition $\alpha\eta \in (0,1)$ is sharp in Theorem 4. There are two reasons: first, when $\alpha = 0$ the three-point boundary value problem (1) "degenerates" into a two-point boundary value problem; secondly, when $\alpha\eta = 1$, when claim that the three-point boundary value problem

$$\begin{cases} -y'' = y^2 & (0 < x < 1), \\ y(0) = 0, \quad y(1) = \dfrac{1}{\eta} y(\eta) \end{cases}$$

has no positive solution. In fact, if the claim is false, i.e., the problem has a positive solution $y(x)$, then the equation implies that $y(x)$ is a strictly concave function on $[0,1]$ and hence $y(\eta) \geq \eta y(1)$, which contradicts the boundary condition $y(1) = \dfrac{1}{\eta} y(\eta)$. This shows that the claim is true.

2 Preliminaries

In this section, we are going to prove Theorems 1 and 2. To this end, we first present a proposition, which will be frequently used later on.

Lemma 1 Let $h(x) \in L^1_{\text{loc}}(0,1)$ with $h(x) \geq 0$ a.e. on $(0,1)$ and

$$\int_0^\eta xh(x)\,\mathrm{d}x + \int_\eta^1 (1-x)h(x)\,\mathrm{d}x < +\infty.$$

Then we have

$$\lim_{x\to 0} x\int_x^\eta h(t)\,\mathrm{d}t = 0 = \lim_{x\to 1}(1-x)\int_\eta^x h(t)\,\mathrm{d}t. \tag{10}$$

Proof of Lemma 1 Put $v(x) = x\int_x^\eta h(t)\,\mathrm{d}t$, $0 < x \leq \eta$. Then

$$0 \leq v(x) \leq \int_0^\eta th(t)\,\mathrm{d}t < +\infty \qquad \text{for all } x \in (0,\eta],$$

$$v'(x) = \int_x^\eta h(t)\,\mathrm{d}t - xh(x) \qquad (0 < x < \eta),$$

and hence for any $\delta \in (0,\eta)$

$$\int_\delta^\eta |v'(x)|\,\mathrm{d}x \leq \int_\delta^\eta \mathrm{d}x \leq \int_x^\eta h(t)\,\mathrm{d}t + \int_0^\eta xh(x)\,\mathrm{d}x =$$

$$\int_\delta^\eta (t-\delta)h(t)\,\mathrm{d}t + \int_0^\eta xh(x)\,\mathrm{d}x \leq$$

$$2\int_\delta^\eta xh(x)\,\mathrm{d}x < +\infty$$

which shows that $v'(x) \in L^1(0,\eta)$ and $v(x) \in AC[0,\eta]$. As a result, we obtain

$$\int_0^s v'(x)\,\mathrm{d}x = \int_0^s \mathrm{d}x \int_x^\eta h(t)\,\mathrm{d}t - \int_0^s xh(x)\,\mathrm{d}x \equiv v(s) \qquad \text{for all } s \in (0,\eta],$$

which implies that $v(0) = 0$, i.e., the first equation is true.

In the same way as above, we can lead to the second equation. The Lemma is thus proved.

Proof of Theorem 1 When $\rho = 0$, all the conclusions of Theorem 1 are fulfilled, of course. We now prove that the initial value problem

$$\begin{cases} w'' = \rho Q(x) w & (0 < x < 1), \\ w(1) = 0, \quad w'(1) = -1 \end{cases} \tag{11}$$

has a unique positive solution for given $\rho > 0$. Put

$$B = \left\{ u(x) \in C[0,1];\ \|u\|_B < +\infty \right\},$$

where $\|u\|_B = \max\limits_{0\leq x\leq 1} |u(x)| \exp\left(-2\rho\int_0^1 s(1-s)Q(s)\,\mathrm{d}s\right)$.

Define a mapping $L : B \to B$ by

$$Lu(1) = 1,$$

$$(Lu)(x) = 1 + \frac{\rho}{1-x}\int_x^1 (t-x)(1-t)Q(t)u(t)\,dt \qquad (0 \le x \le 1).$$

For any $u \in B$, we have

$$\left|\frac{1}{1-x}\int_x^1 (t-x)(1-t)Q(t)u(t)\,dt\right| \le \int_0^1 t(1-t)Q(t)|u(t)|\,dt \le$$

$$\max_{0\le t\le 1}|u(t)|\int_0^1 t(1-t)Q(t)\,dt < +\infty.$$

As a result, we conclude that $L(B) \subset B$. We claim that L is a contraction mapping. For any $u_1(x), u_2(x) \in B$, we have

$$\exp\left(-2\rho\int_x^1 s(1-s)Q(s)\,ds\right)|(Lu_1)(x) - (Lu_2)(x)| \le$$

$$\exp\left(-2\rho\int_x^1 s(1-s)Q(s)\,ds\right)\rho\int_x^1 t(1-t)Q(t)|(u_1)(t) - (u_2)(t)|\,dt \le$$

$$\frac{1}{2}\|u_1 - u_2\|_B \exp\left(-2\rho\int_x^1 s(1-s)Q(s)\,ds\right) \times$$

$$\int_x^1 2\rho t(1-t)Q(t)\exp\left(-2\rho\int_t^1 s(1-s)Q(s)\,ds\right)dt \le \frac{1}{2}\|u_1 - u_2\|_B$$

for all $x \in [0,1]$,

i.e., $\|Lu_1 - Lu_2\|_B \le \frac{1}{2}\|Lu_1 - Lu_2\|_B \ \forall u_1, u_2 \in B$. This shows that the claim is true. From the claim, we know that L has a unique fixed point in B. Let $u_4(x) \in C[0,1]$ be the unique fixed point. Then

$$u_4(x) = 1 + \frac{\rho}{1-x}\int_x^1 (t-x)(1-t)Q(t)u_4(t)\,dt \qquad (0 \le x < 1).$$

Write

$$w_4(x) = (1-x)u_4(x) = 1 - x + \rho\int_x^1 (t-x)Q(t)w_4(t)\,dt \qquad (0 \le x \le 1). \tag{12}$$

Then $w_4(1) = 0$,

$$w_4'(x) = -1 - \rho\int_x^1 Q(t)w_4(t)\,dt \quad (0 < x < 1), \quad w_4'(1) = -1, \tag{13}$$

$$w_4''(x) = \rho Q(x)w_4(x) \qquad \text{a.e. on } x \in (0,1). \tag{14}$$

This shows that $w_4(x)$ is a solution to (11).

Note that

$$\int_0^1 |w_4'(x)|\,dx \le 1 + \rho\int_0^1 dx\int_x^1 (1-t)Q(t)|u_4(t)|\,dt \le$$

$$1 + \rho\int_0^1 t(1-t)Q(t)\,dt\max_{0\le t\le 1}|u_4(t)| < +\infty,$$

which means that $w_4'(x) \in AC_{loc}(0,1] \cap L^1(0,1)$ and $w_4(x) \in AC[0,1]$.

We now claim that $w_4(x) > 0$ for all $x \in [0,1)$ i.e., $w_4(x) = 0$ is the only zero of $w_4(x)$. If it is not the case, then there exists an $x_0 \in [0,1)$, such that

$$w_4'(x) > 0 \text{ on } (x_0,1), \quad w_4(x_0) = w_4(1) = 0$$

since that $w_4'(1) = -1$ together with $w_4(1) = 0$ implies that $w_4(x) > 0$ on a left neighborhood of $x = 1$. By the Rolle's Theorem, there exist a $\xi \in (x_0, 1)$ such that $w_4'(\xi) = 0$. On the other hand, from (13) we lead to

$$w_4'(\xi) = -1 - \rho \int^1 Q(t)\, w_4(t)\, \mathrm{d}t < 0,$$

a contradiction. This shows that the claim is true. That is to say, $w_4(x)$ is the unique positive solution to (11). And (14) tells us that $w_4(x)$ is convex on $[0,1]$.

In the same way as above, we can prove that the initial value problems

$$\begin{cases} w'' = \rho Q(x)\, w & (0 < x < 1), \\ w(0) = 0, \quad w'(0) = 1, \end{cases}$$

$$\begin{cases} w'' = \rho Q(x)\, w & (0 < x < \eta), \\ w(\eta) = 0, \quad w'(\eta) = -1, \end{cases}$$

and

$$\begin{cases} w'' = \rho Q(x)\, w & (\eta < x < 1), \\ w(\eta) = 0, \quad w'(\eta) = 1 \end{cases}$$

have unique positive solutions $w_1(x)$, $w_2(x)$, and $w_3(x)$, which are convex on $[0,\eta]$, $[\eta,1]$, and $[0,1]$, respectively. As a result, we have

$$\begin{cases} x \le w_1(x) \le w_1(\eta)\, x/\eta \text{ and } 0 \le w_2(x) \le w_2(0) \quad \text{on } [\eta,1], \\ 0 \le w_3(x) \le w_3(1) \text{ and } (1-x) \le w_4(x) \le w_4(\eta)(1-x)/(1-\eta) \quad \text{on } [\eta,1]. \end{cases} \tag{15}$$

Put

$$W(x) = \begin{vmatrix} w_4(x) & w_1(x) \\ w_4'(x) & w_1'(x) \end{vmatrix} \qquad (0 < x < 1).$$

Then

$$W'(x) = \begin{vmatrix} w_4'(x) & w_1'(x) \\ w_4'(x) & w_1'(x) \end{vmatrix} + \begin{vmatrix} w_4(x) & w_1(x) \\ \rho Q(x)\, w_4(x) & \rho Q(x)\, w_1(x) \end{vmatrix} = 0 \text{ a.e. on } (0,1).$$

According to Lemma 1, (15), (12), (13), and

$$\begin{cases} w_1(x) = x + \rho \int (x - t)\, Q(t)\, w_1(t)\, \mathrm{d}t & (0 \le x \le 1), \\ w_1'(x) = 1 + \rho \int Q(t)\, w_1(t)\, \mathrm{d}t & (0 \le x < t), \end{cases}$$

we obtain

$$W_x \equiv w_4(0) = w_1(1), \quad \text{on } [0,1].$$

Similarly, we have

$$\begin{vmatrix} w_2(x) & w_1(x) \\ w_2'(x) & w_1'(x) \end{vmatrix} \equiv w_2(0) = w_1(\eta) \quad \text{on} \quad [0,\eta],$$

$$\begin{vmatrix} w_4(x) & w_3(x) \\ w_4'(x) & w_3'(x) \end{vmatrix} \equiv w_4(\eta) = w_3(1) \quad \text{on} \quad [\eta,1]. \tag{16}$$

The proof of Theorem 1 is thus completed.

Proof of Theorem **2** When $\alpha\eta < 1$, we can choose $\rho = 0$. In this case, we have $w_1(1) - \alpha w_1(\eta) = 1 - \alpha\eta > 0$.

When $\alpha\eta = 1$, i.e., $\alpha = 1/\eta$, we can choose $\rho > 0$ sufficiently small. In this case, we have

$$\begin{aligned} w_1(1) - \alpha w_1(\eta) &= 1 + \rho\int_0^1 (1 - t) Q(t) w_1(t)\,\mathrm{d}t - \\ &\alpha\left[\eta + \rho\int_0^\eta (\eta - t) Q(t) w_1(t)\,\mathrm{d}t\right] = \\ &\rho\int_\eta^1 (1 - t) Q(t) w_1(t)\,\mathrm{d}t + \\ &\rho\int_0^\eta (1 - t - \alpha\eta + \alpha t) Q(t) w_1(t)\,\mathrm{d}t > 0. \end{aligned}$$

When $\alpha\eta > 1$, we can choose $\rho > 0$ such that

$$\rho\int_\eta^1 (1 - t) Q(t)\,\mathrm{d}t \geq \alpha.$$

In this case, we have

$$\begin{aligned} w_1(1) - \alpha w_1(\eta) &= w_4(\eta) w_1'(\eta) - w_4'(\eta) w_1(\eta) - \alpha w_1(\eta) > \\ &w_1(\eta)\left[1 + \rho\int_\eta^1 Q(t) w_4(t)\,\mathrm{d}t - \alpha\right] > \\ &w_1(\eta)\left[\rho\int_\eta^1 (1 - t) Q(t)\,\mathrm{d}t - \alpha\right] \geq 0. \end{aligned}$$

To sum up, for each gives $\alpha \in \mathbf{R}$ there exists a $\rho \geq 0$ such that $w_1(1) - \alpha w_1(\eta) > 0$.

By Lemma 1, (15) and (16), it is easy to check that the function $y(x)$ defined by (3) is a solution to the linear three-point boundary value problem

$$\begin{cases} -y'' + \rho Q(x) y = h(x) & (0 < x < 1), \\ y(0) = 0, \quad y(1) = \alpha y(\eta). \end{cases} \tag{17}$$

Next we prove the uniqueness. Let $y_1(x)$ and $y_2(x)$ be solutions to (17). Put $y(x) = y_1(x) - y_2(x)$. Then

$$\begin{cases} y''(x) = \rho Q(x) y(x) & \text{a.e. on } (0,1) \\ y(0) = 0, \quad y(1) = \alpha y(\eta). \end{cases}$$

Note that the homogeneous linear differential equation has a general solution

$$y(x) = C_1 w_1(x) + C_2 w_4(x) \qquad (0 \leq x \leq 1),$$

where C_1 and C_2 are arbitrary constants. From the boundary conditions and (2), it follows that $C_1 = C_2 = 0$. i.e., $y(x) \equiv 0$ on $[0,1]$. The uniqueness is thus proved.

The remainder of Theorem 2 follows from (3). Theorem 2 is thus proved.

In the sequel, we assume that there exists a $\rho \geq 0$ such that $w_1(1) - \alpha w_1(\eta) > 0$. Put

$$\begin{aligned} &Ly = -y'' + \rho Q(x) y, \\ &D(L) = \{y(x) \in AC[0,1];\ y'(x) \in L^1(0,1) \cap AC_{\mathrm{loc}}(0,1), \\ &\qquad y''(x) \in L^*(0,1),\ y(0) = 0,\ y(1) = \alpha y(\eta)\}, \\ &L^*(0,1) = \{h(x) \in L^1_{\mathrm{loc}}(0,1);\ \|h\|^* < +\infty\}, \end{aligned}$$

where $\|h\|^* = \int_0^\eta w_1(x)\,|\,h(x)\,|\,\mathrm{d}x + \int_\eta^1 w_4(x)\,|\,h(x)\,|\,\mathrm{d}x$.

From Theorems 1 and 2, we come to two conclusions. First, $L: D(L) \to L^*(0,1)$ is inverse positive, i.e.,

$$y(x) \in D(L),\ (Ly)(x) \geq 0,\quad \text{a.e. on } (0,1) \Rightarrow y(x) \geq 0 \quad \text{on } [0,1],$$

which is usually called the maximum principle (see [4]). Secondly, there exists a positive number C such that

$$\|L^{-1}h\| \leq C\|h\|^* \qquad (\forall h \in L^*(0,1)),$$

where $\|\cdot\|$ is the usual supremum norm.

3 Proofs of Main Results

In the present section, we give proofs of Theorems 3 and 4.

Proof of Theorem 3 Let us define a mapping $\Phi: K \to K$ by

$$(\Phi y)(x) = \begin{cases} \dfrac{w_4(\eta)\,w_1(\eta)}{w_1(1) - \alpha w_1(\eta)}\left(\displaystyle\int_0^\eta \frac{w_1(t)}{w_1(\eta)}Q(t)f^*(y(t))\,\mathrm{d}t + \int_\eta^1 \frac{w_4(t)}{w_4(\eta)}Q(t)f^*(y(t))\,\mathrm{d}t\right) & (x = \eta), \\ w_2(x)\displaystyle\int_0^x \frac{w_1(t)}{w_1(\eta)}Q(t)f^*(y(t))\,\mathrm{d}t + w_1(x)\int_x^\eta \frac{w_2(t)}{w_1(\eta)}Q(t)f^*(y(t))\,\mathrm{d}t + (\Phi y)(\eta)\frac{w_1(x)}{w_1(\eta)} & (0 \leq x \leq \eta), \\ w_4(x)\displaystyle\int_\eta^x \frac{w_3(t)}{w_4(\eta)}Q(t)f^*(y(t))\,\mathrm{d}t + w_3(x)\int_x^1 \frac{w_4(t)}{w_4(\eta)}Q(t)f^*(y(t))\,\mathrm{d}t + (\Phi y)(\eta)\frac{w_4(x)\,\alpha w_3(x)}{w_4(\eta)} & (\eta \leq x \leq 1), \end{cases}$$

$K = \{y(x) \in [0,1];\ y(x) \geq 0 \text{ on } [0,\eta] \text{ and } y(x) \geq \sigma\|y\| \text{ on } [\eta,1]\}$,

where $\|y\| = \max\{|\,y(x)\,|;\ 0 \leq x \leq 1\}$ and σ is the constant defined by (9). Clearly, K is a cone in $C[0,1]$.

From the definition of Φ, Lemma 1, Theorems 1 and 2, we know that for each fixed $y(x) \in K$

$$\begin{cases} (\Phi y)(0) = 0, \quad (\Phi y)(1) = \alpha(\Phi y)(\eta), \\ (\Phi y)(x) \geq 0, \quad x \in [0,1], \\ (\Phi y)(\eta) \geq \dfrac{\min\{w_4(\eta), w_1(\eta)\}}{w_1(1) - \alpha w_1(\eta)}(I_1 + I_4), \end{cases} \tag{18}$$

$$I_1 = \int_0^\eta w_1(t)\,Q(t)\,f^*(y(t))\,\mathrm{d}t,$$

$$I_4 = \int_\eta^1 w_4(t)\,Q(t)\,f^*(y(t))\,\mathrm{d}t,$$

$$(\Phi y)(\eta) \leq \frac{\max\{w_4(\eta), w_1(\eta)\}}{w_1(1) - \alpha w_1(\eta)}(I_1 + I_4), \tag{19}$$

$$(\Phi y)(x) \leq I_1 + I_4 + (\Phi y)(\eta)\max_{\eta \leq x \leq 1}\left\{\frac{w_4(x) + \alpha w_3(x)}{w_4(\eta)}\right\} \qquad (0 \leq x \leq 1),$$

and hence, by (19)

$$\| \Phi y \| \le \left(1 + \max_{\eta \le x \le 1} \left\{ \frac{w_4(x) + \alpha w_3(x)}{w_4(\eta)} \right\} \frac{\max\{ w_4(\eta), w_1(\eta) \}}{w_1(1) - \alpha w_1(\eta)} \right) (I_1 + I_4) = M(I_1 + I_4), \tag{20}$$

where M is the constant defined by (8). On the other hand, it follows from (18) that

$$\| \Phi y \| \le \left(\frac{w_1(x) - \alpha w_1(\eta)}{\min\{ w_4(\eta), w_1(\eta) \}} + \max_{\eta \le x \le 1} \left\{ \frac{w_4(x) + \alpha w_3(x)}{w_4(\eta)} \right\} \right) (\Phi y)(\eta).$$

Therefore

$$(\Phi y)(x) \ge (\Phi y)(\eta) \min_{\eta \le x \le 1} \left\{ \frac{w_4(x) + \alpha w_3(x)}{w_4(\eta)} \right\}_{(\eta \le x \le 1)} \ge$$

$$\frac{\min_{\eta \le x \le 1} \left\{ \frac{w_4(x) + \alpha w_3(x)}{w_4(\eta)} \right\}}{\frac{w_1(1) - \alpha w_1(\eta)}{\min\{ w_4(\eta), w_1(\eta) \}} + \max_{\eta \le x \le 1} \left(\frac{w_4(x) + \alpha w_3(x)}{w_4(\eta)} \right)} \| \Phi y \| =$$

$$\sigma \| \Phi y \| \quad (\eta \le x \le 1).$$

This shows that $\Phi(K)$ is a subset of K.

We now claim that Φ is a completely continuous mapping. In fact, for any $r > 0$ and $y(x) \in K \cap \bar{\Omega}$,

$$\Omega = \{ y(x) \in C[0,1];\ \| y \| < r \},$$

we have, by (20) and the definition of Φ,

$$\| \Phi y \| \le M(I_1 + I_4) \le$$

$$M \max_{0 \le y \le r} f^*(y) \left(\int_0^{\eta} w_1(t) Q(t) \mathrm{d}t + \int_{\eta}^{1} w_4(t) Q(t) \mathrm{d}t \right) = B_r, \tag{21}$$

$$(\Phi y)'(x) = \begin{cases} w_2'(x) \int_0^x \frac{w_1(t)}{w_1(\eta)} Q(t) f^*(y(t)) \mathrm{d}t + w_1'(x) \int_x^{\eta} \frac{w_2(t)}{w_1(\eta)} Q(t) f^*(y(t)) \mathrm{d}t + \\ \quad (\Phi y)(\eta) \frac{w_1'(x)}{w_1(\eta)} \qquad (0 < x \le \eta), \\ w_4'(x) \int_{\eta}^x \frac{w_3(t)}{w_4(\eta)} Q(t) f^*(y(t)) \mathrm{d}t + w_3'(x) \int_x^1 \frac{w_4(t)}{w_4(\eta)} Q(t) f^*(y(t)) \mathrm{d}t + \\ \quad (\Phi y)(\eta) \frac{w_4'(x) + \alpha w_3'(x)}{w_4(\eta)} \qquad (\eta \le x \le 1), \end{cases} \tag{22}$$

and hence

$$|(\Phi y)'(x)| \le - w_2' \int_0^{\eta} \frac{w_1(t)}{w_1(\eta)} Q(t) f^*(y(t)) \mathrm{d}t + w_1'(x) \int_x^{\eta} Q(t) f^*(y(t)) \mathrm{d}t +$$

$$(\Phi y)(\eta) \frac{w_1'(x)}{w_1(\eta)} \qquad (0 < x \le \eta) \le$$

$$\max_{0 \le y \le r} f^*(y) \left(\frac{-w_2'(x)}{w_1(\eta)} \int_0^{\eta} w_1(t) Q(t) \mathrm{d}t + w_1'(x) \int_x^{\eta} Q(t) \mathrm{d}t \right) +$$

$$B_r \frac{w_1'(x)}{w_1(\eta)} = G_r(x) \qquad (0 < x < \eta),$$

$$|(\Phi y)'(x)| \le - w_4'(x) \int_{\eta}^{x} Q(t) f^*(y(t)) \mathrm{d}t + w_3'(x) \int_x^1 \frac{w_4(t)}{w_4(\eta)} Q(t) f^*(y(t)) \mathrm{d}t +$$

$$(\Phi y)(\eta)\frac{-w_4'(x)+\alpha w_1(x)}{w_4(\eta)}\le$$
$$\max_{0\le y\le r} f^*(y)\left(-w_4'(x)\int_0^\eta Q(t)\,\mathrm{d}t+\frac{w_3'(x)}{w_4(\eta)}\int_\eta^1 w_4(t)Q(t)\,\mathrm{d}t\right)+$$
$$B_r\frac{-w_4'(x)+\alpha w_3'(x)}{w_4(\eta)}=G(x)\qquad(\eta\le x<1).$$

Therefore, we have

$$\int_0^1|(\Phi y)'(x)|\,\mathrm{d}x\le\int_0^1 G_r(x)\,\mathrm{d}r=$$
$$\max_{0\le y\le r} f^*(y)\left(2\int_0^\eta w_1(t)Q(t)\,\mathrm{d}t+2\int_\eta^1 w_4(t)Q(t)\,\mathrm{d}t\right)+(2+\alpha)B_r<+\infty,\qquad(23)$$

which shows that $(\Phi y)'(x)\in L^1(0,1)$ and $(\Phi y)(x)\in AC[0,1]$ for each fixed $y(x)\in K\cap\Omega$ and (22) and (23) imply that $\Phi(K\cap\Omega)$ is relatively compact in K (by the Ascoli-Arzela theorem). Besides, the continuity of Φ on K follows from that of $f^*(y)$ on $\mathbf{R}_+$. To sum up, Φ is a completely continuous mapping.

Moreover, from (22), we know that $(\Phi y)'(\eta-0)=(\Phi y)'(\eta+0)$, and hence $(\Phi y)'(x)\in AC_{\mathrm{loc}}(0,1)$. From the above discussion, we can conclude that each fixed point of Φ in K is exactly a solution to (1).

We are now in the position to prove that Φ has a fixed point in K under the assumptions of Theorem 3.

For given $r_1, r_2>0$, we write

$$\Omega_1=\{y\in C[0,1];\ \|y\|<r_1\},\quad \Omega_2=\{y\in C[0,1]:\ \|y\|<r_2\}.$$

From (6) and (7), we know that there exists a sufficiently small $\varepsilon>0$ such that

$$(\beta+\varepsilon)M\left(\int_0^\eta w_1(t)Q(t)\,\mathrm{d}t+\int_\eta^1 w_1(t)Q(t)\,\mathrm{d}t\right)<1,$$
$$(r-\varepsilon)\frac{\sigma w_1(\eta)}{w_1(1)-\alpha w_1(\eta)}\int_\eta^1 w_4(t)Q(t)\,\mathrm{d}t>1.$$

Now suppose that (4) holds. Since $\limsup_{y\to 0}\frac{f^*(y)}{y}\le\beta$, we can choose $r_1>0$ so that

$$f^*(y)\le(\beta+\varepsilon)y \text{ for all } y\in[0,r_1].$$

In this case, it follows from (20) that for any given $y\in K\cap\partial\Omega_1$

$$\|\Phi y\|\le M\left(\int_0^\eta w_1(t)Q(t)f^*(y(t))\,\mathrm{d}t+\int_\eta^1 w_4(t)Q(t)f^*(y(t))\,\mathrm{d}t\right)\le$$
$$M(\beta+\varepsilon)\left(\int_0^\eta w_1(t)Q(t)\,\mathrm{d}t+\int_\eta^1 w_4(t)Q(t)f^*(y(t)\,\mathrm{d}t\right)r_1<$$
$$r_1=\|y\|.$$

From $\liminf_{y\to\infty}\frac{f^*(y)}{y}\ge\gamma$, we know that there exists an $r_2>r_1$ such that

$$f^*(y)\ge(\gamma-\varepsilon)y \text{ for all } y\ge\sigma r_2.$$

In this case, it follows from the definition of Φ that for any fixed $y\in K\cap\partial\Omega_2$,

$$\|\Phi y\| \geq (\Phi y)(\eta) \geq \frac{w_1(\eta)}{w_1(1) - \alpha w_1(\eta)} \int_\eta^1 w_4(t) Q(t) f^*(y(t))\,dt \geq$$
$$(\gamma - \varepsilon) \frac{\sigma w_1(\eta)}{w_1(1) - \alpha w_1(\eta)} \left(\int_\eta^1 w_4(t) Q(t)\,dt \right) r_2 >$$
$$r_2 = \|y\|.$$

From the first part of Theorem B, we reach the conclusion that Φ has a fixed point in $K \cap (\Omega_2 \setminus \Omega_1)$. Let $y(x)$ be the fixed point. Then the function

$$y(x) = (\Phi y)(x) \quad (0 \leq x \leq 1)$$

is a positive solution to the three-point boundary value problem (1), since

$$y(x) \geq \sigma \|y\| \geq \sigma r_1 \text{ on } [\eta, 1] \text{ and } y(x) \geq y(\eta) \frac{w_1(x)}{w_1(\eta)} \quad \text{on} \quad [0, \eta].$$

Next suppose that (5) holds. From $\liminf_{y \to 0} \frac{f^*(y)}{y} \geq \gamma$, we know that there exists an $r_1 > 0$ such that

$$f^*(y) \geq (\gamma - \varepsilon) y \text{ for any } y \in [0, r_1].$$

In this case, we have that for any given $y \in K \cap \partial\Omega_1$

$$\|\Phi y\| \geq \frac{w_1(\eta)}{w_1(1) - \alpha w_1(\eta)} \int_\eta^1 w_4(t) Q(t) f^*(y(t))\,dt \geq$$
$$(r - \varepsilon) \frac{w_1(\eta)}{w_1(1) - \alpha w_1(\eta)} \left(\int_\eta^1 w_4(t) Q(t)\,dt \right) r_1 >$$
$$r_1 = \|y\|.$$

Since $\limsup_{y \to +\infty} \frac{f^*(y)}{y} \leq \beta$, we can choose an $N > r_1$ so that

$$f^*(y) \leq (\beta + \varepsilon) y \text{ for any } y \geq N.$$

Let $r_2 > N$ be a positive number such that

$$r_2 > \frac{M \max\{ f^*(y); 0 \leq y \leq N\} \left(\int_0^\eta w_1(t) Q(t)\,dt + \int_\eta^1 w_4(t) Q(t)\,dt \right)}{1 - (\beta + \varepsilon) M \left(\int_0^\eta w_1(t) Q(t)\,dt + \int_\eta^1 w_4(t) Q(t)\,dt \right)}.$$

In this case, we have that for any fixed $y \in K \cap \partial\Omega_2$

$$\|\Phi y\| \leq M \left(\int_0^\eta w_1(t) Q(t) f^*(y(t))\,dt + \int_\eta^1 w_1(t) Q(t) f^*(y(t))\,dt \right) \leq$$
$$M \left(\int_{0 \leq y(t) \leq N, 0 \leq t \leq \eta} w_1(t) Q(t) f^*(y(t))\,dt + \int_{0 \leq y(t) \leq N, \eta \leq t \leq 1} w_4(t) Q(t) f^*(y(t))\,dt \right) +$$
$$M \left(\int_{N \leq y(t) \leq r_2, 0 \leq t \leq \eta} w_1(t) Q(t) f^*(y(t))\,dt + \int_{N \leq y(t) \leq r_2, \eta \leq t \leq 1} w_4(t) Q(t) f^*(y(t))\,dt \right) \leq$$
$$M \max\{ f^*(y); 0 \leq y \leq N\} \left(\int_0^\eta w_1(t) Q(t)\,dt + \int_\eta^1 w_4(t) Q(t)\,dt \right) +$$
$$(\beta + \varepsilon) M \left(\int_0^\eta w_1(t) Q(t)\,dt + \int_\eta^1 w_4(t) Q(t)\,dt \right) r_2 <$$

$r_2 = \|y\|$.

The second part of Theorem B tells us that Φ has a fixed point $y(x) \in K \cap (\bar{\Omega}_2 \setminus \Omega_1)$. As before, the fixed point $y(x)$ is a positive solution to (1), of course. The proof of Theorem 3 is completed up to now.

Proof of Theorem **4** Since $\alpha\eta \in (0,1)$, we can choose $\rho = 0$ so that $w_1(1) - \alpha w_1(\eta) = 1 - \alpha\eta > 0$. The assumptions of Theorem 4 imply that β can be arbitrarily small and γ arbitrarily large. Therefore, (6) and (7) are fulfilled. Theorem 4 follows from Theorem 3.

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References:

[1] MA Ru-yun. Positive solutions of a nonlinear three-point boundary value problems[J]. Electron J Differential Equations, 1999, **34**: 1 - 8.

[2] Gupta C P. A sharper condition for the solvability of a three-point second order boundary value problem[J]. J Math Anal Appl, 1997, **205**(2): 586 - 597.

[3] Krasosel'skill M A. Positive Solutions of Operator Equation[M]. Gorningen: Noordhoff, 1964.

[4] Cabada A, Nieto J J. Fixed points and approximate solutions for nonlinear operator equations[J]. J Comput Appl Math, 2000, **113**(1-2): 17 - 25.