A Study on Piecewise Polynomial Smooth Approximation to the Plus Function

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***Abstract*—In smooth support vector machine (SSVM), the plus function must be approximated by some smooth function, and the approximate error will affect the classification ability. This paper studies the smooth approximation to the plus function by piecewise polynomials. First, some standard piecewise polynomial smooth approximation problems are formulated. Then, the existence and uniqueness of solution for these problems are proved and the analytic solutions are achieved. The comparison between the results in this paper and the previous ones shows that the piecewise polynomial functions in this paper achieve better approximation to the plus function.**

***Keywords*—SSVM, PSSVM, smooth approximation to plus function, piecewise polynomial**

I. INTRODUCTION

In 2001, Lee and Mangasarian [1] proposed a smooth support vector machine (SSVM) model to solve classification problems, which was a strongly convex minimization without any constraints. Lee and Mangasarian pointed out that SSVM achieved the same classifying accuracy as the ordinary classification methods did, and additionally, it was much faster than the successive overrelaxation (SOR) algorithm [2], the sequential minimal optimization (SMO) algorithm [3] and SVM[light] [4] particularly. Thus, SSVM is especially suitable for the classification problem with large size.

In SSVM, the plus function $(x)_+$, which is defined as

$$(x_i)_+ = \max\{0, x_i\},\ i = 1,2,\cdots,n, \tag{1}$$

must be approximated by some smooth function. The integral of sigmoid function for neural networks was adopted as the smooth function in [1], which is inherited from the works of Chen & Mangasarian [5] [6].

In 2005, Yuan, Yan and Xu [7] made use of piecewise polynomial to smoothly approximate the plus function, and obtained a polynomial smooth support vector machine (PSSVM) model. It was shown in [7] that PSSVM is better than SSVM in approximate error, classifying ability, computational stability and expansibility. The smooth functions in [7] (piecewise polynomials) are different from that in Chen and Mangasarian's work, and it originates from the polynomial interpolation theory. This class of smooth functions is very attractive because of the excellent characteristics of polynomial interpolation. It has been showed in [7] that the piecewise polynomials achieved the better approximation to the plus function than the integral of sigmoid function did. Two approximate solutions, obtained by trail and error method, were provided in [2]. To improve the classification performance of SSVM, it is important to reduce the approximate error to the plus function. However, without a systematic method, it is very difficult to find more possible solutions with less approximate errors. Thus, in this paper, we first formulate the standard piecewise polynomial smooth approximation problems by the 2-norm measurement of error function, and then give its analytic solution by using the interpolation method. The comparison between the results in this paper and that in [7] shows that the piecewise polynomial functions in this paper approximate the plus function better than that in [7], no matter what measurement, 2-norm or infinity-norm, of error function is adopted.

The rest of the paper is organized as follows. A brief introduction to SSVM and PSSVM is given in Section II. In Section III,we formulate some standard interpolation problems for the piecewise polynomial approximation to the plus function under some given smooth level; the existence and uniqueness of the solution for these problems are proofed, and the analytic solutions are achieved. The comparison between the results in this paper and those in [2] are carried out in Section Ⅳ. Finally, some conclusions and discussions are given in SectionⅤ.

II. SSVM AND PSSVM

We consider the problem of classifying m points in the n-dimensional real space R^n, represented by the $m\times n$ matrix A. We can get a $m\times m$ diagonal matrix D with ones or minus ones along its diagonal, according to the membership of each point A_i in the class 1 or –1. For this problem, the standard support vector machine is given by the following: for some $\nu>0$,

$$\begin{aligned} &\min_{(w,\gamma,y)\in R^{n+1+m}} \nu\vec{1}^T y + \frac{1}{2}w^T w \\ &\text{S.T.}\ \ \mathrm{D}(\mathrm{A}w - \vec{1}\gamma) + \mathrm{y} \ge \vec{1}\ , \\ &\qquad \mathrm{y} \ge 0 \end{aligned} \tag{2}$$

where $\vec{1}$ is the vector in which all elements are 1, w is the normal vector to the bounding planes:

$$\begin{cases} x^T w-\gamma=1 \\ x^T w-\gamma=-1 \end{cases}, \tag{3}$$

and γ determines their location relative to the origin. The first plane above bounds the class 1 points and the second plane bounds the class -1 points. If the two classes are linearly separable, then the slack variable y=0; otherwise, the two planes bound the two classes with a "soft margin", determined by a nonnegative slack variable y, such as

$$x^T w-\gamma+y_i \geq 1, \quad \text{for } x^T = \mathrm{A_i} \text{ and } \mathrm{D_{ii}}=1,$$
$$x^T w-\gamma-y_i \leq -1, \quad \text{for } x^T = \mathrm{A_i} \text{ and } \mathrm{D_{ii}}=-1.$$

In (2), the 1-norm of the slack variable y is minimized with weight v. In our smooth approach, the square of 2-norm of the slack variable y is minimized with weight $v/2$ instead of the 1-norm of y. Additionally, the distance between the planes (3) is measured in the (n+1)-dimensional space of $[w;\gamma]$ by

$$2/\|[w;\gamma]\|_2 .$$

Thus, a modified SVM problem is yielded:

$$\begin{aligned} &\min_{w,\gamma,y} \tfrac{v}{2}\mathrm{y}^T y+\frac{1}{2}(w^T w+\gamma^2) \\ &\text{S.T.} \quad \mathrm{D}(\mathrm{A}w-\vec{1}\gamma)+\mathrm{y} \geq \vec{1} \quad . \\ &\qquad\quad \mathrm{y} \geq 0 \end{aligned} \tag{4}$$

It was shown in [2] that the model (4) has strong convexity and has little or no effect on model (2). As a solution of model (4), y will be given by

$$y=(\vec{1}-D(Aw-\vec{1}\gamma))_+ , \tag{5}$$

where the function $(\cdot)_+$ is the plus function as (1). Replacing y by (5) in (4), we can convert the SVM model (4) into an equivalent SVM as follows:

$$\min_{w,\gamma} \tfrac{1}{2}v\left\|(\vec{1}-D(Aw-\vec{1}\gamma))_+\right\|_2^2+\tfrac{1}{2}(w^T w+\gamma^2), \tag{6}$$

which is a strongly convex minimization problem without any constraints, and thus, will have an unique solution. However, the objective function in (6) is not differentiable which precludes the use of many effective solving methods, for example, famous Newton method. We thus apply some smoothing techniques as in [5],[6], and obtain a general smooth support vector machine (SSVM) by replacing $(\cdot)_+$ with some smooth function $f(\cdot,k)$ (k is a positive constant):

$$\min_{w,\gamma} \tfrac{1}{2}v\left\|f(\vec{1}-D(Aw-\vec{1}\gamma),k)\right\|_2^2+\tfrac{1}{2}(w^T w+\gamma^2) \tag{7}$$

As some particular cases, the smooth function in [1] is the integral of sigmoid function

$$f(x,k)=x+\tfrac{1}{k}\log(1+e^{-kx}), k>0; \tag{8}$$

the smooth functions in [7] are the piecewise polynomial functions

$$f_1(x,k)=\begin{cases} x, & x \geq \frac{1}{k} \\ \frac{k}{4}x^2+\frac{1}{2}x+\frac{1}{4k}, & -\frac{1}{k}<x<\frac{1}{k} \\ 0, & x \leq -\frac{1}{k} \end{cases} \tag{9}$$

and

$$f_2(x,k)=\begin{cases} x, & x \geq \frac{1}{k} \\ -\frac{1}{16k}(kx+1)^3(kx-3), & -\frac{1}{k}<x<\frac{1}{k} \\ 0, & x \leq -\frac{1}{k} \end{cases}, \tag{10}$$

which are respectively the first-order and second-order continuously differentiable.

When $f(\cdot)$ is replaced with piecewise polynomial smooth function, the model (7) is called as a polynomial smooth support vector machine (PSSVM). It is shown in [7] that PSSVM performs better than SSVM in approximate error, classifying ability, computational stability and expansibility.

III. BEST PIECEWISE POLYNOMIAL SMOOTH APPROXIMATION TO THE PLUS FUNCTION

In this section, we select the 2-norm of error function between the piecewise polynomials and the plus function as the measurement of approximate error. By this measurement, some standard approximation problems are formulated under given smoothing level. Then, the existence and uniqueness of solution for these problems are proofed and the analytic solutions are achieved.

A. *The Statements for the Approximation Problem*

By minimizing the 2-norm of error function, we define the piecewise polynomial smooth approximation problems to the plus function under first-order smoothing level and second-order smoothing level as Problem 1 and Problem 2 respectively.

Problem 1: $\min_{p} R(p)=\int_{-\frac{1}{k}}^{\frac{1}{k}}[p(x,k)-(x)_+]^2 dx$

$$\begin{aligned} \text{S.T.} \quad & p(\tfrac{1}{k},k)=\tfrac{1}{k}, p(-\tfrac{1}{k},k)=0 \\ & p'(\tfrac{1}{k},k)=1, p'(-\tfrac{1}{k},k)=0 \\ & p(x,k) \geq 0 \text{ ,if } x \in [-\tfrac{1}{k},\tfrac{1}{k}], \end{aligned}$$

where

$$p(x,k)=\begin{cases} x, & x \geq \frac{1}{k} \\ L_4(x), & -\frac{1}{k}<x<\frac{1}{k} \\ 0, & x \leq -\frac{1}{k} \end{cases},$$

and $L_4(x)$ is a any polynomial function whose degree isn't over 4.

Problem 2: $\min_{p} R(p)=\int_{-\frac{1}{k}}^{\frac{1}{k}}[p(x,k)-(x)_+]^2 dx$

$$\begin{aligned} \text{S.T.} \quad & p(\tfrac{1}{k},k)=\tfrac{1}{k}, p(-\tfrac{1}{k},k)=0 \\ & p'(\tfrac{1}{k},k)=1, p'(-\tfrac{1}{k},k)=0 \\ & p''(\tfrac{1}{k},k)=0, p''(-\tfrac{1}{k},k)=0 \\ & p(x,k) \geq 0, \text{ if } x \in [-\tfrac{1}{k},\tfrac{1}{k}], \end{aligned}$$

where

$$p(x,k)=\begin{cases} x, & x \ge \frac{1}{k} \\ L_6(x), & -\frac{1}{k} < x < \frac{1}{k}, \\ 0, & x \le -\frac{1}{k} \end{cases}$$

and $L_6(x)$ is a any polynomial function whose degree isn't over 6.

B. The Solutions for Problem 1 and Problem 2

Theorem 1. There exists a unique solution

$$L_4(x) = (kx+1)^2(-\tfrac{1}{16}kx^2 + \tfrac{1}{8}x + \tfrac{3}{16k})$$

for Problem 1, which corresponds to the optimal value

$$R(p) = \tfrac{11}{1008}k^{-3}.$$

Proof: From the conditions

$$p(-\tfrac{1}{k},k) = p'(-\tfrac{1}{k},k) = 0,$$

We can set

$$L_4(x) = (kx+1)^2(ax^2+bx+c).$$

So we have

$$L_4'(x) = (kx+1)(4kax^2 + (3kb+2a)x + 2kc + b).$$

From the conditions

$$p(\tfrac{1}{k},k) = \tfrac{1}{k},\ p'(\tfrac{1}{k},k) = 1,$$

we have

$$4a + 4kb + 4k^2c = k \tag{11}$$

$$12a + 8kb + 4k^2c = k. \tag{12}$$

From (11) and (12), we have

$$\begin{aligned} a &= -\tfrac{1}{2}kb \\ c &= -\tfrac{1}{2}k^{-1}b + \tfrac{1}{4}k^{-1} \end{aligned} \tag{13}$$

The condition,

$$\text{if } x \in [-\tfrac{1}{k}, \tfrac{1}{k}],\ p(x,k) \ge 0,$$

is equivalent to

$$\text{if } x \in [-\tfrac{1}{k}, \tfrac{1}{k}],\ Q(x) = ax^2 + bx + c \ge 0.$$

Substituting *a,c* in $Q(x)$ with (13),we can derive out

$$b \le \tfrac{1}{8}. \tag{14}$$

Now $R(p)$ is a function about *b*, set

$$R(b) = \int_{-\frac{1}{k}}^{\frac{1}{k}} [p(x,k) - (x)_+]^2 dx,$$

then

$$R'(b) = \int_{-\frac{1}{k}}^{\frac{1}{k}} \frac{d}{db}[p(x,k) - (x)_+]^2 dx = 2\times(\tfrac{64}{315}k^{-3}b - \tfrac{29}{420}k^{-3}).$$

Set $R'(b)$ =0, we have $b = 609/1792$.

From (13), we gain

$$a = -\tfrac{609}{3584}k,\ c = \tfrac{287}{3584}k^{-1}.$$

Because $b = 609/1792 > 1/8$, which doesn't satisfy (14), it is not a solution for Problem 1.

Noticing that

$$R'(b) < 0, \text{ if } b < 609/1792,$$

R(b) is a strict monotonously decreasing function in $(-\infty, 1/8]$. Hence there exist an unique solution $b = 1/8$, which minimizes *R(b)*,

Now from (13), we have

$$a = -\tfrac{1}{16}k,\ c = \tfrac{3}{16}k^{-1},$$

Substitute *a,b,c* to $L_4(x)$, we gain

$$L_4(x) = (kx+1)^2(-\tfrac{1}{16}kx^2 + \tfrac{1}{8}x + \tfrac{3}{16k}),$$

$$R(p) = \tfrac{11}{1008}k^{-3}.$$

The proof is finished. □

Theorem 2. There exists a unique solution

$$L_6(x) = \tfrac{1}{32}(kx+1)^3(k^2x^3 - 3kx^2 + x + \tfrac{5}{k})$$

for Problem 2, which corresponds to the optimal value

$$R(p) = 0.0064k^{-3}.$$

Proof: From conditions

$$p(-\tfrac{1}{k},k) = p'(-\tfrac{1}{k},k) = p''(-\tfrac{1}{k},k) = 0,$$

we can set

$$L_6(x) = (kx+1)^3(a_3x^3 + a_2x^2 + a_1x + a_0).$$

Then

$$\begin{aligned} L_6'(x) &= (kx+1)^2(6ka_3x^3 + (5ka_2 + 3a_3)x^2 \\ &\quad + (4ka_1 + 2a_2)x + 3ka_0 + a_1), \\ L_6''(x) &= (kx+1)(30k^2a_3x^3 + (20k^2a_2 + 30ka_3)x^2 \\ &\quad + (12k^2a_1 + 16ka_2 + 6a_3)x \\ &\quad + 6k^2a_0 + 6ka_1 + 2a_2). \end{aligned}$$

From conditions

$$p(\tfrac{1}{k},k) = \tfrac{1}{k},\ p'(\tfrac{1}{k},k) = 1,\ p''(\tfrac{1}{k},k) = 0,$$

we have

$$8k^3a_0 + 8k^2a_1 + 8ka_2 + 8a_3 = k^2 \tag{15}$$

$$12k^3a_0 + 20k^2a_1 + 28ka_2 + 36a_3 = k^2 \tag{16}$$

$$6k^3a_0 + 18k^2a_1 + 38ka_2 + 66a_3 = 0. \tag{17}$$

From (15),(16),(17), we gain

$$\begin{aligned} a_0 &= \tfrac{3}{16}k^{-1} - k^{-3}a_3 \\ a_1 &= -\tfrac{1}{16} + 3k^{-2}a_3 \\ a_2 &= -3k^{-1}a_3 \end{aligned} \tag{18}$$

The condition,

$$\text{if } x \in [-\tfrac{1}{k}, \tfrac{1}{k}],\ p(x,k) \ge 0,$$

is equivalent to

$$\text{if } x \in [-\tfrac{1}{k}, \tfrac{1}{k}],\ Q(x) = a_3x^3 + a_2x^2 + a_1x + a_0 \ge 0.$$

Substituting a_0, a_1, a_2 in $Q(x)$ with (18), we can derive that $Q(x) \ge 0$ is equivalent to

$$a_3 \le \tfrac{1}{32}k^2. \tag{19}$$

Set

$$R'(a_3) = 0,$$

we can gain

$$a_3 \approx 0.1222k^2.$$

Because

$$0.1222k^2 > \tfrac{1}{32}k^2,$$

which doesn't satisfy (19), it is not a solution for Problem 2. Noticing that

$$\text{if}\, a_3 < 0.1222k^2\,,\ R'(a_3)<0,$$

$R(a_3)$ is a strict decreasing function in $(-\infty, \frac{1}{32}k^2]$. Hence there exist an unique solution, $a_3 = \frac{1}{32}k^2$. From (18),

$$a_0 = \tfrac{5}{32}k^{-1}, a_1 = \tfrac{1}{32}, a_2 = -\tfrac{3}{32}k\,.$$

Hence we can gain

$$L_6(x) = \tfrac{1}{32}(kx+1)^3(k^2x^3 - 3kx^2 + x + \tfrac{5}{k}),$$

$$R(p) = 0.0064k^{-3}.$$

The proof is finished. □

IV. THE COMPARISON OF APPROXIMATE ERRORS

Now we compare our piecewise polynomial smooth functions with that in [7], under 2-norm and infinity-norm respectively. We denote the optimal solution of Problem 1 and Problem 2 as $p_1(x,k)$ and $p_2(x,k)$ respectively.

A. The Comparison under 2-norm Measurement

For the first-order smooth piecewise polynomial approximation, we can easily see that the function $f_1(x,k)$ in [7] is a feasible solution of Problem 1 when

$$a = 0, b = 0, c = \tfrac{1}{4}k^{-1}.$$

For $f_1(x,k)$, we have $R(f_1) = \frac{1}{40}k^{-3}$, while for the optimal solution $p_1(x,k)$ in this paper,

$$R(p_1) = \tfrac{11}{1008}k^{-3},$$

which is 43.65% of $R(f_1)$. Hence $p_1(x,k)$ is better than $f_1(x,k)$.

Remark 1: From the proof of Theorem 1, it is easily seen that all feasible solutions of Problem 1 for $0 < b \le 1/8$ achieve less approximate error than $f_1(x,k)$ does.

For the second-order smooth piecewise polynomial approximation, the function $f_2(x,k)$ in [7] is also a feasible solution of Problem 2 when

$$a_3 = 0, a_2 = 0, a_1 = -\tfrac{1}{16}, a_0 = \tfrac{3}{16}k^{-1}.$$

For $f_2(x,k)$, we have $R(f_2) = 0.010912k^{-3}$, while for the optimal solution $p_2(x,k)$ in this paper, $R(p_2) = 0.0064k^{-3}$, which is about 58.65% of $R(f_2)$. Hence $p_2(x,k)$ is better than $f_2(x,k)$.

Remark 2: From the proof of Theorem 2, it is easily seen that all feasible solutions of Problem 2 for $0 < a_3 \le \frac{1}{32}k^2$ achieve less approximate error than $f_2(x,k)$ does.

B. The Comparison under Infinity-norm Measurement

For the comparison between piecewise polynomial functions in this paper and those in [7] under infinity-norm measurement, we must firstly analyze the characteristics of $p_1(x,k)$ and $p_2(x,k)$.

Theorem 3. $p_1(x,k)$ satisfies that

1) $0 \le p_1(x,k) \le f_1(x,k)$;
2) if $x \in [-\frac{1}{k}, \frac{1}{k}]$, $p_1(x,k)$ increases monotonously;
3) $p_1(x,k) \ge (x)_+$;
4) $\|p_1(x,k) - (x)_+\|_\infty = \frac{3}{16k}$.

Proof:

1) Case $x < -\frac{1}{k}$ or $x > \frac{1}{k}$:

It is obviously true that $0 \le p_1(x,k) \le f_1(x,k)$.

Case $-\frac{1}{k} \le x \le \frac{1}{k}$:

From Theorem 1,we have $p_1(x,k) \ge 0$. Now proof

$$p_1(x,k) \le f_1(x,k).$$

In fact,

$$p_1(x,k) = \tfrac{1}{4}f_1(x,k)\ (-k^2x^2 + 2kx + 3),$$

It results in that

$$p_1(x,k) \le f_1(x,k)$$
$$\Leftrightarrow \tfrac{1}{4}(-k^2x^2 + 2kx + 3) \le 1$$
$$\Leftrightarrow k^2x^2 - 2kx + 1 \ge 0.$$

Hence

$$p_1(x,k) \le f_1(x,k).$$

2) Now that

$$\text{if } -\tfrac{1}{k} \le x \le \tfrac{1}{k},\ p_1'(x,k) = -\tfrac{1}{4}(kx+1)^2(kx-2) \ge 0,$$

Hence

$p_1(x,k)$ increases monotonously in $[-1/k, 1/k]$.

3) If $x < -\frac{1}{k}$,or $x > \frac{1}{k}$, or $-\frac{1}{k} \le x < 0$, it is evident that $p_1(x,k) \ge (x)_+$.

If $0 \le x \le \frac{1}{k}$, set $Q(x) = p_1(x,k) - (x)_+ = L_4(x) - x$, then

$$Q'(x) = -\tfrac{1}{4}(kx+1)^2(kx-2) - 1 \le Q'(\tfrac{1}{k}) = 0.$$

($\because Q''(x) \ge 0$), which indicates that $Q(x)$ decreases monotonously in $[0, 1/k]$.

Hence $Q(x) \ge Q(\frac{1}{k}) = 0$, that is,

if $0 \le x \le \frac{1}{k}$, then $p_1(x,k) \ge (x)_+$.

4) From 2) and 3), we have at once

$$\|p_1(x,k) - (x)_+\|_\infty = \max_x |\, p_1(x,k) - (x)_+ | = |\, p_1(0,k) | = \tfrac{3}{16k}.$$

The proof is finished. □

Theorem 4. $p_2(x,k)$ satisfies that

1) $0 \le p_2(x,k) \le f_2(x,k)$;
2) $p_2(x,k)$ increases monotonously in $[-1/k, 1/k]$;
3) $p_2(x,k) \ge (x)_+$;
4) $\|p_2(x,k) - (x)_+\|_\infty = \frac{5}{32k}$.

Proof:

1) Case $x < -\frac{1}{k}$ or $x > \frac{1}{k}$:

It is obviously true that $0 \le p_2(x,k) \le f_2(x,k)$.

Case $-\frac{1}{\mathrm{k}} \le x \le \frac{1}{k}$:

From Theorem 2, we have $p_2(x,k) \ge 0$.

Now proof $p_2(x,k) \le f_2(x,k)$.

In fact

$$p_2(x,k) = f_2(x,k)\ \frac{k^3x^3 - 3k^2x^2 + kx + 5}{2(3-kx)},$$

Thus,

$$p_2(x,k) \le f_2(x,k)$$

$$\Leftrightarrow \frac{k^3x^3 - 3k^2x^2 + kx + 5}{2(3-kx)} \le 1$$

$$\Leftrightarrow k^3x^3 - 3k^2x^2 + 3kx - 1 \le 0 \ (\because 3 - kx > 0)$$

$$\Leftrightarrow (kx-1)^3 \le 0 .$$

The last inequality is true if $-\frac{1}{\mathrm{k}} \le x \le \frac{1}{k}$, so

if $-\frac{1}{\mathrm{k}} \le x \le \frac{1}{k}$, $\quad p_2(x,k) \le f_2(x,k)$.

2) If $-\frac{1}{\mathrm{k}} \le x \le \frac{1}{k}$, then

$$p_2'(x,k) = \tfrac{1}{16}(kx+1)^2(3k^3x^3 - 6k^2x^2 - kx + 8).$$

Hence

$$p_2'(x,k) \ge 0$$

$$\Leftrightarrow 3k^3x^3 - 6k^2x^2 - kx + 8 \ge 0 .$$

$$\Leftrightarrow (kx+1)(3k^2x^2 - 9kx + 8) \ge 0$$

$$\Leftrightarrow 3k^2(kx+1)[(x - \tfrac{3}{2}k^{-1})^2 + \tfrac{5}{12}k^{-2}] \ge 0$$

$\Rightarrow$ $p_2(x,k)$ increases monotonously in $[-1/k, 1/k]$.

3) If $x < -\frac{1}{k}$, or $x > \frac{1}{k}$, or $-\frac{1}{\mathrm{k}} \le x < 0$, it is obviously true that $p_2(x,k) \ge (x)_+$.

If $0 \le x \le \frac{1}{k}$, set $Q(x) = p_2(x,k) - (x)_+$,

then

$$Q'(x) = \tfrac{1}{16}(kx+1)^3(3k^2x^2 - 9kx + 8) - 1 .$$

$\Rightarrow$

$$Q'(x) \le Q'(\tfrac{1}{k}) = 0 \ (\because Q''(x) \ge 0),$$

which indicates that $Q(x)$ decreases monotonously in $[0, 1/k]$.

Hence $Q(x) \ge Q(\frac{1}{k}) = 0$, that is,

if $0 \le x \le \frac{1}{k}$, then $p_2(x,k) \ge (x)_+$.

4) From 2) and 3), we have at once

$$\|p_2(x,k) - (x)_+\|_\infty = \max_x |\, p_2(x,k) - (x)_+ | = |\, p_2(0,k) | = \tfrac{5}{32k}$$

The proof is finished. □

In [7],

$$\|f_1(x,k) - (x)_+\|_\infty = \tfrac{1}{4k}, \|f_2(x,k) - (x)_+\|_\infty = \tfrac{3}{16k},$$

whereas in this paper,

$$\|p_1(x,k) - (x)_+\|_\infty = \tfrac{3}{16k}, \|p_2(x,k) - (x)_+\|_\infty = \tfrac{5}{32k}.$$

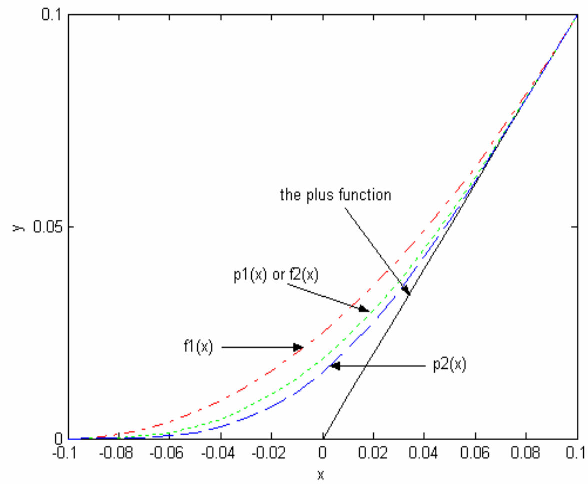

Figure1. The comparisons when k=10

It is clear that the infinity-norm errors for $p_1(x,k)$, $p_2(x,k)$ in this paper are respectively 3/4 and 5/6 of the results in [7], all better than that in [7].

We also notice that $p_1(x,k) = f_2(x,k)$. It indicates that the optimal solution of Problem 1 achieves the second-order smoothing level. In another word, $p_1(x,k)$ achieves the higher smoothing level than that it is requested. For the statement 'the degree of polynomial $L_4(x)$ is not over 4' in Problem 1, it is interesting whether there exists some piecewise polynomial, except for $f_1(x,k)$, which has the degree less than 4 and satisfies the constrained conditions of Problem 1. From (13), the answer is no. Thus, $f_1(x,k)$ is the only feasible solution for Problem 1 with the degree of polynomial function below 4. Similarly, $f_2(x,k)$ is the only feasible solution of Problem 2, in which the degree of polynomial function is less than 6.

The comparison between the results in this paper and the results in [7] is described in Fig.1, where k=10 and f1(x), f2(x), p1(x), p2(x) stand respectively for $f_1(x,k)$, $f_2(x,k)$, $p_1(x,k)$, $p_2(x,k)$.

Ⅴ. CONCLUSIONS AND DISCUSSIONS

In this paper, we formulate the standard piecewise polynomial smooth approximation problems to the plus function. In the proof of the existence and uniqueness of the solution for these problems, their analytic solutions are obtained. By the comparison between our results with the results in [7], we claim that the piecewise polynomials in this paper achieve a better approximation performance than [7].

The piecewise polynomial smooth approximation to the plus function, which is of higher smoothing level, can be carried out according to the method in this paper. But it should be noticed that the polynomial must be nonnegative.

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