

# Sorting photons of different rotational Doppler shifts (RDS) by orbital angular momentum of single-photon with spin-orbit-RDS entanglement

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**Abstract:** We demonstrate that single photons from a rotating  $q$ -plate exhibit an entanglement in three degrees of freedom of spin, orbital angular momentum, and the rotational Doppler shift (RDS) due to the nonconservation of total spin and orbital angular momenta. We find that the rotational Doppler shift  $\Delta\omega = \Omega(\Delta s + \Delta l)$ , where  $s$ ,  $l$  and  $\Omega$  are quantum numbers of spin, orbital angular momentum, and rotating velocity of the  $q$ -plate, respectively. Of interest is that the rotational Doppler shift directly reflects the rotational symmetry of  $q$ -plates and can be also expressed as  $\Delta\omega = \Omega n$ , where  $n = 2(q - 1)$  denotes the fold number of rotational symmetry. Besides, based on this single-photon spin-orbit-RDS entanglement, we propose an experimental scheme to sort photons of different frequency shifts according to individual orbital angular momentum.

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## 1. Introduction

Recently, optical angular momentum is receiving a great deal of interest from both the classical and quantum points of view [1-5]. One of those in focus is the rotational Doppler shift (RDS) [6]. The RDS effect was first described by Garetz and Arnold, who used a rotating half-wave plate of angular velocity  $\Omega$  to imprint a frequency shift  $2\Omega$  to the circularly polarized light [7]. By use of spin Poincaré-sphere, Simon *et al.* [8] demonstrated that the dynamically evolving geometric phase (Berry phase) arising from a rotating quarter-wave plate can offset the frequency of a laser beam. Later, Brettenaker and Le Floch [9] suggested that the frequency shift is a consequence of the conservation of energy and

associated with the spin angular momentum exchange. Unlike the intrinsic photon spin associated with circular polarization, the orbital angular momentum (OAM) was related to helical wavefront or optical vortex with phase singularity, however, much attention is still focused on seeking the analogy between them [10].

After recognizing that a twisted photon of  $l$ -intertwined helical phase front  $\exp(il\phi)$  carries a well-defined OAM of  $l\hbar$  [11], Allen *et al.* also suggested using twisted photons to achieve the azimuthal Doppler shift [12]. Later experimental observation by Courtil *et al.* showed that when a mm-wave beam bearing OAM is rotating with a velocity  $\Omega$ , it is imparted by a frequency shift of  $l\Omega$  [13]. Of striking interest is that the spin and orbital angular momentum were indistinguishable for the RDS effect [14, 15]. It was found that a number of experiments on the RDS can be treated as the special cases of the atomic systems lacking rotational invariance but having stationary state in a rotating frame, as predicted by Bialynicki-Birula and Bialynicka-Birula [16]. Basistiy *et al.* reported their first all-optical detection of the RDS effect using an off-axis optical vortex beam [17]. Barreiro *et al.* demonstrated the first spectroscopic observation of the RDS effect due to OAM via the coherent interaction between light beams and an ensemble of rubidium atoms at room temperature [18]. Recently, a quantum theory of rotating photons with RDS effect was proposed by van Enk and Nienhuis [19]. Here we demonstrate an experiment scheme to show that single photons from a rotating  $q$ -plate exhibit an entanglement in spin, orbital angular momentum, and the rotational Doppler shift, reflecting the rotational symmetry of the  $q$ -plate and, photons with different frequency shifts can be sorted according to individual orbital angular momentum. It has been recognized that the utility of several degrees of freedom of a single-photon to encode multiple qubits opens the possibility for building a deterministic quantum information processor [20-22]. It was also shown that single-photon two-qubit states might be useful for deterministic cryptographic schemes [23]. Recently, beating the channel capacity limit for linear photonic superdense coding was reported, in which single-photon spin-orbital Bell-state analysis played a key role [24]. So, we anticipate that the single-photon spin-orbit-RDS entanglement would have potential applications for quantum computation, quantum communication or optical tweezers system.

## 2. Theory and Application

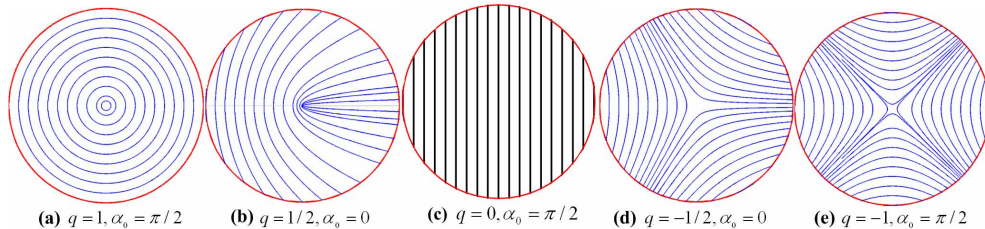


Fig. 1. Illustration of some types of  $q$ -plates, in which the orientation of optical axis is the locally tangent to the line: (a) is rotational invariance; while (b), (c), (d), and (e) are of one-, two-, three-, and four-fold rotational symmetry, respectively.

Recently, Marrucci and colleagues [25] built a novel device called  $q$ -plate (or “Pancharatnam-Berry phase optical element”) that reverses the spin of photons while transferring the change of spin angular momentum into the orbital kind. A general  $q$ -plate is a planar slab of a uniaxial birefringent medium, with an inhomogeneous orientation of the optical axis lying in  $x$ - $y$  plane and a homogeneous phase retardation of  $\chi$  along  $z$ -axis. Suppose the orientation of optical axis has this form in a polar coordinate:  $\alpha(r, \phi) = q\phi + \alpha_0$ , where  $q$  and  $\alpha_0$  are two constant. Some types of  $q$ -plates are illustrated in Fig. 1. On the circular bases, the transmitted matrix for the  $q$ -plate can be expressed as

$$T_q = \frac{1 + \exp(i\chi)}{2} \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} + \frac{1 - \exp(i\chi)}{2} \begin{bmatrix} 0 & \exp[i2\alpha(r, \varphi)] \\ \exp[-i2\alpha(r, \varphi)] & 0 \end{bmatrix}. \quad (1)$$

We find that these  $q$ -plates can be used for the generation of single photons exhibiting the spin-orbit-RDS entanglement, as sketched in Fig. 2. Assume the input are single photons with horizontal linear polarization, namely,  $|\mathbf{in}\rangle = |\mathbf{L}\rangle/\sqrt{2} + |\mathbf{R}\rangle/\sqrt{2}$ , where  $|\mathbf{L}\rangle$  and  $|\mathbf{R}\rangle$  denote left- and right-handed circular polarizations, respectively. A suitable computer-generated phase hologram is employed to serve as an OAM adder, which imprints the  $m$ -intertwined helical phasefront to incoming photons while remains the linear polarization unchanged [26], thus, yielding the state  $|2\rangle = |\mathbf{in}\rangle \otimes |m\rangle = |\mathbf{L}, m\rangle/\sqrt{2} + |\mathbf{R}, m\rangle/\sqrt{2}$ . Subsequently, the photons are directed onto the  $q$ -plate. As the  $q$ -plate is installed in a rotating device, the local orientation of optical axis, with respect to a fixed reference coordinates, is varying along with the time  $t$ , namely,  $\alpha'(r, \varphi, t) = q\varphi + (q-1)\Omega t + \alpha_0$ . Unlike the use of Dove prism for rotating the light beam [13, 14], in present setup the  $q$ -plate is rotating, from which the emerging state becomes

$$|\mathbf{out}\rangle = \alpha_1 |\mathbf{in}, m\rangle + \alpha_2 \exp[i2(q-1)\Omega t] |\mathbf{L}, m+2q\rangle + \alpha_3 \exp[-i2(q-1)\Omega t] |\mathbf{R}, m-2q\rangle. \quad (2)$$

where  $\alpha_1 = [1 + \exp(i\chi)]/2$ ,  $\alpha_2 = [1 - \exp(i\chi)]\exp(i2\alpha_0)/2\sqrt{2}$ , and  $\alpha_3 = [1 - \exp(i\chi)]\exp(-i2\alpha_0)/2\sqrt{2}$ . Of particular interest is that Eq. (2) shows the correlations among three degrees of freedom of spin, orbital angular momentum, and rotational Doppler shift. Actually, this state consists of three orthogonal OAM states, i.e.,  $|l=m\rangle$ ,  $|l=m-2q\rangle$ , and  $|l=m+2q\rangle$ , which correspond respectively to three different Pancharatnam-Berry phases,  $0$ ,  $2(q-1)\Omega t$ , and  $-2(q-1)\Omega t$ , therefore, three different frequency shifts,  $\Delta\omega=0$ ,  $2(q-1)\Omega$ , and  $-2(q-1)\Omega$ . The two nonzero frequency shifts originate from the nonconservation of total spin and orbital angular momenta. Detailedly, for the state  $|l=m\rangle$ , as the angular momentum change  $\Delta j=0$ , no RDS is expected; while for  $|l=m+2q\rangle$ ,  $\Delta j = \Delta s + \Delta l = 2(q-1)$  and the frequency shift  $\Delta\omega = 2(q-1)\Omega$ ; and for  $|l=m-2q\rangle$ ,  $\Delta j = -2(q-1)$  and  $\Delta\omega = -2(q-1)\Omega$ . Table 1 lists some example input states and  $q$  against resulting frequency shifts.

Table 1. The example input states and  $q$  against resulting frequency shifts.

Input states q-plates	$s = +1$ (SAM) $m = -1$ (OAM)	$s = +1$ $m = +1$	$s = -1$ $m = -1$	$s = -1$ $m = +1$
$q=0$	$\Delta\omega = -2\Omega$	$\Delta\omega = -2\Omega$	$\Delta\omega = 2\Omega$	$\Delta\omega = 2\Omega$
$q=1/2$	$\Delta\omega = -\Omega$	$\Delta\omega = -\Omega$	$\Delta\omega = \Omega$	$\Delta\omega = \Omega$
$q=1$	$\Delta\omega = 0$	$\Delta\omega = 0$	$\Delta\omega = 0$	$\Delta\omega = 0$

Obviously, the rotational Doppler shift is only determined by the change of total spin and orbital angular momenta, irrelative of the absolute orbital angular momentum ( $m$ ) bore by the incoming photons. The single-photon spin-orbit-RDS entanglement can be understood as follows:  $\exp(i0t)$ ,  $\exp(in\Omega t)$ , and  $\exp(-in\Omega t)$  are orthogonal to each other in the space  $[0, T]$  [ $n = 2(q-1), T = 2\pi/(n\Omega)$ ], so they can be taken as a set of bases. By denoting  $\exp(i0t)/\sqrt{2\pi} = |\Delta\omega = 0\rangle$ ,  $\exp(in\Omega t)/\sqrt{2\pi} = |\Delta\omega = n\Omega\rangle$ , and  $\exp(-in\Omega t)/\sqrt{2\pi} = |\Delta\omega = -n\Omega\rangle$ , Eq. (2) can be written as

$$|\mathbf{out}\rangle = \alpha_1 |\mathbf{in}, m\rangle |\Delta\omega = 0\rangle + \alpha_2 |\mathbf{L}, m+2q\rangle |\Delta\omega = n\Omega\rangle + \alpha_3 |\mathbf{R}, m-2q\rangle |\Delta\omega = -n\Omega\rangle. \quad (3)$$

The state described by Eq. (3) is an entangled state, which is achieved among three different degrees of freedom of spin, orbital angular momentum, and the rotational Doppler shift in single photons. Also, Equation (3) could be considered as a standard Schmidt decomposition, whose Schmidt number [27]  $\kappa = (|\alpha_1|^4 + |\alpha_2|^4 + |\alpha_3|^4)^{-1} = 8/[3(\cos\chi + 1/3)^2 + 8/3] \in (1,3)$  ( $\chi \neq 2\pi$ ) and, when  $\cos\chi = -1/3$  ( $\chi = 109.5^\circ$ ), it reaches its maximum value  $\kappa = 3$ . One can describe the single-photon state in terms of a tensor product Hilbert space, namely  $H = H_1 \otimes H_2 \otimes H_3$ , where  $H_1$ ,  $H_2$ , and  $H_3$  are respectively disjoint Hilbert spaces corresponding to three different degrees of freedom of spin, orbital angular momentum, and rotational Doppler shift. As well known, the entanglement is not limited to different particles, but generally applicable to different degrees of freedom in single particles [28-30]. In fact, single-photon spin-orbit entangled state (two-qubit) has been recently reported [31-33] and has been utilized to implement a simple quantum Deutsch algorithm [34]. However, it would be more proper that the single-photon spin-orbit or spin-orbit-RDS entanglement described by Eq. (3) is called classical entanglement [35].

We know from Eq. (3) that, the single-photon RDS is proportional to the change of its total angular momentum, namely,  $\Delta\omega = \Omega\Delta j = \Omega(\Delta s + \Delta l)$ . The rotational Doppler shift can be understood in terms of ray optics and the torque action on the rotating optical components [36] or the accompanying energy exchange [37]. It is also illuminating for us to attribute this effect to the rotational symmetry of the system. It is known that if the rotational invariance around the propagation direction of light holds, then the angular momentum along this direction is expected to be conserved [38]. This is the case for  $q=1$ . For  $q \neq 1$ , the nonconservation of angular momentum as well as the frequency shift will happen synchronously. As shown in Fig. 1, it is evident that the  $q$ -plate with parameter  $q$  has  $2(q-1)$ -fold rotational symmetry, just equal to the angular momentum increment. So we argue that the RDS can be considered as a nature consequence of the rotational symmetry of the  $q$ -plate and the RDS can be also simply expressed by  $\Delta\omega = n\Omega$ , where  $n = 2(q-1)$  denotes the fold number of rotational symmetry. For the special case of  $q=0$ , the  $q$ -plate will reduce to the half-wave plate, which is of two-fold rotational symmetry [see Fig. 1(c)] and  $\Delta\omega = \pm 2\Omega$ , due to pure spin angular momentum [7-9], where only the spin-RDS entanglement can exist.

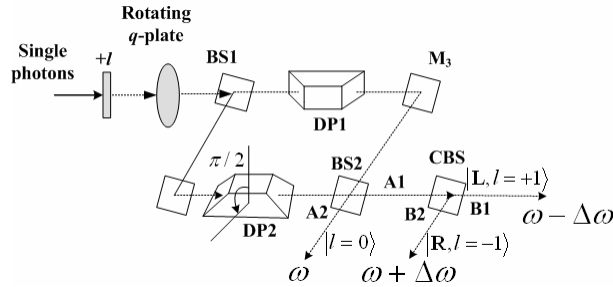


Fig. 2. The experimental scheme using  $q$ -plate to generate single-photon spin-orbit-RDS entanglement; and sort photons of different RDS according to individual OAM: BS, 50:50 non-polarizing beam splitter; DP, Dove prism; CBS, circularly polarizing beam splitter; M, mirror.

Of particular interest is that the single-photon spin-orbit-RDS entanglement provides us an approach to sort photons of different rotational Doppler shifts according to individual orbital angular momentum. Assume  $q=1/2$  and  $\alpha_0=0$  is considered [39] and the orbital angular momentum adder is trivially absent ( $m=0$ ), then the orbital angular momentum to be sorted are  $|l=-1\rangle$ ,  $|l=0\rangle$ , and  $|l=1\rangle$ . The Mach Zehnder interferometer with two Dove

prisms having a relative rotation angle performs the OAM sorting, which can work at a single photon level [40]. In general, to sort single photons with three different OAM states, two stages of cascaded Mach Zehnder interferometers are required. However, due to the fact that single photons exhibit the spin-orbit entanglement in present case, only one stage of Mach Zehnder interferometer together with a circularly polarizing beam splitter (CBS) is enough. With the relative angle of two Dove prisms set as  $\pi/2$  (resulting in a  $\pi$  rotation to the beam), a relative phase difference  $\pi$  between two arms is introduced for the state  $|l = \pm 1\rangle$ , as shown in Fig. 3, therefore, the resulted destructive interference will make photons of  $|l = \pm 1\rangle$  emerge from port A1 (Here, phase retardation  $\chi = 109.5^\circ$  is assumed to reach  $\kappa = 3$ ); while for photons of  $|l = 0\rangle$ , the condition of constructively interference always holds, therefore they come out from port A2. Subsequently, as photons from Port A1 have the spin-orbit-RDS entanglement, namely  $|\text{out}\rangle_{A1} = \langle \mathbf{L} | l = 1 \rangle |\Delta\omega = -\Omega\rangle + |\mathbf{R}\rangle | l = -1 \rangle |\Delta\omega = \Omega\rangle / \sqrt{2}$ , the CBS, serving as a photon spin sorter, is capable to separate photons of  $|l = +1\rangle$  and  $|l = -1\rangle$  according to individual circular polarizations. Then photons of  $|l = +1\rangle$  and  $|l = -1\rangle$  come out from B1 and B2 port, respectively. For the state  $|l = 0\rangle$ , we know  $\Delta\omega = 0$ , without RDS; while for  $|l = \pm 1\rangle$ ,  $\Delta\omega = \mp\Omega$ , with down- and up-shift, respectively. Besides, if the input linearly polarized light is replaced by an arbitrarily polarized one, then the yield of photons with different RDS could be manipulated.

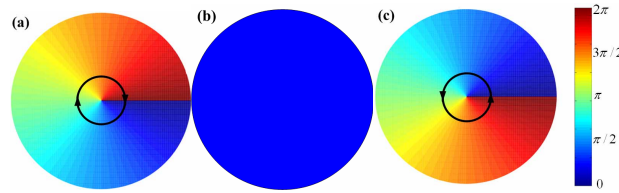


Fig. 3. Phase diagrams of photons with OAM (a)  $l = -1$ , (b)  $l = 0$ , and (c)  $l = +1$ , respectively.

### 3. Conclusion

We have demonstrated that the single-photon entanglement in three degrees of freedom of spin, orbital angular momentum, and rotational Doppler shift can be prepared from a rotating  $q$ -plate and based on this type of entanglement the photons with different rotational Doppler shifts can be sorted according to individual orbital angular momentum. We anticipate the single-photon spin-orbit-RDS entanglement would have potential applications for quantum computation, quantum communication or optical tweezers system.

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