Finite-size effects in a *D*-dimensional ideal Fermi gas

2009 Chinese Phys. B 18 5189

(http://iopscience.iop.org/1674-1056/18/12/015)

Vol 18 No 12, December 2009
1674-1056/2009/18(12)/5189-07

Chinese Physics B

Finite-size effects in a D-dimensional ideal Fermi gas*

Su Guo-Zhen(苏国珍)[a)†], Ou Cong-Jie(欧聪杰)[b)], Wang A Qiu-Ping[b)], and Chen Jin-Can(陈金灿)[a)]

[a)] *Department of Physics and Institute of Theoretical Physics and Astrophysics,
Xiamen University, Xiamen 361005, China*

[b)] *Institut Supérieur des Matériaux et Mécaniques Avancées du Mans,
44 Av. Bartholdi, 72000 Le Mans, France*

(Received 22 January 2009; revised manuscript received 17 March 2009)

By using the Euler–MacLaurin formula, this paper studies the thermodynamic properties of an ideal Fermi gas confined in a D-dimensional rectangular container. The general expressions of the thermodynamic quantities with the finite-size corrections are given explicitly and the effects of the size and shape of the container on the properties of the system are discussed. It is shown that the corrections of the thermodynamic quantities due to the finite-size effects are significant to be considered for the case of strong degeneracy but negligible for the case of weak degeneracy or non-degeneracy. It is important to find that some familiar conclusions under the thermodynamic limit are no longer valid for the finite-size systems and there are some novel characteristics resulting from the finite-size effects, such as the nonextensivity of the system, the anisotropy of the pressure, and so on.

Keywords: finite-size effect, ideal Fermi gas, thermodynamic property

PACC: 0520, 0530F

1. Introduction

The properties of an ideal Fermi gas confined in a certain volume have been extensively studied in the textbooks.[1,2] Most of the investigations are conventionally carried out in the so-called thermodynamic limit. It is well known that there are two classes of quantities describing the thermodynamic properties of the system which satisfies the thermodynamic limit, i.e., the extensive and intensive quantities. The extensive quantities are proportional to the size of the system while the intensive quantities are size-independent. All the intensive properties of the system satisfying the thermodynamic limit can be well settled by any two of the intensive quantities, e.g., the particle density and temperature of the system, and are independent of the size of the system and the shape of the boundary. These familiar conclusions, however, will not be true for the system of a finite number of particles enclosed in a finite volume.

In recent years, the investigation on the properties of finite-size systems has become one of the important topics[3−14] in thermodynamics and statistical physics. For example, Sisman and Muller[3,4] studied the boundary effects of the ideal classical gas and discussed the Casimir-like size effect in a confined space; Dai *et al*[5−7] researched into the properties of the ideal quantum gases in a confined space and obtained some significant results; Begun and Gorenstein[8] investigated the Bose–Einstein condensation in the relativistic pion gas, in which the finite-size effects were taken into account.

In this paper, we will study the properties of an ideal Fermi gas confined in a D-dimensional container and discuss the effects of the size and shape of the container on the properties of the system. It finds that the finite-size effects result in several novel characteristics, such as the nonextensive property and anisotropic pressure of the system, and so on.

2. General expressions of thermodynamic quantities

We consider an ideal Fermi gas confined in a D-dimensional rectangular container with the i-th side length L_i (i = 1, 2, ..., D). By solving the Schrödinger equation, the energy spectrum of a particle is found to be

$$\varepsilon(n_1, n_2, \ldots, n_D) = \frac{h^2}{8m}\sum_{i=1}^{D}\frac{n_i^2}{L_i^2},$$
$$(n_i = 1,\ 2,\ 3,\ \ldots), \qquad (1)$$

*Project supported by the National Natural Science Foundation of China (Grant No 10875100).
†Corresponding author. E-mail: gzsu@xmu.edu.cn

where m is the mass of a particle and h is the Planck's constant. Using Eq.(1), we can express the grand partition function Ξ of the system as

$$\ln \Xi = \sum_{n_1=1}^{\infty}\sum_{n_2=1}^{\infty}\cdots\sum_{n_D=1}^{\infty}\ln\left[1+z\exp\left(-\frac{\beta h^2}{8m}\sum_{i=1}^{D}\frac{n_i^2}{L_i^2}\right)\right], \quad (2)$$

where $\beta = 1/k_{\rm B}T$, $k_{\rm B}$ is the Boltzman's constant, T is the temperature, $z = \exp(\beta\mu)$ is the fugacity, and μ is the chemical potential of the system. For simplicity the degeneracy related to the internal structure of the particles is assumed to be unit.

The summations of Eq.(2) are conventionally replaced by the integrals and the grand partition function can be readily found under the thermodynamic limit. However the approximation may cause noticeable deviations for the systems with small scales. A more precise solution of the grand partition function can be found by using the following Euler–MacLaurin formula:[15]

$$\sum_{n=0}^{\infty}F(n) = \int_0^{\infty}F(n)\,{\rm d}n + \frac{1}{2}F(0) - \frac{1}{12}F'(0) + \frac{1}{720}F'''(0) + \cdots . \quad (3)$$

On condition that only the first-order correction due to the finite-size effects is considered, the grand partition function is found to be (a detailed derivation is given in the Appendix)

$$\ln\Xi = \frac{V_D}{\lambda^D}\left[f_{D/2+1}(z) - \frac{1}{2}\frac{\lambda}{\tilde{L}}f_{D/2+1/2}(z)\right], \quad (4)$$

where $\lambda = h/\sqrt{2\pi m k_{\rm B}T}$ is the thermal wavelength, $V_D = \prod_{i=1}^{D}L_i$ is the D-dimensional volume of the container, $\tilde{L} = \left(\sum_{i=1}^{D}1/L_i\right)^{-1}$, and

$$f_l(z) = \frac{1}{\Gamma(l)}\int_0^{\infty}\frac{x^{l-1}{\rm d}x}{z^{-1}\exp(x)+1} \quad (5)$$

is the Fermi integral.[1]

According to Eq.(4), one can obtain the total number of particles and internal energy as

$$N = z\left(\frac{\partial\ln\Xi}{\partial z}\right)_{\beta,L_i} = \frac{V_D}{\lambda^D}\left[f_{D/2}(z) - \frac{1}{2}\frac{\lambda}{\tilde{L}}f_{D/2-1/2}(z)\right] \quad (6)$$

and

$$E = -\left(\frac{\partial\ln\Xi}{\partial\beta}\right)_{z,L_i} = \frac{D}{2}\frac{V_D k_{\rm B}T}{\lambda^D}\left[f_{D/2+1}(z) - \frac{D-1}{2D}\frac{\lambda}{\tilde{L}}f_{D/2+1/2}(z)\right], \quad (7)$$

respectively.

Using Eq.(4), one can find the pressure of the system as well. It should be noted here that the familiar formulas such as $p = (1/\beta)(\partial\ln\Xi/\partial V_D)_{z,\beta}$ and $pV_D/k_{\rm B}T = \ln\Xi$ are no longer valid for the finite-size system. Generally, the pressure of the finite-size system is anisotropic and should be expressed as a tensor. If the i-th coordinate axis is set along the i-th side of the container, the tensor may be diagonalized and p_{ii} represents the pressure acting on the wall perpendicular to the i-th side of the container. According to Eq.(4), it can be found that

$$p_{ii} = \frac{1}{\beta}\frac{L_i}{V_D}\left(\frac{\partial\ln\Xi}{\partial L_i}\right)_{z,\beta,L_{j\neq i}} = \frac{k_{\rm B}T}{\lambda^D}\left[f_{D/2+1}(z) - \frac{1}{2}\left(1-\frac{\tilde{L}}{L_i}\right)\frac{\lambda}{\tilde{L}}f_{D/2+1/2}(z)\right]. \quad (8)$$

It is seen from Eqs.(7) and (8) that there exists an important relationship between the internal energy and the D components of the pressure tensor as

$$E = \frac{V_D}{2}\sum_{i=1}^{D}p_{ii}. \quad (9)$$

Equation (9) is similar to the familiar relation $E = (D/2)pV_D$ under the thermodynamic limit, but they are not identical with each other unless all the side lengths of the container are equal.

According to Eqs.(6) and (7), one can further derive the heat capacity at the given side lengths of the container as

$$C_{L_i} = \left(\frac{\partial E}{\partial T}\right)_{L_i} = \left(\frac{\partial E}{\partial T}\right)_{z,L_i} + \left(\frac{\partial E}{\partial z}\right)_{T,L_i}\left(\frac{\partial z}{\partial T}\right)_{L_i}$$

$$= \frac{D}{2}\frac{V_D k_{\rm B}}{\lambda^D}\left\{\left(\frac{D}{2}+1\right)f_{D/2+1}(z) - \frac{D}{2}\frac{f_{D/2}^2(z)}{f_{D/2-1}(z)} - \frac{D}{4}\left[\frac{D^2-1}{D^2}f_{D/2+1/2}(z) - \frac{2(D-1)}{D}\frac{f_{D/2}(z)f_{D/2-1/2}(z)}{f_{D/2-1}(z)} + \frac{f_{D/2}^2(z)f_{D/2-3/2}(z)}{f_{D/2-1}^2(z)}\right]\frac{\lambda}{\tilde{L}}\right\}. \quad (10)$$

In order to expound more clearly the corrections of the above thermodynamic quantities due to the finite-size effects, we introduce a parameter z_0 which is determined by

$$N = \frac{V_D}{\lambda^D}f_{D/2}(z_0). \quad (11)$$

It is obvious that z_0 represents the fugacity of the system under the thermodynamic limit, which depends on the particle density and temperature but is independent of the size and shape of the system. If one only considers the first-order finite-size effects, then from Eqs.(6) and (11) one can get that the relation between z and z_0 as

$$z = z_0\left[1 + \frac{1}{2}\frac{\lambda}{\tilde{L}}\frac{f_{D/2-1/2}(z_0)}{f_{D/2-1}(z_0)}\right]. \quad (12)$$

Substituting Eq.(12) into $\mu = k_{\rm B}T\ln z$ and Eqs.(6)–(8) and (10) and keeping only the first power in $\lambda/\tilde{L}$, one can derive the chemical potential, internal energy, pressure and heat capacity at the given side lengths of the container respectively as

$$\mu = k_{\rm B}T\left[\ln z_0 + \frac{1}{2}\frac{f_{D/2-1/2}(z_0)}{f_{D/2-1}(z_0)}\frac{\lambda}{\tilde{L}}\right], \quad (13)$$

$$E = \frac{D}{2}Nk_{\rm B}T\left\{\frac{f_{D/2+1}(z_0)}{f_{D/2}(z_0)} + \frac{1}{2}\left[\frac{f_{D/2-1/2}(z_0)}{f_{D/2-1}(z_0)} - \frac{D-1}{D}\frac{f_{D/2+1/2}(z_0)}{f_{D/2}(z_0)}\right]\frac{\lambda}{\tilde{L}}\right\}, \quad (14)$$

$$p_{ii} = \frac{Nk_{\rm B}T}{V_D}\left\{\frac{f_{D/2+1}(z_0)}{f_{D/2}(z_0)} + \frac{1}{2}\left[\frac{f_{D/2-1/2}(z_0)}{f_{D/2-1}(z_0)} - \left(1-\frac{\tilde{L}}{L_i}\right)\frac{f_{D/2+1/2}(z_0)}{f_{D/2}(z_0)}\right]\frac{\lambda}{\tilde{L}}\right\}, \quad (15)$$

and

$$C_{L_i} = \frac{D}{2}Nk_{\rm B}\left\{\left(\frac{D}{2}+1\right)\frac{f_{D/2+1}(z_0)}{f_{D/2}(z_0)} - \frac{D}{2}\frac{f_{D/2}(z_0)}{f_{D/2-1}(z_0)} - \frac{D}{4}\left[\frac{D^2-1}{D^2}\frac{f_{D/2+1/2}(z_0)}{f_{D/2}(z_0)} - \frac{f_{D/2-1/2}(z_0)}{f_{D/2-1}(z_0)} + \frac{f_{D/2}(z_0)f_{D/2-3/2}(z_0)}{f_{D/2-1}^2(z_0)} - \frac{f_{D/2}(z_0)f_{D/2-1/2}(z_0)f_{D/2-2}(z_0)}{f_{D/2-1}^3(z_0)}\right]\frac{\lambda}{\tilde{L}}\right\}. \quad (16)$$

Equations (13)–(16) are the main results of this paper, from which one can deduce the following important conclusions:

(i) The corrections of the thermodynamic quantities due to the finite-size effects are increased with λ and decreased with $\tilde{L}$. For a small system that the thermal wavelength λ is comparable with the linear order of the container characterized by $\tilde{L}$, the finite-size effects may be significant. It is also observed that the properties of the system are dependent not only on the size but also on the shape of the container, since $\tilde{L} = (\sum_{i=1}^{D} 1/L_i)^{-1}$ is determined by both the size and shape of the container. For a large-scale system of $\tilde{L} \gg \lambda$, the above equations will be reduced to the familiar results under the thermodynamic limit.[16]

(ii) From Eq.(15) one can see that $p_{ii} \neq p_{jj}$ if $L_i \neq L_j$. This indicates that the pressure of the system is anisotropic, i.e., the pressure acting on the different wall of the container may be different. The difference may be significant for the small container with significantly different side lengths, such as the tube-like or disk-like containers.

(iii) Equations (14) and (16) show that E and C_{L_i}, which are extensive quantities under the thermodynamic limit, are not proportional to the size of the system at the given particle density and temperature (and hence given z_0) because $\tilde{L}$ in the equations is size- and shape-dependent. On the other hand, according to Eqs.(13) and (15), μ and p_{ii}, which are both intensive quantities under the thermodynamic limit, are dependent not only on the particle density and temperature, but also on the size and shape of the container. This indicates that for a small system in a confined space, the extensive property is eliminated due to the finite-size effects.

3. The case of strong degeneracy

In the case of strong degeneracy, $\ln z_0 \gg 1$ and the function $f_l(z_0)$ can be expanded as

$$f_l(z_0) = \frac{(\ln z_0)^l}{\Gamma(l+1)}\left[1 + l(l-1)\frac{\pi^2}{6}\frac{1}{(\ln z_0)^2} + \cdots\right], \tag{17}$$

according to the Sommerfeld's lemma.[1] Substituting Eq.(17) into Eqs.(11) and (13)–(16) and keeping only the terms up to the second power of $k_{\rm B}T/\varepsilon_{\rm F0}$, one can obtain the expressions of μ, E, p_{ii} and C_{L_i} respectively as the explicit functions of temperature, i.e.,

$$\mu = \varepsilon_{\rm F0}\left\{1 - \frac{\pi^2(D-2)}{12}\left(\frac{k_{\rm B}T}{\varepsilon_{\rm F0}}\right)^2 + \frac{\Gamma(D/2)}{2\Gamma(D/2+1/2)}\left[1 + \frac{\pi^2(D-3)}{24}\left(\frac{k_{\rm B}T}{\varepsilon_{\rm F0}}\right)^2\right]\frac{\lambda_{\rm F0}}{\tilde{L}}\right\}, \tag{18}$$

$$E = \frac{D}{D+2}N\varepsilon_{\rm F0}\left\{1 + \frac{\pi^2(D+2)}{12}\left(\frac{k_{\rm B}T}{\varepsilon_{\rm F0}}\right)^2 + \frac{D+2}{4}\frac{\Gamma(D/2)}{\Gamma(D/2+3/2)}\left[1 - \frac{\pi^2(D+1)}{24}\left(\frac{k_{\rm B}T}{\varepsilon_{\rm F0}}\right)^2\right]\frac{\lambda_{\rm F0}}{\tilde{L}}\right\}, \tag{19}$$

$$p_{ii} = \frac{2}{D+2}\frac{N\varepsilon_{\rm F0}}{V_D}\left\{1 + \frac{\pi^2(D+2)}{12}\left(\frac{k_{\rm B}T}{\varepsilon_{\rm F0}}\right)^2 + \frac{D+2}{8}\frac{\Gamma(D/2)}{\Gamma(D/2+3/2)}\right.$$
$$\left.\times\left[1 + \frac{D\tilde{L}}{L_i} - \frac{\pi^2(D+1)}{8}\left(1 - \frac{D\tilde{L}}{3L_i}\right)\left(\frac{k_{\rm B}T}{\varepsilon_{\rm F0}}\right)^2\right]\frac{\lambda_{\rm F0}}{\tilde{L}}\right\}, \tag{20}$$

and

$$C = \frac{\pi^2 D}{6}Nk_{\rm B}\left[1 - \frac{\Gamma(D/2)}{4\Gamma(D/2+1/2)}\frac{\lambda_{\rm F0}}{\tilde{L}}\right]\left(\frac{k_{\rm B}T}{\varepsilon_{\rm F0}}\right), \tag{21}$$

where

$$\varepsilon_{\rm F0} = \frac{h^2}{2\pi m}\left[\frac{N\Gamma(D/2+1)}{V_D}\right]^{2/D} \tag{22}$$

is the Fermi energy under the thermodynamic limit[16] and $\lambda_{\rm F0} = h/\sqrt{2\pi m\varepsilon_{\rm F0}}$.

Setting $T \to 0$ K in Eqs.(18)–(20), one can obtain the Fermi energy, ground-state energy and ground-state pressure respectively as

$$\varepsilon_{\rm F} = \varepsilon_{\rm F0}\left[1 + \frac{\Gamma(D/2)}{2\Gamma(D/2+1/2)}\frac{\lambda_{\rm F0}}{\tilde{L}}\right], \tag{23}$$

$$E_{\rm G} = \frac{D}{D+2}N\varepsilon_{\rm F0}\left[1 + \frac{D+2}{4}\frac{\Gamma(D/2)}{\Gamma(D/2+3/2)}\frac{\lambda_{\rm F0}}{\tilde{L}}\right], \tag{24}$$

and

$$p_{iiG} = \frac{2}{D+2}\frac{N\varepsilon_{\rm F0}}{V_D}\left\{1 + \frac{D+2}{8}\frac{\Gamma(D/2)}{\Gamma(D/2+3/2)}\times\left(1 + \frac{D\tilde{L}}{L_i}\right)\frac{\lambda_{\rm F0}}{\tilde{L}}\right\}. \tag{25}$$

From Eqs.(18)–(25) one can find that the corrections due to the finite-size effects are of the order of

$$\lambda_{\rm F0}/\tilde{L} = \left[\frac{V_D}{\tilde{L}^D N\Gamma(D/2+1)}\right]^{1/D} \sim \sum_{i=1}^{D}\frac{\bar{l}}{L_i} = \sum_{i=1}^{D}\frac{1}{N_i}, \tag{26}$$

where $\bar{l} = (V_D/N)^{1/D}$ is the average distance between the particles and $N_i = L_i/\bar{l}$ represents the particle number in a row along the i-th side of the container. From Eq.(26) one can see that the finite size in any direction will give rise to the finite-size effects in the system and the total finite-size effects may be considered as the sum of the finite-size effects in each direction. The order of Eq.(26) is about $1/N^D$ if the side lengths of the system are not significantly different.

4. The case of weak degeneracy

In the case of weak degeneracy, $z_0 \ll 1$ and $f_l(z_0)$ can be expanded as

$$f_l(z_0) = \sum_{j=1}^{\infty}(-1)^{j-1}\frac{z_0^j}{j^l} = z_0 - \frac{z_0^2}{2^l} + \cdots. \tag{27}$$

If one only keeps the first power of $\rho_D\lambda^D$, then one can derive the expressions of μ, E, p_{ii} and C_{L_i} by substituting Eq.(27) into Eqs.(11) and (13)–(16), which are respectively, given by

$$\mu = k_{\rm B}T\left\{\ln(\rho_D\lambda^D) + \frac{1}{2^{D/2}}\rho_D\lambda^D + \frac{1}{2}\left(1 + \frac{2-2^{1/2}}{2^{D/2}}\rho_D\lambda^D\right)\frac{\lambda}{\tilde{L}}\right\}, \tag{28}$$

$$E = \frac{D}{2} N k_{\rm B} T \left\{ 1 + \frac{1}{2^{D/2+1}} \rho_D \lambda^D + \frac{1}{2D} \left[1 + \frac{2 - 2^{1/2}}{2^{D/2+1}} (D+1) \rho_D \lambda^D \right] \frac{\lambda}{\tilde{L}} \right\}, \tag{29}$$

$$p_{ii} = \frac{N k_{\rm B} T}{V_D} \left\{ 1 + \frac{1}{2^{D/2+1}} \rho_D \lambda^D + \frac{1}{2} \left[\frac{\tilde{L}}{L_i} + \frac{2 - 2^{1/2}}{2^{D/2+1}} \left(1 + \frac{\tilde{L}}{L_i} \right) \rho_D \lambda^D \right] \frac{\lambda}{\tilde{L}} \right\}, \tag{30}$$

and

$$C_{L_i} = \frac{D}{2} N k_{\rm B} \left\{ 1 - \frac{D-2}{2^{D/2+2}} \rho_D \lambda^D + \frac{1}{4D} \left[1 - \frac{2 - 2^{1/2}}{2^{D/2+1}} (D^2 - 1) \rho_D \lambda^D \right] \frac{\lambda}{\tilde{L}} \right\}, \tag{31}$$

where $\rho_D = N/V_D$ is the D-dimensional density of particles.

In the case of $\rho_D \lambda^D \to 0$, equations (28)–(31) are reduced respectively to

$$\mu = k_{\rm B} T \left[\ln(\rho_D \lambda^D) + \frac{\lambda}{2\tilde{L}} \right], \tag{32}$$

$$E = \frac{D}{2} N k_{\rm B} T \left(1 + \frac{\lambda}{2\tilde{L}} \right), \tag{33}$$

$$p_{ii} = \frac{N k_{\rm B} T}{V_D} \left(1 + \frac{\lambda}{2L_i} \right), \tag{34}$$

and

$$C_{L_i} = \frac{D}{2} N k_{\rm B} \left(1 + \frac{\lambda}{4\tilde{L}} \right). \tag{35}$$

Equations (32)–(35) are just the results for the finite-size systems of ideal Boltzman's gases. When $D = 3$ is chosen, some interesting results obtained in Ref.[3] can be directly derived from Eqs.(32)–(34).

It is seen from the above results that the corrections due to the finite-size effects are of the order of

$$\lambda/\tilde{L} = (\rho_D \lambda^D)^{1/D} \left(\sum_{i=1}^{D} \frac{1}{N_i} \right) \ll \sum_{i=1}^{D} \frac{1}{N_i}, \tag{36}$$

in the case of $\rho_D \lambda^D \ll 1$ and hence is negligible. Therefore, it seems less important to consider the finite-size effects in the Boltzman's gases.

5. Conclusions

We have studied the finite-size effects in a D-dimensional ideal Fermi gas based on the Euler–MacLaurin formula and some important conclusions are obtained as follows: (i) the properties of a finite-size Fermi system are closely dependent on the size of the system and shape of the boundary; (ii) the corrections of the thermodynamic quantities due to the finite-size effects are of the order of $\sum_{i=1}^{D} \bar{l}/L_i = \sum_{i=1}^{D} 1/N_i$ for the case of strong degeneracy but negligible for the case of weak degeneracy or non-degeneracy; (iii) the pressure of the system is, in general, anisotropic in the small system with the anisotropic geometric boundary; (iv) the extensive property under the thermodynamic limit is eliminated due to the finite-size effects.

Appendix

Equation (2) may be rewritten as

$$\ln \Xi = \sum_{n_1=1}^{\infty} \sum_{n_2=1}^{\infty} \cdots \sum_{n_D=1}^{\infty} \ln \left[1 + z \exp \left(-\beta \varepsilon_0 \sum_{i=1}^{D} \gamma_i^2 n_i^2 \right) \right], \tag{A1}$$

where $\gamma_i = V_D^{1/D}/L_i$, $V_D = (\prod_{i=1}^{D} L_i)$ and $\varepsilon_0 = h^2/(8mV_D^{2/D})$. Using the Euler–MacLaurin formula given by Eq.(3) and considering only the first and second terms in Eq.(3), we have

$$\begin{aligned} & \sum_{n_1=1}^{\infty} \ln \left[1 + z \exp \left(-\beta \varepsilon_0 \sum_{i=1}^{D} \gamma_i^2 n_i^2 \right) \right] \\ = & \sum_{n_1=0}^{\infty} \ln \left[1 + z \exp \left(-\beta \varepsilon_0 \sum_{i=1}^{D} \gamma_i^2 n_i^2 \right) \right] - \ln \left[1 + z \exp \left(-\beta \varepsilon_0 \sum_{i=2}^{D} \gamma_i^2 n_i^2 \right) \right] \\ = & \int_0^{\infty} \ln \left[1 + z \exp \left(-\beta \varepsilon_0 \sum_{i=1}^{D} \gamma_i^2 n_i^2 \right) \right] {\rm d}n_1 - \frac{1}{2} \ln \left[1 + z \exp \left(-\beta \varepsilon_0 \sum_{i=2}^{D} \gamma_i^2 n_i^2 \right) \right], \end{aligned} \tag{A2}$$

$$\sum_{n_1=1}^{\infty}\sum_{n_2=1}^{\infty}\ln\left[1+z\exp\left(-\beta\varepsilon_0\sum_{i=1}^{D}\gamma_i^2n_i^2\right)\right]$$
$$=\int_0^{\infty}\int_0^{\infty}\ln\left[1+z\exp\left(-\beta\varepsilon_0\sum_{i=1}^{D}\gamma_i^2n_i^2\right)\right]\mathrm{d}n_1\mathrm{d}n_2$$
$$-\frac{1}{2}\left\{\begin{array}{l}\int_0^{\infty}\ln\left\{1+z\exp\left[-\beta\varepsilon_0\left(\sum\limits_{i=3}^{D}\gamma_i^2n_i^2+\gamma_1^2n_1^2\right)\right]\right\}\mathrm{d}n_1\\+\int_0^{\infty}\ln\left\{1+z\exp\left[-\beta\varepsilon_0\left(\sum\limits_{i=3}^{D}\gamma_i^2n_i^2+\gamma_2^2n_2^2\right)\right]\right\}\mathrm{d}n_2\end{array}\right\}$$
$$+\frac{1}{4}\ln\left[1+z\exp\left(-\beta\varepsilon_0\sum_{i=3}^{D}\gamma_i^2n_i^2\right)\right], \tag{A3}$$

$$\cdots$$

$$\sum_{n_1=1}^{\infty}\sum_{n_2=1}^{\infty}\cdots\sum_{n_D=1}^{\infty}\ln\left[1+z\exp\left(-\beta\varepsilon_0\sum_{i=1}^{D}\gamma_i^2n_i^2\right)\right]$$
$$=\int_0^{\infty}\int_0^{\infty}\cdots\int_0^{\infty}\ln\left[1+z\exp\left(-\beta\varepsilon_0\sum_{i=1}^{D}\gamma_i^2n_i^2\right)\right]\prod_{i}^{D}\mathrm{d}n_i$$
$$-\frac{1}{2}\sum_{j=1}^{D}\int_0^{\infty}\int_0^{\infty}\cdots\int_0^{\infty}\ln\left[1+z\exp\left(-\beta\varepsilon_0\sum_{i=1}^{D}{}'\gamma_i^2n_i^2\right)\right]\prod_{i=1}^{D}{}'\mathrm{d}n_i$$
$$+\frac{1}{4}\sum_{j>k=1}^{D}\int_0^{\infty}\int_0^{\infty}\cdots\int_0^{\infty}\ln\left[1+z\exp\left(-\beta\varepsilon_0\sum_{i=1}^{D}{}''\gamma_i^2n_i^2\right)\right]\prod_{i=1}^{D}{}''\mathrm{d}n_i$$
$$+\cdots$$
$$+\left(-\frac{1}{2}\right)^{D-1}\sum_{l=1}^{D}\int_0^{\infty}\ln[1+z\exp(-\beta\varepsilon_0\gamma_l^2n_l^2)]\mathrm{d}n_l$$
$$+\left(-\frac{1}{2}\right)^{D}\ln(1+z), \tag{A4}$$

where $\sum\limits_{i=1}^{D}{}'$ and $\prod\limits_{i=1}^{D}{}'$ represent the sum and product over $i\neq j$, respectively, and $\sum\limits_{i=1}^{D}{}''$ and $\prod\limits_{i=1}^{D}{}''$ represent the sum and product over $i\neq j$ and $i\neq k$, respectively.

Letting $\beta\varepsilon_0\gamma_i^2n_i^2=X_i^2$ and $r=(\sum_{i=1}^{D}X_i^2)^{1/2}$, one has

$$\int_0^{\infty}\int_0^{\infty}\cdots\int_0^{\infty}\ln\left[1+z\exp\left(-\beta\varepsilon_0\sum_{i=1}^{D}\gamma_i^2n_i^2\right)\right]\prod_{i}^{D}\mathrm{d}n_i$$
$$=\frac{1}{(\beta\varepsilon_0)^{D/2}}\int_0^{\infty}\int_0^{\infty}\cdots\int_0^{\infty}\ln\left[1+z\exp\left(-\sum_{i=1}^{D}X_i^2\right)\right]\prod_{i}^{D}\mathrm{d}X_i$$
$$=\frac{DC_D}{2^D(\beta\varepsilon_0)^{D/2}}\int_0^{\infty}\ln[1+z\exp(-r^2)]\,\mathrm{d}r$$
$$=\frac{V_D}{\lambda^D}f_{D/2+1}(z), \tag{A5}$$

where $C_D=\pi^{D/2}/\Gamma(D/2+1)$, $\lambda=h/\sqrt{2\pi mk_{\mathrm{B}}T}$ and $f_l(z)$ is the Fermi integral given by Eq.(5). Similarly,

$$\sum_{j=1}^{D}\int_0^{\infty}\int_0^{\infty}\cdots\int_0^{\infty}\ln\left[1+z\exp\left(-\beta\varepsilon_0\sum_{i=1}^{D}{}'\gamma_i^2n_i^2\right)\right]\prod_{i=1}^{D}{}'\mathrm{d}n_i=\frac{V_D}{\lambda^{D-1}}\left(\sum_{j=1}^{D}\frac{1}{L_j}\right)f_{D/2+1/2}(z), \tag{A6}$$
$$\sum_{j>k=1}^{D}\int_0^{\infty}\int_0^{\infty}\cdots\int_0^{\infty}\ln\left[1+z\exp\left(-\beta\varepsilon_0\sum_{i}^{D}{}''\gamma_i^2n_i^2\right)\right]\prod_{i=1}^{D}{}''\mathrm{d}n_i=\frac{V_D}{\lambda^{D-2}}\left(\sum_{j>k=1}^{D}\frac{1}{L_jL_k}\right)f_{D/2}(z), \tag{A7}$$

$$\cdots$$

$$\sum_{l=1}^{D}\int_0^{\infty}\ln\left[1+z\exp\left(-\beta\varepsilon_0\gamma_l^2n_l^2\right)\right]\mathrm{d}n_l=\frac{1}{\lambda}\left(\sum_{l=1}^{D}L_l\right)f_{3/2}(z), \tag{A8}$$

$$\ln(1+z)=f_1(z). \tag{A9}$$

Substituting Eqs.(A5)–(A9) into Eq.(A4), we get the grand partition function as

$$\ln\varXi=\frac{V_D}{\lambda^D}f_{D/2+1}(z)-\frac{1}{2}\frac{V_D}{\lambda^{D-1}}\left(\sum_{j=1}^{D}\frac{1}{L_j}\right)f_{D/2+1/2}(z)+\frac{1}{4}\frac{V_D}{\lambda^{D-2}}\left(\sum_{j>k=1}^{D}\frac{1}{L_jL_k}\right)f_{D/2}(z)$$
$$+\cdots+\left(-\frac{1}{2}\right)^{D-1}\frac{1}{\lambda}\left(\sum_{l=1}^{D}L_l\right)f_{3/2}(z)+\left(-\frac{1}{2}\right)^{D}f_1(z). \tag{A10}$$

When the first-order correction due to the finite-size effects is only considered, we obtain Eq.(4) from Eq.(A10).

References

[1] Huang K 1963 *Statistical Mechanics* (New York: Wiley) p224

[2] Pathria R K 1972 *Statistical Mechanics* (Oxford: Pergamon) p215

[3] Sisman A and Muller I 2004 *Phys. Lett.* A **320** 360

[4] Sisman A 2004 *J. Phys.* A: *Math. Gen.* **37** 11353

[5] Pang H, Dai W S and Xie M 2006 *J. Phys.* A: *Math. Gen.* **39** 2563

[6] Dai W S and Xie M 2003 *Phys. Lett.* A **311** 340

[7] Dai W S and Xie M 2004 *Phys. Rev.* E **70** 016103

[8] Begun V V and Gorenstein M I 2008 *Phys. Rev.* C **77** 064903

[9] Nash C and O'Connor D 1999 *Ann. Phys.* (NY) **273** 72

[10] Leboeuf P and Monastra A G 2002 *Ann. Phys.* (NY) **297** 127

[11] Chamati H 2008 *J. Phys.* A: *Math. Theor.* **41** 375002

[12] Sun J R, Wei Y N and Pu F C 1995 *Acta Phys. Sin.* (Overseas Edition) **4** 542

[13] Wu S Q, Wang S J, Sun W L and Yu W L 2004 *Chin. Phys.* **13** 510

[14] Cui H T, Wang L C and Yi X X 2004 *Acta Phys. Sin.* **53** 991 (in Chinese)

[15] Abramowitz M and Stegun C A 1972 *Handbook of Mathematical Functions with Formulas, Graphs, and Mathematical Tables* (New York: Dover Press)

[16] Li M, Chen L, Chen J, Yan Z and Chen C 1999 *Phys. Rev.* A **60** 4168