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INTERVAL INCLUSION COMPUTATION FOR THE SOLUTIONS OF THE BURGERS EQUATION*

QUN LIN† AND LUNG-AN YING‡

Abstract. In this paper we study the interval computation for the solutions of the Burgers equation. For the initial-boundary value problems of the Burgers equation by using the technique of the Green function, a new kind of interval method is proposed. Both algorithm and computational examples are given. Convergence is proved. From the results we see that this interval method can get a better solution with our corroboration.

Key words. interval computation, Burgers equation, Green function

AMS subject classifications. 65M99, 35Q53

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1. Introduction.

The problem of convection-diffusion conservation laws is very important for a variety of physical phenomena in fluid dynamics. Up to now there has been a lot of efficient work in this area. The finite difference methods are based on regular grids; their TVD properties can be presented by means of numerical flux designs. This information may be found in books such as Kroner [7]. The finite volume methods and discontinuous Galerkin methods can be set on unstructured grids and are of the same good properties as finite difference methods. In addition the a priori error estimations with L^1-norm are taken into account. This information may be found in Kroner [7] and Cockburn [2]. From the works such as by Cockburn [3] and Olhberger [8], [15], [16], a posteriori error estimations related to L^1-norm were established as monitors in terms of the discrete solution and data that measure the actual computational errors without the knowledge of the exact solution, which enable us to determine a grid adaptive strategy to refine or coarsen the mesh according to the changes of those monitors for practical computation of the approximate schemes.

This paper will also focus on the same kind of problems, that is, the Burgers equation. But our aim and result are different from those above. Our aim is to search an interval which contains the exact solution. To our knowledge, interval computation methods were mainly restricted to the algebraic areas in past studies. At the present time we are going to apply them to the partial differential equations. There is still limited research results for this area. In [10], [12], [13], [14], some computable interval algorithms are constructed. In [11], [17], [6], engineering applications of interval computations are considered. In [20], [21], computer-assisted proofs by using interval analysis appear for mathematical theory such as periodic orbits and Smale's 14th problem.

On the other hand, besides a posteriori error computation, our result is to introduce a new kind of approach for the interval methods, which consists of two parts:

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†School of Mathematical Sciences, Xiamen University, Xiamen, 361005, People's Republic of China (linqun@xmu.edu.cn).

‡Corresponding author. School of Mathematical Sciences, Xiamen University, Xiamen, 361005, People's Republic of China and School of Mathematical Sciences, Peking University, Beijing, 100871, People's Republic of China (yingla@xmu.edu.cn).

both an approximate solution and a set of pointwise intervals covering the exact solution. To obtain the intervals we evaluate the errors. The error calculation is explicit, so it is different from the a priori error estimations. To guarantee the intervals also cover the exact solution, we calculate the upper and lower bounds of the solution pointwisely. Pointwise upper bounds and pointwise lower bounds can be used to draw figures pointwisely. Thus we can "look at" the solution under an arbitrary scale in the case that the exact solution cannot be expressed, so this error calculation is different from the conventional a posteriori error estimations too. The key idea of our method is deriving an implicit formula for the solution at first. Using the Green function we are able to derive this formula. With the aid of this formula, we design a convergent numerical scheme and a set valued mapping around the approximate solution. Thus we can provide a clearer region image covering the figure of the exact solution.

Although our approach is restricted to the Burgers equation for definiteness, we believe that this approach can be extended to a class of equations. Moreover, the Green function representations have some advantages in computation according to our experiments. We will give some comparison on theory and computation between our schemes described in section 2 and some fashionable approximate methods. The related conclusions will be discussed in other papers.

The paper is organized as follows. We derive an equivalence form for the initial-boundary value problem in the next section, and then we present our approximate schemes with the proof of convergence. In section 3 we present two interval computation schemes with the proof of the intervals covering the exact solution. In section 4 some auxiliary results regarding the properties of the Green function are listed for our algorithm. Then truncations of error are derived which are necessary for the computing of the intervals. Finally we summarize the above schemes and present the complete algorithm. In section 6 we show the numerical examples to verify the efficiency of the method.

2. Approximation schemes.

2.1. An equivalence form. In this paper we deal with the initial-boundary value problem of the Burgers equation as follows:

$$\left\{\begin{array}{l} \frac{\partial u}{\partial t}+u\frac{\partial u}{\partial x}=a\frac{\partial^2 u}{\partial x^2}, \\ u(x,0)=u_0(x), \quad x\in[0,1], \\ u(0,t)=u(1,t)=0. \end{array}\right. \tag{1}$$

For the following initial-boundary value problem of the heat equation

$$\frac{\partial v}{\partial t}=a\frac{\partial^2 v}{\partial x^2}, \quad v(x,0)=v_0(x), \quad v(0,t)=v(1,t)=0 \tag{2}$$

both odd continuation and periodic continuation are used to get

$$\begin{array}{l} \frac{\partial v}{\partial t}=a\frac{\partial^2 v}{\partial x^2}, \\ v(x,0)=\varphi(x)=\left\{\begin{array}{ll} v_0(x-2n), & x\in(2n,\ 2n+1), \\ -v_0(-x+2n), & x\in(2n-1,\ 2n), \end{array}\right. \\ n=0,\ \pm1,\ \pm2,\ \dots . \end{array}$$

By means of the fundamental solution

$$K(x,\ t)=\frac{1}{2\sqrt{\pi a t}}e^{-\frac{x^2}{4at}}$$

we can obtain

$$v(x,t) = \sum_{n=-\infty}^{+\infty} \left(\int_0^1 K(x-\xi-2n,\ t)\, v_0(\xi)d\xi \ - \int_0^1 K(x+\xi-2n,\ t)\, v_0(\xi)d\xi \right).$$

The Green function is defined by

$$G(x,\ \xi,\ t) = \sum_{n=-\infty}^{+\infty} \{\, K(x-\xi-2n,\ t) - K(x+\xi-2n,\ t)\,\}$$

and thus

$$v(x,t) = \int_0^1 G(x,\ \xi,\ t)\, v_0(\xi)d\xi. \tag{3}$$

By Duhamel's principle it follows that for the problem (1)

$$u(x,t) = \int_0^1 G(x,\xi,t)\ u_0(\xi)d\xi \ - \int_0^t \int_0^1 G(x,\xi,t-\tau)\ \left(u\frac{\partial u}{\partial \xi}\right)_{\xi,\tau} d\xi d\tau. \tag{4}$$

Based upon integration by parts, we have

$$u(x,t) = \int_0^1 G(x,\xi,t)\ u_0(\xi)d\xi \ + \frac{1}{2}\int_0^t \int_0^1 G_\xi(x,\xi,t-\tau)\ u^2(\xi,\tau)\, d\xi d\tau.$$

2.2. Discrete solutions. Given a partition

$$\begin{aligned} &X_j = [x_{j-1},\ x_j], \qquad \bigcup_{j=1}^{N} X_j = [0,1], \qquad \Delta x_j = x_j - x_{j-1},\\ &T_k = [t_{k-1},\ t_k], \qquad \bigcup_{k} T_k = [0,\ T], \qquad \Delta t_k = t_k - t_{k-1} \end{aligned}$$

we introduce the piecewise linear space

$$V = \{\ f \in C[0,1];\ \ f|_{X_j} \in P_1(X_j)\,\}$$

and linear interpolation operator $\Pi : C[0,1] \to V$. Then we define an interval

$$[w(x)]_k := [w_k^-(x), w_k^+(x)] \qquad \forall k \geq 0. \tag{5}$$

Assume that there exist $m_{k-1} \in V$, $[w]_{k-1,j}$, $j = 1,\ldots,N$, such that the exact solution of (1) satisfies

$$u(x,t_{k-1}) \in m_{k-1}(x) + [w(x)]_{k-1},\ \ x \in [0,1].$$

Our methods are recursively to find m_k and $[w(x)]_k$ such that

$$u(x,t_k) \in m_k(x) + [w(x)]_k,\ \ x \in [0,1]. \tag{6}$$

This section is devoted to the schemes of the approximate solutions m_k, and the intervals $[w(x)]_k$ will be designed in section 3.

2.3. Scheme 1. We set

$$u_1(x,t) = \int_0^1 G(x,\xi,t-t_{k-1})\, m_{k-1}(\xi) d\xi \ + \frac{1}{2}\int_{t_{k-1}}^t \int_0^1 G_\xi(x,\xi,t-\tau)\, u_1^2(\xi,\tau) d\xi d\tau, \quad t > t_{k-1},$$
$$m_k = \Pi\, u_1(\,\cdot\,,\ t_k). \tag{7}$$

This is Scheme 1, which is unable to be solved explicitly, but it is a basis of the following scheme.

2.4. Scheme 2. For this scheme we set the predictor

$$u_2(x,t) = \int_0^1 G(x,\xi,t-t_{k-1})\, m_{k-1}(\xi) d\xi \ + \frac{1}{2}\int_{t_{k-1}}^t \int_0^1 G_\xi(x,\xi,t-\tau)\, m_{k-1}^2(\xi) d\xi d\tau \tag{8}$$

and the corrector

$$u_3(x,t) = \int_0^1 G(x,\xi,t-t_{k-1})\, m_{k-1}(\xi) d\xi$$
$$+ \frac{1}{4}\int_{t_{k-1}}^t \int_0^1 G_\xi(x,\xi,t-\tau)\, \{m_{k-1}^2(\xi) \ + \ ((\Pi u_2)(\xi,t_k))^2\}\, d\xi d\tau,$$
$$m_k = \Pi\, u_3(\,\cdot\,,\ t_k). \tag{9}$$

2.5. Convergence. We prove the convergence of the above schemes. For the sake of simplicity we assume that the lengths of spacial steps and time steps are constants in this section, $\Delta x_j = \Delta x$ and $\Delta t_k = \Delta t$. It is straightforward to generalize the results to variable lengths. We denote by C a generic constant in the following, which may depend on different parameters but is independent of the mesh size.

THEOREM 2.1. *Let T be an arbitrary positive number. We assume that the solution u to* (1) *is sufficiently smooth and $\triangle x^\alpha \le C\Delta t,\ \alpha < 2$; then*

$$|u - u_1| \le C\triangle x^{2-\alpha}\, e^{CT} \tag{10}$$

on the domain $[0,1]\times[0,T]$, where the constant C depends on the solution u.

Proof. (1) and (7) imply that

$$\frac{\partial(u-u_1)}{\partial t} \ + (u-u_1)\frac{\partial u}{\partial x} \ + \ u_1\frac{\partial(u-u_1)}{\partial x} \ = a\frac{\partial^2(u-u_1)}{\partial x^2}.$$

Since u is smooth, $\left|\frac{\partial u}{\partial x}\right| \le C$. We take the transformation $v = e^{C(t-t_k)}(u-u_1)$; then from the maximum norm principle it follows that

$$\max_{x\in[0,1]} |v(x,t_k)| \le \max_{x\in[0,1]} |v(x,t_{k-1})|\,,$$

$$\max_{x\in[0,1]} |(u-u_1)(x,t_k)| \le \ e^{C\Delta t_k} \max_{x\in[0,1]} |(u-u_1)(x,t_{k-1})|\,.$$

On the other hand,

$$\max_{x\in[0,1]} |\,\Pi(u-u_1)(x,t_k)| \le \ \max_{x\in[0,1]} |(u-u_1)(x,t_k)|\,.$$

Thus

$$\max_{x\in[0,1]} |\,\Pi(u-u_1)(x,t_k)\,| \\ \leq e^{C\Delta t_k}\left\{\max_{x\in[0,1]} |\,\Pi(u-u_1)(x,t_{k-1})| + \max_{x\in[0,1]} |\,(u-\Pi u)(x,t_{k-1})\,|\right\}.$$

Since the solution u is smooth, that is, $\left|\frac{\partial^2 u}{\partial x^2}\right|$ is bounded, we have $|u-\Pi u|\leq C\triangle x^2$. It is obtained recursively that

$$\max_{x\in[0,1]} |\,\Pi(u-u_1)(x,t_k)| \leq \; C\triangle x^2\left(1+e^{C\Delta t_k}+\cdots+e^{C\Delta t_1}\right)\leq C\triangle x^{2-\alpha}\,e^{CT}.$$

Thus the proof is complete. ∎

THEOREM 2.2. *Let T be an arbitrary positive number. We assume that $\triangle x\to 0$, $\triangle t\to 0$, and $\frac{\triangle x^2}{\triangle t}\to 0$; then the solutions u_3 and u_2 converge uniformly on the domain $[0,1]\times[0,T]$ to the solution of the problem* (1). *Moreover, they also converge weakly in $L^2(0,T;H^2(0,1))$ and $H^1(0,T;L^2(0,1))$.*

Proof. Let $[t_{k-1},t_k]\subset[0,T]$. By (8) we have the equation

(11)
$$\frac{\partial u_2}{\partial t}+m_{k-1}m'_{k-1}=a\frac{\partial^2 u_2}{\partial x^2},\quad u_2(x,t_{k-1})=m_{k-1}(x),\quad x\in[0,1],\quad t\in[t_{k-1},t_k).$$

Multiply it by $\frac{\partial u_2}{\partial t}$, and take the integral over the domain $[0,1]\times[t_{k-1},t_k)$. We assume a priori that $|u_3|\leq M$, where the constant M will be fixed later on. We have the following:

$$a\int_{t_{k-1}}^{t_k}\int_0^1\frac{\partial u_2}{\partial t}\frac{\partial^2 u_2}{\partial x^2}\,dx\,dt=-\frac{a}{2}\int_0^1\left(\frac{\partial u_2}{\partial x}\right)^2 dx\Bigg|_{t_{k-1}}^{t_k-0}$$

and

$$\left|\int_{t_{k-1}}^{t_k}\int_0^1\frac{\partial u_2}{\partial t}m_{k-1}m'_{k-1}\,dx\,dt\right| \\ \leq\frac{1}{2}\int_{t_{k-1}}^{t_k}\int_0^1\left(\frac{\partial u_2}{\partial t}\right)^2 dx\,dt+\frac{M^2}{2}\int_{t_{k-1}}^{t_k}\int_0^1(m'_{k-1})^2\,dx\,dt.$$

Therefore it holds that

$$\frac{1}{2}\int_{t_{k-1}}^{t_k}\int_0^1\left(\frac{\partial u_2}{\partial t}\right)^2 dx\,dt+\frac{a}{2}\int_0^1\left(\frac{\partial u_2}{\partial x}\right)^2 dx\Bigg|_{t_k-0}\leq\left(\frac{a}{2}+\frac{M^2}{2}\triangle t\right)\int_0^1(m'_{k-1})^2\,dx.$$

Since m_k is a piecewise linear interpolation of $u_3(x,t_k)$, the inequality

$$\int_0^1(m'_{k-1})^2\,dx\leq\int_0^1\left(\frac{\partial u_3(x,t_{k-1})}{\partial x}\right)^2 dx$$

holds. We obtain an estimate as follows:

$$\frac{1}{2}\int_{t_{k-1}}^{t_k}\int_0^1\left(\frac{\partial u_2}{\partial t}\right)^2 dx\,dt+\frac{a}{2}\int_0^1\left(\frac{\partial u_2}{\partial x}\right)^2 dx\Bigg|_{t_k-0} \\ \leq\frac{a}{2}\int_0^1\left(\frac{\partial u_3}{\partial x}\right)^2 dx\Bigg|_{t_{k-1}-0}+\frac{M^2}{2}\triangle t\int_0^1(m'_{k-1})^2\,dx. \tag{12}$$

By (9) we have the equation

$$\tag{13} \begin{aligned} &\frac{\partial u_3}{\partial t} + \frac{1}{2} m_{k-1} m'_{k-1} + \frac{1}{2} \Pi u_2(x, t_k - 0) \frac{\partial \Pi u_2(x, t_k - 0)}{\partial x} = a \frac{\partial^2 u_3}{\partial x^2}, \\ &u_3(x, t_{k-1}) = m_{k-1}(x), \quad x \in [0,1], \quad t \in [t_{k-1}, t_k), \end{aligned}$$

for u_3. Then similar to (12) and following the same argument we obtain the estimate for u_3 that

$$\tag{14} \begin{aligned} &\frac{1}{2} \int_{t_{k-1}}^{t_k} \int_0^1 \left(\frac{\partial u_3}{\partial t} \right)^2 dx\, dt + \frac{a}{2} \int_0^1 \left(\frac{\partial u_3}{\partial x} \right)^2 dx \bigg|_{t_k - 0} \\ &\leq \left(\frac{a}{2} + \frac{M^2}{4} \triangle t \right) \int_0^1 (m'_{k-1})^2\, dx + \frac{M^2}{4} \triangle t \int_0^1 \left(\frac{\partial u_2}{\partial x} \right)^2 dx \bigg|_{t_k - 0}. \end{aligned}$$

By plugging (12) into (14) we get

$$\tag{15} \begin{aligned} &\frac{1}{2} \int_{t_{k-1}}^{t_k} \int_0^1 \left(\frac{\partial u_3}{\partial t} \right)^2 dx\, dt + \frac{a}{2} \int_0^1 \left(\frac{\partial u_3}{\partial x} \right)^2 dx \bigg|_{t_k - 0} \\ &\leq \frac{a}{2} \int_0^1 \left(\frac{\partial u_3}{\partial x} \right)^2 dx \bigg|_{t_{k-1} - 0} + \left(\frac{M^2}{2} \triangle t + \frac{M^4}{4a} \triangle t^2 \right) \int_0^1 (m'_{k-1})^2\, dx. \end{aligned}$$

Taking the sum with respect to k leads to

$$\tag{16} \begin{aligned} &\frac{1}{2} \int_0^{t_k} \int_0^1 \left(\frac{\partial u_3}{\partial t} \right)^2 dx\, dt + \frac{a}{2} \int_0^1 \left(\frac{\partial u_3}{\partial x} \right)^2 dx \bigg|_{t_k - 0} \\ &\leq \sum_{j=1}^{k} \left(\frac{M^2}{2} \triangle t + \frac{M^4}{4a} \triangle t^2 \right) \int_0^1 (m'_{j-1})^2\, dx + \frac{a}{2} \int_0^1 (m'_0)^2\, dx, \end{aligned}$$

where the derivatives with respect to t are defined on the open set $\bigcup_j (t_{j-1}, t_j)$. We apply the discrete Gronwall's inequality and obtain an estimate as follows:

$$\tag{17} \int_0^1 (m'_k)^2\, dx \leq e^{\frac{M^2 t_k}{a} \left(1 + \frac{M^2}{2a} \triangle t\right)} \int_0^1 (m'_0)^2\, dx,$$

which is denoted by $|m_k|_1 \leq K_1(M,T) |m_0|_1$, where

$$K_1(M,T) = e^{\frac{M^2 T}{2a} \left(1 + \frac{M^2}{2a} \triangle t\right)}.$$

Then we apply it to (11) and (13) to obtain

$$\int_0^{t_k} \int_0^1 \left(\frac{\partial^2 u_2}{\partial x^2} \right)^2 dx\, dt + \int_0^{t_k} \int_0^1 \left(\frac{\partial^2 u_3}{\partial x^2} \right)^2 dx\, dt \leq K_2(M,T) |m_0|_1^2.$$

Owing to the embedding theorem of the Sobolev spaces, we get

$$\|u_2\|_{L^2(0,t_k,C^1[0,1])} + \|u_3\|_{L^2(0,t_k,C^1[0,1])} \leq K_3(M,T) |m_0|_1.$$

Differentiate (11) and (13) with respect to t, and denote $v = \frac{\partial u_2}{\partial t}$ or $v = \frac{\partial u_3}{\partial t}$. Then v satisfies the heat equation,

$$\frac{\partial v}{\partial t} = a\frac{\partial^2 v}{\partial x^2}.$$

Multiply it by v, and integrate over the domain $[0,1] \times (t, t_k)$; then we have the estimate for the L^2-norm:

$$\int_0^1 v^2(x, t_k - 0)\, dx \le \int_0^1 v^2(x,t)\, dx, \qquad t \in (t_{k-1}, t_k).$$

Then we integrate it over the interval (t_{k-1}, t_k) for t, take the sum with respect to k, and obtain

$$\sum_{j=1}^{k} \int_0^1 v^2(x, t_j - 0)\, dx(t_j - t_{j-1}) \le \int_0^{t_k} \int_0^1 v^2(x,t)\, dx\, dt.$$

Noting (11) and (13) again we find the following is also true:

$$\sum_{j=1}^{k} \int_0^1 \left(\frac{\partial^2 u_2(x, t_j - 0)}{\partial x^2}\right)^2 dx(t_j - t_{j-1}) + \sum_{j=1}^{k} \int_0^1 \left(\frac{\partial^2 u_3(x, t_j - 0)}{\partial x^2}\right)^2 dx(t_j - t_{j-1}) \le K_2(M,T)|m_0|_1^2.$$

We set $M_0 = \max_x |u_0(x)|$, $M_1 = |u_0|_1$, and $d_{k-1} = \max_x |\frac{\partial \Pi u_2(x, t_k - 0)}{\partial x}| + \max_x |m'_{k-1}(x)|$. Then we define a variable transform $u_2 = v_2 + M d_{k-1}(t - t_{k-1})$ in (11) and get the equation for v_2 as

$$\frac{\partial v_2}{\partial t} + M d_{k-1} + m_{k-1} m'_{k-1} = a\frac{\partial^2 v_2}{\partial x^2}.$$

Noting $v_2(0,t) = v_2(1,t) \le 0$, we get by applying the maximum principle that $v_2 \le \max_x |m_{k-1}(x)|$. The estimate for the lower bound is the same. Thus we have

$$|u_2| \le \max_x |m_{k-1}(x)| + M d_{k-1}(t - t_{k-1}).$$

Since Π is a piecewise linear interpolation operator, the inequality $\max|\Pi u_2| \le \max|u_2|$ holds. Applying the maximum principle and following the same lines we get the bound for u_3:

$$|u_3| \le \max_x |m_{k-1}(x)| e^{\frac{1}{2} d_{k-1}(t - t_{k-1})} + \frac{1}{2} M d_{k-1}(t - t_{k-1}) + \frac{1}{2} M d_{k-1}^2 (t - t_{k-1})^2.$$

By induction, we get

$$|u_3(x, t_k)| \le e^{\frac{1}{2}\sum_{j=1}^{k} d_{j-1}(t_j - t_{j-1})} \left\{ \max_x |m_0(x)| + \frac{1}{2} M \sum_{j=1}^{k} d_{j-1}(t_j - t_{j-1}) + \frac{1}{2} M \sum_{j=1}^{k} d_{j-1}^2 (t_j - t_{j-1})^2 \right\}.$$

The Cauchy–Schwarz inequality leads to

$$\sum_{j=1}^{k} d_{j-1}(t_j - t_{j-1}) \le \left(\sum_{j=1}^{k} d_{j-1}^2 (t_j - t_{j-1})\right)^{1/2} \left(\sum_{j=1}^{k} (t_j - t_{j-1})\right)^{1/2}$$
$$\le K_3^{1/2}(M,T)|m_0|_1 t_k^{\frac{1}{2}}.$$

Therefore

$$(18)\qquad |u_3(x,t_k)| \le e^{\frac{1}{2}K_3^{\frac{1}{2}}(M,T)|m_0|_1 t_k^{\frac{1}{2}}} \left\{ \max_x |m_0(x)| + \frac{1}{2} M K_3^{\frac{1}{2}}(M,T)|m_0|_1 t_k^{\frac{1}{2}} + \frac{1}{2} M K_3(M,T)|m_0|_1^2 \triangle t \right\}.$$

Letting $M > M_0$, the equality

$$M = e^{\frac{1}{2}K_3^{\frac{1}{2}}(M,T)M_1(T')^{\frac{1}{2}}} \left\{ M_0 + \frac{1}{2} M K_3^{\frac{1}{2}}(M,T) M_1 (T')^{\frac{1}{2}} + \frac{1}{2} M K_3(M,T) M_1^2 T' \right\}$$

defines a positive T':

$$(19)\qquad T' = K_4(M, T, M_0, M_1).$$

Now we fix $M = 2M_0$; then for small $\triangle t$ it is easy to see that $|u_3| \le M$ as $t \le t_1$, so the above estimation takes place for small t. Then (18) and (19) show that if $t_k \le T'$, then $|u_3| \le M$, so the above estimation is valid for $t_k \le T'$.

Next, let us study the continuity of the solutions with respect to t. Suppose $t, \tau \in [0, T']$ and $\tau < t$; then

$$u_3(x,t) - u_3(x,\tau) = \int_\tau^t \frac{\partial u_3}{\partial t}\, dt + \sum_{t_k \in (\tau, t)} (\Pi u_3(x, t_k - 0) - u_3(x, t_k - 0)).$$

Then by the Cauchy–Schwarz inequality

$$(20)\qquad \|u_3(\cdot,t) - u_3(\cdot,\tau)\|_0^2 \le C(t-\tau) + C \frac{\triangle x^4}{\triangle t^2} K_2(M,T) M_1^2 (t - \tau + \triangle t).$$

We consider a converging series of meshes. By the estimate (17), there is a subsequence of u_3 converging in L^2 for each $t \in [0, T']$. The uniform continuity (20) implies that there is a subsequence converging uniformly with respect to t, that is, converging in $L^\infty(0, T'; L^2(0,1))$. Again using the estimate (17) and the interpolation between Sobolev spaces, we get the convergence in $L^\infty(0, T'; H^\alpha(0,1))$ with $\alpha \in (0,1)$. Applying the embedding theorem of Sobolev spaces, the convergence in $L^\infty(0, T'; C[0,1])$ is obtained. Thus the subsequence of u_3 converges uniformly on the domain $[0,1] \times [0, T']$. The subsequences of functions m_k and u_2 also converge to the same limit. Moreover, by the estimation, the subsequence can also converge weakly in $L^2(0, T'; H^2(0,1))$ and $H^1(0, T'; L^2(0,1))$.

Let us prove the limit of the subsequence is a weak solution to the problem (1). Taking $\varphi \in C_0^\infty$ and the inner product of φ with (13), we get

$$-\int_0^{T'} \int_0^1 \left\{ u_3 \frac{\partial \varphi}{\partial t} + a u_3 \frac{\partial^2 \varphi}{\partial x^2} \right\} dx\, dt - \sum_k \int_{t_{k-1}}^{t_k} \int_0^1 \frac{1}{4} \left\{ m_{k-1}^2 + (\Pi u_2(x, t_k - 0))^2 \right\} \frac{\partial \varphi}{\partial x}\, dx\, dt$$
$$= \sum_k \int_0^1 (\Pi u_3 - u_3)(x, t_k) \varphi(x, t_k)\, dx.$$

The right-hand side can be estimated as follows:

$$\left|\sum_k \int_0^1 (\Pi u_3 - u_3)(x,t_k)\varphi(x,t_k)\,dx\right| \le \frac{\triangle x^2}{\triangle t} K_2^{1/2}(M,T)|m_0|_1 \|\varphi\|_{L^2(0,T';L^2(0,1))}.$$

By the assumption of the theorem it converges to zero. So the same limit of u_3, u_2, m_k, and Πu_2 is a weak solution. The solution of (1) uniquely exists [9], so the original series converges.

By the maximum norm principle, the solution u to (1) satisfies $|u| \le M_0$. Therefore $|u_3| \le \frac{3}{2}M_0$, provided $\triangle x$ and $\triangle t$ are small enough. We set $M_0' = \frac{3}{2}M_0$, $M_1' = K_1(M,T)M_1$, and $T'' = K_4(M,T,M_0',M_1')$; then the solutions to the scheme can be extended to $[T',T'']$ with the initial data on $t = T'$. Then let $\triangle x$ and $\triangle t$ be small enough so that $|u_3| \le M_0'$ on $[0, T'+T'']$. After finite steps we reach $t = T$, and the proof is complete. ∎

3. Interval computation schemes. We are going to introduce two interval computation schemes. The first one consists of a computation of local a posteriori error computation, which leads to the intervals directly, while for the second one we need to solve two sets of axillary solutions for the upper and lower bounds, and an iterative scheme is applied to improve the results. In our numerical experiments we found that the second one gave sharper results, and thus we will present the algorithm of the second scheme in section 5.

3.1. A posteriori error computation. For the intervals (5) we assume that

$$[w(x)]_k = [w]_{k,j}, \quad x \in X_j,$$

which are constant intervals for each j. Let

$$u_{k-1}^{\pm}(x) = m_{k-1}(x) \ \pm \ w_{k-1,j}, \quad x \in X_j,$$

$$u^{\pm}(x,t) = \int_0^1 G(x,\xi,t-t_{k-1})\, u_{k-1}^{\pm}(\xi)d\xi \ + \frac{1}{2}\int_{t_{k-1}}^t \int_0^1 G_\xi(x,\xi,t-\tau)\,(u^{\pm}(\xi,\tau))^2 d\xi d\tau;$$

then all of u, u^+, and u^- satisfy (1). From the theorem of comparison it follows that

$$u^-(x,t) \le u(x,t) \le u^+(x,t).$$

Let $w_{k,j} = \max_{x\in X_j} \{\,|\,u^+(x,t_k) - m_k(x)\,|, \ \ |\,m_k(x) - u^-(x,t_k)\,|\,\}$; then from scheme 1 we have

$$u(x,t_k) \in m_k(x) \ + \ [w]_{k,j}, \quad x \in X_j. \tag{21}$$

From scheme 2 we note that for $x \in X_j$

$$\left|u^+(x,t_k) - m_k(x)\right| \le \left|u^+(x,t_k) - u_3(x,t_k)\right| + |u_3(x,t_k) - \Pi u_3(x,t_k)|\,,$$
$$\left|u^-(x,t_k) - m_k(x)\right| \le \left|u^-(x,t_k) - u_3(x,t_k)\right| + |u_3(x,t_k) - \Pi u_3(x,t_k)|\,.$$

Here

$$\begin{aligned} u^+(x,t) - u_3(x,t) &= \int_0^1 G(x,\xi,t-t_{k-1})\, w_{k-1}(\xi)d\xi \\ &+\frac{1}{4}\int_{t_{k-1}}^t \int_0^1 G_\xi(x,\xi,t-\tau) \ \left\{\,(u^+(\xi,\tau))^2 - m_{k-1}^2(\xi)\,\right\}\, d\xi d\tau \\ &+\frac{1}{4}\int_{t_{k-1}}^t \int_0^1 G_\xi(x,\xi,t-\tau) \ \left\{ (u^+(\xi,\tau))^2 - (\Pi u_2(\xi,t_k))^2\right\} d\xi d\tau, \end{aligned}$$

where $w_{k-1}(\xi) = w_{k-1,j},\ \xi \in X_j$. We have

$$(u^+(\xi,\tau))^2 - m_{k-1}^2(\xi) = (u^+(\xi,\tau) - m_{k-1}(\xi))^2 + 2(u^+(\xi,\tau) - m_{k-1}(\xi))\, m_{k-1}(\xi),$$

$$(u^+(\xi,\tau))^2 - (\Pi u_2(\xi,t_k))^2 = \left(u^+(\xi,\tau) - \Pi u_2(\xi,t_k)\right)^2 + 2\left(u^+(\xi,\tau) - \Pi u_2(\xi,t_k)\right) \cdot \Pi u_2(\xi,t_k)$$

and

$$u^+(x,t) - m_{k-1}(x) = \left\{ \int_0^1 G(x,\xi,t-t_{k-1})\, m_{k-1}(\xi) d\xi - m_{k-1}(x) \right\} + \int_0^1 G(x,\xi,t-t_{k-1})\, w_{k-1}(\xi) d\xi + \frac{1}{2} \int_{t_{k-1}}^t \int_0^1 G_\xi(x,\xi,t-\tau)\, (u^+(\xi,\tau))^2 d\xi d\tau.$$

Let $M_{k-1} = \max_{x\in[0,1]} \left\{ \left|u_{k-1}^+(x)\right|, \left|u_{k-1}^-(x)\right| \right\}$; then

$$\begin{aligned} |u^+(x,t) - m_{k-1}(x)| &\le \max_{\substack{x\in X_j\\ t\in T_k}} \left| \int_0^1 G(x,\xi,t-t_{k-1})\, m_{k-1}(\xi) d\xi - m_{k-1}(x) \right| \\ &+ \max_{\substack{x\in X_j\\ t\in T_k}} \left| \int_0^1 G(x,\xi,t-t_{k-1})\, w_{k-1}(\xi) d\xi \right| \\ &+ \tfrac{1}{2} M_{k-1}^2 \max_{\substack{x\in X_j\\ t\in T_k}} \int_{t_{k-1}}^t \int_0^1 |G_\xi(x,\xi,t-\tau)|\, d\xi d\tau =: \delta_1. \end{aligned} \tag{22}$$

Thus

$$\left|(u^+(\xi,\tau))^2 - m_{k-1}^2(\xi)\right| \le (2\,|m_{k-1}(\xi)| + \delta_1) \cdot \delta_1.$$

On the other hand, we have

$$\begin{aligned} &u^+(x,t) - u_2(x,t_k) \\ &= \int_0^1 \{G(x,\xi,t-t_{k-1}) - G(x,\xi,t_k-t_{k-1})\}\, m_{k-1}(\xi) d\xi \\ &\quad + \int_0^1 G(x,\xi,t-t_{k-1})\, w_{k-1}(\xi) d\xi \\ &\quad + \frac{1}{2} \int_{t_{k-1}}^t \int_0^1 G_\xi(x,\xi,t-\tau)\ \left\{ (u^+(\xi,\tau))^2 - m_{k-1}^2(\xi) \right\} d\xi d\tau \\ &\quad + \frac{1}{2} \int_{t_{k-1}}^t \int_0^1 \{G_\xi(x,\xi,t-\tau) - G_\xi(x,\xi,t_k-\tau)\}\ m_{k-1}^2(\xi)\, d\xi d\tau \end{aligned}$$

and

$$\begin{aligned} &\left|u^+(x,t) - \Pi u_2(x,t_k)\right| \le \left|u^+(x,t) - u_2(x,t_k)\right| + |u_2(x,t_k) - \Pi u_2(x,t_k)| \\ &\le \max_{\substack{x\in X_j\\ t\in T_k}} \left| \int_0^1 \{G(x,\xi,t-t_{k-1}) - G(x,\xi,t_k-t_{k-1})\}\ m_{k-1}(\xi) d\xi \right| \\ &\quad + \max_{\substack{x\in X_j\\ t\in T_k}} \left| \int_0^1 G(x,\xi,t-t_{k-1})\, w_{k-1}(\xi) d\xi \right| \\ &\quad + \delta_1 \max_{\substack{x\in X_j\\ t\in T_k}} \int_{t_{k-1}}^t \int_0^1 |G_\xi(x,\xi,t-\tau)| \cdot \left|m_{k-1}(\xi)\right| d\xi d\tau \end{aligned}$$

$$
\begin{aligned}
&+\frac{1}{2}\delta_1^2 \max_{\substack{x\in X_j\\ t\in T_k}} \int_{t_{k-1}}^{t}\int_0^1 |G_\xi(x,\xi,t-\tau)|\, d\xi d\tau\\
&+\frac{1}{2}\max_{\substack{x\in X_j\\ t\in T_k}} \int_{t_{k-1}}^{t}\int_0^1 |G_\xi(x,\xi,t-\tau)-G_\xi(x,\xi,t_k-\tau)|\cdot |m_{k-1}(\xi)|^2\, d\xi d\tau \qquad (23)\\
&+\max_{x\in X_j}|u_2(x,t_k)-\Pi u_2(x,t_k)| =: \delta_2.
\end{aligned}
$$

Thus

$$\left|(u^+(\xi,\tau))^2 - (\Pi u_2(\xi,t_k))^2\right| \le (2\,|\Pi u_2(\xi,t_k)|+\delta_2)\cdot \delta_2.$$

Hence, for $x\in X_j$ we have

$$
\begin{aligned}
(24)\quad &|u^+(x,t)-u_3(x,t)| \le \max_{x\in X_j}\left|\int_0^1 G(x,\xi,t-t_{k-1})\, w_{k-1}(\xi) d\xi\right|\\
&\quad+\frac{1}{4}(\delta_1^2+\delta_2^2)\max_{\substack{x\in X_j\\ t\in T_k}} \int_{t_{k-1}}^{t}\int_0^1 |G_\xi(x,\xi,t-\tau)|\, d\xi d\tau\\
&\quad+\frac{\delta_1}{2}\max_{\substack{x\in X_j\\ t\in T_k}} \int_{t_{k-1}}^{t}\int_0^1 |G_\xi(x,\xi,t-\tau)|\cdot |m_{k-1}(\xi)|\, d\xi d\tau\\
&\quad+\frac{\delta_2}{2}\max_{\substack{x\in X_j\\ t\in T_k}} \int_{t_{k-1}}^{t}\int_0^1 |G_\xi(x,\xi,t-\tau)|\cdot |\Pi u_2(\xi,t_k)|\, d\xi d\tau =: \chi_{k,j}.
\end{aligned}
$$

The estimation of $|u^-(x,t)-u_3(x,t)|$ is similar. Therefore from (21) we have the following theorem.

THEOREM 3.1. *Let*

$$w_{k,j} = \chi_{k,j} + \max_{x\in X_j}|u_3(x,t_k)-\Pi u_3(x,t_k)|,$$

where δ_1, δ_2, *and* $\chi_{k,j}$ *are determined, respectively, in* (22), (23), *and* (24); *then*

$$u(x,t_k)\in m_k(x) + [w]_{k,j} := [u]_{k,j}, \quad x\in X_j.$$

3.2. Continuous bounds. For the intervals (5) we assume that $w_k^+, w_k^- \in V$; then $m_k+w_k^+, m_k+w_k^- \in V$. We set

$$\overline{u}_k = m_k + w_k^+, \qquad \underline{u}_k = m_k + w_k^-.$$

Let

$$\underline{u}_0(x) \le u_0(x) \le \bar{u}_0(x), \qquad \underline{u}_0(x), \bar{u}_0(x) \in V \subseteq C[0,1].$$

We suppose further that

$$\underline{u}_{k-1}(x) \le u(x,t_{k-1}) \le \bar{u}_{k-1}(x), \qquad \underline{u}_{k-1}(x),\ \bar{u}_{k-1}(x)\in V.$$

In the following we take u^+ and u^- satisfying

$$\begin{cases} \frac{\partial u^+}{\partial t} + u^+\frac{\partial u^+}{\partial x} = a\frac{\partial^2 u^+}{\partial x^2}, \qquad u^+(0,t)=u^+(1,t)=0,\\ u^+(x,t_{k-1}) = \bar{u}_{k-1}(x) \end{cases}$$

and

$$\begin{cases} \frac{\partial u^-}{\partial t} + u^- \frac{\partial u^-}{\partial x} = a\frac{\partial^2 u^-}{\partial x^2}, & u^-(0,t) = u^-(1,t) = 0, \\ u^-(x,t_{k-1}) = \underline{u}_{k-1}(x). \end{cases}$$

Then by the theorem of comparison it holds that

$$u^-(x,t_k) \le u(x,t_k) \le u^+(x,t_k).$$

The scheme (8) is applied to get the following approximations:

$$\bar{u}(x,t) = \int_0^1 G(x,\xi,t-t_{k-1})\bar{u}_{\,k-1}(\xi)d\xi + \frac{1}{2}\int_{t_{k-1}}^t \int_0^1 G_\xi(x,\xi,t-\tau)\bar{u}_{\,k-1}^{\,2}(\xi)d\xi d\tau \tag{25}$$

and

$$\underline{u}(x,t) = \int_0^1 G(x,\xi,t-t_{k-1})\underline{u}_{\,k-1}(\xi)d\xi \;+\; \frac{1}{2}\int_{t_{k-1}}^t \int_0^1 G_\xi(x,\xi,t-\tau)\underline{u}_{\,k-1}^{\,2}(\xi)d\xi d\tau. \tag{26}$$

It is necessary to compute the error $u^+ - \overline{u}$ and $u^- - \underline{u}$. Since u^+ can be expressed in terms of the Green function as

$$u^+(x,t) = \int_0^1 G(x,\xi,t-t_{k-1})\bar{u}_{\,k-1}(\xi)d\xi + \frac{1}{2}\int_{t_{k-1}}^t \int_0^1 G_\xi(x,\xi,t-\tau)\left(u^+(\xi,\tau)\right)^2 \,d\xi\,d\tau,$$

we obtain the difference

$$u^+(x,t) - \bar{u}(x,t) = \frac{1}{2}\int_{t_{k-1}}^t \int_0^1 G_\xi(x,\xi,t-\tau)\left\{(u^+(\xi,\tau))^2 - \bar{u}_{\,k-1}^{\,2}(\xi)\right\} d\xi\,d\tau.$$

So we need to estimate $u^+(x,t) - \bar{u}_{k-1}(x)$. We have

$$\begin{aligned} u^+(x,t) - \bar{u}_{k-1}(x) &= \int_0^1 G(x,\xi,t-t_{k-1})\,\bar{u}_{k-1}(\xi)\,d\xi - \bar{u}_{k-1}(x) \\ &\quad + \frac{1}{2}\int_{t_{k-1}}^t \int_0^1 G_\xi(x,\xi,t-\tau)\cdot\left\{u^+(\xi,\tau)^2 - \bar{u}_{k-1}(\xi)^2\right\} d\xi d\tau \\ &\quad + \frac{1}{2}\int_{t_{k-1}}^t \int_0^1 G_\xi(x,\xi,t-\tau)\;\bar{u}_{k-1}^2(\xi)d\xi d\tau. \end{aligned}$$

Let

$$\begin{aligned} \bar{f}_{k-1}(x,t) &= \int_0^1 G(x,\xi,t-t_{k-1})\,\bar{u}_{k-1}(\xi)\,d\xi - \bar{u}_{k-1}(x) \\ &\quad + \frac{1}{2}\int_{t_{k-1}}^t \int_0^1 G_\xi(x,\xi,t-\tau)\;\bar{u}_{k-1}^2(\xi)d\xi d\tau; \end{aligned}$$

then we get

$$u^+(x,t) - \bar{u}_{k-1}(x) = \bar{f}_{k-1}(x,t) + \frac{1}{2}\int_{t_{k-1}}^t \int_0^1 G_\xi(x,\xi,t-\tau)\cdot\left\{u^+(\xi,\tau)^2 - \bar{u}_{k-1}^2(\xi)\right\} d\xi\,d\tau. \tag{27}$$

It is a nonlinear integral equation. Since we need only to get an upper bound of the norm, we consider an approximate linear equation. Set $\max_\xi |\bar{u}_{k-1}(\xi)| = \bar{M}_{0,k-1}$, and find ν satisfying

$$\nu(x,t) = |\bar{f}_{k-1}(x,t)| + \bar{M}_{0,k-1} \int_{t_{k-1}}^{t} \int_0^1 |G_\xi(x,\xi,t-\tau)| \nu(\xi,\tau)\, d\xi\, d\tau. \tag{28}$$

The Picard iterative scheme to (28) is

$$\nu^{(l)}(x,t) = |\bar{f}_{k-1}(x,t)| + \bar{M}_{0,k-1} \int_{t_{k-1}}^{t} \int_0^1 |G_\xi(x,\xi,t-\tau)| \nu^{(l-1)}(\xi,\tau)\, d\xi\, d\tau, \quad \nu^{(0)} = 0. \tag{29}$$

We assume that $\triangle t_k$ is small enough so that

$$\bar{\rho}_k := \bar{M}_{0,k-1} \cdot \max_{x\in[0,1]} \left(\int_{t_{k-1}}^{t_k} \int_0^1 |G_\xi(x,\xi,t_k-\tau)|\; d\xi\, d\tau \right) < 1; \tag{30}$$

then it is easy to see that the scheme (29) converges, and $\lim_{l\to\infty} \nu^{(l)} = \nu$. Moreover, we multiply (29) by $|G_\xi(x,\xi,t-\tau)|$ and take integration to obtain

$$\begin{aligned} &\int_{t_{k-1}}^{t} \int_0^1 |G_\xi(x,\xi,t-\tau)| \nu^{(l)}(\xi,\tau)\, d\xi\, d\tau \\ &= \int_{t_{k-1}}^{t} \int_0^1 |G_\xi(x,\xi,t-\tau)| |\bar{f}_{k-1}(\xi,\tau)|\, d\xi\, d\tau \\ &\quad + \bar{M}_{0,k-1} \int_{t_{k-1}}^{t} \int_0^1 |G_\xi(x,\xi,t-\tau)| \int_{t_{k-1}}^{\tau} \int_0^1 |G_{\xi_1}(\xi,\xi_1,\tau-\tau_1)| \nu^{(l-1)}(\xi_1,\tau_1)\, d\xi_1\, d\tau_1\, d\xi\, d\tau. \end{aligned} \tag{31}$$

Then we define a sequence $\bar{Y}^{(l)}(x,t)$, $l = 0,1,2,\ldots$, from the following recursions:

$$\begin{aligned} \bar{Y}^{(l)}(x,t) \geq& \int_{t_{k-1}}^{t} \int_0^1 |G_\xi(x,\xi,t-\tau)| \cdot \left| \int_{t_{k-1}}^{\tau} \int_0^1 G_\tau(\xi,\xi_1,\tau-t_{k-1})\, \bar{u}_{k-1}(\xi_1) \right| d\xi_1 d\tau_1 d\xi d\tau \\ &+ \frac{1}{2} \int_{t_{k-1}}^{t_k} \int_0^1 |G_\xi(x,\xi,t-\tau)| \cdot \int_{t_{k-1}}^{\tau} \int_0^1 |G_\xi(\xi,\xi_1,\tau-\tau_1)|\; \bar{u}_{k-1}^2(\xi_1) d\xi_1 d\tau_1 d\xi d\tau \\ &+ \bar{M}_{0,k-1} \int_{t_{k-1}}^{t_k} \int_0^1 |G_\xi(x,\xi,t-\tau)| \cdot \bar{Y}^{(l-1)}(\xi,\tau) d\xi d\tau \end{aligned} \tag{32}$$

and $\bar{Y}^{(0)}(x,t) = 0$. Then we have the following theorem.

THEOREM 3.2. *Under the assumption* (30) *it holds that*

$$|u^+(x,t) - \overline{u}_{k-1}(x)| \leq |\bar{f}_{k-1}(x,t)| + \bar{M}_{0,k-1} \underline{\lim}_{l\to\infty} \bar{Y}^{(l)}(x,t). \tag{33}$$

Proof. Letting $\varepsilon > 0$ be an arbitrary constant, we consider an axillary equation

$$\nu^\varepsilon(x,t) = |\bar{f}_{k-1}(x,t)| + \bar{M}_{0,k-1} \int_{t_{k-1}}^{t} \int_0^1 |G_\xi(x,\xi,t-\tau)| \nu^\varepsilon(\xi,\tau)\, d\xi\, d\tau + \varepsilon.$$

It is easy to see that $|u^+(x,t_{k-1}) - \overline{u}_{k-1}(x)| \leq \nu^\varepsilon(x,t_{k-1})$. By continuity

$$|u^+(x,t) - \overline{u}_{k-1}(x)| \leq \nu^\varepsilon(x,t) \tag{34}$$

holds for small $t > t_{k-1}$. Let $T_0 = \sup\{t;\ (34) \text{ holds}\}$. We claim that $T_0 = t_k$. If not, $T_0 < t_k$; then

$$|u^+(x,T_0) - \bar{u}_{k-1}(x)| \le |\bar{f}_{k-1}(x,T_0)| \\ + \frac{1}{2}\int_{t_{k-1}}^{T_0}\int_0^1 |G_\xi(x,\xi,T_0-\tau)| \cdot \left| \left\{ u^+(\xi,\tau)^2 - \bar{u}_{k-1}^2(\xi) \right\} \right| d\xi\, d\tau.$$

By the maximum norm principle, $|u^+| \le \bar{M}_{0,k-1}$, and thus

$$\begin{aligned} &|u^+(x,T_0) - \bar{u}_{k-1}(x)| \\ &\quad\le |\bar{f}_{k-1}(x,T_0)| + \bar{M}_{0,k-1}\int_{t_{k-1}}^{T_0}\int_0^1 |G_\xi(x,\xi,T_0-\tau)| \cdot |u^+(\xi,\tau) - \bar{u}_{k-1}(\xi)|\, d\xi\, d\tau \\ &\quad\le |\bar{f}_{k-1}(x,T_0)| + \bar{M}_{0,k-1}\int_{t_{k-1}}^{T_0}\int_0^1 |G_\xi(x,\xi,T_0-\tau)| \nu^\varepsilon(\xi,\tau)\, d\xi\, d\tau = \nu^\varepsilon(x,T_0) - \varepsilon. \end{aligned}$$

By continuity (34) holds for some $t > T_0$, which leads to a contradiction. Since (34) holds for all ε, we take the limit as $\varepsilon \to 0$ and find that it also holds for $\varepsilon = 0$, so

$$|u^+(x,t) - \overline{u}_{k-1}(x)| \le \nu = \lim_{l\to\infty} \nu^{(l)}, \quad x \in [0,1].$$

By induction we find

$$\bar{Y}^{(l)}(x,t) \ge \int_{t_{k-1}}^{t}\int_0^1 |G_\xi(x,\xi,t-\tau)| \nu^{(l)}(\xi,\tau)\, d\xi\, d\tau, \quad x \in [0,1];$$

then it yields

$$\underline{\lim}_{l\to\infty} \bar{Y}^{(l)}(x,t) \ge \int_{t_{k-1}}^{t}\int_0^1 |G_\xi(x,\xi,t-\tau)| \nu(\xi,\tau)\, d\xi\, d\tau, \quad x \in [0,1].$$

Consequently (33) follows from (29), and the proof is complete. □

(32) is an inequality, so we don't need to evaluate the exact integrals and just do some estimating. It is convenient in real computation.

Certainly the procedure (32) should be terminated. To this end we first prove a continuity estimation for the heat equation as the following.

THEOREM 3.3. *Let $I = (x-d, x+d)$, with $0 < d \le +\infty$. Suppose that $|\varphi(\xi)| \le M_0$ on $[0,1]$, $\varphi(0) = \varphi(1) = 0$, and $|\varphi'(\xi)| \le M_1$ on $I \bigcap [0,1]$. Moreover, let*

$$v = \int_0^1 G(x,\xi,t)\, \varphi(\xi)\, d\xi;$$

then for $t_2 > t_1 \ge 0$ we have

$$|v(x,t_2) - v(x,t_1)| \le \sqrt{\frac{a}{\pi}} \left\{ 2M_1 + \frac{e^{-1}}{d} M_0 \right\} \sqrt{t_2 - t_1} + \frac{8\sqrt{2}ae^{-1}}{d^2} M_0 (t_2 - t_1).$$

In particular when $d = +\infty$,

$$|v(x,t_2) - v(x,t_1)| \le 2\sqrt{\frac{a}{\pi}} M_1 \sqrt{t_2 - t_1}.$$

Proof. We take odd extension and periodic extension for φ as in section 2 to obtain that

$$v = \int_{-\infty}^{+\infty} K(x-\xi,t)\,\varphi(\xi)\,d\xi$$

and then

$$v(x,t_2) - v(x,t_1) = a\int_{t_1}^{t_2}\left\{\int_I + \int_{\mathrm{R}\setminus I}\right\}\ \frac{\partial^2 K(x-\xi,t)}{\partial\xi^2}\varphi(\xi)\,d\xi dt$$

where we have

$$\left| a\int_{t_1}^{t_2}\int_I \frac{\partial^2 K(x-\xi,t)}{\partial\xi^2}\varphi(\xi)\,d\xi dt\ \right| \le \sqrt{\frac{a}{\pi}}\left\{2M_1 + \frac{e^{-1}}{d}M_0\right\}\sqrt{t_2-t_1}$$

and

$$\begin{aligned}&\left| a\int_{t_1}^{t_2}\int_{\mathrm{R}\setminus I}\ \frac{\partial^2 K(x-\xi,t)}{\partial\xi^2}\varphi(\xi)\,d\xi dt\ \right|\\ &\quad\le \left| 2a\int_{t_1}^{t_2}\int_d^{+\infty}\ \left\{\frac{\eta^2}{8\sqrt{\pi}(at)^{\frac{5}{2}}} - \frac{1}{4\sqrt{\pi}(at)^{\frac{3}{2}}}\right\} e^{-\frac{\eta^2}{4at}}\ \varphi(x+\eta)\,d\eta dt\ \right|.\end{aligned}$$

It is noted that

$$\int_0^{+\infty}\ \eta^2 e^{-\frac{\eta^2}{4at}}\,d\eta\ = 2at\int_0^{+\infty}\ e^{-\frac{\eta^2}{4at}}\,d\eta,$$

and also using $\max_{x>0}(xe^{-x}) = e^{-1}$, then we continue that

$$\left| a\int_{t_1}^{t_2}\int_{\mathrm{R}\setminus I}\ \frac{\partial^2 K(x-\xi,t)}{\partial\xi^2}\varphi(\xi)\,d\xi dt\ \right| \le\ \frac{8\sqrt{2}a\,e^{-1}}{d^2}M_0\ (t_2-t_1).$$

Therefore we get the result. ∎

Applying Theorem 3.3 we can estimate $\overline{f}_{k-1}$. Let $\overline{M}_{1,k-1} = \max_{\xi\in I}|\overline{u'}_{k-1}(\xi)|$; then

$$\begin{aligned}|\overline{f}_{k-1}| \le& \sqrt{\frac{a}{\pi}}\left(2\bar{M}_{1,k-1} + \frac{e^{-1}}{d}\bar{M}_{0,k-1}\right)\sqrt{t-t_{k-1}}\\ &+ \frac{8\sqrt{2}ae^{-1}}{d^2}\bar{M}_{0,k-1}(t-t_{k-1}) + \frac{1}{2}\bar{\rho}_k\bar{M}_{0,k-1}.\end{aligned} \tag{35}$$

Here we notice that $\overline{M}_{1,k-1}$ depends on d, and we choose different d's for different x's. If x is suited in a neighborhood of a shock wave, $\overline{M}_{1,k-1}$ is large. We prefer not to apply the theorem and simply take

$$|\overline{f}_{k-1}| \le 2\bar{M}_{0,k-1} + \frac{1}{2}\bar{\rho}_k\bar{M}_{0,k-1}. \tag{36}$$

By (35) and (36) we can evaluate the bound:

$$\int_{t_{k-1}}^{t_k}\int_0^1 |G_\xi(x,\xi,t_k-\tau)|\cdot|\overline{f}_{k-1}(\xi,\tau)|\,d\xi\,d\tau \le \overline{M}_2. \tag{37}$$

(31) can be written in the form

$$\mathcal{Y}^{(l)} = \mathcal{F} + L\mathcal{Y}^{(l-1)}, \quad \mathcal{Y}^{(0)} = 0,$$

where L is a linear operator in the space $S = C([0,1] \times [t_{k-1}, t_k])$. We have

$$\mathcal{Y} - \mathcal{Y}^{(l)} = (L^l + L^{l+1} + \cdots)\mathcal{F}.$$

Therefore the conclusion is the following theorem.

THEOREM 3.4. *Under the assumption* (30) *it holds that*

$$\|\mathcal{Y} - \mathcal{Y}^{(l)}\|_S \le \frac{\bar{\rho}_k^l}{1-\bar{\rho}_k} \overline{M}_2. \tag{38}$$

The above theorem gives an error estimate.

Thus for $x \in X_j$ we define

$$\bar{u}_k(x) = \Pi\bar{u}(x,t_k) + \theta_j(x)\max\left\{\bar{\delta}_{k,j-1},\ \bar{\delta}_{k,j}\right\} + (1-\theta_j(x))\max\left\{\bar{\delta}_{k,j},\ \bar{\delta}_{k,j+1}\right\}, \tag{39}$$

where

$$\theta_j(x) = \frac{x - x_{j-1}}{x_j - x_{j-1}}, \quad 0 \le \theta_j(x) \le 1, \quad x \in X_j,$$

(40)

$$\begin{aligned}\bar{\delta}_{k,j} &= \max_{x\in X_j} |\bar{u}(x,t_k) - \Pi\bar{u}(x,t_k)| \\ &\quad + \frac{1}{2}\max_{x\in X_j}\int_{t_{k-1}}^{t_k}\int_0^1 |G_\xi(x,\xi,t_k-\tau)| \cdot \left|(u^+(\xi,\tau))^2 - \bar{u}_{k-1}^2(\xi)\right| d\xi d\tau \\ &\le \max_{x\in X_j} |\bar{u}(x,t_k) - \Pi\bar{u}(x,t_k)| + \overline{M}_{0,k-1}\max_{x\in X_j} Y^{(l)}(x,t_k) + \frac{\bar{\rho}_k^l}{1-\bar{\rho}_k}\overline{M}_{0,k-1}\overline{M}_2.\end{aligned}$$

Analogously we set

$$\begin{aligned}\underline{f}_{k-1}(x,t) &= \int_0^1 G(x,\xi,t-t_{k-1})\,\underline{u}_{k-1}(\xi)\,d\xi - \underline{u}_{k-1}(x) \\ &\quad + \frac{1}{2}\int_{t_{k-1}}^t\int_0^1 G_\xi(x,\xi,t-\tau)\,\underline{u}_{k-1}^2(\xi)d\xi d\tau,\end{aligned}$$

$\underline{M}_{0,k-1} = \max_{\xi\in I}|\underline{u}_{k-1}(\xi)|, \quad \underline{M}_{1,k-1} = \max_{\xi\in I}|\underline{u'}_{k-1}(\xi)|$, and

$$\begin{aligned}\underline{Y}^{(l)}(x,t) &\ge \int_{t_{k-1}}^{t_k}\int_0^1 |G_\xi(x,\xi,t-\tau)| \cdot \left|\int_{t_{k-1}}^{\tau}\int_0^1 G_\tau(\xi,\xi_1,\tau-t_{k-1})\,\underline{u}_{k-1}(\xi_1)\right| d\xi_1 d\tau_1 d\xi d\tau \\ &\quad + \frac{1}{2}\int_{t_{k-1}}^{t_k}\int_0^1 |G_\xi(x,\xi,t-\tau)| \cdot \int_{t_{k-1}}^{\tau}\int_0^1 |G_\xi(\xi,\xi_1,\tau-\tau_1)|\ \underline{u}_{k-1}^2(\xi_1) d\xi_1 d\tau_1 d\xi d\tau \\ &\quad + \underline{M}_{0,k-1}\int_{t_{k-1}}^{t_k}\int_0^1 |G_\xi(x,\xi,t-\tau)| \cdot \underline{Y}^{(l-1)}(\xi,\tau) d\xi d\tau.\end{aligned}$$

Then we can get an estimate

$$\int_{t_{k-1}}^{t_k}\int_0^1 |G_\xi(x,\xi,t_k-\tau)| \cdot |\underline{f}_{k-1}(\xi,\tau)|\,d\xi\,d\tau \le \underline{M}_2.$$

Thus for $x \in X_j$ we define

(41) $$\underline{u}_k(x) = \Pi\underline{u}(x,t_k) - \theta_j(x)\max\left\{\underline{\delta}_{k,j-1},\ \underline{\delta}_{k,j}\right\} - (1-\theta_j(x))\max\left\{\underline{\delta}_{k,j},\ \underline{\delta}_{k,j+1}\right\},$$

where

(42)
$$\begin{aligned}\underline{\delta}_{k,j} &= \max_{x\in X_j}|\underline{u}(x,t_k) - \Pi\underline{u}(x,t_k)| \\ &\quad + \frac{1}{2}\max_{x\in X_j}\int_{t_{k-1}}^{t_k}\int_0^1 |G_\xi(x,\xi,t_k-\tau)|\cdot\left|(u^-(\xi,\tau))^2 - \underline{u}_{k-1}^2(\xi)\right| d\xi d\tau \\ &\le \max_{x\in X_j}|\underline{u}(x,t_k) - \Pi\underline{u}(x,t_k)| + \underline{M}_{0,k-1}\cdot\max_{x\in X_j}\underline{Y}^{(l)}(x,t_k) + \frac{\underline{\rho}_k^l}{1-\underline{\rho}_k}\underline{M}_{0,k-1}\underline{M}_2.\end{aligned}$$

4. Algorithm. Here we will discuss the realistic computation for our interval method. First, some properties of Green functions are considered which are very efficient to our schemes. Then the truncations are needed for integrals and infinite series. Finally we propose our procedure of interval algorithm.

4.1. Properties of Green functions. We need some axillary lemmas about the properties of Green functions of the heat equation (see, for example, [4]) for our algorithm. From these we know that the values of Green functions will be exponentially descending as x becomes infinity, which are very useful in practice to compute Green functions in our schemes. The deduction of them is straightforward and thus omitted.

LEMMA 4.1. *$G(x,\xi,t) \ge 0$; and $0 \le \int_0^1 G(x,\xi,t)\,d\xi \le 1$.*

LEMMA 4.2. *Let $X_j = [a_j,\ b_j] \subset [0,\ 1]$; then if $n_0 \ge 2 + \sqrt{\frac{a\Delta t_k}{2}}$, then*

$$\sum_{n=n_0}^{+\infty}\int_{a_j}^{b_j}|K(x-\xi-2n,\ \Delta t_k) - K(x+\xi-2n,\ \Delta t_k)|\,d\xi \le \frac{b_j-a_j}{2\sqrt{\pi}}\int_{\frac{2n_0-x-a_j-2}{2\sqrt{a\Delta t_k}}}^{\frac{2n_0-x+b_j-2}{2\sqrt{a\Delta t_k}}} e^{-s^2}ds.$$

Also if $n_1 \le -1 - \sqrt{\frac{a\Delta t_k}{2}}$, then

$$\sum_{n=n_1}^{-\infty}\int_{a_j}^{b_j}|K(x-\xi-2n,\ \Delta t_k) - K(x+\xi-2n,\ \Delta t_k)|\,d\xi \le \frac{b_j-a_j}{2\sqrt{\pi}}\int_{\frac{x-a_j-2-2n_1}{2\sqrt{a\Delta t_k}}}^{\frac{x+b_j-2-2n_1}{2\sqrt{a\Delta t_k}}} e^{-s^2}ds.$$

LEMMA 4.3. *Let $X_j = [a_j,\ b_j] \subset [0,\ 1]$, $x \in X_i = [a_i,\ b_i] \subset [0,\ 1]$; then if $n_1 \ge 1 + \sqrt{\frac{a\Delta t_k}{2}}$, then*

$$\begin{aligned}&\sum_{n=n_1}^{+\infty}\int_{t_{k-1}}^{t_k}\int_{a_j}^{b_j}|K_\xi(x-\xi-2n,\ t_k-\tau) - K_\xi(x+\xi-2n,\ t_k-\tau)|\ d\xi d\tau \\ &\quad\le \frac{b_j-a_j}{2\sqrt{\pi a}}\int_{t_{k-1}}^{t_k}\frac{1}{\sqrt{t_k-\tau}}e^{-\frac{(2n_1+2)^2}{4a(t_k-\tau)}}d\tau\end{aligned}$$

and

$$\begin{aligned}&\sum_{n=-n_1}^{-\infty}\int_{t_{k-1}}^{t_k}\int_{a_j}^{b_j}|K_\xi(x-\xi-2n,\ t_k-\tau) - K_\xi(x+\xi-2n,\ t_k-\tau)|\ d\xi d\tau \\ &\quad\le \frac{b_j-a_j}{2\sqrt{\pi a}}\int_{t_{k-1}}^{t_k}\frac{1}{\sqrt{t_k-\tau}}e^{-\frac{(2n_1-4)^2}{4a(t_k-\tau)}}d\tau.\end{aligned}$$

LEMMA 4.4. *It holds that*

$$\int_{t_{k-1}}^{t_k}\int_{a_j}^{b_j} |G_\xi(x,\ \xi,\ t_k-\tau)|\ d\xi d\tau \le \frac{\sqrt{t_k-t_{k-1}}}{2\sqrt{\pi a}}, \quad x\in X_i.$$

4.2. Truncation errors. We continue to study the implementation of schemes 1 and 2. Except for using interval arithmetic to overcome the effect of round off error, we must still apply quadrature schemes to compute the integrals, and some truncations are needed for the infinite series.

1. The quadrature quantities involved in this algorithm are computed by methods such as the Simpson formula.

2. For a quadrature containing some parameter involved in the algorithm, such as $\tilde{G}(x)=\int_{a_j}^{b_j} G(x,\xi,\Delta t_k)d\xi$, in order to find its interval extension, we need to compute the upper bound of the maximum and the lower bound of the minimum for the parameter on $x\in X_j$.

Letting $x\in X_j$, we fix two constants k_1 and k_2 such that $\tilde{G}(x)\le \tilde{G}(a_j)+k_1(x-a_j)$ and $\tilde{G}(x)\le\tilde{G}(b_j)+k_2(x-b_j)$; then we solve $y=\tilde{G}(a_j)+k_1(x-a_j)=\tilde{G}(b_j)+k_2(x-b_j)$ to obtain

$$\bar{x}=\tfrac{\tilde{G}(b_j)-\tilde{G}(a_j)+k_1a_j-k_2b_j}{k_1-k_2},$$
$$\bar{y}=\tfrac{k_1\tilde{G}(b_j)-k_2\tilde{G}(a_j)-k_1k_2(b_j-a_j)}{k_1-k_2}.$$

We compare $\{\ \bar{y},\ \tilde{G}(a_j),\ \tilde{G}(b_j)\ \}$ to get an upper bound.

4.3. Algorithm. Finally we propose an *interval algorithm* as follows:

Let $[u_0](x):=[\underline{u}_0(x),\ \ \bar{u}_0(x)]$ such that $\underline{u}_0(x)\le u_0(x)\le\ \bar{u}_0(x), x\in[0,1]$, and $\underline{u}_0,\bar{u}_0\in V$. Then for $k=1,\ 2,\dots$ we compute along the following steps recursively:

1. Assume that m_{k-1}, $\underline{u}_{k-1}$, and $\bar{u}_{k-1}$ are known. (8) and (9) are applied to compute m_k. Or if we don't need the corrector, we can simply set $m_k=\Pi\, u_2$. In these formulas we use the Simpson formula for integrals, and we truncate the infinite series by the properties of G stated in Lemmas 4.2, 4.3, and 4.4. The errors are obtained.

2. Apply (25) and (26) to get $\bar{u}$ and $\underline{u}$. Errors of the Simpson formula and the truncation of the infinite series are also obtained.

3. We evaluate $\bar{M}_{0,k-1}=\max_\xi|\bar{u}_{k-1}(\xi)|$.

4. Given l, we use (32) to evaluate $\bar{Y}^{(l)}$ by recursion. Using (30) and Lemma 4.4 we evaluate $\bar{\rho}_k$.

5. For a given $x\in[0,1]$, we take a suitable distance $d>0$, evaluate $\bar{M}_{1,k-1}$, then apply (35) to estimate $\bar{f}_{k-1}(x,t)$. Near a shock wave the value of $\bar{M}_{1,k-1}$ may be too large, so we estimate $\bar{f}_{k-1}(x,t)$ from (36) as well and take the smaller one. Then we obtain $\bar{M}_2$ by (37).

6. Analogously we evaluate $\underline{M}_{0,k-1}$, $\underline{Y}^{(l)}$, $\underline{\rho}_k$, and $\underline{M}_2$.

7. We evaluate $\bar{\delta}_{k,j}$ and $\underline{\delta}_{k,j}$ by (40) and (42).

8. We evaluate $\bar{u}_k$ and $\underline{u}_k$ by (39) and (41).

9. Taking the truncation error into account, (40) and (42) need to be modified. In fact the right-hand side of (40) should have three more terms $+R_1+R_2+R_3$, where R_1 is the error of the Simpson formula, R_2 is the error of the truncation of infinite series, and R_3 is the upper bound of the round off error. Being the same, (42) should have three more terms $-R_1-R_2-R_3$. In our numerical computation these are achieved by an interval arithmetic software.

10. Replace k by $k+1$, and return to the first step.

5. Computational examples. In the initial-boundary value problem of the Burgers equation, the initial condition is taken as

$$u(x,0)=\frac{1}{2}\sin\pi x+\sin 2\pi x.$$

Two cases related to different Reynolds numbers are considered.

Case I (Re= $a^{-1}=10$). In this case, equidistant steps are chosen as $\Delta x_j=0.005$ and $\Delta t_k=0.04$.

We first use scheme (8), and $u_2(x,t_k)$ is replaced by $m_k=\Pi\, u_2(x,t_k)$. Some representative results are exhibited in Figure 1 where $k=10,13,20,40$.

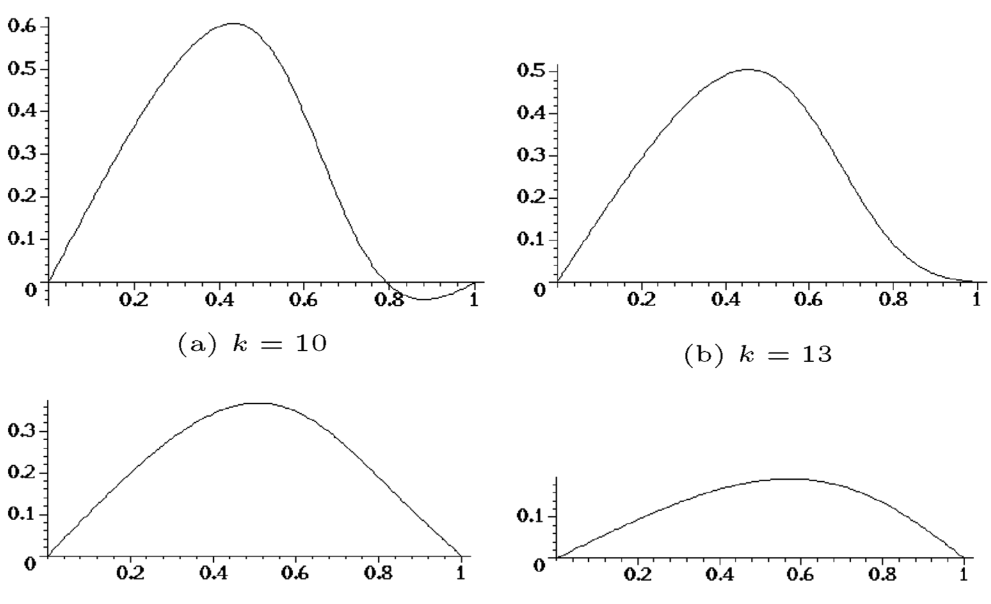
Second, we use the interval algorithm in section 4 combined with the maximum principle at each step. The process of the algorithm depends on the iterative times l and d chosen. On each element the iterated times l are determined to be slightly different so that the widths for u are small enough and then may be adjusted in global to be uniform. Some representative results are exhibited in Figure 2 where $k=4,11,20,30$. From the theory above, the exact solution is in the zonary region.

Case II (Re= $a^{-1}=100$). In this case, different equidistant steps from above are chosen, respectively, as $\Delta x_j=0.004$ and $\Delta t_k=0.006$ in order to guarantee the precision and stability.

Also using scheme (8), some representative results are exhibited in Figure 3 where $k=29,50,99,175$.

Then, by means of the interval algorithm in section 4 combined with the maximum principle at each step, some representative results are exhibited in Figure 4 where $k=29,50,84,157$.

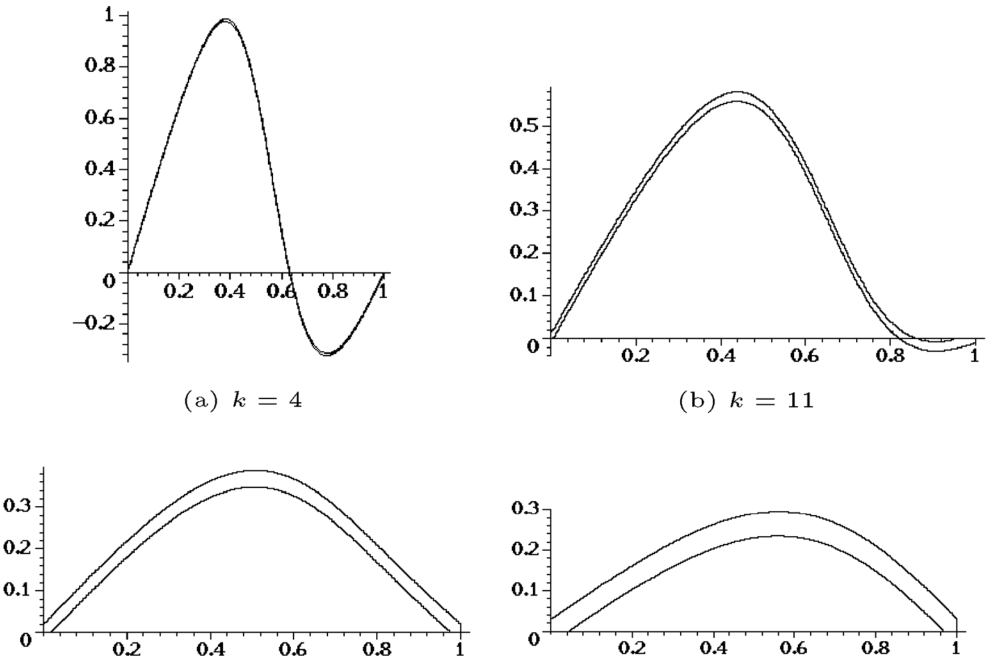
In the case of large Reynolds number, to obtain a perfect result the computational quantities of the present algorithms will increase rapidly. The question of how to improve the efficiency of our algorithms will be answered in our forthcoming papers.



(a) $k=10$ (b) $k=13$

(c) $k=20$ (d) $k=40$

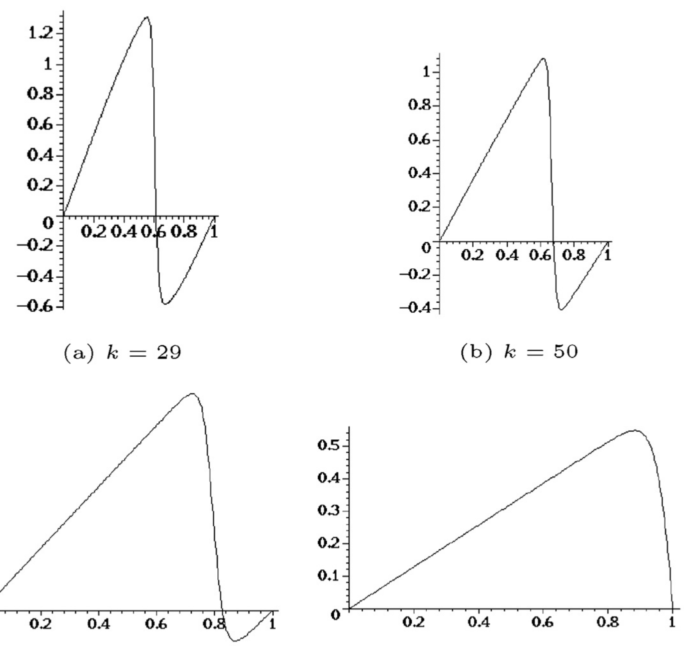
FIG. 1. *Approximate solution when Re*= 10.



(a) $k = 4$ (b) $k = 11$

(c) $k = 20$ (d) $k = 30$

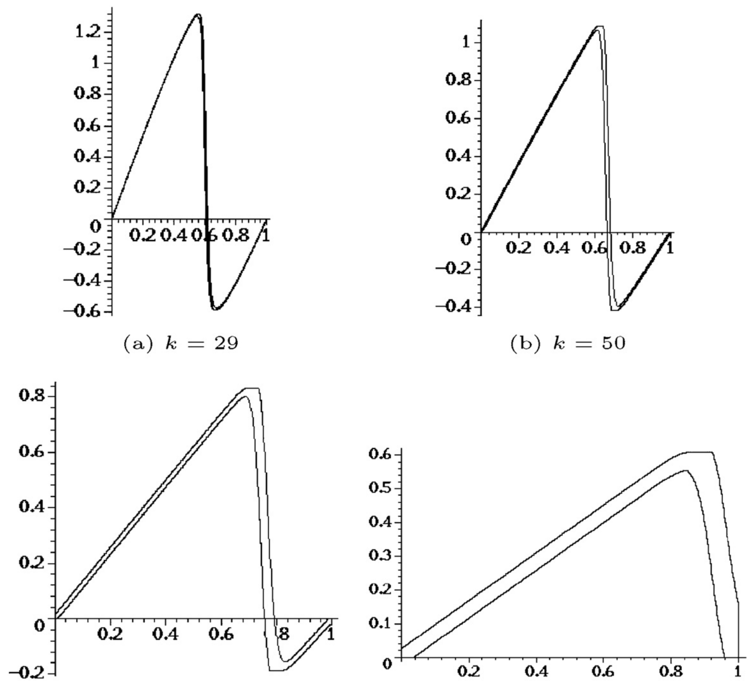
FIG. 2. *Interval solution when Re*= 10.



(a) $k = 29$ (b) $k = 50$

(c) $k = 99$ (d) $k = 175$

FIG. 3. *Approximate solution when Re*= 100.



(a) $k = 29$ (b) $k = 50$

(c) $k = 84$ (d) $k = 157$

FIG. 4. *Interval solution when Re*= 100.

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