Precise linear internal friction expression for a freely decaying vibrational system

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From a linear motion equation, the precise linear internal friction expression for the free decay method has been deduced for the first time. A detailed comparison between the newly deduced expression and the currently used approximation has also been addressed. © *1997 American Institute of Physics.* [S0034-6748(97)01608-0]

I. INTRODUCTION

Internal friction is an important mechanical property in modern materials development as well as in the increasing investigation of seismic wave attenuation of the earth's mantle.[1–6] Most critically, internal friction can nondestructively reveal important structure-property relationships in materials because it is sensitive to the dynamic behavior of defects in the materials under test. However, all internal friction calculation expressions available presently are approximate and only useful for linear behavior at very small values of internal fraction (Q^{-1}), usually below the order of 10^{-3}. Therefore they are not suitable to calculate the high internal frictions that arise in the emerging new materials such as polymers, glasses, amorphous materials, composites, new alloys, nanomaterials, and geomaterials at high temperatures and pressures. Also such approximations in the internal friction values lead to serious discrepancies for the same material measured by different methods, particularly when the internal friction is large.

The freely decaying vibration pendulum is a conventional method widely used in the measurement of low, sonic-frequency internal friction and ultrasonic attenuation.[7] In recent years, some authors[8] have used the approximate expression for Q^{-1} to calculate the very high internal friction measured by a freely decaying vibrational pendulum associated with alloy phase transformation processes, without regard to checking whether or not the internal friction was nonlinear or linear. Others[9] have also used this expression to calculate the nonlinear internal friction of dislocations in metals because no suitably accurate expression is available. Therefore, it is imperative to derive a precise internal friction expression for the free decay method in such situations. For this purpose, we have deduced, in this article, a precise and still very explicit internal friction expression for the free decay method and highlighted, as well, the limitations of the existing approximate expression for the method.

II. ACCURATE LINEAR INTERNAL FRICTION FORMULA FOR FREELY DECAYING VIBRATIONAL SYSTEM

A. Equation of motion and its solution

We consider here a linear equation of motion for a free vibrational system:

$$m\ddot{x}+\eta\dot{x}+kx=0, \tag{1}$$

where the first term is an inertial term, the second comes from a damping term, and the third from a restoring force term. In Eq. (1), it has been assumed that the mass m, the damping coefficient η, and the elastic constant k are all independent of the displacement x, its rate $\dot{x}$, and its acceleration $\ddot{x}$.

Provided that the solution of the Eq. (1) is of a form of e^{-rt}, its characteristic equation should be of the form:

$$mr^2-\eta r+k=0, \tag{2}$$

and its solution should be

$$r_{1,2}=[\eta\pm(\eta^2-4mk)^{1/2}]/(2m). \tag{3}$$

However, the vibrational solution for x occurs only when

$$\eta^2-4mk\leqslant 0$$

or

$$k/m-(\eta/2m)^2\geqslant 0.$$

Let (see the Appendix)

$$\omega=[k/m-(\eta/2m)^2]^{1/2} \tag{4}$$

and

$$\delta=\pi\eta/\omega m. \tag{5}$$

Then Eq. (3) becomes

$$r_{1,2}=(\omega/2\pi)\delta\pm i\omega, \tag{6}$$

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and the solution to Eq. (1) is

$$x = c_1 \exp(-r_1 t) + c_2 \exp(-r_2 t). \tag{7}$$

According to the initial conditions $x|_{t=0} = x_0$ and $\dot{x}|_{t=0} = 0$, parameters c_1 and c_2 can be found to give

$$x = [(-r_2 x_0)/(r_1 - r_2)]\exp(-r_1 t) + [(r_1 x_0)/(r_1 - r_2)] \times \exp(-r_2 t). \tag{8}$$

Substituting Eq. (6) into Eq. (8), we get an accurate solution to Eq. (1):

$$x = x_0 \exp[-(\delta/2\pi)\omega t][1 + (\delta/2\pi)^2]^{1/2} \cos(\omega t - \psi), \tag{9}$$

where $\tan \psi = \delta/2\pi$.

B. Logarithmic decrement

From Eq. (9), the amplitude of the vibrational solution is

$$A(t) = x_0 \exp[-(\delta/2)\omega t][1 + (\delta/2\pi)^2]^{1/2}. \tag{10}$$

Let the time when measuring the amplitude A_n be t_n and the time when measuring the next amplitude A_{n+1} (after one period) be $t_n + T$ (where $T = 2\pi/\omega$). Then, from Eq. (10), we can obtain a ratio of the amplitudes for two successive vibrations as

$$A_n / A_{n+1} = A(t_n)/A(t_n + T) = e^{\delta}$$

or

$$\delta = \ln(A_n / A_{n+1}). \tag{11}$$

A similar procedure can also be applied to any other period: if we let the time when measuring the amplitude A_n be t_n, the time when measuring the amplitude A_{n+m} at m periods after A_n be $t + mT$, or $t + m(2/\omega)$, we can then obtain the ratio of the two measured amplitudes as

$$A_n / A_{n+m} = A(t_n)/A(t_n + mT) = e^{m\delta}$$

or

$$\delta = (1/m)\ln(A_n / A_{n+m}). \tag{12}$$

Equations (11) and (12) are then expressions defining the logarithmic decrement for a freely decaying vibrational system.

C. Internal friction expression for a freely decaying vibrational system

In terms of the original definition of internal friction

$$Q^{-1} = (1/2\pi)(\Delta W / W), \tag{13}$$

where ΔW is the mechanical energy dissipated in a full cycle and W is the system's energy at the beginning of the cycle. Equation (13) implies that the internal friction equals the ratio of the mechanical energy dissipated per arc of the circle to the total energy of the system.

Since the total energy of the system is comprised of kinetic energy and potential energy,

$$W = m\dot{x}^2/2 + kx^2/2,$$

from Eq. (1), we have

$$dW = -\eta \dot{x}^2 dt.$$

The energy dissipated within a period from $t = 0$ to $t = T$ is

$$\Delta W = \int_0^T |dW| = \int_0^T \eta \, \dot{x} \, dt. \tag{14}$$

Here, from Eq. (9), we have

$$\dot{x} = dx/dt = -x_0(1 + \delta^2/4\pi^2)^{1/2} e^{-(\delta/2\pi)\omega t} \omega \sin \omega t / \cos \psi.$$

Further substituting the above result for $\dot{x}$ into Eq. (14) and then integrating we have

$$\Delta W = x_0^2(1 + \delta^2/4\pi^2)\omega\pi/2\delta(1 - e^{2\delta}). \tag{15}$$

Since, at the beginning of a circle, the energy of the system is

$$W = kx_0^2/2, \tag{16}$$

substitution of Eqs. (15) and (16) into Eq. (13), yields

$$Q^{-1} = (\omega\eta/k)(1 + \delta^2/4\pi^2)1/2\delta(1 - e^{2\delta}).$$

From Eqs. (4) and (5), we obtain the relation $(\omega\eta/k)(1 + \delta^2/4\pi^2)(1/2\delta) = 1/2\pi$. Therefore, we finally arrive at

$$Q^{-1} = 1/2\pi(1 - e^{2\delta}). \tag{17}$$

Equation (17) is now a precise expression for the linear internal friction in a freely decaying vibrational system.

From Eqs. (11), (12), and (17), we can also obtain two alternative expressions for Q^{-1}, namely

$$Q^{-1} = 1/2\pi[(A_n^2 - A_{n+1}^2)/A_{n+1}^2] \tag{18}$$

and

$$Q^{-1} = 1/2\pi[(A_n^{2/m} - A_{n+m}^{2/m})/A_{n+m}^{2/m}]. \tag{19}$$

Because at the two successive amplitudes A_n and A_{n+1}, the total energy of the system equals its potential energy and is proportional to the square of its amplitude, Eq. (18) can be construed as

$$Q^{-1} = (1/2\pi)(\Delta W/W), \quad (\Delta W = W_n - W_{n+1}, \quad W = W_n).$$

The above relation is totally compatible with the original definition of internal friction in Eq. (13) and indicates the accuracy of Eq. (17).

Although Eq. (18) can also be obtained directly from the original definition of internal friction [i.e., Eq. (13)] without the solution to the original differential equation [Eq. (1)] and has been known for a long time, it cannot be employed to calculate the internal friction easily because it is difficult to measure accurately the two successive amplitudes, especially at higher frequency. Also, through Eq. (19) as deduced above only involves the two amplitudes separated by m vibration periods, and can avoid the inconvenience of measuring two successive amplitudes, the parameters $A_n^{2/m}$ and $A_{n+m}^{2/m}$ in Eq. (19) are not readily calculated without using numerical methods. Equation (17) is a simple and practical formula to calculate the internal friction and it can be used as long as we calculate the logarithmic decrement first via Eq.

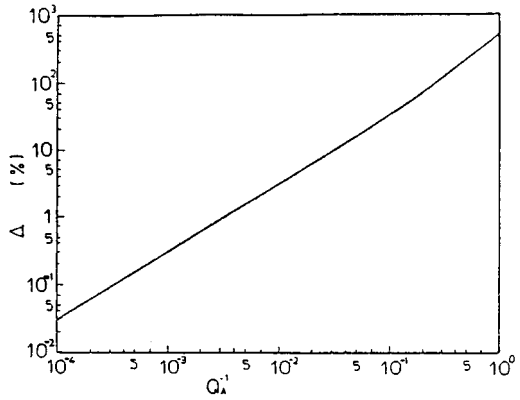

FIG. 1. Variation in relative error $\Delta=(Q_A^{-1}-Q^{-1})Q^{-1}$ with Q_A^{-1}.

(12) and then obtain the accurate internal friction value by substituting the logarithmic decrement into Eq. (17).

III. COMPARISON BETWEEN ACCURATE AND APPROXIMATE FORMULA

It can be seen from Eq. (17) that, if there is no internal friction, $A_n=A_{n+1}$, $\delta=0$, and $Q^{-1}=0$. Conversely, if the internal friction is very large, $A_{n+1}/A_n=0$, $\delta\to\infty$, $Q^{-1}=1/(2\pi)$. Hence, the measurable range of internal friction, theoretically, should never exceed $1/(2\pi)$.

When the internal friction is very small, i.e., $\delta\ll 1$, Eq. (17) can be expanded. Neglecting the second and higher order terms, we can obtain

$$Q_A^{-1}=\delta/\pi. \tag{20}$$

Equation (20) is the approximate equation currently used to calculate the internal friction in the case of freely decaying vibrational measurements.[10]

Equation (20) can be derived from a linear motion equation,[10] but only in the approximate sense of small damping[10] and, therefore, it can only be utilized to calculate small linear internal frictions produced frequently by low-dimensional defects in crystals, e.g., point defects or dislocations. When calculating a larger internal friction, one should consider correcting the higher order terms when using Eq. (20). Since no accurate equation was previously available, such a correction is not straightforward. In recent years, Eq. (20) has often been used to calculate the nonlinear or amplitude-dependent internal friction, often without prior verification of the amplitude dependency of the internal friction under study.[9] In this case, the error is conceptual and may invalidate the results. Indeed, it can be seen that the relative error in using Eq. (20), $\Delta=(Q_A^{-1}-Q^{-1})/Q^{-1}$, increases almost exponentially with Q_A^{-1} as shown in Fig. 1. When $Q_A^{-1}=5\times10^{-3}$, the error has already approached 1%, and when $Q_A^{-1}=0.1$, the error is >50%. Therefore, Eq. (20) is only suitable for calculation of linear internal frictions $<5\times10^{-3}$.

In some recent work,[11,12] the new formula in Eq. (17) has been used to reinterpret some previously reported results which have presented false volume-phase transformation damping mechanisms as derived from Eq. (20). One such case is shown in Figs. 2 and 3. Figure 2 gives the internal

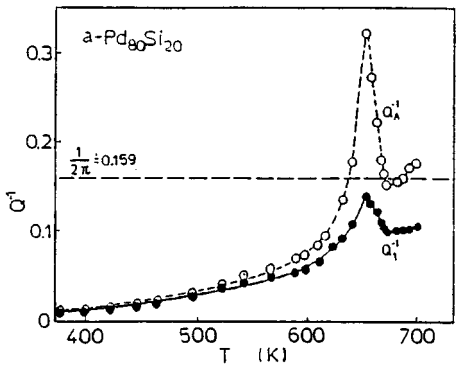

FIG. 2. Internal friction curves with increasing temperature for an amorphous PdSi alloy, which was measured by using a low frequency torsion pendulum.

friction curves with increasing temperature for an amorphous PdSi alloy, as was measured by using a low frequency torsion pendulum. The solid line is the internal friction calculated using Eq. (17) which, overall, is less than $1/(2\pi)$. However, the dotted line represents the internal friction evaluated by using Eq. (20). At or near the internal friction peak (which is associated with the glass transition in the alloy), it exceeds the limit of $1/(2\pi)$. This results are obviously contradictory to the definition of Eq. (13).

Figure 3 gives the dependency of $Q_{\max}^{-1}$ (the peak internal friction in Fig. 2) on v/f, where v is the rate of temperature rise and f the frequency of the specimen at the internal friction peak. Similarly, the solid line is the curve of the internal friction value at peak, $Q_{\max}^{-1}$, obtained by Eq. (17) versus v/f. This is a nonlinear relation and implies an interface mechanism in which the damping is proportional to the interfacial area of the material under transformation in a period $\Delta t(\propto 1/f)$ of vibration, with the corresponding range of temperature being $\Delta T(=v\Delta t\propto v/f)$. The dotted line corresponds to that of the peak value $Q_{\max}^{-1}$, estimated from Eq. (20), which assumes a linear dependence and implies that the internal friction is proportional to the volume of the material transformed in a period of the vibration or in the corresponding temperature range $\Delta T(=v\Delta T)$. Thus the two analysis methods will lead to two absolutely different transformation mechanisms. It is obvious that the linear relation between $Q_{\max}^{-1}$ (process internal friction peak value) and v/f and its

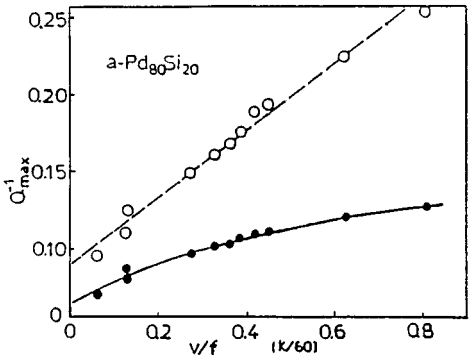

FIG. 3. Relation of value $Q_{\max}^{-1}$ of internal friction peak in Fig. 2 with v/f.

implied volume mechanism are false: the true relation between $Q_{\max}^{-1}$ and v/f is nonlinear, implying an interface mechanism. This nonlinear relation will be expected to reveal the fractal nucleation and growth of a crystalline phase in the amorphous matrix, which will be detailed in a future publication.[11]

Another example of a nonlinear relationship is obtained from an earlier study[12] on the martensite transformation in a MnCu alloy. In this case, the relations between $Q_{\max}^{-1}$ and v/f, v, and $1/f$, measured by a forced-vibration pendulum, are all nonlinear. This nonlinear relation can be confirmed by other experimental evidences[11,12] and can be explained using a twin interface damping mechanism, in the sense that the increase rate of the transforming phase interface area is slower than the rate of increase of the transformed volume for the growth of marstensitic phase. Even so, the volume of the transformed martensitic phase is always proportional to the temperature change $\Delta T(=v\Delta t)$ in a vibration period Δt. Also, the result measured by a forced-vibration pendulum should be experimentally correct since, in this case, the internal friction is calculating using the equation $Q^{-1}=\tan(\varnothing)$ where $\varnothing$ is the phase angle by which the strain of specimen lags the applied stress, and hence is not subject to the error caused by using Eq. (20).

As a consequence of the above, many reports in the literature, concerning relations of $Q_{\max}^{-1}$ versus v, $1/f$, or v/f calculated from Eq. (20) should be carefully checked with Eq. (17) or against a forced-vibration method, as these relations can be used to critically judge the mechanisms of the transformation process giving rise to the measured internal friction peak. Indeed, such a nonlinear relation of internal friction can often occur in phase transitions in many noncrystalline alloys and many crystalline alloys such as MnCu, NiTi, and CuZnAl.

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APPENDIX

From Eqs. (4) and (5), we can obtain

$$k/m=\omega^2(1+\delta^2/4\pi^2).$$

Let the resonant frequency of the system without any damping be $\omega_0=(k/m)^{1/2}$. Thus,

$$\omega_0^2/\omega^2=1+\delta^2/4\pi^2.$$

This latter relation predicts that ω_0, the resonant frequency without damping ($\delta=0$), is always larger than that ω with damping ($\delta\neq 0$). Furthermore, ω will decrease when δ increases. This prediction is compatible with experimental results of damping.[13]

In contrast, the approximate linear motion equation presented by Nowick and Berry[10] gives

$$\omega_0^2/\omega^2=1-\delta^2/4\pi^2,$$

which predicts that the resonant frequency without damping ($\delta=0$) is always lower than that with damping ($\delta\neq 0$): hence ω will increase when damping or δ increases. This prediction is contraction with experimental damping results. Therefore, the approximate formula is unable to describe correctly the experimental behavior of the resonant frequency in material damping.

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