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Simple approach to the creation of a strange nonchaotic attractor in any chaotic system

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A simple approach to the creation of a strange nonchaotic attractor in any chaotic system is described. The main idea is to control the parameter of the system in such a manner that the system dynamics is expanding at some times, but converging at others. With this approach, a strange nonchaotic attractor can be found in a large region in the parameter space near the boundaries between chaotic and regular phases or within the chaotic region far from the regular one. The maximum nontrivial Lyapunov exponent of the system can pass through zero nonsmoothly and cannot be fitted by a linear function. [S1063-651X(99)13805-4]

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I. INTRODUCTION

Recently there has been much interest in the study of strange nonchaotic attractors (SNAs) that are commonly found in quasiperiodically forced systems [1–10]. The geometry of an SNA exhibits a fractal structure, but its typical trajectories do not show sensitive dependence on initial conditions. SNAs have been observed in a number of physical systems [11–13] and one of their potential applications is for secure communications [14,15].

One of the major research topics on SNAs is the study of dynamical routes for the creation of the SNA. A mechanism is investigated by Heagy and Hammel [16]. In quasiperiodically driven maps, when a period-doubled torus collides with its unstable parent torus, it becomes extremely wrinkled and develops into a fractal set at the collision, while the corresponding Lyapunov exponent remains negative. Feudel *et al.* [17] find that the collision between a stable torus and an unstable one at a dense set of points can lead to an SNA. Kaneko and Nishikawa [18,19] describe the fractalization of a torus, namely, the increasing wrinkling of tori that leads to the appearance of an SNA without any interaction with a nearby unstable periodic orbit. Yalcinkaya and Lai [20,21] show that, for systems with an invariant subspace in which there is a torus, the loss of the transverse stability of the torus can lead to the birth of an SNA. A physical phenomenon accompanying this route to SNAs is the on-off intermittency. Prasad *et al.*[22,23] investigate an intermittency route to SNAs that arises in the neighborhood of a saddle-node bifurcation, whereby a strange attractor is replaced by a periodic torus attractor. These reports [16–23] describe the dynamical mechanisms for the creation of SNAs that typically occur in a narrow vicinity along the transition boundary of the chaotic and quasiperiodic phases [23]. It is shown that an SNA can also appear in a narrow area within the periodic region [5,17]. There is another quite different approach to the creation of the SNA. In particular, by controlling the system's parameter, one can drive the system repeatedly to visit regions of chaotic dynamics in the phase space of the autonomous system. Kapitaniak [24] describes a control technique to generate the strange nonchaotic trajectory by making small changes in the parameters of a three-dimensional system. The method is applicable to systems whose periodic attractor has a strange repeller that exhibits transient chaos. In Ref. [25], the implications of fluctuations of finite-time Lyapunov exponents are discussed for nonchaotic systems. For a nonchaotic system driven by a low-frequency quasiperiodical force, the resultant attractor is strange but nonchaotic if its finite-time Lyapunov exponents largely fluctuate to be positive repeatedly.

SNAs can be quantitatively characterized by a variety of methods, including the estimation of Lyapunov exponents and fractal dimension [1,2,26], the spectral properties [2,27], the phase-sensitivity exponent [17,28], and the examination of time series [29]. One of the important observations is that typical trajectories of an SNA are characterized by finite-time Lyapunov exponents that can be positive for sufficiently long time intervals, although the time-independent Lyapunov exponent is asymptotically negative [17,22,23,25,28]. In general, the phase space of a dynamical system can be divided into three typical regions where a trajectory either experiences pure expansion, pure contraction, or remains unchanged. In particular, the pure expanding (contracting) region is the region where an infinitesimal vector in the tangent space expands (contracts) under the dynamics. If the trajectory of a nonchaotic system runs into the expanding region during a time interval τ, the corresponding dynamics is expanding and the time-τ Lyapunov exponent is positive. Meanwhile, the quasiperiodic force (i.e., the two incommensurate sine functions in continuous equations or an irrational-frequency sine function in a discrete map) always provides different values to the system at different times. These values are enlarged individually during different expanding time intervals. As a result, a strange attractor can be created. Furthermore, if the time intervals with pure expanding dynamics are long enough, the system will possess an intermittently bursting behavior, as discussed in Refs. [20–25]. As a consequence, the SNA can be found in any quasiperiodically driven system if its dynamics are repeatedly expanding for sufficiently long time intervals. Thus, to create an SNA in a nonchaotic system, the major problem is how to construct an

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expanding region in its phase space.

Inspired by the works [24,25], a simple controlling approach to the creation of SNAs with any chaotic system is presented here. The main idea is described in Sec. II. One can control the parameter of a system in such a manner that expanding dynamics occur at some times while contracting dynamics appear at others. Simulation results of a quasiperiodically forced logistic map are discussed in Sec. III. The phase diagram and the transition properties from strange nonchaotic to chaotic attractors are studied in detail. The robustness of SNAs of this type against noise is shown in Sec. IV. Discussions are presented in Sec. V.

II. CONTROL METHOD

Consider a system $\mathbf{x}(t+1)=\mathbf{F}(\mathbf{x}(t),\beta)$ with a control parameter β. Suppose that the attractor is chaotic when $\beta=\beta_1$ and periodic when $\beta=\beta_2$. To construct an SNA, the first step is to add a small-amplitude quasiperiodical wave $A\sin(2\pi\omega_1 t)$ to the system. As both the regular and the chaotic attractors are robust against small perturbations [32], the chaotic or nonchaotic property of the system can still be maintained when $A\ll 1$. In particular, now we have a chaotic attractor $\mathbf{x}(t+1)=\mathbf{F}(\mathbf{x}(t),\beta_1)+A\sin(2\pi\omega_1 t)$ with the maximum positive Lyapunov exponent λ_1 and a torus $\mathbf{x}(t+1)=\mathbf{F}(\mathbf{x}(t),\beta_2)+A\sin(2\pi\omega_1 t)$ with the maximum negative Lyapunov exponent λ_2. The second step is to control the parameter β to switch periodically between β_1 and β_2 with time intervals T_1 and T_2, respectively. Then, the system $\mathbf{x}(t+1)=\mathbf{F}(\mathbf{x}(t),\beta(t))+A\sin(2\pi\omega_1 t)$ becomes the chaotic system $\mathbf{x}(t+1)=\mathbf{F}(\mathbf{x}(t),\beta_1)+A\sin(2\pi\omega_1 t)$ or the periodic system $\mathbf{x}(t+1)=\mathbf{F}(\mathbf{x}(t),\beta_2)+A\sin(2\pi\omega_1 t)$ periodically with frequency $\omega_0=1/(T_1+T_2)$. In each T_2 time interval contracting dynamics occur and the corresponding finite-time Lyapunov exponent is negative. In each T_1 time interval chaotically expanding dynamics occur and the finite-time Lyapunov exponent is positive.

Note that $\sin(2\pi\omega_1 t_1)\neq\sin(2\pi\omega_1 t_2)$ for any $t_1\neq t_2$, since ω_1 is irrational. As a result, expanding dynamics for different values of $A\sin(2\pi\omega_1 t)$ in different T_1 time intervals can generate different diverging orbits, and thus the geometric structure of the attractor is strange. The Lyapunov exponent is the average of the exponents of the expanding and contracting dynamics. One can adjust the lengths of the time intervals T_1 and T_2 and let the contracting dynamics be stronger than the expanding ones. The trajectories with different initial conditions are then converged together asymptotically and the maximum nontrivial Lyapunov exponent of the system is negative. The resulting attractor is an SNA. From this point, we deal only with the maximal nontrivial exponents, and refer to these as the Lyapunov exponent.

Unlike the method proposed by Kapitaniak [24], the periodic attractor in our approach does not require strange repellers in order to exhibit transient chaos. Also, in contrast with the on-off or type-I intermittency described in Refs. [20–23], intermittency can be observed periodically with frequency ω_0. In our approach, by employing a suitable square function, an expanding region is constructed in the phase space while the system remains nonchaotic. By applying a quasiperiodic wave, a noiselike signal is provided to lead the trajectory run into different diverging orbits, resulting in a strange structure. The present approach provides a simple picture to show how an attractor can be strange but nonchaotic. As one can always find a parameter to control, SNAs can be constructed in any chaotic system with this approach.

In the following, we suppose that the time intervals T_1 and T_2 are long enough. Then the finite-time maximum nontrivial Lyapunov exponents during these time intervals can be approximated by λ_1 and λ_2, respectively. The maximum nontrivial Lyapunov exponent of the system, i.e., Λ, can then be approximated by

$$\Lambda=\frac{T_1\lambda_1+T_2\lambda_2}{T_1+T_2}. \tag{1}$$

In general, the Lyapunov exponent of the system will depend not only on the values λ_1 and λ_2, but also on the mutual orientation of Lyapunov vectors associated with the maximal nontrivial Lyapunov exponents during the transient process. However, the last will have an influence only on the preexponential factors; apparently, in a typical case, this may be neglected if the periods T_1 and T_2 are long enough. This equation indicates that whether the attractor is chaotic or not is determined by the competition between the expanding rate $T_1\lambda_1$ and the contracting rate $T_2\lambda_2$. When the contracting dynamics dominates, the attractor is an SNA.

Equation (1) shows that, if we fix T_1 and let T_2 be the control parameter, a nonlinear dependence, as governed by Eq. (1), is obtained at the transition of the system from strange nonchaotic to chaotic attractors. If we fix ω_0 and let T_2 be the control parameter, then

$$\Lambda=\lambda_1+\omega_0(\lambda_2-\lambda_1)T_2. \tag{2}$$

The Lyapunov exponent passes through zero linearly with the negative slope $k=\omega_0(\lambda_2-\lambda_1)$. Lai showed that during the transition from strange nonchaotic to chaotic attractors in quasiperiodically driven systems, the Lyapunov exponent often passes through zero linearly [30]. With the present method, if another control parameter is chosen, e.g., β_1 or β_2, the function $\lambda(\beta)$ of the system $\mathbf{x}(t+1)=\mathbf{F}(\mathbf{x}(t),\beta)+A\sin(2\pi\omega_1 t)$ must be considered for Λ. The bifurcation and crisis phenomena in nonlinear systems indicate that the Lyapunov exponent generally has a complex dependence on the control parameter. As a result, the Lyapunov exponent can be nonsmooth at the transition point, which is difficult to fit by a linear function.

III. SIMULATION RESULTS

To show the feasibility of our approach, we consider a quasiperiodically forced logistic map defined as follows:

$$x(t+1)=ax(t)[1-x(t)]+A\sin(2\pi\omega_1 t)+B, \tag{3}$$

with $a=3.6$, $A=0.001$, and $\omega_1=(\sqrt{5}-1)/2$. The attractor is chaotic with nontrivial Lyapunov exponent $\lambda_1=0.172$ (± 0.001) when $B=B_1=0$, while it is a torus with nontrivial Lyapunov exponent $\lambda_2=-0.114$ when $B=B_2=-0.02$. To construct an SNA with the above two attractors, let B be the control parameter, denoted as $B(t)$:

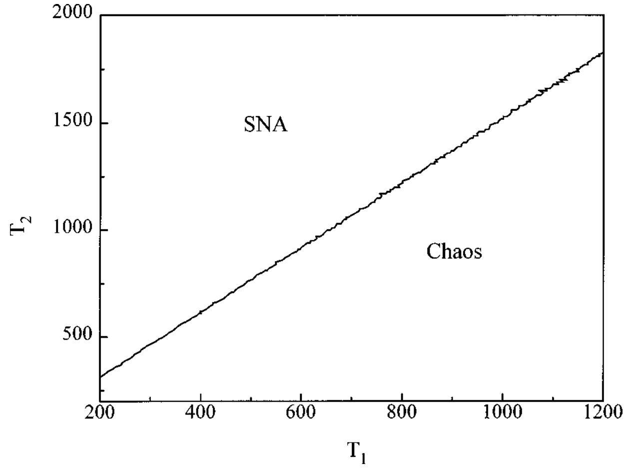

FIG. 1. Phase diagram of the system (4) in the T_1-T_2 plane with $B_1=0$ and $B_2=-0.02$. The slope of the transition line is fitted to be 1.510.

$$x(t+1)=ax(t)[1-x(t)]+A\sin(2\pi\omega_1 t)+B(t),$$
$$B(t)=\begin{cases} B_1 & \text{if} \quad \theta<T_1, \\ B_2 & \text{otherwise,} \end{cases} \tag{4}$$

with $\theta=t\,\mathrm{mod}(T_1+T_2)$. If T_1 and T_2 are large enough, the Lyapunov exponent can be approached by Eq. (1). [Driven by two periodic forces, system (4) is a three-dimensional map. The two additional Lyapunov exponents are zero, associated with the periodic forces.] Simulation results show that, compared with the exact values of the Lyapunov exponent, the results given by Eq. (1) have an error smaller than 6% when $T_1,T_2\approx 500$ and an error smaller than 2% when $T_1,T_2\approx 1000$.

According to Eq. (1), the phases of the SNA and the chaotic attractor are separated by the transition line $T_2=1.51T_1$. The phase diagram of the system (4) is calculated in the T_1-T_2 plane and shown in Fig. 1. As expected, a transition line is obtained with the slope $k=1.510$ (± 0.001). According to Eq. (2), if we fix $T_1+T_2=1000$ and make T_2 adjustable, the Lyapunov exponent should pass through zero linearly with $k=-2.86\times 10^{-4}$ when T_2 is approximately 600. Simulation results given in Fig. 2(a) show that the transition occurs at $T_2=606$ with $k=-2.89\times 10^{-4}$. If we fix $T_1=500$ and vary T_2, a nonlinear dependence of the Lyapunov exponent at the zero-crossing region should occur when T_2 is approximately 755. Figure 2(b) shows that the transition appears at $T_2=769$.

As expected, an SNA is obtained with $\Lambda=-0.005$ when $T_1=450$ and $T_2=750$, while a chaotic attractor with $\Lambda=0.006$ is obtained when $T_1=500$ and $T_2=700$. Their geometric structures in the x-θ plane are given in Fig. 3. It is shown that the trajectory is converging during the T_2 time intervals, while a strange structure is created during the T_1 time intervals. The geometric properties of the SNA shown in Fig. 3(a) are similar to those of the chaotic attractor presented in Fig. 3(b), in both the regular and the strange parts. Their dynamical difference arises only from the different time intervals of the contracting and expanding dynamics.

In general, if all of its Lyapunov exponents are zero, the attractor is a torus. It is shown that the attractor in skew

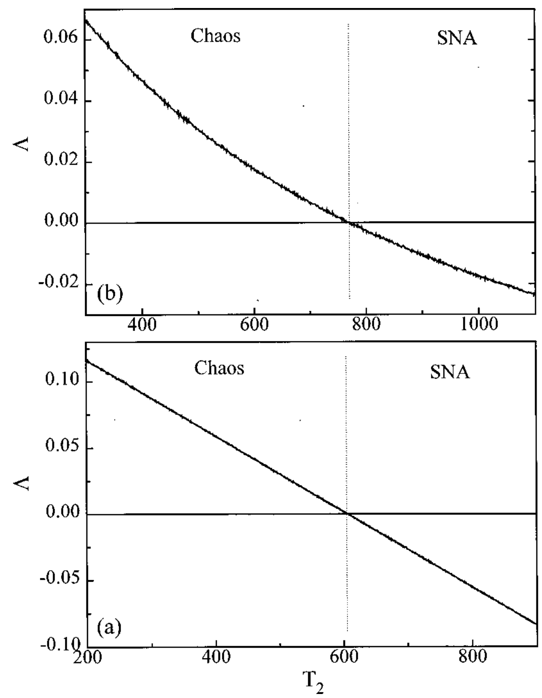

FIG. 2. Near the transition from chaos to the SNA, the Lyapunov exponent versus the control parameter T_2 for (a) $T_1+T_2=1000$ and (b) $T_1=500$.

translation systems can be strange, while all of the Lyapunov exponents are zero [31]. For such an attractor, the rate of the expanding dynamics that corresponds to the creation of strange structure equals the contracting rate so that all the Lyapunov exponents are zero. Theoretically, the attractor at the transition point between the chaotic attractor and the SNA is strange, while all the Lyapunov exponents are zero for a quasiperiodically driven logistic map. It is a little dif-

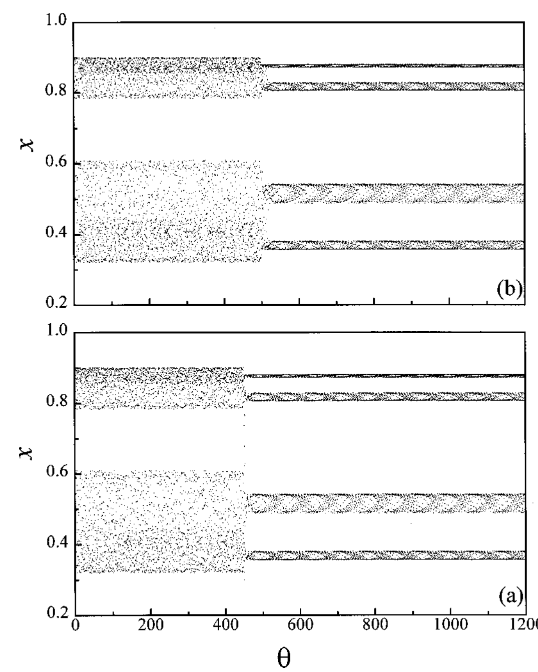

FIG. 3. (a) Strange nonchaotic trajectory in the x-θ plane with $T_1=450$ and $T_2=750$. (b) Chaotic trajectory in the x-θ plane with $T_1=500$ and $T_2=700$.

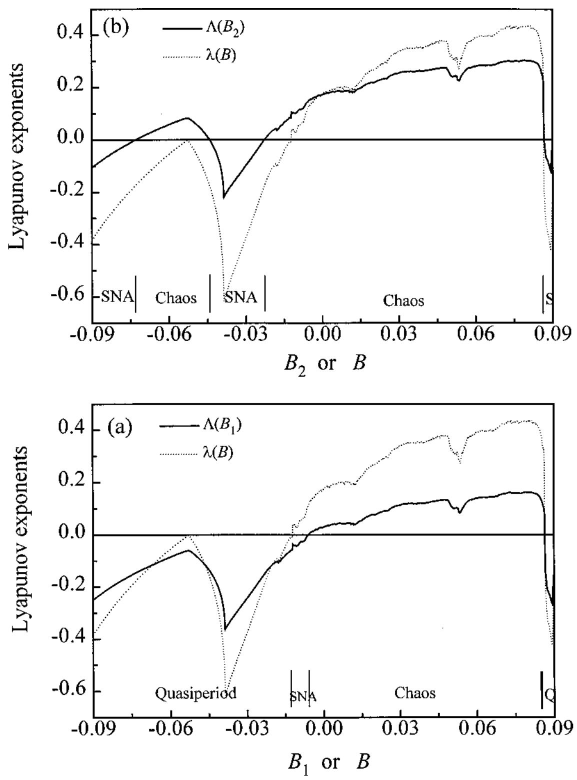

FIG. 4. (a) A plot of $\Lambda(B_1)$ versus B_1 with $B_2=-0.02$ for system (4). (b) A plot of $\Lambda(B_2)$ versus B_2 with $B_1=0$. Here, $T_1=T_2=1000$. For comparison, a plot of $\lambda(B)$ versus B for the system (3) is also given in the figure.

ficult to obtain such an attractor numerically with the methods used in Refs. [16–23], due to fluctuation of the Lyapunov exponents in numerical simulations, as pointed out in Refs. [16,23]. However, the transition line in Fig. 1 suggests that it is easily generated with the present method.

Now we discuss the properties of Eq. (4) with other control parameters. For example, consider B_1. The Lyapunov exponent is expressed as

$$\Lambda(B_1)=\Lambda_0+\beta\lambda_1(B_1), \qquad (5)$$

with $\Lambda_0=T_2\lambda_2/(T_1+T_2)$ and $\beta=T_1/(T_1+T_2)$. The fact that λ_1 is still positive when $\Lambda(B_1)$ just passes through zero from the chaotic phase indicates that the chaotic and regular phases are always separated by the SNA phase. Equation (5) shows that, near those points B with $\lambda(B)=-T_2\lambda_2/T_1$, the attractor always has a transition between strange nonchaotic and chaotic attractors. A similar conclusion is drawn as B_2 varies. In Figs. 4(a) and 4(b), plots of $\Lambda(B_1)$ versus B_1 with $B_2=-0.02$ and $\Lambda(B_2)$ versus B_2 with $B_1=0$ are presented with $T_1=T_2=1000$, respectively. For comparison, a plot of $\lambda(B)$ versus B for Eq. (3) is also given in the figure. One can see that $\Lambda(B_1)$ and $\Lambda(B_2)$ are quite similar to $\lambda(B)$. For the example corresponding to Fig. 4(a), as $\lambda_2=-0.114$, the SNA phase occurs only in a narrow vicinity between chaotic and quasiperiodic attractors with $0.086\,521<B_1<0.086\,526$, although it cannot be shown in the figure. In Fig. 4(b), we fix $\lambda_1=0.172$ so that the structure of the attractor is always strange. Therefore the attractors are always strange and nonchaotic once $\Lambda(B_2)<0$.

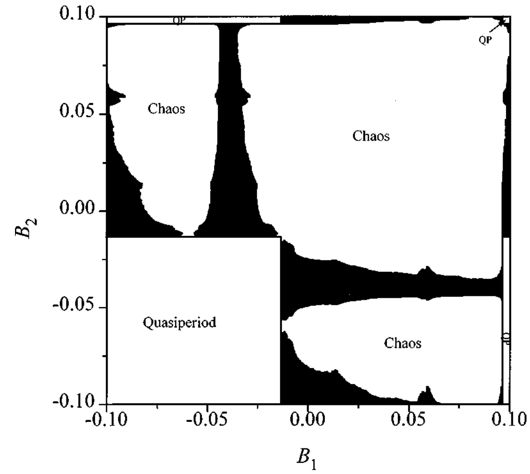

FIG. 5. Phase diagram of the system (4) in the B_1-B_2 plane with $T_1=T_2$. The shaded region corresponds to the SNA. Here QP denotes the torus.

For a set of large T_1 and T_2, the phase diagram of Eq. (4) in the B_1-B_2 plane can be easily obtained from $\lambda(B)$ using Eq. (1). For those systems with $T_1=T_2$, the phase diagram is given in Fig. 5, in which the shaded region corresponds to the SNA. Because $T_1=T_2$, we have

$$\Lambda(B_1,B_2)=\tfrac{1}{2}[\lambda(B_1)+\lambda(B_2)]=\Lambda(B_2,B_1), \qquad (6)$$

so the phase diagram is symmetric along the line $T_1=T_2$. From Fig. 5, one can see that the SNA not only occurs in the vicinity of the chaotic and regular phases, but also within the chaotic region far from the regular boundary.

Equation (5) shows that the shape of the curve $\Lambda(B_1)$ is quite similar to that of $\lambda(B)$. Furthermore, one can adjust the values of T_1 and T_2 so as to locate the zero point of Λ at any desired point in its parameter space. At some points, such as the point of bifurcation, the curve $\lambda(B)$ versus B is not smooth enough to be fitted by any simple smooth function. Thus, with properly selected T_1 and T_2, the Lyapunov exponent Λ passing through zero via B can experience a sudden change. Figure 6 shows that the Lyapunov exponents of

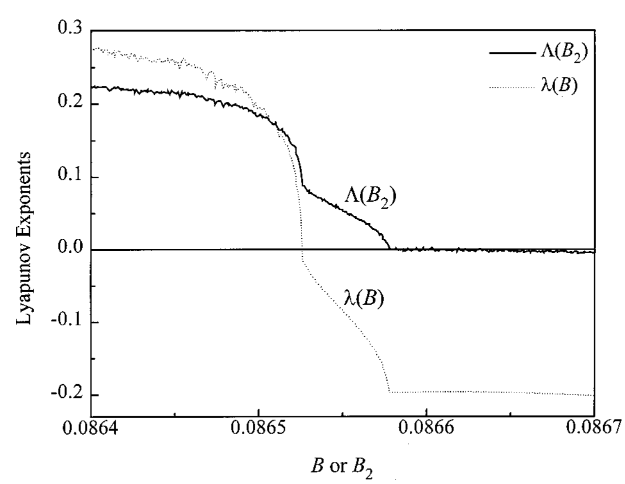

FIG. 6. Near the transition of the system (4) from chaos to the SNA, the Lyapunov exponent versus the control parameter B_2 in the region $0.0864<B_2<0.0867$. For comparison, $\lambda(B)$ versus B for the system (3) is also given out.

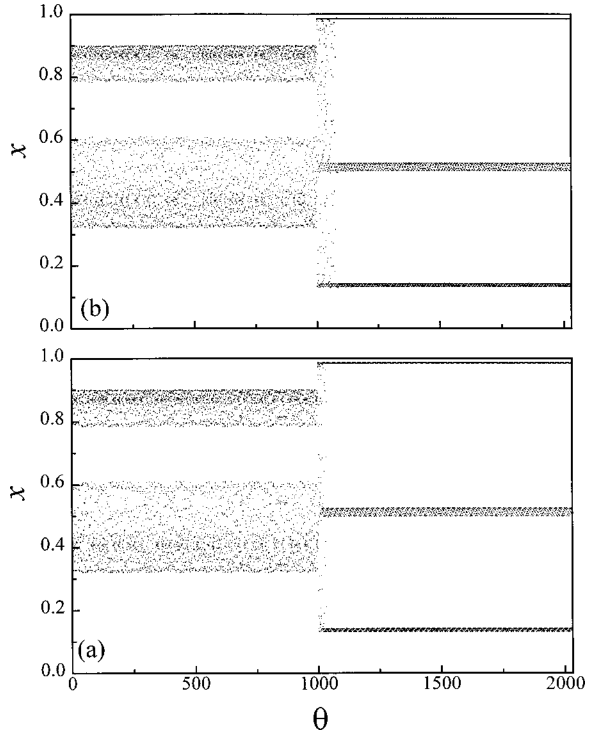

FIG. 7. (a) Strange nonchaotic trajectory in the x-θ plane with $B_2=0.086\,61$. (b) Chaotic trajectory in the x-θ plane with $B_2=0.086\,55$. The other parameters are $T_1=1000$, $T_2=1030$, and $B_1=0$.

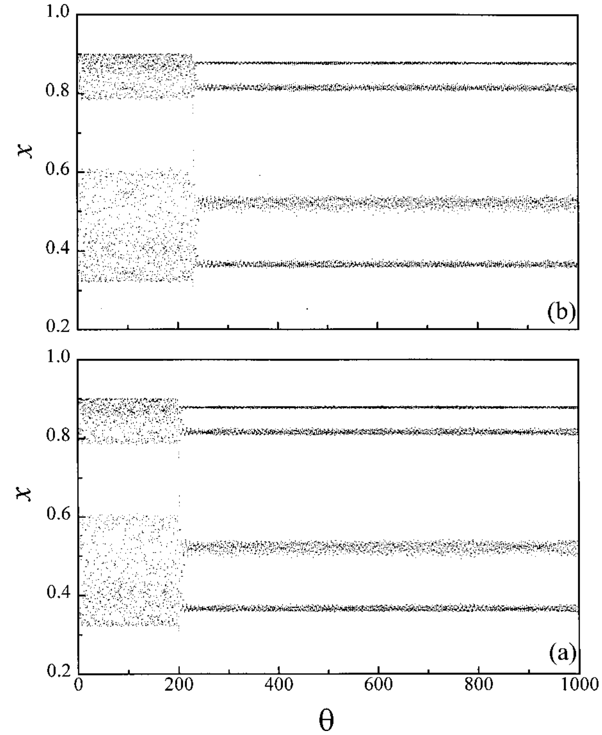

FIG. 8. (a) Noisy strange nonchaotic trajectory in the x-θ plane with $T_1=200$ and $T_2=800$. (b) Noisy chaotic trajectory with $T_1=230$ and $T_2=770$. Here, the maximum amplitude A of the noise is equal to 0.001.

Eq. (3) have a nonsmooth change when the control parameter B passes through 0.086 58.... Now let $T_1=1000$, $T_2=1030$, $B_1=0$, and vary B_2. The system of Eq. (4) then has a transition from the SNA to the chaotic attractor when $B_2=0.086\,58$. At the zero-crossing point, the Lyapunov exponent changes nonsmoothly. In particular, when the transition is about to occur, the Lyapunov exponent encounters a rapid change in the chaotic region but a slow change in the SNA region. For example, an SNA with $\Lambda=-0.002$ is obtained when $B_2=0.086\,61$. However, when $B_2=0.086\,55$, which is at the same small distance at another side of the transition point in the parameter space, a chaotic attractor is obtained with $\Lambda=0.05$. Figure 7 shows these two attractors in the x-θ plane. In contrast to that given in Fig. 3, the dynamical difference between these two attractors comes from the different contracting dynamics in the same T_2 time intervals.

IV. EFFECT OF NOISE

An important consideration in the study of SNAs is their robustness. This is of particular relevance with respect to the experimental observation of the SNA [11–13]. It is shown that, upon the addition of noise, the global structure of different types of SNAs remains approximately the same [23]. However, in some systems, if a small noise disturbs the quasiperiodic force, the SNA is destroyed and a chaotic or periodic attractor is observed [1,2].

As not only the regular but also the chaotic attractors are robust against small noise [32], noisy chaos and noisy tori are always obtained with the addition of small noise. Thus, a noisy SNA occurs if the system $\mathbf{x}(t+1)=\mathbf{F}(\mathbf{x}(t),\beta(t))+A\sin(2\pi\omega_1 t)$ is disturbed with noise. In our numerical simulation of Eq. (4), no matter how a, $x(t)$, A, $B(t)$, ω_0, or ω_1 are disturbed by small noise, SNAs can still be observed.

As discussed in Ref. [25], one of the important roles that the quasiperiodic wave plays is to provide different values at different times so as to cause the trajectory to run into different diverging orbits and create a strange structure. In fact, instead of the small-amplitude quasiperiodical wave $A\sin(2\pi\omega_1 t)$, one can apply a small noise with maximum amplitude A. In this case, the square function still drives the trajectory periodically into the expanding region, while the noise leads the trajectory into different orbits in different periods with expanding dynamics. A noisy SNA is thus generated. Figure 8(a) gives a noisy strange nonchaotic trajectory in the x-θ plane with $T_1=200$, $T_2=800$, and $A=0.001$. Its Lyapunov exponent is -0.003 (± 0.001). For comparison, a noisy chaotic trajectory with $\Lambda=0.004$ is shown in Fig. 8(b) with $T_1=230$ and $T_2=770$.

V. DISCUSSION

In this paper, a simple control method to create an SNA with any chaotic system is described. The main idea is to control the parameter of the system so that the system acts sometimes in a chaotic manner and sometimes in a periodic manner. We show that (i) the method provides a simple picture how an attractor may be strange while nonchaotic, (ii) the SNA can occur near the boundaries between chaotic and quasiperiodic motion or within the chaotic region far from a boundary, and (iii) by the proposed method the SNA may be realized in many nonlinear systems in large regions of parameter space. The maximum nontrivial Lyapunov exponent of the system can pass through zero nonsmoothly. The resultant SNA is robust against noise.

With the method discussed in Refs. [16–23], simulation results show that SNAs typically appear in a narrow parameter region. Thus SNAs are observed in several physical systems [11–13]. The proposed approach suggests an easy way to construct SNAs with chaotic physical systems: If a nonchaotic physical system driven by quasiperiodic force experiences expanding dynamics from time to time with long enough time intervals, the resultant attractor is an SNA. In particular, instead of applying a square function, a natural way to control the parameter of the system is to use a low-frequency sine force. A suitable low-frequency sine force can also ensure that the system oscillates to the expanding dynamics repeatedly with sufficiently long time intervals while the asymptotic dynamics is contracting.

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