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Local Discontinuous Galerkin Finite Element Method and Error Estimates for One Class of Sobolev Equation

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Abstract In this paper we present a numerical scheme based on the local discontinuous Galerkin (LDG) finite element method for one class of Sobolev equations, for example, generalized equal width Burgers equation. The proposed scheme will be proved to have good numerical stability and high order accuracy for arbitrary nonlinear convection flux, when time variable is continuous. Also an optimal error estimate is obtained for the fully discrete scheme, when time is discreted by the second order explicit total variation diminishing (TVD) Runge-Kutta time-marching. Finally some numerical results are given to verify our analysis for the scheme.

Keywords Sobolev equation · Local discontinuous Galerkin method · Fully-discrete · Stability analysis · Error estimate

1 Introduction

In this paper, we will present a numerical scheme based on local discontinuous Galerkin (LDG) finite element method for the equation

$$u_t + f(u)_x - \delta u_{xx} - \mu u_{xxt} = 0, \quad x \in I, t \in (0, T], \tag{1.1a}$$

$$u(x, 0) = u_0(x), \quad x \in I, \tag{1.1b}$$

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where $I = (0, 1)$, and $f(u)$ is a smooth function of u. Here, μ is a given positive constant, and δ is an arbitrary nonnegative constant. We do not pay attention to boundary condition in this paper; hence the solution is considered to be either periodic or compactly supported.

This type of equation is also called as Sobolev equation, for the occurrence of the mixed derivative with respect to temporal and spatial variables. It includes many classical equations arising in different fields, for example, thermodynamics, shear in second-order fluids, and consolidation of clay. If $f(u) = \alpha(p+1)^{-1}u^{p+1}$ with given constants α and p, (1.1a) is referred as the generalized equal width Burgers equation; if $\delta = 0$ and $f(u) = \alpha u + \frac{1}{2}\beta u^2$ with given numbers α and β, (1.1a) is referred as the regularized long-wave equation, or Benjamin-Bona-Mahony (BBM) equation [10]. In general, (1.1) not only features a balance between nonlinear and dispersive effects, but also takes into account mechanisms dissipation, if $u(x, t)$ is looked upon as the amplitude of the long wave.

Many numerical approximations based on finite difference (and/or element) method have been considered in the literature, for instance, [2, 4, 13, 16, 20]. In this paper we continue this work and develop a class of the LDG method to solve (1.1), which uses completely discontinuous piecewise polynomial space for the numerical solution and the test functions in space, coupled with explicit total variation diminishing (TVD) Runge-Kutta (RK) time discretization. We will show by theory analysis and numerical experiments that this proposed scheme has strong stability and optimal accuracy, inheriting the advantages of Discontinuous Galerkin (DG) method to capture discontinuous jump and/or sharp transient layer.

The LDG method is a particular version of DG method, which was introduced firstly in 1973 by Reed and Hill [18], in the framework of neutron linear transport. Then it was developed into Runge-Kutta DG (RKDG) method [6, 7] by Cockburn et al. for nonlinear hyperbolic systems, by using a strong stable time-marching scheme, numerical flux and slope limiter. Later, the LDG method was introduced by Cockburn and Shu in [8] as an extension of the DG method to general convection-diffusion problems, inspired by the work of Bassi and Rebay [3] for compressible Navier-Stokes equation. After that, many work on the develop of LDG method for higher order derivatives was carried out. For example, Yan and Shu developed a series of LDG method for general KdV type equation containing third derivatives in [23], and for some type of PDEs with fourth and fifth spatial derivatives in [24]. Levy and Shu developed the LDG methods for nonlinear dispersive equations with compactly supported traveling wave solutions in [15]. Xu and Shu further developed the LDG methods for a series of nonlinear wave equations; please see [22] and the included references. For a fairly complete set of references on DG methods as well as their implementation and applications, see the review paper by Cockburn and Shu [9].

In this paper we will propose a LDG method for (1.1) by defining three auxiliary variables, and transforming spatial derivative into temporal derivative. The main computation is to solve an elliptic equation by the traditional LDG method, and the definition of numerical flux is very nature and clear. We will prove this scheme is stable for any flux $f(u)$. Similar work can be found in [11, 12, 22], where the auxiliary variables are introduced in different way. The another highlight in this paper is the optimal error estimates for semi-discrete and fully-discrete scheme with the second TVD RK time-marching. As far as the authors know, till now there are few works about error estimates for fully-discrete LDG scheme to solve smooth solution of those equations with high order derivatives, such as (1.1). For semi-discrete schemes, Cockburn and Shu [8] have discussed for the standard linear diffusion equation, and Xu and Shu [21] have discussed for three kinds of nonlinear equations with high order derivatives. In this paper we will adopt those tricks in [25, 26] to obtain optimal error for the fully-discrete scheme with the second order explicit TVD RK time-marching.

The rest content of this paper is organized as follows. In Sect. 2, we will describe the scheme for (1.1) based on the LDG method, and present the stability analysis for the semi-discrete scheme. Time discrete is also considered in this section, and the fully-discrete scheme with the second order explicit TVD RK time-marching is given. In Sect. 3, the optimal error estimates are obtained for the proposed schemes. Both semi-discrete and fully-discrete schemes are considered in the simple case that the convection direction is assumed to be from left to right. Some more technical proofs of two lemmas are collected in Appendix A. In Sect. 4, three numerical experiments are presented to verify our error estimate is optimal. Concluding remarks are given in Sect. 5.

2 Schemes Based on LDG Method

2.1 Equivalent Formulation

We will propose the scheme for (1.1) along the same line as in designing the general LDG method. By introducing three auxiliary variables

$$w = u_t, \qquad p = w_x, \qquad q = u_x, \tag{2.1}$$

(1.1) can be rewritten into the following equivalent first-order differential system with regard to the solution $\mathbf{w} = (u, q, w, p)$. It reads, for $(x, t) \in I \times (0, T)$, that

$$u_t = w, \tag{2.2a}$$

$$q_t = p, \tag{2.2b}$$

$$w + (f(u) - \delta q - \mu p)_x = 0, \tag{2.2c}$$

$$p - w_x = 0, \tag{2.2d}$$

with initial value (denote $F_{0,x} = (F_0)_x$ for simplicity)

$$u(x, 0) = u_0(x), \qquad q(x, 0) = u_{0,x}(x), \quad x \in I. \tag{2.2e}$$

Remark that (2.2b) follows from the relation between three auxiliary variables, say, (2.1). By transferring the derivative from space to time, we will avoid the spatial DG discrete directly for equation $q = u_x$, and design the correct numerical fluxes for (2.2c) and (2.2d) in a nature way. By using the routine of LDG method for elliptic equation, this treatment also helps us to code the program easily.

In what follows, we denote the solution in a compact form $\mathbf{w} = (\mathbf{w}_{uq}, \mathbf{w}_{wp})$, where $\mathbf{w}_{uq} = (u, q)$ and $\mathbf{w}_{wp} = (w, p)$. Further, we write (2.2c) and (2.2d) into a compact form $\mathbf{w}_{wp} + \mathbf{h}(\mathbf{w})_x = 0$, with the flux

$$\mathbf{h}(\mathbf{w}) = (h_w(\mathbf{w}), h_p(\mathbf{w})) = (f(u) - \delta q - \mu p, -w). \tag{2.3}$$

Obviously this is an elliptic system with respect to the variable w, if $\mathbf{w}_{uq}$ is given.

Thus we can obtain a scheme for (1.1) by simply discretizing the above system (2.2), especially for the last two equations, (2.2c) and (2.2d), with the traditional LDG method.

2.2 Finite Element Space and Notations

Let $0 = x_{1/2} < x_{3/2} < \cdots < x_{N-1/2} < x_{N+1/2} = 1$ be any regular partition of I, and denote each cell $I_j = [x_{j-1/2}, x_{j+1/2}]$ of length $h_j = x_{j+1/2} - x_{j-1/2}$, for $j = 1, 2, \ldots, N$. The mesh parameter is defined as $h = \max_{1 \le j \le N} h_j < 1$.

We would like to find the LDG approximation solution of (1.1), denoted by $\mathbf{w}_h = (u_h, q_h, p_h, w_h)$, in which each component belongs to the discontinuous finite element space

$$V_h = \left\{ v \in L^2(0,1) : v|_{I_j} \in \mathbb{P}_k(I_j),\ j = 1, 2, \ldots, N \right\}, \tag{2.4}$$

where $\mathbb{P}_k(I_j)$ denotes the space of polynomials in the cell I_j of degree at most $k \ge 1$. We do not consider the piecewise constants in this paper, since the proposed scheme is the finite volume method. Note that functions in V_h are allowed to have discontinuities across element interfaces, which is very useful to enhance the numerical behaving.

Before presenting the detailed implementation of scheme, we first explain some notations used in this paper. For any function $z(x) \in L^2(I)$, we denote by $z^{\pm}_{j+1/2}$ the limit of $z(x)$ at the interface point $x_{j+1/2}$, from the right and left direction, respectively. Dropping the subscript $j + 1/2$, we denote the jump and average, at each interface point, by $[\![z]\!] = z^+ - z^-$, and $\bar{z} = (z^+ + z^-)/2$, respectively.

Further, we will use three local projections in this paper. One is the local L^2-projection of $z(x)$, denoted by $\mathbb{P}_h z(x)$, which is defined in each element I_j, $j = 1, 2, \ldots, N$, as the unique function in V_h such that

$$\int_{I_j} \mathbb{P}_h z(x) v_h(x)\,\mathrm{d}x = \int_{I_j} z(x) v_h(x)\,\mathrm{d}x, \quad \forall v_h(x) \in \mathbb{P}_k(I_j). \tag{2.5}$$

The others are two kinds of local Gauss-Raudu projection [14] of $z(x)$, denoted by $\mathbb{Q}_h^{\pm} z(x)$, which is defined in each element I_j, $j = 1, 2, \ldots, N$, as the unique function in V_h such that

$$\int_{I_j} \mathbb{Q}_h^{\pm} z(x) v_h(x)\,\mathrm{d}x = \int_{I_j} z(x) v_h(x)\,\mathrm{d}x, \quad \forall v_h(x) \in \mathbb{P}_{k-1}(I_j), \tag{2.6}$$

together with boundary condition $\mathbb{Q}_h^{\pm} z(x^{\mp}_{j\pm 1/2}) = z(x^{\mp}_{j\pm 1/2})$ solely at an endpoint of the cell I_j. This projection has an advantage that the interpolation solution is exact at one endpoint of each cell.

2.3 LDG Scheme

Now we keep time in continuous and present the semi-discrete scheme of LDG method with piecewise polynomials of degree at most k, which is referred to as LDG(k). It is given as follows: for any time $t \in (0, T)$, find the approximate solution $\mathbf{w}_h = (u_h, q_h, p_h, w_h) \in (V_h)^4$, such that

$$\int_{I_j} u_{h,t}(x,t) v(x)\,\mathrm{d}x - \int_{I_j} w_h(x,t) v(x)\,\mathrm{d}x = 0, \tag{2.7a}$$

$$\int_{I_j} q_{h,t}(x,t) z(x)\,\mathrm{d}x - \int_{I_j} p_h(x,t) z(x)\,\mathrm{d}x = 0, \tag{2.7b}$$

$$\int_{I_j} w_h(x,t) s(x)\,\mathrm{d}x - \int_{I_j} h_w(\mathbf{w}_h) s_x(x)\,\mathrm{d}x + \hat{h}_w(\mathbf{w}_h) s(x)\Big|_{j-1/2}^{j+1/2} = 0, \tag{2.7c}$$

$$\int_{I_j} p_h(x,t)r(x)\,\mathrm{d}x - \int_{I_j} h_p(\mathbf{w}_h)r_x(x)\,\mathrm{d}x + \hat{h}_p(\mathbf{w}_h)r(x)\big|_{j-1/2}^{j+1/2} = 0, \tag{2.7d}$$

hold for any test function $(v,z,s,r)\in(V_h)^4$ and $j=1,2,\ldots,N$. Here and below we would like to use the vertical line after a multi-valued function $\mathbb{p}$, to denote its difference between two endpoints, namely, $\mathbb{p}|_{j-1/2}^{j+1/2}=\mathbb{p}_{j+1/2}^{-}-\mathbb{p}_{j-1/2}^{+}$.

We would like to point out that the first two equations, (2.7a) and (2.7b), are not finite element schemes in essence. They are equivalent to say $u_{h,t}=w_h$ and $q_{h,t}=p_h$ for any time $t\in(0,T]$, since the trial functions and the test functions are in the same finite element space. Thus we can compute the approximation solution by directly advancing every freedoms of u_h and q_h by the freedoms of w_h and p_h, if the same bases of finite element space are used for every trial functions. This property also plays an important role in our analysis.

To design a successful LDG scheme, it is important to define the numerical flux $\hat{\mathbf{h}}(\mathbf{w}_h)$ in a correct way to ensure good stability. In this paper we take it as

$$\hat{h}_w(\mathbf{w}_h) = \hat{f}(u_h^-,u_h^+) - \delta q_h^+ - \mu p_h^+, \qquad \hat{h}_p(\mathbf{w}_h) = -w_h^-, \tag{2.8}$$

at each interface point, where the numerical flux $\hat{f}(u_h^-,u_h^+)$ is any locally Lipschitz E-flux [17] consistent with the flux function $f(u)$. In order to get optimal error estimate, the natural choice is upwind flux, which satisfies

$$\widehat{f}(u^-,u^+) = \begin{cases} f(u^-) & \text{if } f'(u)\ge 0 \quad \forall u\in[\min(u^-,u^+),\max(u^-,u^+)], \\ f(u^+) & \text{if } f'(u)< 0 \quad \forall u\in[\min(u^-,u^+),\max(u^-,u^+)]. \end{cases} \tag{2.9}$$

The best-known examples are the Godunov flux, the Engquist-Osher flux, and the Roe flux with an entropy fix; for more details, please see [9].

The feature to define the numerical flux, (2.8), is that we use alternative direction for w and p, and same direction for p and q. The former restriction for w and p is necessary for the LDG method to solve elliptic equation, and the latter one for p and q is very natural since $q_{h,t}=p_h$. In this paper, we also demand the same direction used in $\hat{\mathbf{h}}_p(\mathbf{w})$ as in the upwind numerical flux $\hat{f}(u^-,u^+)$. This restriction is only for optimal error, but not necessary for stability. That is to say, the given numerical flux, (2.8), is very suitable when the numerical flux is taken as $\hat{f}(u^-,u^+)=f(u^-)$; otherwise, it is the best choice to change the signs of direction for p_h, q_h and w_h, in (2.8), if $\hat{f}(u^-,u^+)=f(u^+)$.

The initial solution for (2.7) is given as an approximation of $u_0(x)$ and $u_{0,x}(x)$. In this paper, we would like to take them depending on the definition of numerical flux. According to the numerical flux (2.8), we take

$$u_h(x,0)=\mathbb{Q}_h^+u_0(x) \quad \text{and} \quad q_h(x,0)=\mathbb{P}_h u_{0,x}(x), \tag{2.10}$$

just for simplicity in stability analysis and error estimates. Now the semi-discrete LDG method is described completely.

In the next paragraph we are going to prove that the LDG(k) scheme has a strong stability for any numerical E-flux $\hat{f}(u_h^-,u_h^+)$. For convenience, we would like, for any piecewise functions φ and ψ whose restriction in each cell I_j belong to $H^1(I_j)$, to abbreviate the discretization in the cell I_j by virtue of the DG method, in the form

$$H_j^{\pm}(\varphi,\psi) = \int_{I_j}\varphi\psi_x\,\mathrm{d}x - (\varphi^{\pm}\psi^-)_{j+1/2} + (\varphi^{\pm}\psi^+)_{j-1/2}, \tag{2.11a}$$

$$G_j(\varphi,\psi) = \int_{I_j} f(\varphi)\psi_x\,\mathrm{d}x - (\hat{f}(\varphi)\psi^-)_{j+1/2} + (\hat{f}(\varphi)\psi^+)_{j-1/2}. \tag{2.11b}$$

The first operator, $H_j^{\pm}(\cdot,\cdot)$, is used for linear variables p_h, q_h and w_h, depending on the direction of them in the numerical flux. The second one, $G_j(\cdot,\cdot)$, is used for nonlinear (maybe linear) function $f(u)$.

It is worthy to mention that the above linear operators $H_j^{\pm}(\cdot,\cdot)$ are not symmetric for their arguments. However, their combination has a very nice property.

Lemma 2.1 *For any piecewise functions φ and ψ whose restriction in each cell I_j belong to $H^1(I_j)$, we have, for $j=1,2,\dots,N$, that*

$$H_j^+(\varphi,\psi)+H_j^-(\psi,\varphi)=-\Delta_j(\varphi^+\psi^-)\equiv-\big[(\varphi^+\psi^-)_{j+1/2}-(\varphi^+\psi^-)_{j-1/2}\big], \tag{2.12}$$

where $\Delta_j\mathbb{p}$ denotes the center difference for a signal-valued function $\mathbb{p}$ at two endpoints of the cell I_j.

Proof Since $\varphi\psi_x+\psi\varphi_x=(\varphi\psi)_x$, a simple manipulation deduces that

$$H_j^+(\varphi,\psi)+H_j^-(\psi,\varphi)=(\varphi\psi)\Big|_{j-\frac{1}{2}}^{j+\frac{1}{2}}-(\varphi^+\psi)\Big|_{j-\frac{1}{2}}^{j+\frac{1}{2}}-(\psi^-\varphi)\Big|_{j-\frac{1}{2}}^{j+\frac{1}{2}}=-\Delta_j(\varphi^+\psi^-),$$

which completes the proof of this lemma. □

Theorem 2.1 *For the numerical solution to scheme* (2.7) *with the initial setting* (2.10), *there is a good stability as follows*

$$\frac{\mathrm{d}}{\mathrm{dt}}\int_0^1\big[u_h^2(x,t)+\mu q_h^2(x,t)\big]\,\mathrm{d}x\le 0. \tag{2.13}$$

Proof Firstly we point out an interesting property from (2.10). This setting of initial solution implies that $q_h(x,0)$ satisfies the equation paralleled to (2.7d), of the form

$$\int_{I_j}q_h(x,0)v_h(x)\,\mathrm{d}x+H_j^-(u_h(x,0),v_h(x))=0,\quad \forall v_h\in V_h,\ j=1,2,\dots,N. \tag{2.14}$$

It follows from an integration by parts for $q_h(x,0)=\mathbb{P}_h u_{0,x}$, since $u_h(x,0)=\mathbb{Q}^+u_0(x)$ indicates $H_j^-(u_0(x),v_h(x))=H_j^-(u_h(x,0),v_h(x))$ for any $v_h\in V_h$.

Therefore, integration of equation (2.7d) in time from 0 to $t\in(0,T]$, yields that

$$\int_{I_j}q_hv_h\,\mathrm{d}x+H_j^-(u_h,v_h)=0,\quad \forall v(x)\in V_h,\ j=1,2,\dots,N, \tag{2.15}$$

holds for any time $t\in(0,T]$, since $u_{h,t}=w_h, q_{h,t}=p_h$ and (2.14). It is equivalent to the DG discrete of equation $q=u_x$. Remark that the above result (2.15) is also true for $t=0$. This equation plays very important role in stability analysis here and error estimate below.

Let the test function $s(x)=u_h$ in (2.7c), and let $v_h(x)=\delta q_h$ and $v_h(x)=\mu p_h$ in (2.15), respectively. By Lemma 2.1, the sum of these new equations yields that

$$\begin{aligned}0=&\int_{I_j}[u_hw_h+\delta q_h^2+\mu q_hp_h]\,\mathrm{d}x+\delta\big[H_j^+(q_h,u_h)+H_j^-(u_h,q_h)\big]\\&+\mu\big[H_j^+(p_h,u_h)+H_j^-(u_h,p_h)\big]-G_j(u_h,u_h)\end{aligned}$$

$$= \int_{I_j} [u_h u_{h,t} + \delta q_h^2 + \mu q_h q_{h,t}]\,\mathrm{d}x - \delta \Delta_j (q_h^+ u_h^-)$$
$$- \mu \Delta_j (p_h^+ u_h^-) - G_j(u_h, u_h), \tag{2.16}$$

where $u_{h,t} = w_h$ and $q_{h,t} = p_h$ are used again.

Next, we define the entropy flux for $f(u)$, as usual, $F(u) = \int^u f(s)\,\mathrm{d}s$, to deal with the last term in (2.16). Then it follows from a series of manipulation that

$$-G_j(u_h, u_h) = -\Delta_j (F(u_h^-) - \hat{f}(u_h) u_h^-) + \big([\![F(u_h)]\!] - \hat{f}(u_h)[\![u_h]\!] \big)_{j-1/2}, \tag{2.17}$$

where the details of analysis are omitted, since they can be found in any paper about the theory analysis of the DG method for conservation law, for example, [9]. The property of E-flux implies that $[\![F(u_h)]\!] - \hat{f}(u_h)[\![u_h]\!] = \int_{u_h^-}^{u_h^+} (f(s) - \hat{f}(u_h^-, u_h^+))\,\mathrm{d}s \geq 0$. Consequently, the last term in (2.17), denoted by $\Theta_{j-1/2}$, is not less than zero.

We sum up (2.16) over all elements I_j, $j = 1, 2, \dots, N$. By using (2.17) and periodic boundary condition, we obtain

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\int_0^1 (u_h^2 + \mu q_h^2)\,\mathrm{d}x + \delta \int_0^1 q_h^2\,\mathrm{d}x + \sum_{1 \le j \le N} \Theta_{j+1/2} = 0. \tag{2.18}$$

Then it completes the proof of this theorem, since $\Theta_{j+1/2} \geq 0$. □

2.4 Time Marching

After we have chosen the basis functions of finite element space V_h, the semi-discrete scheme (2.7) can be written in an ODE system with regard to the freedoms of $\mathbf{w}_{uq,h} = (u_h, q_h)$. For simplicity of notations, we still use the solution to represent its freedoms. The system is given as follows:

$$[\mathbf{w}_{uq,h}]_t = \mathcal{L}(\mathbf{w}_{uq,h}) = \mathbf{w}_{wp,h}, \tag{2.19}$$

where $\mathbf{w}_{wp,h} = (w_h, p_h)$ is coupled with $\mathbf{w}_{uq,h}$ by (2.7c) and (2.7d). It only takes a small quantity of algebra manipulations in advancing directly the freedoms of approximation solutions, by any standard ODE solver, for example, the r-order explicit TVD RK time-marching, denoted by RK(r). Please refer to [19] for more details.

In this paper we would like to adopt RK(2) time-marching, which is defined as follows:

$$\mathbf{w}_{uq,h}^{n\star} = \mathbf{w}_{uq,h}^{n} + \tau^n \mathcal{L}(\mathbf{w}_{uq,h}^{n}), \tag{2.20a}$$
$$\mathbf{w}_{uq,h}^{n+1} = \frac{1}{2}\mathbf{w}_{uq,h}^{n} + \frac{1}{2}\mathbf{w}_{uq,h}^{n\star} + \frac{1}{2}\tau^n \mathcal{L}(\mathbf{w}_{uq,h}^{n\star}), \tag{2.20b}$$

where $\mathcal{L}(\mathbf{w}_{uq,h}) = \mathbf{w}_{wp,h}$, and $\tau^n = t^{n+1} - t^n$ is the time step. We refer to this fully-discrete scheme as RKLDG(2,k), if the finite element space V_h is of piecewise polynomials of degree at most k. The detail implementation is given as follows:

1. compute the initial solution u_h^0 and q_h^0 by (2.10);
2. until the final time T, compute for $n = 0, 1, 2, \dots$
   - get w_h^n and p_h^n from u_h^n and q_h^n, by (2.7c) and (2.7d);
   - get $u_h^{n\star}$ and $q_h^{n\star}$ by (2.20a);

- get $w_h^{n\star}$ and $p_h^{n\star}$ from $u_h^{n\star}$ and $q_h^{n\star}$, by (2.7c) and (2.7d);
- get u_h^{n+1} and q_h^{n+1} by (2.20b).

Since (2.7c) and (2.7d) are coupled into an algebraic system of linear equations, we need use a linear equation solver twice in each complete step of RK(2). In this paper we use Gaussian elimination method with partial pivoting.

To ensure the stability of scheme RKLDG(2,k), time step $\tau^n(<1)$ has to be restricted under a suitable time-spatial condition. In this paper it is enough to demand

$$\delta^2\tau^n \le M_\delta h^2, \quad \text{and} \quad \max_{1\le j\le N} |f'(u^n_{j+\frac{1}{2}})|\tau^n \le M_f h, \tag{2.21}$$

for arbitrary positive constant M_δ, however, and a fixed CFL number $M_f = 1/(4M)$. The constant M will be given in error estimate to the fully discrete scheme, see (3.35) below. In particular, for convenience we would like to take each time step in same length $\tau(<1)$, under the following restriction

$$\delta^2\tau \le M_\delta h^2, \quad \text{and} \quad S_{\max}\tau \le M_f h, \tag{2.22}$$

with the same constants as those in (2.21), where $S_{\max}$ is the maximum absolution of $f'(u)$ taken over a relevant range of $u(x,t)$, the exact solution of (1.1).

3 Error Estimates

In this section we will present *a priori* error estimates to the semi-discrete scheme LDG(k) and the fully-discrete scheme RKLDG(2,k) for smooth solution of (1.1). Below we denote by C (maybe with indicates) a positive constant depending solely on the exact solution, which may have a different value in each occurrence.

3.1 Preliminaries

3.1.1 Notations of Sobolev Space

For $1 \le r \le +\infty$ and integers $s \ge 0$, let $W^{s,r}(\Omega)$ represent the well-known Sobolev spaces consisting of functions with (distributional) derivatives of order less than or equal to s in $L^r(\Omega)$. Also, let $\|\cdot\|_{s,r,\Omega}$ denote the usual norm. Next, let the scalar inner product on $L^2(\Omega)$ be denoted by $(\cdot,\cdot)_\Omega$, and the associated norm by $\|\cdot\|_\Omega$. Further, $\|\cdot\|_{\infty,\Omega}$ represents the norm on $L^\infty(\Omega)$, and $\|\cdot\|_{L^\infty(W^{s,r}(\Omega))}$ the norm on $L^\infty([0,T], W^{s,r}(\Omega))$. If $\Omega = I$ we omit it. See Adams [1] for more details.

In this paper we also use some notations for vector-valued function. For any $\mathbb{p} = (\mathbb{p}_1, \mathbb{p}_2)$ and $\mathbb{q} = (\mathbb{q}_1, \mathbb{q}_2)$, we denote the μ-inner product by $(\mathbb{p},\mathbb{q})_\mu = \mathbb{p}_1\mathbb{q}_1 + \mu\mathbb{p}_2\mathbb{q}_2$, where μ is a given positive constant. The corresponding norm is denoted by $\|\mathbb{p}\|_\mu^2 = \|\mathbb{p}_1\|^2 + \mu\|\mathbb{p}_2\|^2$. We use $\mathcal{K}(\mathbb{p},\mathbb{q})$ to denote a vector composed of the scalar inner product of those components with same-index, i.e., $\mathcal{K}(\mathbb{p},\mathbb{q}) = ((\mathbb{p}_1,\mathbb{q}_1),(\mathbb{p}_2,\mathbb{q}_2))$.

3.1.2 Smoothness Assumptions and Remarks on the Flux

In this paper we assumed that each component of the solution, $\mathbf{w}_{\mathcal{F}} = \mathcal{F}$, is smooth enough, for example, $\mathbf{w}_{\mathcal{F}} \in L^\infty([0,T]; W^{k+1,\infty})$ for $\mathcal{F} = u, q, p$ and w. The assumptions to w and p are equivalent to say that u_t and q_t are in $L^\infty([0,T]; W^{k+1,\infty})$. It follows from Sobolev embedding theorem $H^{2,\infty}(I) \hookrightarrow C^1(I)$ in one dimension that both $u(x,t)$ and its first order spatial derivative are continuous and bounded in $I \times [0,T]$, since $k \geq 1$.

Also, we assume that the flux function $f(u)$ itself and up to the second order derivatives are all bounded on $\mathbb{R}$. If not, we can modify the flux $f(u)$ to satisfy this assumption, as we have done in [25]. To emphasize the nonlinearity of the flux $f(u)$, we use $C_\star$ to denote the positive constant depending solely on the maximum of $|f''|$. Remark that $C_\star = 0$ for a linear flux $f = cu$.

In addition we assume that the convection direction contains $f'(u) \geq 0$, in order to present error estimate in a clear fashion. Now the numerical flux in the proposed scheme is defined by (2.8) with $\widehat{f}(u^-,u^+) = f(u^-)$. There is no essential difficulty in obtaining error estimate for general cases, if we introduce the quantity $\alpha(\hat{f}; u^-, u^+)$ to measure the diffusion of numerical flux; please refer to [25].

3.1.3 Short Notations with Respect to the Flux

Below we will use some compact notations with regard to $f(u)$. For any function $\mathbb{p}$, we denote the bounded constant in error estimates by a short form $C(\mathbb{p}) = C + C_\star h^{-2}\|\mathbb{p}\|_\infty^2$, where C and $C_\star$ are independent of h and $\mathbb{p}$. Further, we define two compact operators for any functions $\mathbb{p}$ and $\mathbb{q}$, in the form

$$\mathcal{A}(\mathbb{p},\mathbb{q}) = \sum_{1\leq j\leq N} |f'(\mathbb{p}_{j+\frac{1}{2}})| [\![\mathbb{q}]\!]_{j+\frac{1}{2}}^2, \qquad \mathcal{B}(\mathbb{p},\mathbb{q}) = \sum_{1\leq j\leq N} \|f'(\mathbb{p}_{j+\frac{1}{2}})\mathbb{q}_x\|_{I_j}^2, \tag{3.1}$$

where $\mathbb{p}_{j+1/2}$ is the (left-limiting) value of $\mathbb{p}$ at the right endpoint of the cell I_j. The first one is used to denote the sum of jump of $\mathbb{q}$ at every interface points, corresponding to the diffusion of numerical flux $\widehat{f}$. Similar notation has been used in [25]. The second one is used to denote the sum of L^2-norm of first order spatial derivative of $\mathbb{q}$ over all elements. The relation between two operators will be given in Lemma 3.7 and Corollary 3.1.

3.1.4 Properties of Finite Element Space

Gauss-Raudu projection (2.6) is a very useful tool to obtain the optimal error estimate for the DG method. By virtue of the scaling technique [5], it is easy to get, for the interpolation error $\eta_z^\pm = z(x) - \mathbb{Q}_h^\pm z(x)$, that

$$\|\eta_z^\pm\| + h^{\frac{1}{2}}\|\eta_z^\pm\|_\infty + h^{\frac{1}{2}}\|\eta_z^\pm\|_{\Gamma_h} \leq C\|z(x)\|_{k+1,\infty}h^{k+1}, \tag{3.2}$$

where Γ_h is the union of the interface point of every elements, and C is a positive constant independent of h and $z(x)$.

Finally, we list some inverse properties of the finite element space V_h that will be used in our error analysis. For any $v_h \in V_h$, there exists a positive constant C, independent of v_h and h, such that

$$\begin{aligned} &\text{(i)} && \|v_{h,x}\| \leq Ch^{-1}\|v_h\|;\\ &\text{(ii)} && \|v_h\|_{\Gamma_h} \leq Ch^{-1/2}\|v_h\|;\\ &\text{(iii)} && \|v_h\|_\infty \leq Ch^{-1/2}\|v_h\|. \end{aligned} \tag{3.3}$$

For more details of these inverse properties, we refer the reader to [5].

3.2 Error Estimate for the Semi-Discrete LDG Method

By $e_F = F_h - F$ we denote the error for each component of the solution of (1.1), where $F = u, q, w$ and p. Further, we use a short notation $e_{uq} = (e_u, e_q)$ and $e_{wp} = (e_w, e_p)$. Similar notations are used for short below. According to the definition of numerical flux (2.8), we divide each component of error into two parts by using different kind of Gauss-Raudu projection. They reads

$$e_F = (F_h - \mathbb{Q}_h^+ F) - (F - \mathbb{Q}_h^+ F) = \xi_F - \eta_F, \quad \text{for } F = u, w; \tag{3.4a}$$

$$e_F = (F_h - \mathbb{Q}_h^- F) - (F - \mathbb{Q}_h^- F) = \xi_F - \eta_F, \quad \text{for } F = q, p; \tag{3.4b}$$

where $\xi_F \in V_h$ need to be estimated below, and η_F is the interpolation error. Remark that $\eta_u^- = \eta_w^- = \eta_p^+ = \eta_q^+ = 0$ at each interface point. From (3.2), we have that

$$\|\eta_F\| + h^{1/2}\|\eta_F\|_\infty + h^{1/2}\|\eta_F\|_{\Gamma_h} \le C\|F\|_{L^\infty(W^{k+1,\infty})} h^{k+1}, \quad t \in [0, T], \tag{3.5}$$

where C is a positive constant independent of h and F.

3.2.1 *Error Equation*

Now we are going to estimate the error $\xi_F \in V_h$ by virtue of the energy analysis. To this end, we first multiply the test functions v_u, v_q, v_w and v_p, respectively, on both hand side of each equation in (2.2), and then make some integrations by parts. This process gives us four almost same equations as (2.7), except the diminishing subscript h, since $\hat{f}(u^-, u^+) = f(u)$ for the continuous function u at each interface point. Next, we subtract them from the corresponding equation in (2.7). Finally, we sum up the resulted equations over all elements $I_j, j = 1, 2, \ldots, N$, to deduce that

$$\mathcal{K}(e_{uq,t}, v_{uq}) = \mathcal{K}(e_{wp}, v_{uq}), \qquad \mathcal{K}(e_{wp}, v_{wp}) = \mathcal{H}(e_{uq}, v_{wp}), \quad \forall t \in (0, T], \tag{3.6}$$

hold for any test function $v_{uq} = (v_u, v_q)$ and $v_{wp} = (v_w, v_p)$ in $(V_h)^2$. Here $\mathcal{K}(\cdot, \cdot)$ has been defined in Sect. 3, and $\mathcal{H}(\cdot, \cdot)$ is used to denote the LDG spatial discrete for $\mathbf{w}_{wp}$, in the form

$$\mathcal{H}(e_{uq}, v_{wp}) = \left(\sum_{1 \le j \le N} \left[K_j(e_u, v_w) - \mu H_j^+(e_p, v_w) - \delta H_j^+(e_q, v_w)\right], - \sum_{1 \le j \le N} H_j^-(e_w, v_p) \right),$$

where $K_j(e_u, v_w) = G_j(u_h, v_w) - G_j(u, v_w)$ is relevant to the convection flux $f(u)$, and not equal to $G_j(u - u_h, v_w)$ in general.

By taking the test function $(v_{uq}, v_{wp}) = (\xi_u, \mu\xi_q, \xi_u, 0)$ in (3.6), we will get the following error equation

$$(e_{u,t}, \xi_u) + \mu(e_{q,t}, \xi_q) = \mathcal{K} \otimes \mathcal{H}(e_\mathbf{w}, \xi_{uq}) \equiv \sum_{1 \le j \le N} \mathcal{K}_j \otimes \mathcal{H}_j(e_\mathbf{w}, \xi_{uq}), \tag{3.7}$$

where $e_\mathbf{w} = (e_u, e_q, e_w, e_p)$, $\xi_{uq} = (\xi_u, \xi_q)$ and

$$\mathcal{K}_j \otimes \mathcal{H}_j(e_\mathbf{w}, \xi_{uq}) = \int_{I_j} \mu e_p \xi_q \,\mathrm{d}x - \mu H_j^+(e_p, \xi_u) - \delta H_j^+(e_q, \xi_u) + K_j(e_u, \xi_u). \tag{3.8}$$

In what follows we will analyze the right-hand side term of (3.7), in order to obtain the error estimate. Remark that $\mathcal{K}_j \otimes \mathcal{H}_j(e_\mathbf{w}, \xi_{uq})$ includes two kinds of terms, where $H_j^\pm(\cdot, \cdot)$ is linear for both arguments, and $K_j(\cdot, \cdot)$ is linear only for the second argument.

Lemma 3.1 *Suppose the interpolation property* (3.5) *is satisfied*; *then we have that*

$$\mathcal{K}\otimes\mathcal{H}(e_{\mathbf{w}},\xi_{uq})\le Ch^{2k+2}+\frac{1}{2}(\mu-\delta)\|\xi_q\|^2+\sum_{1\le j\le N}K_j(e_u,\xi_u)+\mu\int_0^1\eta_q\xi_p\,\mathrm{d}x, \tag{3.9}$$

where C is the positive constant independent of h and $\mathbf{w}_h$.

Proof Since (2.15) is consistent with $q=u_x$, we have, for any time $t\in(0,T]$, that

$$\int_{I_j}e_q v_h\,\mathrm{d}x+H_j^-(e_u,v_h)=0,\qquad v_h\in V_h,\ \ j=1,2,\dots,N. \tag{3.10}$$

This fact is the foundation of our proof to this lemma.

Take the test function $v_h=-\delta\xi_q$ and $v_h=-\mu\xi_p$ in (3.10), respectively. By adding two new equations to the formula of $\mathcal{K}_j\otimes\mathcal{H}_j(e_{\mathbf{w}},\xi_{uq})$, we get that

$$\begin{aligned}
&\mathcal{K}_j\otimes\mathcal{H}_j(e_{\mathbf{w}},\xi_{uq})\\
&\quad=\underbrace{\int_{I_j}[\mu e_p\xi_q-\delta e_q\xi_q-\mu e_q\xi_p]\,\mathrm{d}x}_{\Lambda_0}+K_j(e_u,\xi_u)\\
&\qquad-\underbrace{\delta(H_j^+(\xi_q,\xi_u)+H_j^-(\xi_u,\xi_q))}_{\Lambda_1}-\underbrace{\mu(H_j^+(\xi_p,\xi_u)+H_j^-(\xi_u,\xi_p))}_{\Lambda_2}\\
&\qquad+\underbrace{\delta(H_j^+(\eta_q,\xi_u)+H_j^-(\eta_u,\xi_q))+\mu(H_j^+(\eta_p,\xi_u)+H_j^-(\eta_u,\xi_p))}_{\Lambda_3}.
\end{aligned} \tag{3.11}$$

Below we will estimate each above term, separately.

By arithmetic mean inequality and Schwartz inequality, it is easy to get that

$$\begin{aligned}
\Lambda_0&=\int_{I_j}\big[\delta(\eta_q\xi_q-\xi_q^2)+\mu(\eta_q\xi_p-\eta_p\xi_q)\big]\,\mathrm{d}x\\
&\le\mu\int_{I_j}\eta_q\xi_p\,\mathrm{d}x+\frac{1}{2}(\mu-\delta)\|\xi_q\|_{I_j}^2+\frac{1}{2}\big[\mu\|\eta_p\|_{I_j}^2+\delta\|\eta_q\|_{I_j}^2\big].
\end{aligned} \tag{3.12}$$

Next, it follows from Lemma 2.1 that $\Lambda_1+\Lambda_2=-\delta\varDelta_j(\xi_q^+\xi_u^-)-\mu\varDelta_j(\xi_p^+\xi_u^-)$. Further, we assert that $\Lambda_3=0$ since each term included there is equal to zero. To show that, we take $H_j^+(\eta_q,\xi_u)$ as an example. Obviously it is true, as an immediate consequence of Gauss-Raudu projection (2.6).

We substitute all estimates about $\Lambda_i, i=0,1,2,3$ into (3.11), and sum up the resulted inequality over all elements I_j, $j=1,2,\dots,N$. Finally, by using the interpolation property (3.5) and periodic boundary condition, we complete the proof of this lemma. □

Along the same line as [25], we can obtain the following lemma to estimate $K_j(e_u,\xi_u)$. For the completeness of this paper, the proof will be given in the appendix.

Lemma 3.2 *Suppose the interpolation property* (3.5) *is satisfied*; *then we have that*

$$\sum_{1\le j\le N}K_j(e_u,\xi_u)\le C(e_u)\big[\|\xi_u\|^2+h^{2k+2}\big]-\frac{1}{2}\mathcal{A}(u,\xi_u),$$

where C and $C_\star$ in $C(e_u)$ are the positive constants independent of h and $\mathbf{w}_h$.

Based on the above lemmas, we can get the optimal error estimate for the semi-discrete scheme.

Theorem 3.1 *Assume that each component of the exact solution of the problem* (1.1) *satisfy* $\mathbf{w}_F \in L^\infty([0,T]; W^{k+1,\infty}(I))$ *for* $F = u, q, w$ *and* p; *Let* $\mathbf{w}_h = (u_h, q_h, w_h, p_h)$ *be the numerical solution of* (2.7) *with the initial setting* (2.10), *where the finite element space* V_h *is of piecewise polynomials of degree at most* $k \ge 1$, *defined on arbitrary regular triangulations of* $I = [0,1]$. *Then there exists a positive constant* $C > 0$ *independent of* h *and* $\mathbf{w}_h$, *such that*

$$\|u - u_h\|^2_{L^\infty(L^2)} + \mu\|q - q_h\|^2_{L^\infty(L^2)} \le Ch^{2k+2}. \tag{3.13}$$

Proof To deal with the nonlinearity of the flux $f(u)$, we would like to use *a priori* assumption as follows: for h small enough, there exists a constant $C_0 > 0$ independent of h, such that

$$\|e_u\|_{L^\infty(L^\infty)} \le C_0 h. \tag{3.14}$$

For linear convection flux this assumption is not necessary. We will verify this assumption is reasonable at the end of this proof.

The *a priori* assumption (3.14) implies that $C(e_u) \le C$ for any time $t \in [0,T]$, where C is a positive constant independent of t, h and $\mathbf{w}_h$. Together with Lemmas 3.1 and 3.2, (3.7) yields that

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|\xi_{uq}\|^2_\mu + \frac{\delta}{2}\|\xi_q\|^2 + \frac{1}{2}\mathcal{A}(u,\xi_u) \le Ch^{2k+2} + C\|\xi_{uq}\|^2_\mu + \mathcal{S}, \quad \forall t \in (0,T], \tag{3.15}$$

where $\mathcal{S} = \int_0^1 [\eta_{u,t}\xi_u + \mu\eta_{q,t}\xi_q + \mu\eta_q\xi_p]\,\mathrm{d}x$. We do not estimate $\mathcal{S}$ at any certain time t, but tend to estimate its integration in the time interval $[0,t]$. Since Gauss-Raudu projection (2.6) is linear for time, it is obvious that $\eta_{u,t} = \eta_w$; thus it follows $\xi_{q,t} = \xi_p$ from the semi-discrete scheme (2.7b). Then Young's inequality yields that

$$\begin{aligned}
\int_0^t \mathcal{S}(s)\,\mathrm{d}s &= \mu\int_0^1 \big[\eta_q(t)\xi_q(t) - \eta_q(0)\xi_q(0)\big]\,\mathrm{d}x + \int_0^t\int_0^1 \eta_{u,t}\xi_u(s)\,\mathrm{d}x\,\mathrm{d}s \\
&\le \frac{\mu}{4}\|\xi_q\|^2(t) + \frac{\mu}{4}\|\xi_q\|^2(0) + \mu\|\eta_q\|^2(t) + \mu\|\eta_q\|^2(0) \\
&\quad + C\int_0^t \big[\|\xi_u\|^2 + \|\eta_{u,t}\|^2\big](s)\,\mathrm{d}s \\
&\le \frac{\mu}{4}\|\xi_q\|^2(t) + \frac{\mu}{4}\|\xi_q\|^2(0) + C\int_0^t \|\xi_u\|^2(s)\,\mathrm{d}s + Ch^{2k+2},
\end{aligned} \tag{3.16}$$

where the interpolation property (3.5) is used.

We integrate inequality (3.15) in time, and substitute (3.16) into the new inequality. Then we obtain, for any time $t \in (0,T]$, that there exists a positive constant C independent of t, h and $\mathbf{w}_h$, such that

$$\|\xi_{uq}\|^2_\mu(t) + \delta\int_0^t \big[\|\xi_q\|^2 + \mathcal{A}(u,\xi_u)\big](s)\,\mathrm{d}s \le C\left[\int_0^t \|\xi_{uq}\|^2_\mu(s)\,\mathrm{d}s + \|\xi_{uq}\|^2_\mu(0) + h^{2k+2}\right].$$

An application of the classical Gronwall lemma for the above inequality implies that

$$\|\xi_{uq}\|^2_\mu(t) \le C\big[h^{2k+2} + \|\xi_{uq}\|^2_\mu(0)\big] \le Ch^{2k+2}, \quad \forall t \in (0,T], \tag{3.17}$$

where we have used the approximation property of initial solutions

$$\xi_u(0) = 0, \qquad \|\xi_q(0)\| \le C\|u_{0,x}\|_{k+1}h^{k+1}, \tag{3.18}$$

since $u_h(x,0) = \mathbb{Q}_h^+ u_0(x)$ and $q_h(x,0) = \mathbb{P}_h u_{0,x}(x)$. Finally, (3.13) follows from triangular inequality combing (3.17) with the interpolation property (3.5).

Before we complete this proof, we have to verify the *a priori* assumption is reasonable. Obviously the initial solution can be bounded by Ch for h small enough. By the interpolation property (3.5) and inverse property (iii) in (3.3), we can get from inequality (3.17) that

$$\|e_u\|_\infty \le Ch^{-1/2}\|\xi_u\| + \|\eta_u\|_\infty \le Ch^{k+1/2}, \quad \forall t \in (0, T],$$

where C is the positive constant independent of h. Thus the *a priori* assumption (3.14) holds for h small enough, since $k \ge 1$. Now we complete the proof of this theorem. □

3.3 Error Estimate for the Fully-Discrete LDG Method

In this subsection we would like to obtain the optimal error estimate for the fully-discrete LDG scheme, RKLDG(2,k). To do that, the solution of (1.1) is assumed to have same smoothness as that for the semi-discrete scheme, and $\digamma_{ttt} \in L^\infty([0,T], L^2)$ in addition, for $\digamma = u$ and q.

Let $[T/\tau]$ be the maximum integer not greater than T/τ. Following [25] we define the reference function with respect to the second stage of RK(2), of the form

$$\digamma^\star(t) = \digamma(t) + \tau\digamma_t(t), \quad \text{for } \digamma = u, q, p, w, \tag{3.19}$$

and denote $\digamma^n = \digamma(t^n)$ and $\digamma^{n\star} = \digamma^\star(t^n)$, for any $n \le [T/\tau]$. Denote the error at each stage of RK(2) by $e_\digamma^n = \digamma_h^n - \digamma^n$ and $e_\digamma^{n\star} = \digamma_h^{n\star} - \digamma^{n\star}$, respectively.

Below we use notation $\sharp$ to represent n and $n\star$. According to the definition of numerical flux (2.8), we use different projections to divide each component of the error $e_\digamma$ into two parts, as we have done for semi-discrete scheme. They reads

$$e_\digamma^\sharp = (\digamma_h^\sharp - \mathbb{Q}_h^+\digamma^\sharp) - (\digamma^\sharp - \mathbb{Q}_h^+\digamma^\sharp) = \xi_\digamma^\sharp - \eta_\digamma^\sharp, \quad \text{for } \digamma = u, w; \tag{3.20a}$$

$$e_\digamma^\sharp = (\digamma_h^\sharp - \mathbb{Q}_h^-\digamma^\sharp) - (\digamma^\sharp - \mathbb{Q}_h^-\digamma^\sharp) = \xi_\digamma^\sharp - \eta_\digamma^\sharp, \quad \text{for } \digamma = q, p; \tag{3.20b}$$

where $\xi_\digamma^\sharp \in V_h$ need to be estimated below, and $\eta_\digamma^\sharp$ is the interpolation error. Remark that $(\eta_u^\sharp)^- = (\eta_w^\sharp)^- = (\eta_p^\sharp)^+ = (\eta_q^\sharp)^+ = 0$ at each interface point. From (3.2), we have that

$$\|\eta_\digamma^\sharp\| + h^{1/2}\|\eta_\digamma^\sharp\|_\infty + h^{1/2}\|\eta_\digamma^\sharp\|_{\Gamma_h} \le C\|\digamma\|_{k+1,\infty}h^{k+1}, \quad \forall n \le [T/\tau], \tag{3.21}$$

where C is a positive constant independent of n, h, τ and $\digamma$.

3.3.1 Error Equations

Now we want to get the error equation about $\xi_\digamma^\sharp$. To this end, we firstly consider the time-marching of exact solution of (1.1). By a series of Taylor expansion [25] in time for u and q, one can deduce that the solution $\mathbf{w}^\sharp$ satisfies, for any test function $(v_{uq}, v_{wp}) \in (V_h)^4$ and

$n < [T/\tau]$, that

$$\mathcal{K}(\mathbf{w}_{uq}^{n\star}, v_{uq}) = \mathcal{K}(\mathbf{w}_{uq}^{n}, v_{uq}) + \tau\mathcal{K}(\mathbf{w}_{wp}^{n}, v_{uq}), \tag{3.22a}$$

$$\mathcal{K}(\mathbf{w}_{wp}^{n}, v_{wp}) = \mathcal{H}(\mathbf{w}_{uq}^{n}, v_{wp}), \tag{3.22b}$$

$$\begin{aligned}\mathcal{K}(\mathbf{w}_{uq}^{n+1}, v_{uq}) &= \frac{1}{2}\mathcal{K}(\mathbf{w}_{uq}^{n}, v_{uq}) + \frac{1}{2}\mathcal{K}(\mathbf{w}_{uq}^{n\star}, v_{uq}) \\ &\quad + \frac{\tau}{2}\mathcal{K}(\mathbf{w}_{wp}^{n\star}, v_{uq}) + \mathcal{K}(\mathcal{E}^n, v_{uq}),\end{aligned} \tag{3.22c}$$

$$\mathcal{K}(\mathbf{w}_{wp}^{n\star}, v_{wp}) = \mathcal{H}(\mathbf{w}_{uq}^{n\star}, v_{wp}) + \mathcal{K}(\widetilde{\mathcal{E}}_x^n, v_{wp}), \tag{3.22d}$$

where $\mathcal{E}^n$ is the local time-discrete error about $\mathbf{w}_{uq}$ after one complete step of RK(2), and $\widetilde{\mathcal{E}}^n$ is resulted from the nonlinear perturbation of convection flux $f(u)$ after the first stage of RK(2). In detail, they are given by

$$\mathcal{E}^n = \frac{1}{2}\int_0^\tau s(s-\tau)\mathbf{w}_{uq,ttt}(t^n+s)\,\mathrm{d}s, \qquad \widetilde{\mathcal{E}}^n = \left((u_t^n)^2\int_0^\tau (\tau-s)f''(u^n+su_t^n)\,\mathrm{d}s, 0\right),$$

hence $\|\mathcal{E}^n\| = \mathcal{O}(\tau^3)$ and $\|\widetilde{\mathcal{E}}_x^n\| = \mathcal{O}(\tau^2)$ for any $n < [T/\tau]$.

By subtracting (3.22) from the fully-discrete scheme RKLDG(2,k), we obtain the following equation about $\xi_F^\sharp$. It reads, for any $n < [T/\tau]$, that

$$\mathcal{K}(\xi_{uq}^{n\star}, v_{uq}) = \mathcal{K}(\xi_{uq}^{n}, v_{uq}) + \tau\mathcal{K}(e_{wp}^{n}, v_{uq}) + \mathcal{K}(\mathcal{D}_{uq}^{n\star}, v_{uq}), \tag{3.23a}$$

$$\mathcal{K}(e_{wp}^{n}, v_{wp}) = \mathcal{H}(e_{uq}^{n}, v_{wp}), \tag{3.23b}$$

$$\begin{aligned}\mathcal{K}(\xi_{uq}^{n+1}, v_{uq}) &= \frac{1}{2}\mathcal{K}(\xi_{uq}^{n}, v_{uq}) + \frac{1}{2}\mathcal{K}(\xi_{uq}^{n\star}, v_{uq}) \\ &\quad + \frac{\tau}{2}\mathcal{K}(e_{wp}^{n\star}, v_{uq}) + \mathcal{K}(\mathcal{D}_{uq}^{n+1} + \mathcal{E}^n, v_{uq}),\end{aligned} \tag{3.23c}$$

$$\mathcal{K}(e_{wp}^{n\star}, v_{wp}) = \mathcal{H}(e_{uq}^{n\star}, v_{wp}) + \mathcal{K}(\widetilde{\mathcal{E}}_x^n, v_{wp}), \tag{3.23d}$$

hold for any v_{uq} and v_{wp} in $(V_h)^2$, where $\mathcal{D}_{uq}^{n\star} = \eta_{uq}^{n\star} - \eta_{uq}^{n}$ and $\mathcal{D}_{uq}^{n+1} = \eta_{uq}^{n+1} - (\eta_{uq}^{n} + \eta_{uq}^{n\star})/2$. Since Gauss-Raudu projection (2.6) is linear for time, we also have

$$\|\mathcal{D}_{uq}^{n\star}\| + \|\mathcal{D}_{uq}^{n+1}\| + \|\eta_{uq}^{n+1} - \eta_{uq}^{n}\| \le Ch^{k+1}\tau, \quad \forall n < [T/\tau], \tag{3.24}$$

where C is the positive constant independent of n, h and τ.

We are going to obtain the optimal error estimate for the RKLDG(2,k) scheme, by virtue of energy analysis for (3.23), along the same line as in [25]. To do that, we take different test function in each equation of (3.23). Let $v_{uq} = (\xi_u^n, \mu\xi_q^n)$ in (3.23a), $v_{wp} = (\xi_u^n, 0)$ in (3.23b), $v_{uq} = (\xi_u^{n\star}, \mu\xi_q^{n\star})$ in (3.23c), and $v_{wp} = (\xi_u^{n\star}, 0)$ in (3.23d), respectively. By summing up the above new equations, after a simple manipulation we can finally get the following energy equation

$$\|\xi_{uq}^{n+1}\|_\mu^2 - \|\xi_{uq}^{n}\|_\mu^2 = \underbrace{\mathcal{J}\otimes\mathcal{H}(e_{\mathbf{w}}^n, \xi_{uq}^n)}_{\mathcal{R}_1} + \underbrace{\mathcal{J}^\star\otimes\mathcal{H}^\star(e_{\mathbf{w}}^{n\star}, \xi_{uq}^{n\star})}_{\mathcal{R}_2} + \underbrace{\|\xi_{uq}^{n+1} - \xi_{uq}^{n\star}\|_\mu^2}_{\mathcal{R}_3}, \tag{3.25a}$$

where $\mathcal{R}_1$ and $\mathcal{R}_2$ are inheriting from the semi-discrete scheme, of the form

$$\mathcal{J}\otimes\mathcal{H}(e_{\mathbf{w}}^n, \xi_{uq}^n) = \tau\mathcal{K}\otimes\mathcal{H}(e_{\mathbf{w}}^n, \xi_{uq}^n) + (\mathcal{D}_{uq}^{n\star}, \xi_{uq}^n)_\mu, \tag{3.25b}$$

$$\mathcal{J}^{\star} \otimes \mathcal{H}^{\star}(e_{\mathbf{w}}^{n\star}, \xi_{uq}^{n\star}) = \tau \mathcal{K} \otimes \mathcal{H}(e_{\mathbf{w}}^{n\star}, \xi_{uq}^{n\star}) + (2\mathcal{D}_{uq}^{n+1} + 2\mathcal{E}^{n} + \widetilde{\mathcal{E}}_{x}^{n}, \xi_{uq}^{n\star})_{\mu}, \tag{3.25c}$$

and $\mathcal{R}_3$ is resulted from the second order explicit RK time-marching. Below we will estimate them separately.

3.3.2 Estimates to $\mathcal{R}_1$ and $\mathcal{R}_2$

The main term in formula (3.25b) is $\mathcal{K} \otimes \mathcal{H}(e_{\mathbf{w}}^{n}, \xi_{uq}^{n})$, which is easy to estimate as we have done for the semi-discrete scheme. Thus we only present the conclusion and the sketch of proof.

Lemma 3.3 *Suppose the interpolation property* (3.21) *is satisfied; then we have, for any $n < [T/\tau]$, that*

$$\mathcal{R}_1 \le C(e_u^n)\big[h^{2k+2} + \|\xi_u^n\|^2\big]\tau + \mu\tau \int_0^1 \eta_q^n \xi_p^n \,\mathrm{d}x + \frac{1}{2}(\mu - \delta)\|\xi_q^n\|^2\tau - \frac{1}{2}\mathcal{A}(u^n, \xi_u^n)\tau,$$

where C and $C_\star$ in $C(e_u^n)$ are the positive constants independent of n, h, τ and $\mathbf{w}_h$.

Proof Recall that (3.10) plays an important role in the proof for Lemma 3.1. We will show this property is also true for the fully-discrete scheme.

In fact, it follows from an inductive analysis. By substituting the time-marching relation, between $\mathbf{w}_{uq,h}^{\sharp}$ and $\mathbf{w}_{wp,h}^{\sharp}$, into the DG spatial discrete of getting $p_h^{\sharp}$ from $w_h^{\sharp}$, we deduce that

$$\int_{I_j} q_h^{\sharp} v_h(x)\,\mathrm{d}x + H_j^{-}(u_h^{\sharp}, v_h) = 0, \quad \forall v_h \in V_h,\ j = 1, 2, \ldots, N,$$

since the initial solutions u_h^0 and q_h^0 are given by (2.10). As an immediate consequence of equation $q^{\sharp} = u_x^{\sharp}$, we have the following important equation

$$\int_{I_j} e_q^{\sharp} v_h(x)\,\mathrm{d}x + H_j^{-}(e_u^{\sharp}, v_h) = 0, \quad \forall v_h \in V_h,\ j = 1, 2, \ldots, N. \tag{3.26}$$

Please keep in mind that the errors we want to estimate and the test functions are both staying at the same time now. Along the same line as in Lemma 3.1, we add two equations by taking $v_h = -\delta\xi_q^n$ and $v_h = -\mu\xi_p^n$ in (3.26), respectively. It yields an almost same estimate as (3.9), except the appending supscript n. Also the estimate to the convection flux is almost same as Lemma 3.2. The left inner product is easy to estimate by using Young's inequality and the interpolation property (3.24). Finally, we collect up the above estimates and complete the proof of this lemma. □

There is a similar conclusion for the second term $\mathcal{R}_2$, along the same line. Here we only present the result without any proof.

Lemma 3.4 *Suppose the interpolation property* (3.21) *is satisfied; then we have, for any $n < [T/\tau]$, that*

$$\mathcal{R}_2 \le C\tau^6 + C(e_u^{n\star})\big[h^{2k+2} + \|\xi_u^{n\star}\|^2\big]\tau$$
$$+ \mu\tau \int_0^1 \eta_q^{n\star} \xi_p^{n\star} \,\mathrm{d}x + \frac{1}{2}(\mu - \delta)\|\xi_q^{n\star}\|^2\tau - \frac{1}{2}\mathcal{A}(u^{n\star}, \xi_u^{n\star})\tau,$$

where C and $C_\star$ in $C(e_u^{n\star})$ are the positive constants independent of n, h, τ and $\mathbf{w}_h$.

3.3.3 Estimate to $\mathcal{R}_3$

We can estimate the third term $\mathcal{R}_3$ under a suitable time-space restriction, namely, (2.22). To describe this more clearly, we use notation M to emphasize the positive constant independent of $n, h, \tau, \mathbf{w}$ and $\mathbf{w}_h$, which may have a different value in each occurrence.

We utilize the structure of RK(2) time-marching to estimate $\mathcal{R}_3$. Subtracting equation (3.23c) from (3.23a), we have, for any $v_{uq} \in (V_h)^2$, that

$$\mathcal{K}(\xi_{uq}^{n+1} - \xi_{uq}^{n\star}, v_{uq}) = \frac{\tau}{2}\mathcal{K}(e_{wp}^{n\star} - e_{wp}^{n}, v_{uq}) + \mathcal{K}(\mathcal{D}_{uq}^{n+1} - \frac{1}{2}\mathcal{D}_{uq}^{n\star} + \mathcal{E}^n, v_{uq}).$$

Let $v_{uq} = (\xi_u^{n+1} - \xi_u^{n\star}, \mu(\xi_q^{n+1} - \xi_q^{n\star}))$ in the above equation, and sum up the new equations. A simple application of Young's inequality yields that

$$\|\xi_{uq}^{n+1} - \xi_{uq}^{n\star}\|_\mu^2 \le \frac{1}{2}\|\xi_{wp}^{n\star} - \xi_{wp}^{n}\|_\mu^2 \tau^2 + Ch^{2k+2}\tau^2 + C\tau^6, \tag{3.27}$$

where the interpolation property (3.24) and $\|\mathcal{E}^n\| = \mathcal{O}(\tau^3)$ are used.

Now we have to estimate $\|\xi_{wp}^{n\star} - \xi_{wp}^{n}\|_\mu$. To do that, we subtract (3.23b) from (3.23d), with the same test function $v_{wp} = \xi_{wp}^{n\star} - \xi_{wp}^{n}$. It deduces that

$$\begin{aligned}
\|\xi_{wp}^{n\star} - \xi_{wp}^{n}\|_\mu^2 = {} & \underbrace{\sum_{1\le j\le N}\big[K_j(e_u^{n\star}, v_w) - K_j(e_u^{n}, v_w)\big]}_{T_1(v_{wp})} - \underbrace{\sum_{1\le j\le N}\big[\delta H_j^+(e_q^{n\star} - e_q^{n}, v_w)\big]}_{T_2(v_{wp})} \\
& \underbrace{- \sum_{1\le j\le N}\big[\mu H_j^+(e_p^{n\star} - e_p^{n}, v_w) + \mu H_j^-(e_w^{n\star} - e_w^{n}, v_p)\big]}_{T_3(v_{wp})} \\
& \underbrace{+(\eta_{wp}^{n\star} - \eta_{wp}^{n} + \widetilde{\mathcal{E}}_x^n, v_{wp})_\mu}_{T_4(v_{wp})}.
\end{aligned} \tag{3.28}$$

Next we will estimate each above term, separately.

It is easy to get an estimate to $T_1(v_{wp})$, along the same line as [25]. For completeness of this paper, we give the proof in the appendix.

Lemma 3.5 *Let ε be any given small positive constant. Suppose that the interpolation properties (3.21) and (3.24) are satisfied; then we have, for any $v_w \in V_h$ and $n < [T/\tau]$, that*

$$\begin{aligned}
T_1(v_{wp}) \le {} & \varepsilon\|v_w\|^2 + \big[C(e_u^n) + C(e_u^{n\star})\big]h^{2k+2} + C(e_u^n)\|\xi_u^n\|^2 + C(e_u^{n\star})\|\xi_u^{n\star}\|^2 \\
& + MS_{\max}h^{-1}\mathcal{A}(u^n, \xi_u^n) + MS_{\max}h^{-1}\mathcal{A}(u^{n\star}, \xi_u^{n\star}) + M\mathcal{B}(u^n, \xi_u^n) + M\mathcal{B}(u^{n\star}, \xi_u^{n\star}),
\end{aligned}$$

where $\mathcal{A}(\cdot,\cdot)$ and $\mathcal{B}(\cdot,\cdot)$ have been defined in Sect. 3.1, and the positive constants C, $C_\star$ and M are independent of n, h, τ, and $\mathbf{w}_h$; however, maybe depending on ε.

Let ε be any given small positive constant, and keep in mind below that $v_{wp} = \xi_{wp}^{n\star} - \xi_{wp}^{n}$. Since $H_j^+(\eta_q^{n\star} - \eta_q^n, v_w) = 0$ for Gauss-Raudu projection (2.6), we can use the inverse properties (i) and (ii) in (3.3), and Young's inequality, to have that

$$T_2(v_{wp}) = -\delta \sum_{1\le j\le N} H_j^+(\xi_q^{n\star} - \xi_q^n, v_w) \le \delta\|\xi_q^{n\star} - \xi_q^n\|\|v_{w,x}\| + \delta\|\xi_q^{n\star} - \xi_q^n\|_{\Gamma_h}\|[\![v_w]\!]\|_{\Gamma_h}$$
$$\le \varepsilon\|v_w\|^2 + C\delta^2 h^{-2}\big(\|\xi_q^{n\star}\|^2 + \|\xi_q^n\|^2\big). \tag{3.29a}$$

Also there holds $H_j^+(\eta_p^{n\star} - \eta_p^n, v_w) = H_j^-(\eta_w^{n\star} - \eta_w^n, v_w) = 0$, hence Lemma 2.1 together with periodic boundary condition yields that

$$T_3(v_{wp}) = -\mu \sum_{1\le j\le N} \big[\Delta_j(v_p^+ v_w^-) - H_j^+(\eta_p^{n\star} - \eta_p^n, v_w) - H_j^-(\eta_w^{n\star} - \eta_w^n, v_p)\big] = 0. \tag{3.29b}$$

By interpolation property (3.24) and $\|\widetilde{\mathcal{E}}_x^n\| = \mathcal{O}(\tau^2)$, Young's inequality yields that

$$T_4(v_{wp}) \le \varepsilon\|v_{wp}\|_\mu^2 + Ch^{2k+2} + C\tau^4. \tag{3.29c}$$

Finally substituting estimate (3.29) and Lemma 3.5 into (3.28), we can obtain an estimate to $\|\xi_{wp}^{n\star} - \xi_{wp}^n\|_\mu$, if ε is small enough, for instance, $\varepsilon = 1/8$. Then an immediate consequence of this result and (3.27) is the estimate we wanted. We conclude it in the next lemma.

Lemma 3.6 *Suppose the interpolation properties* (3.21) *and* (3.24) *are satisfied*; *then we have*, *for any* $n < [T/\tau]$, *that*

$$\mathcal{R}_3 \le \big[C(e_u^n) + C(e_u^{n\star})\big]h^{2k+2}\tau^2 + C\tau^6$$
$$+ C(e_u^n)\|\xi_u^n\|^2\tau^2 + C(e_u^{n\star})\|\xi_u^{n\star}\|^2\tau^2 + C\delta^2 h^{-2}\|\xi_q^n\|^2\tau^2 + C\delta^2 h^{-2}\|\xi_q^{n\star}\|^2\tau^2$$
$$+ MS_{\max}h^{-1}\big[\mathcal{A}(u^n, \xi_u^n) + \mathcal{A}(u^{n\star}, \xi_u^{n\star})\big]\tau^2 + M\mathcal{B}(u^n, \xi_u^n)\tau^2 + M\mathcal{B}(u^{n\star}, \xi_u^{n\star})\tau^2,$$

where C *and* $C_\star$ *are the positive constants independent of* n, h, τ, *and* $\mathbf{w}_h$.

In Lemma 3.6 there emerges a new term $\mathcal{B}(u^\sharp, \xi_u^\sharp)$, so we have to get an upper bound for it. The next lemma shows the relation between the spatial derivative and the jumps at boundary points, namely, $\mathcal{B}(u^\sharp, \xi_u^\sharp)$ can be controlled by $\mathcal{A}(u^\sharp, \xi_u^\sharp)$ in some sense.

Lemma 3.7 *Suppose that the interpolation property* (3.21) *is satisfied*; *then we have*, *for any* $n < [T/\tau]$, *that*

$$\mathcal{B}(u^\sharp, \xi_u^\sharp) \le MS_{\max}^\sharp h^{-1}\mathcal{A}(u^\sharp, \xi_u^\sharp) + C\|\xi_u^\sharp\|^2 + C\|\xi_q^\sharp\|^2 + Ch^{2k+2}, \quad \sharp = n, n\star, \tag{3.30}$$

where $S_{\max}^\sharp$ *is the maximum of convection speed* $|f'(u^\sharp)|$ *over all boundary points*, *and* C *and* M *are the positive constants independent of* n, h *and* τ.

Proof Recall $H_j^-(\eta_u^{n\star} - \eta_u^n, v_h) = 0$ for any $v_h \in V_h$ and $j = 1, 2, \ldots, N$, since Gauss-Raudu projection (2.6). Integration by parts for (3.26) yields that

$$\int_{I_j} \xi_{u,x}^\sharp v_h \,\mathrm{d}x = \int_{I_j} e_q^\sharp v_h \,\mathrm{d}x - [\![\xi_u]\!]_{j-\frac12}^\sharp v_{h,j-\frac12}^+, \quad \forall v_h \in V_h,\ j = 1, 2, \ldots, N.$$

We take the test function $v_h = \xi^\sharp_{u,x}$ in this equation, and control the trace $v^+_{h,j-1/2}$ by the inverse inequality (ii) in (3.3). Then Young's inequality yields that

$$\|\xi^\sharp_{u,x}\|^2_{I_j} \le C\|\xi^\sharp_q\|^2_{I_j} + Mh^{-1}[\![\xi^\sharp_u]\!]^2_{j-\frac{1}{2}} + C\|\eta^\sharp_q\|^2_{I_j}.$$

We can complete the proof of this lemma, by substituting the above inequality and the interpolation property (3.21), into the formula of $\mathcal{B}(u^\sharp, \xi^\sharp_u)$, since u is smooth enough. □

Corollary 3.1 *Suppose that the interpolation property* (3.21), *and the general time-spatial condition $\tau = \mathcal{O}(h)$ are satisfied*; *then we have*, *for any $n < [T/\tau]$*, *that*

$$\mathcal{B}(u^\sharp, \xi^\sharp_u) \le MS_{\max}h^{-1}\mathcal{A}(u^\sharp, \xi^\sharp_u) + C\|\xi^\sharp_u\|^2 + C\|\xi^\sharp_q\|^2 + Ch^{2k+2}, \quad \sharp = n, n\star, \tag{3.31}$$

where C and M are the positive constants independent of n, h and τ.

Proof We only need to point out that $S^n_{\max} \le S_{\max}$ and $S^{n\star}_{\max} \le S_{\max} + C\tau^2$. The former is obvious, and the latter follows from $f'(u^{n\star}) - f(u^{n+1}) = \mathcal{O}(u^n + \tau u^n_t - u^{n+1}) = \mathcal{O}(\tau^2)$, since f and u are assumed to be smooth enough. Then (3.31) is an immediate consequence of Lemma 3.7 and the inverse property (ii) in (3.3). □

To obtain the error estimate for the fully-discrete scheme RKLDG(2,k), we also need to build up the relation between errors ξ^n_{uq} and $\xi^{n\star}_{uq}$. The proof for this conclusion is very similar as that for Lemma 3.6, so omitted here.

Lemma 3.8 *Suppose that the interpolation property* (3.21) *is satisfied*; *then we have*, *for any $n < [T/\tau]$*, *that*

$$\|\xi^{n\star}_{uq} - \xi^n_{uq}\|^2_\mu \le C(e^n_u)\big[h^{2k+2} + \|\xi^n_u\|^2\big]\tau^2 + C\delta^2h^{-2}\|\xi^n_q\|^2\tau^2 + C\mathcal{A}(u^n, \xi_u)\tau^2 + C\mathcal{B}(u^n, \xi_u)\tau^2,$$

where C and $C_\star$ are the positive constants independent of n, h, τ, and $\mathbf{w}_h$.

3.3.4 Main Conclusion

Based on the above lemmas, we can obtain the optimal error estimate for the fully-discrete scheme RKLDG(2,k). The result is given in the following theorem.

Theorem 3.2 *Assume each component of the exact solution of the problem* (1.1) *satisfy $\mathbf{w}_F \in L^\infty(W^{k+1,\infty}(I))$ for $F = u, q, w, p$, and $F_{ttt} \in L^\infty(L^2)$ for $F = u, q$*; *Let $\mathbf{w}_h = (u_h, q_h, w_h, p_h)$ be the numerical solution of the fully-discrete scheme*, *RKLDG*(2, k), *with the initial setting* (2.10), *where the finite element space V_h is of piecewise polynomials of degree at most $k \ge 1$*, *defined on arbitrary regular triangulations of $I = [0, 1]$*, *and the time step is restricted under the CFL condition* (2.22). *Then there exists a positive constant C independent of n, h, τ and $\mathbf{w}_h$ such that*

$$\|u(t^n) - u^n_h\|^2 + \mu\|q(t^n) - q^n_h\|^2 \le Ch^{2k+2} + C\tau^4, \quad \forall n \le [T/\tau]. \tag{3.32}$$

Proof As we have done before, to deal with the nonlinearity of convection flux $f(u)$, we use the following *a priori* assumption: for h small enough, there exists a positive constant C_1 independent of n, h, τ and $\mathbf{w}_h$ such that

$$\|e^m_u\|_\infty \le C_1h, \quad \|e^{m\star}_u\|_\infty \le C_1h, \quad \forall m \le n. \tag{3.33}$$

For linear flux $f(u)$, this assumption is not necessary. It is obvious that this assumption is true for $n=0$, since (3.18) and the interpolation property (3.21). Later we will prove it is also reasonable for $n+1$.

The *a priori* assumption (3.33) implies that $C(e_u^n)+C(e_u^{n\star})\le C$, where C is a positive constant independent of n,h and τ. By using the inverse inequalities (i) and (ii) in (3.3), Lemma 3.8 implies that, if $\tau=\mathcal{O}(h)$ then

$$\|\xi_{uq}^{n\star}\|_\mu^2 \le C\|\xi_{uq}^n\|_\mu^2 + Ch^{2k+2}\tau^2. \tag{3.34}$$

Now we substitute those estimates for $\mathcal{R}_1,\mathcal{R}_2$ and $\mathcal{R}_3$, given in Lemmas 3.3, 3.4 and 3.6, into the energy equation (3.25a). Then by using Corollary 3.1 and inequality (3.34), we obtain that, if $\tau=\mathcal{O}(h)$, there exists a positive constant C independent of n,h,τ and $\mathbf{w}_h$ such that

$$\begin{aligned}
&\|\xi_{uq}^{n+1}\|_\mu^2 - \|\xi_{uq}^n\|_\mu^2 + \frac{\delta}{2}\|\xi_q^n\|^2\tau + \frac{\delta}{2}\|\xi_q^{n\star}\|^2\tau + \frac{1}{2}\mathcal{A}(u^n,\xi_u^n)\tau + \frac{1}{2}\mathcal{A}(u^{n\star},\xi_u^{n\star})\tau\\
&\quad\le C(h^{2k+2}\tau+\tau^6) + C\|\xi_{uq}^n\|_\mu^2\tau + \underbrace{\mu(\eta_q^n,\xi_p^n)\tau + \mu(\eta_q^{n\star},\xi_p^{n\star})\tau}_{\mu\mathcal{T}_{RK}^n}\\
&\qquad+\underbrace{C\delta^2h^{-2}\|\xi_q^n\|^2\tau^2}_{\mathcal{G}_1^n} + \underbrace{MS_{\max}h^{-1}\big[\mathcal{A}(u^n,\xi_u^n)+\mathcal{A}(u^{n\star},\xi_u^{n\star})\big]\tau^2}_{\mathcal{G}_2^n},
\end{aligned}\tag{3.35}$$

where M is the positive constant independent of $n,h,\tau,\mathbf{w}$ and $\mathbf{w}_h$.

The last two terms in (3.35) can be controlled by using the time-spatial restriction (2.22). Since $\delta^2\tau=\mathcal{O}(h^2)$, we have $\mathcal{G}_1^n\le C\|\xi_q^n\|^2\tau$. Since $S_{\max}\tau\le h/(4M)$, we can control $\mathcal{G}_2^n$ by the interface's jumps on the left-hand side of (3.35). Consequently, by summing up the inequality (3.35) over the time level from 0 to n, we finally get that

$$\begin{aligned}
&\|\xi_{uq}^{n+1}\|_\mu^2 + \frac{\delta}{2}\sum_{m\le n}\big[\|\xi_q^m\|^2+\|\xi_q^{m\star}\|^2\big]\tau + \frac{1}{4}\sum_{m\le n}\big[\mathcal{A}(u^m,\xi_u^m)+\mathcal{A}(u^{m\star},\xi_u^{m\star})\big]\tau\\
&\quad\le C\sum_{m\le n}\big[\|\xi_{uq}^m\|_\mu^2\tau + h^{2k+2}\tau+\tau^6\big] + \|\xi_{uq}(0)\|_\mu^2 + \mu\sum_{m\le n}\mathcal{T}_{RK}^n,
\end{aligned}\tag{3.36}$$

under the time-spatial restriction (2.22), where C is the positive constant independent of n,h,τ, and $\mathbf{w}_h$.

The last term in (3.36) can be looked upon as the discretization of $\mu\int_{t^n}^{t^{n+1}}T(t)\,\mathrm{d}t$, where $T(t)=\int_0^1\eta_q(x,t)\xi_p(x,t)\,\mathrm{d}x$. Recall that we have used an integration by parts to transfer the position of time derivative, in the error estimate for the semi-discrete scheme. So we rearrange the order of sum in time to give a discrete version of integration by parts. After a simple manipulation, it is easy to get the following identity

$$\begin{aligned}
\mathcal{T}_{RK}^n &= (\eta_q^n,\xi_p^n)\tau + (\eta_q^{n\star},\xi_p^{n\star})\tau = (\eta_q^n,\xi_p^{n\star}+\xi_p^n)\tau + (\eta_q^{n\star}-\eta_q^n,\xi_p^{n\star})\tau\\
&= (\eta_q^n,\xi_q^{n+1}-\xi_q^n) - (\eta_q^n,(\xi_q^{n+1}-\xi_q^n)-\tau(\xi_p^{n\star}+\xi_p^n)) + (\eta_q^{n\star}-\eta_q^n,\xi_p^{n\star})\tau\\
&= (\eta_q^{n+1},\xi_q^{n+1}) - (\eta_q^n,\xi_q^n)\\
&\quad\underbrace{-(\eta_q^{n+1}-\eta_q^n,\xi_q^{n+1}) + (\eta_q^{n\star}-\eta_q^n,\mathbb{a})}_{\Pi_1}\underbrace{-(\eta_q^{n\star}-\eta_q^n,\mathbb{b}) - (\eta_q^n,\mathbb{c})}_{\Pi_2},
\end{aligned}\tag{3.37}$$

where $\mathbb{a} = 2\xi_q^{n+1} - \xi_q^{n\star} - \xi_q^n$, $\mathbb{b} = \mathbb{a} - \tau\xi_p^{n\star}$, and $\mathbb{c} = (\xi_q^{n+1} - \xi_q^n) - \tau(\xi_p^{n\star} + \xi_p^n)$. By Young's inequality and the interpolation property (3.24), it is easy to see

$$\Pi_1 \le \frac{1}{4}\|\xi_q^{n+1}\|^2\tau + C\big[\|\xi_q^{n\star}\|^2 + \|\xi_q^n\|^2 + h^{2k+2}\big]\tau. \tag{3.38}$$

To estimate Π_2, we need to estimate $\mathbb{b}$ and $\mathbb{c}$ by the structure of RK(2) time-marching. It is easy to get from (3.23a) and (3.23c), respectively, that

$$(\mathbb{b}, v_h) = -\tau(\eta_p^{n\star}, v_h) + 2(\mathcal{D}_q^{n+1} + \mathcal{E}_q^n, v_h), \quad \forall v_h \in V_h; \tag{3.39a}$$

$$(\mathbb{c}, v_h) = -\tau(\eta_p^{n\star} + \eta_p^n, v_h) + (2\mathcal{D}_q^{n+1} - \mathcal{D}_q^{n\star} + 2\mathcal{E}_q^n, v_h), \quad \forall v_h \in V_h. \tag{3.39b}$$

Take the test function $v_h = \mathbb{b}$ in (3.39a), and $v_h = \mathbb{c}$ in (3.39b), respectively. Then Young's inequality yields that $\|\mathbb{b}\|^2 + \|\mathbb{c}\|^2 \le Ch^{2k+2}\tau^2 + C\tau^6$, where the interpolation properties (3.21) and (3.24), and $\mathcal{E}_q^n = \mathcal{O}(\tau^3)$ are used. Therefore, Young's inequality yields that

$$\Pi_2 \le \frac{1}{2}\big(\|\eta_q^{n\star} - \eta_q^n\|^2 + \tau\|\eta_q^n\|^2 + \|\mathbb{b}\|^2 + \tau^{-1}\|\mathbb{c}\|^2\big) \le Ch^{2k+2}\tau + C\tau^5. \tag{3.40}$$

Then together with inequalities (3.38) and (3.40), equation (3.37) yields that

$$\begin{aligned}\sum_{m\le n}\mathcal{T}_{RK}^m &\le (\eta_q^{n+1}, \xi_q^{n+1}) - (\eta_q^0, \xi_q^0) + \frac{1}{4}\|\xi_q^{n+1}\|^2\tau + C\sum_{m\le n}\big[\|\xi_q^{n\star}\|^2 + \|\xi_q^n\|^2 + h^{2k+2} + \tau^4\big]\tau\\ &\le \frac{1}{2}\|\xi_q^{n+1}\|^2 + C\|\xi_q^0\|^2 + C\sum_{m\le n}\big[\|\xi_q^{n\star}\|^2 + \|\xi_q^n\|^2 + h^{2k+2} + \tau^4\big]\tau, \end{aligned} \tag{3.41}$$

since $\tau < 1$, where Young's inequality and interpolation property (3.21) are used.

We substitute inequality (3.41) into (3.36). By using inequality (3.34) and the initial approximation property (3.18), we can obtain that there exists a positive constant C independent of n, h, τ and $\mathbf{w}_h$, such that

$$\frac{1}{2}\|\xi_{uq}^{n+1}\|_\mu^2 \le C\sum_{m\le n}\|\xi_{uq}^m\|_\mu^2\tau + C(h^{2k+2} + \tau^4), \quad \forall n < [T/\tau].$$

An application of discrete Gronwall lemma for the above inequality implies that

$$\|\xi_{uq}^{n+1}\|_\mu^2 \le Ch^{2k+2} + C\tau^4, \quad \forall n < [T/\tau]. \tag{3.42}$$

Finally, triangular inequality combining (3.42) with the interpolation property (3.21) gives estimate (3.32).

In order to complete the proof of this theorem, we need to verify the *a priori* assumption (3.14) is reasonable for $n+1$. By the interpolation property (3.21) and the inverse property (iii) in (3.3), we have, from the conclusion (3.42), that

$$\|e_u^{n+1}\|_\infty \le Ch^{-1/2}\|\xi_u^{n+1}\| + \|\eta_u^{n+1}\| \le Ch^{-\frac{1}{2}}(h^{k+1} + \tau^2).$$

It implies $\|e_u^{n+1}\|_\infty \le C_1 h$ if h small enough, since $k \ge 1$ and $\tau = \mathcal{O}(h)$. Based on this new result, we use Lemma 3.8 to get $\|\xi_u^{(n+1)\star}\| \le C(h^{k+1} + \tau^2)$. Repeating above analysis, we can see that $\|e_u^{(n+1)\star}\|_\infty \le C_1 h$ if h small enough. Thus the *a priori* assumption also holds for $(n+1)\star$. Till now we complete the proof of this theorem. □

4 Numerical Experiment

In this section, we will present some numerical examples to demonstrate the error order of the proposed LDG method. The second order explicit Runge-Kutta time discretization and uniform meshes are used in the calculation. In Tables from 1 to 3 we list the numerical error and corresponding order in different norms, which show the scheme has an optimal error in L^2-norm, as we have proved in this paper.

Example 1 Let $\delta = \mu = 1$ and $f(u) = 0$, consider (1.1) with periodic boundary condition $u(0,t) = u(2\pi,t)$, and initial solution $u(x,0) = \sin x$. This is a pure parabolic-type Sobolev equation, with the exact solution $u(x,t) = e^{-t/2}\sin x$. We compute this equation till the final time $T = 1$. The time-space restriction is taken as $\tau = h^2$ for piecewise linear polynomials, and $\tau = 0.09h^2$ for piecewise quadratic polynomials, respectively.

Example 2 Let $\mu = 1, \delta = 2$ and $f(u) = 2u$, consider (1.1) with periodic boundary condition $u(0,t) = u(2\pi,t)$, and initial solution $u(x,0) = \sin x$. Its exact solution is $u(x,t) = e^{-t}\sin(x-t)$. We compute this equation till the final time $T = 1$. The time-space restriction is taken as $\tau = \min\{0.15h, h^2\}$ for piecewise linear polynomials, and $\tau = 0.09h^2$ for piecewise quadratic polynomials, respectively.

Example 3 Let $\delta = 0$ and $f(u) = \alpha u + \frac{1}{2}\beta u^2$ with given numbers α, β, (1.1) is just a regular long wave equation (RLW). Consider it with periodic boundary condition $u(a,t) = u(b,t)$, with the exact solution

$$u(x,t) = 3(c-1)\mathrm{sech}^2\left[\sqrt{\frac{c-1}{4\mu c}}(x-ct-d)\right], \tag{4.1}$$

where c and d are two given parameters. The initial condition $u(x,0)$ is given by the above formula. In our simulation, we set $\alpha = \beta = 1, \mu = 0.1, a = 0, b = 20, c = 2$ and $d = 8$, and compute this equation till the final time $T = 1$. The time-space restriction is taken as $\tau = 0.15h$ for piecewise linear polynomials, and $\tau = 0.09h^{3/2}$ for piecewise quadratic polynomials, respectively. Remark that the latter setting is only for getting the optimal error order in space.

Table 1 Example 1. Errors and orders for $\mathbb{P}_1$ and $\mathbb{P}_2$ with RK(2); $T = 1$

	N	L^1 error	L^1 order	L^2 error	L^2 order	L^∞ error	L^∞ order
$\mathbb{P}_1$	10	4.94E–03		6.61E–03		2.26E–02	
	20	1.25E–03	1.98	1.64E–03	2.01	5.40E–03	2.07
	40	3.16E–04	1.99	4.11E–04	2.00	1.34E–03	2.01
	80	7.91E–05	2.00	1.03E–04	2.00	3.34E–04	2.00
	160	1.98E–05	2.00	2.57E–05	2.00	8.34E–05	2.00
$\mathbb{P}_2$	10	2.61E–04		3.36E–04		8.95E–04	
	20	3.34E–05	2.96	4.22E–05	3.00	1.17E–04	2.94
	40	4.17E–06	3.00	5.28E–06	3.00	1.47E–05	2.99
	80	5.21E–07	3.00	6.60E–07	3.00	1.84E–06	3.00
	160	6.51E–08	3.00	8.25E–08	3.00	2.31E–07	3.00

Table 2 Example 2. Errors and orders for $\mathbb{P}_1$ and $\mathbb{P}_2$ with RK(2); $T = 1$

	N	L^1 error	L^1 order	L^2 error	L^2 order	L^∞ error	L^∞ order
$\mathbb{P}_1$	10	4.41E–03		5.48E–03		1.66E–02	
	20	1.09E–03	2.02	1.36E–03	2.01	4.14E–03	2.00
	40	2.72E–04	2.00	3.39E–04	2.01	1.03E–03	2.01
	80	6.44E–05	2.08	8.21E–05	2.04	2.71E–04	1.92
	160	1.59E–05	2.02	2.04E–05	2.01	6.89E–05	1.98
$\mathbb{P}_2$	10	2.52E–04		3.08E–04		6.83E–04	
	20	2.68E–05	3.23	3.38E–05	3.19	9.06E–05	2.91
	40	3.22E–06	3.06	4.06E–06	3.06	1.20E–05	2.91
	80	3.97E–07	3.02	5.01E–07	3.02	1.55E–06	2.96
	160	4.94E–08	3.01	6.23E–08	3.01	1.96E–07	2.98

Table 3 Example 3. Errors and orders for $\mathbb{P}_1$ and $\mathbb{P}_2$ with RK(2); $T = 1$

	N	L^1 error	L^1 order	L^2 error	L^2 order	L^∞ error	L^∞ order
$\mathbb{P}_1$	20	4.06E–02		9.98E–02		5.24E–01	
	40	1.01E–02	2.00	2.70E–02	1.88	1.68E–01	1.64
	80	2.84E–03	1.84	7.12E–03	1.92	5.51E–02	1.61
	160	7.78E–04	1.87	1.91E–03	1.90	1.59E–02	1.79
	320	2.08E–04	1.90	5.05E–04	1.92	4.33E–03	1.88
$\mathbb{P}_2$	20	1.11E–02		2.89E–02		1.56E–01	
	40	1.64E–03	2.75	4.31E–03	2.75	3.10E–02	2.33
	80	2.27E–04	2.86	5.86E–04	2.88	4.18E–03	2.89
	160	2.86E–05	2.99	7.38E–05	2.99	5.35E–04	2.96
	320	3.55E–06	3.01	9.17E–06	3.01	6.76E–05	2.99

5 Concluding Remarks

In this paper we presented a LDG method to solve one class of Sobolev equation (1.1), by introducing three auxiliary variables. For semi-discrete scheme, it is proved to be of good stability and high order accuracy. Time-discrete is also considered by virtue of the second order explicit TVD Runge-Kutta time-marching, and an optimal error estimate is also obtained for this fully-discrete scheme. Finally, numerical experiment verifies the optimal order of the proposed scheme. In the ongoing work, we will discuss how to extend and analyze this LDG method to high dimensional nonlinear Sobolev equations, and with another time-marching.

Appendix A

A.1 Proof of Lemma 3.2

We can prove this lemma by the following Taylor expansions [25] for the convection function $f(u)$, in the cell I_j and at the interface points, respectively. They read

$$f(u) - f(u_h) = f'(u)\xi_u - f'(u)\eta_u + \frac{1}{2}f''_u \cdot (e_u)^2, \quad \text{in the cell } I_j; \tag{A.1a}$$

$$f(u) - f(u_h^-) = f'(u)\xi_u^- - f'(u^n)\eta_u^- + \frac{1}{2}\tilde{f}''_u \cdot (e_u^-)^2, \quad \text{at the endpoint } x_{j+\frac{1}{2}}. \tag{A.1b}$$

Here $f'' = f''(\theta_1 u + (1-\theta_1)u_h)$ and $\tilde{f}'' = f''(\theta_2 u_{j+\frac{1}{2}} + (1-\theta_2)u_{h,j+\frac{1}{2}})$, with $\theta_i \in (0,1)$, are the mean value of the second order derivative of $f(u)$.

Since we have assumed $\widehat{f}(u_h^-, u_h^+) = f(u_h^-)$ in this paper, the proof is more easier than that in [25]. After some simple manipulations, we can have that

$$\begin{aligned}
\sum_{1\le j\le N} K_j(e_u, \xi_u) = &\sum_{1\le j\le N}\int_{I_j} f'(u)\xi_u\xi_{u,x}\,\mathrm{d}x + \sum_{1\le j\le N} f'(u_{j+\frac{1}{2}})\xi^-_{u,j+\frac{1}{2}}[\![\xi_u]\!]_{j+\frac{1}{2}}\\
&- \sum_{1\le j\le N}\int_{I_j} f'(u)\eta_u\xi_{u,x}\,\mathrm{d}x - \sum_{1\le j\le N} f'(u_{j+\frac{1}{2}})\eta^-_{u,j+\frac{1}{2}}[\![\xi_u]\!]_{j+\frac{1}{2}}\\
&+ \frac{1}{2}\sum_{1\le j\le N}\int_{I_j} f''_u\cdot(e_u)^2\xi_{u,x}\,\mathrm{d}x + \frac{1}{2}\sum_{1\le j\le N}\tilde{f}''\cdot(e^-_{u,j+\frac{1}{2}})^2[\![\xi_u]\!]_{j+\frac{1}{2}},
\end{aligned}$$

where each above term is denoted by $\mathcal{Y}_i$, $i = 1, \ldots, 6$.

Firstly, a simple integration by parts yields that

$$\begin{aligned}
\mathcal{Y}_1 + \mathcal{Y}_2 &= -\frac{1}{2}\sum_{1\le j\le N}\int_{I_j}\partial_x f'(u)(\xi_u)^2\,dx - \sum_{1\le j\le N} f'(u_{j+\frac{1}{2}})(\{\!\{\xi_u\}\!\} - \xi_u^-)_{j+\frac{1}{2}}[\![\xi_u]\!]_{j+\frac{1}{2}}\\
&\le C_\star\|\xi_u\|^2 - \frac{1}{2}\sum_{1\le j\le N}|f'(u_{j+\frac{1}{2}})|[\![\xi_u]\!]^2_{j+\frac{1}{2}}.
\end{aligned}$$

Recall that $|f'(u) - f'(u_{j+1/2})| = \mathcal{O}(h)$ in the cell I_j for smooth solution u. By using the interpolation property (3.5), and the inverse property (i) in (3.3), we can get that

$$\begin{aligned}
\mathcal{Y}_3 + \mathcal{Y}_4 &= -\sum_{1\le j\le N}\int_{I_j}\big[f'(u) - f'(u_{j+\frac{1}{2}})\big]\eta_u\xi_{u,x}\,\mathrm{d}x - \sum_{1\le j\le N}\int_{I_j} f'(u_{j+\frac{1}{2}})\eta_u\xi_{u,x}\,\mathrm{d}x\\
&\le C\|\eta_u\|^2 + Ch^2\|\xi_{u,x}\|^2 \le Ch^{2k+2} + C\|\xi_u\|^2,
\end{aligned}$$

where the definition of projection (2.6) is used twice. Along the same line, we use the interpolation property (3.5), and the inverse properties (i) and (ii) in (3.3), to get that

$$\begin{aligned}
\mathcal{Y}_5 + \mathcal{Y}_6 &\le C_\star\|e_u\|_\infty\big[\|\xi_{u,x}\|\|e_u\| + \|\xi_u\|_{\Gamma_h}\|e_u\|_{\Gamma_h}\big]\\
&\le \big[C_\star + C_\star h^{-2}\|e_u\|^2_\infty\big]\big[\|\xi_u\|^2 + h^{2k+2}\big].
\end{aligned}$$

Thus we complete the proof of Lemma 3.2 by summing up the above inequalities.

A.2 Proof of Lemma 3.5

Let $\mathcal{Q}^\sharp(v_w) = \sum_{1\le j\le N} K_j(e^\sharp_{uq}, v_w)$, then $T_1(v_{wp}) = \mathcal{Q}^{n\star}(v_w) - \mathcal{Q}^n(v_w)$. In this paper we need not to give sharp estimate as [25], to the difference of $\mathcal{Q}^{n\star}(v_w)$ and $\mathcal{Q}^n(v_w)$, and only analyze each of them along the same line as for Lemma 3.2. Below we give the estimate for $\mathcal{Q}^\sharp(v_w)$, and drop the supscript $\sharp$ for convenience.

By using the same Taylor expansion as (A.1), and noticing $\widehat{f}(u_h^-, u_h^+) = f(u_h^-)$, after a simple manipulation we can divide $\mathcal{Q}_1(v_w)$ into six terms, of the form

$$\begin{aligned}\mathcal{Q}^\sharp(v_w) = &\sum_{1\le j\le N}\int_{I_j} f'(u)\xi_u v_{w,x}\,\mathrm{d}x + \sum_{1\le j\le N} f'(u_{j+\frac12})\xi^-_{u,j+\frac12}[\![v_w]\!]_{j+\frac12}\\ &-\sum_{1\le j\le N}\int_{I_j} f'(u)\eta_u v_{w,x}\,\mathrm{d}x - \sum_{1\le j\le N} f'(u_{j+\frac12})\eta^-_{u,j+\frac12}[\![v_w]\!]_{j+\frac12}\\ &+\frac12\sum_{1\le j\le N}\int_{I_j} f''\cdot(e_u)^2 v_{w,x}\,\mathrm{d}x + \frac12\sum_{1\le j\le N}\tilde{f}''\cdot(e^-_{u,j+\frac12})^2[\![v_w]\!]_{j+\frac12},\end{aligned}$$

where each above term is denoted as $\mathcal{Z}_i, i = 1, \ldots, 6$.

Firstly, an integration by parts yields that

$$\begin{aligned}\mathcal{Z}_1 + \mathcal{Z}_2 = &-\sum_{1\le j\le N} f'(u_{j+\frac12})[\![\xi_u]\!]_{j+\frac12} v_{w,j+\frac12} - \sum_{1\le j\le N}\int_{I_j} f'(u_{j+\frac12})\xi_{u,x} v_w\,\mathrm{d}x\\ &-\sum_{1\le j\le N}\int_{I_j}\left[f'(u)\right]_x \xi_u v_w\,\mathrm{d}x - \sum_{1\le j\le N}\int_{I_j}\left[f'(u) - f'(u_{j+\frac12})\right]\xi_{u,x} v_w\,\mathrm{d}x.\end{aligned}$$

Since $|f'(u) - f'(u_{j+1/2})| = \mathcal{O}(h)$ in each cell I_j, by Young's inequality and the inverse properties (i) and (ii) in (3.3), we have

$$\begin{aligned}\mathcal{Z}_1 + \mathcal{Z}_2 &\le \varepsilon\|v_u\|^2 + C\|\xi_u\|^2 + M_\varepsilon\sum_{1\le j\le N}\|f'(u_{j+\frac12})\xi_{u,x}\|^2_{I_j}\\ &\quad + M_\varepsilon h^{-1}\sum_{1\le j\le N}|f'(u_{j+\frac12})|^2[\![\xi_u]\!]^2_{j+\frac12}\\ &\le \frac{\varepsilon}{6}\|v_u\|^2 + C\|\xi_u\|^2 + M_\varepsilon\mathcal{B}(u,\xi_u) + M_\varepsilon S_{\max} h^{-1}\mathcal{A}(u,\xi_u),\end{aligned}$$

where M_ε solely depends on ε. The left four terms, $\mathcal{Z}_i, i = 3, 4, 5, 6$, can be estimated as $\mathcal{Y}_i$. By Gauss-Raudu projection (2.6), and the inverse property (i) and (ii) in (3.3), we have

$$\begin{aligned}\mathcal{Z}_3 + \mathcal{Z}_4 &\le \frac{\varepsilon}{6}\|v_u\|^2 + Ch^{2k+2},\\ \mathcal{Z}_5 + \mathcal{Z}_6 &\le \frac{\varepsilon}{6}\|v_u\|^2 + C_\star h^{-2}\|e_u\|^2_\infty(\|\xi_u\|^2 + h^{2k+2}).\end{aligned}$$

Summing up the above conclusions we can get an estimate to $\mathcal{Q}^\sharp(v_w)$, and then complete the proof of this lemma.

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