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GLOBAL CLASSICAL SOLUTIONS TO THE 3-D ISENTROPIC COMPRESSIBLE NAVIER-STOKES EQUATIONS WITH GENERAL INITIAL ENERGY*

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Abstract We establish the global existence and uniqueness of classical solutions to the Cauchy problem for the 3-D compressible Navier-Stokes equations under the assumption that the initial density $\|\rho_0\|_{L^\infty}$ is appropriate small and $1<\gamma<\frac{6}{5}$. Here the initial density could have vacuum and we do not require that the initial energy is small.

Key words compressible Navier-Stokes equations; global classical solutions; general initial energy

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1 Introduction

The time evolution of the density and the velocity of a general viscous isentropic compressible fluid occupying a domain $\Omega\subset\mathbb{R}^3$ is governed by the compressible Navier-Stokes equations

$$\begin{cases}\rho_t+\operatorname{div}(\rho u)=0,\\ (\rho u)_t+\operatorname{div}(\rho u\otimes u)-\mu\triangle u-(\mu+\lambda)\nabla\operatorname{div}u+\nabla P(\rho)=0,\end{cases}\tag{1.1}$$

where $\rho\ge 0$, $u=(u^1,u^2,u^3)$ and $P=a\rho^\gamma$ $(a>0,\gamma>1)$ are the fluid density, velocity and pressure, respectively. The constant viscosity coefficients μ and λ satisfy the physical restrictions

$$\mu>0,\ \mu+\frac{3}{2}\lambda\ge 0.\tag{1.2}$$

Let $\Omega=\mathbb{R}^3$. We look for the solutions, $(\rho(x,t),u(x,t))$, to the Cauchy problem for (1.1) with the far field behavior:

$$u(x,t)\to 0,\ \rho(x,t)\to 0 \text{ as } |x|\to\infty,\tag{1.3}$$

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and initial data,

$$(\rho, u)|_{t=0} = (\rho_0, u_0), \ x \in \mathbb{R}^3. \tag{1.4}$$

Much efforts were devoted to study the global existence and behavior of solutions to (1.1). The one dimensional problem was studied extensively by many people, see [1–4]. For the multi-dimensional case, the local existence and uniqueness of smooth solutions were known in [5, 6] in the absence of vacuum and in [7–12] for the case that the initial density need not be positive and may vanish in an open sets. The global classical solutions were first obtained by Matsumura-Nishida [11] for initial data close to a nonvacuum equilibrium in some Sobolev space H^s. Later, Hoff [12, 13] studied the problem for discontinuous initial data. For the existence of solutions for arbitrary data (the far field is vacuum, that is $\tilde{\rho} = 0$), the major breakthough is due to Lions [14] (see also Feireisl [15]), where he obtained the global existence of weak solutions—defined as solutions with finite energy when the exponent γ is suitably large. The main restriction on initial data is that the initial energy is finite, so that the density vanishes at far fields, or even has compact support. However, little was known on the structure of such weak solutions. Recently, Zhang [16, 17] studied the existence of global classical (weak) solutions in $\mathbb{R}^n$ $(n = 2, 3)$ with small initial energy, the initial density away from the vacuum and the viscous coefficient λ depending density; Huang-Li-Xin [18] established the well-posedness of the global classical solutions with nonnegative initial density under the assumption that the initial energy is small. The proofs in [18] supplied a method to deal with the initial density having vacuum. Deng-Zhang-Zhao [19] established the global existence and uniqueness of classical solutions to the Cauchy problem for the isentropic compressible Navier-Stokes equations in the 3-D space with general initial data which could be either vacuum or non-vacuum under the assumption that the viscosity coefficient μ is large enough.

In this paper we are interested to study the global existence and uniqueness of classical solutions to Cauchy problem (1.1) with general initial energy. Enlightening by [18], we obtain the well-posedness of global classical solutions with general initial energy which is allowed to be vanish, under the assumption that the initial density $\|\rho_0\|_{L^\infty}$ is small enough and the index γ satisfies $1 < \gamma < \frac{6}{5}$. In proof, some ideals in [18] are used.

Before stating the main result, we explain the notations and conventions used throughout this paper. We denote

$$\int f \mathrm{d}x = \int_{\mathbb{R}^3} f \mathrm{d}x. \tag{1.5}$$

For $1 < r < \infty$, we denote the standard homogeneous and inhomogeneous Sobolev spaces as follows:

$$\begin{cases} L^r = L^r(\mathbb{R}^3), \ D^{k,r} = \{u \in L^1_{\mathrm{loc}}(\mathbb{R}^3) | \quad \|\nabla^k u\|_{L^r} < \infty\}, \ \|u\|_{D^{k,r}} := \|\nabla^k u\|_{L^r}, \\ W^{k,r} = L^r \cap D^{k,r}, \ H^k = W^{k,2}, \ D^k = D^{k,2}, \ D^1 = \{u \in L^6 | \ \|\nabla u\|_{L^2} < \infty\}. \end{cases} \tag{1.6}$$

The initial energy is defined as follows:

$$C_0 = \int \left(\frac{1}{2} \rho_0 |u_0|^2 + G(\rho_0) \right) \mathrm{d}x, \tag{1.7}$$

where G denotes the potential energy density given by

$$G(\rho) := \rho \int_0^\rho \frac{P(s)}{s^2} \mathrm{d}s. \tag{1.8}$$

It is easy to see

$$G(\rho)=\frac{P}{\gamma-1}. \tag{1.9}$$

Then the main result in this paper can be stated as follows:

Theorem 1.1 Assume that (1.2) holds. For given appropriate small positive numbers M and not necessarily small positive numbers M_1 and M_2, suppose that the initial data (ρ_0,u_0) satisfy

$$0\le\inf\rho_0\le\sup\rho_0\le M,\ \|\nabla u_0\|_{L^2}^2\le M_1, \tag{1.10}$$

$$u_0\in D^1\cap D^3,\ (\rho_0,P(\rho_0))\in H^3, \tag{1.11}$$

and the compatibility condition

$$-\mu\triangle u_0-(\mu+\lambda)\nabla\mathrm{div}u_0+\nabla P(\rho_0)=\rho_0 g \tag{1.12}$$

for some $g\in D^1$ with $\int\rho_0|g|^2\mathrm{d}x\le M_2$. Then if

$$1<\gamma<\frac{6}{5}, \tag{1.13}$$

the Cauchy problem (1.1), (1.3)–(1.4) has a unique global classical solution (ρ,u) satisfying for any $0<\tau<T<\infty$,

$$0\le\rho(x,t)\le 2M,\ x\in\mathbb{R}^3,\ t\ge 0, \tag{1.14}$$

and

$$\begin{cases}(\rho,P(\rho))\in C([0,T]:H^3),\\ u\in C([0,T];D^1\cap D^3)\cap L^2(0,T;D^4)\cap L^\infty(\tau,T;D^4),\\ u_t\in L^\infty(0,T;D^1)\cap L^2(0,T;D^2)\cap L^\infty(\tau,T;D^2)\cap H^1(\tau,T;D^1),\\ \sqrt{\rho}u_t\in L^\infty(0,T;L^2),\end{cases} \tag{1.15}$$

and the following large-time behavior:

$$\lim_{t\to\infty}\int\left(|\rho|^q+\rho^{1/2}|u|^4+|\nabla u|^2\right)(x,t)\mathrm{d}x=0 \tag{1.16}$$

for all $q\in(\gamma,\infty)$.

Remark It is easy to show that the solution obtained in Theorem 1.1 is a classical solution for positive time. Moreover, in Theorem 1.1 we have not require that the initial energy is small.

The rest of the paper is organized as follows: In Section 2, we state some elementary facts and inequalities which will be needed in later analysis. Section 3 is devoted to derive the necessary a priori estimates on classical solutions which are needed to extend the local existence of solution to all the time. Section 4 gives out the proof of main theorem.

2 Preliminaries

In this section, we will recall some known facts and elementary inequalities which will be used frequently later.

First, the following well-known Gagliardo-Nirenberg inequality will be used.

Lemma 2.1 For $p \in [2,6]$, $q \in (1,\infty)$ and $r \in (3,\infty)$, there exists some generic constant $C > 0$ which may depend on q, r such that for $f \in H^1(\mathbb{R}^3)$ and $g \in L^q(\mathbb{R}^3) \cap D^{1,r}(\mathbb{R}^3)$, we have

$$\|f\|_{L^p} \le C\|f\|_{L^2}^{(6-p)/2p}\|\nabla f\|_{L^2}^{(3p-6)/2p}, \tag{2.1}$$

$$\|g\|_{C(\mathbb{R}^3)} \le C\|g\|_{L^q}^{q(r-3)/(3r+q(r-3))}\|\nabla g\|_{L^r}^{3r/(3r+q(r-3))}. \tag{2.2}$$

Next, the following Zlotnik inequality will be used to get the uniform (in time) upper bound of the density ρ.

Lemma 2.2 Let the function y satisfy

$$y'(t) = g(y) + b'(t) \text{ on } [0,T],\ y(0) = y^0$$

with $g \in C(\mathbb{R})$ and $y,\ b \in W^{1,1}(0,T)$. If $g(\infty) = -\infty$ and

$$b(t_2) - b(t_1) \le N_0 + N_1(t_2 - t_1) \tag{2.3}$$

for all $0 \le t_1 < t_2 \le T$ with some $N_0 \ge 0$ and $N_1 \ge 0$, then

$$y(t) \le \max\{y^0, \bar{\zeta}\} + N_0 < \infty \text{ on } [0,T],$$

where $\bar{\zeta}$ is a constant such that

$$g(\zeta) \le N_1 \quad \text{for} \quad \zeta \ge \bar{\zeta}. \tag{2.4}$$

The following lemma is the local existence and uniqueness of classical solutions when the initial density may not be positive and may vanish in an open set.

Lemma 2.3 [8] Assume that the initial data (ρ_0, u_0) with $\rho_0 \ge 0$ satisfy (1.10)–(1.12). Then there exist a small time $T_* > 0$ and a unique classical solution (ρ, u) to the Cauchy problem (1.1), (1.3)–(1.4) such that

$$\begin{cases} (\rho, P(\rho)) \in C([0,T_*]; H^3), \\ u \in C([0,T_*]; D^1 \cap D^3) \cap L^2(0,T_*; D^4), \\ u_t \in L^\infty(0,T_*; D^1) \cap L^2(0,T_*; D^2),\ \sqrt{\rho}u_t \in L^\infty(0,T_*; L^2), \\ \sqrt{\rho}u_{tt} \in L^2(0,T_*; L^2),\ t^{1/2}u \in L^\infty(0,T_*; D^4), \\ t^{1/2}\sqrt{\rho}u_{tt} \in L^\infty(0,T_*; L^2),\ tu_t \in L^\infty(0,T_*; D^3), \\ tu_{tt} \in L^\infty(0,T_*; D^1) \cap L^2(0,T_*; D^2). \end{cases} \tag{2.5}$$

We now state some elementary estimates which follow from Gagliardo-Nirenberg inequalities and the standard L^p-estimate for the following elliptic system derived from the momentum equations in (1.1):

$$\triangle F = \operatorname{div}(\rho\dot{u}),\ \mu\triangle w = \nabla \times (\rho\dot{u}), \tag{2.6}$$

where

$$\dot{f} := f_t + u \cdot \nabla f,\ F := (2\mu + \lambda)\operatorname{div}u - P(\rho),\ w := \nabla \times u \tag{2.7}$$

are the material derivative of f, the effective viscous flux and the vorticity respectively.

Lemma 2.4 Let (ρ, u) be a smooth solutions of (1.1) and (1.3). Then there exists a generic positive constant C, such that for any $p \in [2,6]$,

$$\|\nabla F\|_{L^p} \le C\|\rho\dot{u}\|_{L^p}, \quad \|\nabla w\|_{L^p} \le C\|\rho\dot{u}\|_{L^p}, \tag{2.8}$$

$$\|F\|_{L^p} \le C\|\rho\dot{u}\|_{L^2}^{\frac{3p-6}{2p}}(\|\nabla u\|_{L^2} + \|P(\rho)\|_{L^2})^{\frac{6-p}{2p}}, \tag{2.9}$$

$$\|w\|_{L^p} \le C\|\rho\dot{u}\|_{L^2}^{\frac{3p-6}{2p}}\|\nabla u\|_{L^2}^{\frac{6-p}{2p}}, \tag{2.10}$$

$$\|\nabla u\|_{L^p} \le C(\|F\|_{L^p} + \|w\|_{L^p} + \|P(\rho)\|_{L^p}), \tag{2.11}$$

$$\|\nabla u\|_{L^p} \le C\|\nabla u\|_{L^2}^{\frac{6-p}{2p}}(\|\rho\dot{u}\|_{L^2} + \|P(\rho)\|_{L^6})^{\frac{3p-6}{2p}}. \tag{2.12}$$

Proof The standard L^p-estimate for the elliptic system (2.6) yields directly (2.8), which together with (2.1) and (2.7) gives (2.9) and (2.10).

Note that $-\triangle u = -\nabla \mathrm{div} u + \nabla \times w$, which implies that

$$\nabla u = -\nabla(-\triangle)^{-1}\nabla \mathrm{div} u + \nabla(-\triangle)^{-1}\nabla \times w.$$

Thus the standard L^p-estimates shows that

$$\|\nabla u\|_{L^p} \le C(\|\mathrm{div} u\|_{L^p} + \|w\|_{L^p}) \le C(\|F\|_{L^p} + \|w\|_{L^p} + \|P(\rho)\|_{L^p}). \tag{2.13}$$

That is, (2.11) holds.

By Hölder inequality, (2.2) and the second inequality of (2.8), one has

$$\begin{aligned}
\|\nabla u\|_{L^p} &\le \|\nabla u\|_{L^2}^{(6-p)/2p}\|\nabla u\|_{L^6}^{(3p-6)/2p} \\
&\le C\|\nabla u\|_{L^2}^{(6-p)/2p}(\|F\|_{L^6} + \|w\|_{L^6} + \|P(\rho)\|_{L^6})_{L^6}^{(3p-6)/2p} \\
&\le C\|\nabla u\|_{L^2}^{(6-p)/2p}(\|\rho\dot{u}\|_{L^2} + \|P(\rho)\|_{L^6})^{(3p-6)/2p}.
\end{aligned} \tag{2.14}$$

This finishes the proof of Lemma. □

Finally, we state the following Beal-Kato-Majda type inequality, see [18, 20].

Lemma 2.5 For $3 < q < \infty$, there is a constant $C(q)$ such that the following estimate holds for all $\nabla u \in L^2(\mathbb{R}^3) \cap D^{1,q}(\mathbb{R}^3)$,

$$\|\nabla u\|_{L^\infty(\mathbb{R}^3)} \le C\left(\|\mathrm{div}\ u\|_{L^\infty(\mathbb{R}^3)} + \|w\|_{L^\infty(\mathbb{R}^3)}\right)\log\left(e + \|\nabla^2 u\|_{L^q(\mathbb{R}^3)}\right) + C\|\nabla u\|_{L^2(\mathbb{R}^3)} + C. \tag{2.15}$$

3 A Priori Estimates

To extend the local classical solution to all time, in this section, we will establish necessary a priori estimates for smooth solutions to the Cauchy problem (1.1), (1.3)–(1.4). Let $T > 0$ be a fixed time and (ρ, u) be the smooth solution to (1.1), (1.3)–(1.4), on $\mathbb{R}^3 \times (0, T]$ in the class (2.5) with smooth initial data (ρ_0, u_0) satisfying (1.10)–(1.12). To estimate this solution, we define

$$A_1(T) := \sup_{t\in[0,T]}\|\nabla u\|_{L^2}^2 + \int_0^T\int \rho|\dot{u}|^2 \mathrm{d}x\mathrm{d}t,$$

$$A_2(T) := \sup_{t\in[0,T]}\int \rho|\dot{u}|^2 \mathrm{d}x + \int_0^T\int |\nabla\dot{u}|^2 \mathrm{d}x\mathrm{d}t.$$

We have the following key a priori estimates on (ρ, u).

Proposition 3.1 For given numbers $M > 0$, assume that (ρ_0, u_0) satisfy (1.10)–(1.12). Then there exist positive constants K_1, K_2 depending only on C_0, a, γ, M_1 and M_2, such that if (ρ, u) is a smooth solution of (1.1), (1.3)–(1.4) on $\mathbb{R}^3 \times (0, T]$ satisfying

$$\begin{cases} \sup\limits_{\mathbb{R}^3\times[0,T]} \rho \le 2M, \\ A_1(T) \le 2K_1, \\ A_2(T) \le 2K_2, \end{cases} \tag{3.1}$$

the following estimates hold

$$\sup_{\mathbb{R}^3\times[0,T]} \rho \le \frac{7}{4}M,\ \ A_1(T) \le K_1,\ \ A_2(T) \le K_2, \tag{3.2}$$

provided $1 < \gamma < \frac{6}{5}$ and M small enough.

Proposition 3.1 is an easy consequence of the following Lemmas 3.2–3.4.

In the following, we will use the convention that C denotes a generic positive constant depending on a, γ, C_0, μ, λ, M_1 and M_2, and we write $C(\alpha)$ to emphasize that C depends on α.

We start with the following standard energy estimate for (ρ, u) and preliminary L^2 bounds for ∇u and $\rho\dot{u}$.

Lemma 3.2 Let (ρ, u) be a smooth solution of (1.1), (1.3)–(1.4). Then there is a constant C depending on a, C_0, μ, λ such that

$$\sup_{0\le t\le T} \int \left(\frac{1}{2}\rho|u|^2 + G(\rho)\right) \mathrm{d}x + \int_0^T \int \left(\mu|\nabla u|^2 + (\lambda+\mu)(\mathrm{div}u)^2\right) \mathrm{d}x\mathrm{d}t \le C_0, \tag{3.3}$$

$$A_1(T) \le CM_1 + CM^\gamma + C\int_0^T \int |\nabla u|^3 \mathrm{d}x\mathrm{d}t, \tag{3.4}$$

and

$$A_2(T) \le M_2 + CM^{2\gamma} + C\int_0^T \int |\nabla u|^4 \mathrm{d}x\mathrm{d}t. \tag{3.5}$$

Proof Multiplying the first equation of (1.1) by $G'(\rho)$ and the second by u^j and integrating, applying the far field condition (1.3), one shows easily the energy inequality (3.3).

Multiplying $(1.1)_2$ by $\dot{u}$ then integrating the resulting equality over $\mathbb{R}^3$ leads to

$$\int \rho|\dot{u}|^2 \mathrm{d}x = \int \left(-\dot{u}\cdot\nabla P(\rho) + \dot{u}\cdot\triangle u + \dot{u}\cdot\nabla\mathrm{div}u\right) \mathrm{d}x := \sum_{i=1}^3 M_i'. \tag{3.6}$$

Using $(1.1)_1$ and integrating by parts give

$$\begin{aligned} M_1' &= -\int \dot{u}\cdot\nabla P(\rho)\mathrm{d}x \\ &= \int \left((\mathrm{div}u)_t P(\rho) - (u\cdot\nabla u)\cdot\nabla P\right) \mathrm{d}x \\ &= \left(\int \mathrm{div}uP(\rho)\mathrm{d}x\right)_t + \int (P'\rho(\mathrm{div}u)^2 - P(\mathrm{div}u)^2 + P\partial_i u^j \partial_j u^i)\mathrm{d}x \\ &\le \left(\int \mathrm{div}uP(\rho)\mathrm{d}x\right)_t + CM^\gamma\|\nabla u\|_{L^2}^2. \end{aligned} \tag{3.7}$$

Integration by parts implies

$$\begin{aligned} M_2' &= \mu \int \dot{u} \cdot \triangle u \mathrm{d}x \\ &= -\frac{\mu}{2} \left(\|\nabla u\|_{L^2}^2\right)_t + \mu \int \partial_i u^j \partial_i (u^k \partial_k u^j) \mathrm{d}x \\ &\le -\frac{\mu}{2} \left(\|\nabla u\|_{L^2}^2\right)_t + C \int |\nabla u|^3 \mathrm{d}x, \end{aligned} \tag{3.8}$$

and similarly,

$$\begin{aligned} M_3' &= (\mu+\lambda) \int \dot{u} \cdot \nabla \mathrm{div} u \mathrm{d}x \\ &= -\frac{\mu+\lambda}{2} \left(\|\mathrm{div} u\|_{L^2}^2\right)_t - (\mu+\lambda) \int \mathrm{div} u \mathrm{div}(u \cdot \nabla u) \mathrm{d}x \\ &\le -\frac{\mu+\lambda}{2} \left(\|\mathrm{div} u\|_{L^2}^2\right)_t + C \int |\nabla u|^3 \mathrm{d}x. \end{aligned} \tag{3.9}$$

Combining (3.6)–(3.9) leads to

$$B'(t) + \int \rho |\dot{u}|^2 \mathrm{d}x \le CM^\gamma \|\nabla u\|_{L^2}^2 + C \int |\nabla u|^3 \mathrm{d}x, \tag{3.10}$$

where

$$\begin{aligned} B(t) &= \frac{\mu}{2} \|\nabla u\|_{L^2}^2 + \frac{\mu+\lambda}{2} \|\mathrm{div} u\|_{L^2}^2 - \int \mathrm{div} u P(\rho) \mathrm{d}x \\ &\ge \frac{\mu}{4} \|\nabla u\|_{L^2}^2 + \frac{\mu+\lambda}{2} \|\mathrm{div} u\|_{L^2}^2 - CM^\gamma C_0. \end{aligned} \tag{3.11}$$

Integrating (3.10) over $(0,T)$, and using (3.3), one has

$$\begin{aligned} B(t) + \int_0^T \int \rho |\dot{u}|^2 \mathrm{d}x \mathrm{d}t &\le B(0) + CM^\gamma C_0 + C \int_0^T \int |\nabla u|^3 \mathrm{d}x \mathrm{d}t \\ &\le CM_1 + CM^\gamma + C \int_0^T \int |\nabla u|^3 \mathrm{d}x \mathrm{d}t, \end{aligned} \tag{3.12}$$

i.e., (3.4) holds.

Next, operating $\dot{u}^j(\partial/\partial t + \mathrm{div}(u\cdot))$ to $(1.1)_2^j$, summing with respect to j, and integrating the resulting equation over $\mathbb{R}^3$, one obtains after integration by parts

$$\begin{aligned} \left(\frac{1}{2} \int \rho |\dot{u}|^2 \mathrm{d}x\right)_t &= -\int \dot{u}^j \left(\partial_j P_t + \mathrm{div}(\partial_j P u)\right) \mathrm{d}x + \mu \int \dot{u}^j \left(\triangle u_t^j + \mathrm{div}(u \triangle u^j)\right) \mathrm{d}x \\ &\quad +(\mu+\lambda) \int \dot{u}^j \left(\partial_t \partial_j \mathrm{div} u + \mathrm{div}(u \partial_j \mathrm{div} u)\right) \mathrm{d}x \\ &:= \sum_{i=1}^3 N_i. \end{aligned} \tag{3.13}$$

It follows from integration by parts and using equation $(1.1)_1$ that

$$\begin{aligned} N_1 &= -\int \dot{u}^j \left(\partial_j P_t + \mathrm{div}(\partial_j P u)\right) \mathrm{d}x \\ &= \int \left(-P' \rho \mathrm{div} u \partial_j \dot{u}^j + P \partial_k (\partial_j \dot{u}^j u^k) - P \partial_j (\partial_k \dot{u}^j u^k)\right) \mathrm{d}x \\ &\le CM^\gamma \|\nabla u\|_{L^2} \|\nabla \dot{u}\|_{L^2} \\ &\le \delta \mu \|\nabla \dot{u}\|_{L^2}^2 + CM^{2\gamma} \|\nabla u\|_{L^2}^2. \end{aligned} \tag{3.14}$$

Integration by parts leads to

$$\begin{aligned} N_2 &= \mu \int \dot{u}^j \left(\triangle u_t^j + \mathrm{div}(u \triangle u^j) \right) \mathrm{d}x \\ &= -\mu \int \left(|\nabla \dot{u}^j|^2 + \partial_i \dot{u}^j \partial_k u^k \partial_i u^j - \partial_i \dot{u}^j \partial_i u^k \partial_k u^j - \partial_k \dot{u}^j \partial_i u^k \partial_i u^j \right) \mathrm{d}x \\ &\le -\mu \|\nabla \dot{u}\|_{L^2}^2 + C \int |\nabla u|^4 \mathrm{d}x. \end{aligned} \tag{3.15}$$

Similarly,

$$\begin{aligned} N_3 &= (\mu+\lambda) \int \dot{u}^j \left(\partial_t \partial_j \mathrm{div} u + \mathrm{div}(u \partial_j \mathrm{div} u) \right) \mathrm{d}x \\ &= (\mu+\lambda) \|\mathrm{div}\dot{u}\|_{L^2}^2 + (\mu+\lambda) \int (\mathrm{div}\dot{u} \partial_i u^k \partial_k u^i - \mathrm{div}\dot{u} (\mathrm{div} u)^2 + \partial_k \dot{u}^j \partial_j u^k \mathrm{div} u) \mathrm{d}x \\ &\le -(\mu+\lambda) \|\mathrm{div}\dot{u}\|_{L^2}^2 + \delta \|\nabla \dot{u}\|_{L^2}^2 + C(\delta) \int |\nabla u|^4 \mathrm{d}x. \end{aligned} \tag{3.16}$$

Substituting (3.14)–(3.16) into (3.13) shows that for δ suitably small, it holds that

$$\left(\int \rho |\dot{u}|^2 \mathrm{d}x \right)_t + \mu \|\nabla \dot{u}\|_{L^2}^2 + (\mu+\lambda) \|\mathrm{div}\dot{u}\|_{L^2}^2 \le C M^{2\gamma} \|\nabla u\|_{L^2}^2 + C \int |\nabla u|^4 \mathrm{d}x. \tag{3.17}$$

Integrating (3.17) over $(0,T)$ gives

$$\begin{aligned} &\int \rho |\dot{u}|^2 \mathrm{d}x + \mu \int_0^T \|\nabla \dot{u}\|_{L^2}^2 \mathrm{d}t + (\mu+\lambda) \int_0^T \|\mathrm{div}\dot{u}\|_{L^2}^2 \mathrm{d}t \\ &\le M_2 + C M^{2\gamma} + C \int_0^T \int |\nabla u|^4 \mathrm{d}x \mathrm{d}t, \end{aligned} \tag{3.18}$$

where we have used the compatibility condition, which implies that $\sqrt{\rho}\dot{u}(x, t=0) = \sqrt{\rho_0} g$. Thus one finishes the proof of this Lemma. □

The following lemma will give more accurate estimates with respect to $A_1(T)$ and $A_2(T)$.

Lemma 3.3 There exist positive constants K_1 and K_2 depending on C_0, a, μ, λ, M_1 and M_2 such that, if (ρ, u) is a smooth solution of (1.1), (1.3)–(1.4) satisfying (3.1), then

$$A_1(T) \le K_1, \; A_2(T) \le K_2, \tag{3.19}$$

provided M appropriate small.

Proof Using Hölder inequality and Young's inequality, it follows from (3.3) and (3.4) that

$$\begin{aligned} A_1(T) &\le C M_1 + C M^{\gamma} + C \int_0^T \int |\nabla u|^3 \mathrm{d}x \mathrm{d}t \\ &\le C + C M_1 + C M^{\gamma} + C \int_0^T \int |\nabla u|^4 \mathrm{d}x \mathrm{d}t. \end{aligned} \tag{3.20}$$

Due to (2.11)

$$\int_0^T \|\nabla u\|_{L^4}^4 \mathrm{d}t \le C \int_0^T \|F\|_{L^4}^4 \mathrm{d}t + C \int_0^T \|w\|_{L^4}^4 \mathrm{d}t + C \int_0^T \|P(\rho)\|_{L^4}^4 \mathrm{d}t. \tag{3.21}$$

It follows from (2.9) that

$$\begin{aligned}\int_0^T \|F\|_{L^4}^4 \mathrm{d}t &\le C\int_0^T (\|\nabla u\|_{L^2}+\|P(\rho)\|_{L^2})\|\rho\dot{u}\|_{L^2}^3 \mathrm{d}t\\ &\le C\Big(\int_0^T \|\nabla u\|_{L^2}^2\mathrm{d}t\Big)^{1/2}\Big(\int_0^T \|\rho\dot{u}\|_{L^2}^6\mathrm{d}t\Big)^{1/2}\\ &\quad +CM^{\frac32}\sup_{t\in[0,T]}(\|P(\rho)\|_{L^2}\|\sqrt{\rho}\dot{u}\|_{L^2})\int_0^T\int \rho|\dot{u}|^2\mathrm{d}x\mathrm{d}t\\ &\le CM^{\frac32}A_1^{\frac12}(T)A_2(T)+CM^{\frac{\gamma+3}{2}}A_1(T)A_2^{\frac12}(T),\end{aligned} \tag{3.22}$$

duo to (2.10), (3.3) and Hölder inequality

$$\int_0^T \|w\|_{L^4}^4\mathrm{d}t \le C\int_0^T \|\nabla u\|_{L^2}\|\rho\dot{u}\|_{L^2}^3\mathrm{d}t \le CM^{\frac32}A_1^{\frac12}(T)A_2(T). \tag{3.23}$$

To estimate the third term on the right side of (3.21), one deduces from $(1.1)_1$ that $P(\rho)$ satisfies

$$(P(\rho))_t + u\cdot\nabla(P(\rho)) + \gamma P(\rho)\mathrm{div}u = 0. \tag{3.24}$$

Multiplying (3.24) by $3(P(\rho))^2$ and integrating the resulting equality over $\mathbb{R}^3$, one gets after using $\mathrm{div}u = \frac{1}{2\mu+\lambda}(F+P(\rho))$ that

$$\begin{aligned}\frac{3\gamma-1}{2\mu+\lambda}\|P(\rho)\|_{L^4}^4 &= -\Big(\int (P(\rho))^3\mathrm{d}x\Big)_t - \frac{3\gamma-1}{2\mu+\lambda}\int (P(\rho))^3F\mathrm{d}x\\ &\le -\Big(\int (P(\rho))^3\mathrm{d}x\Big)_t + \frac{\delta(3\gamma-1)}{2\mu+\lambda}\|P(\rho)\|_{L^4}^4 + \frac{C(\delta)}{2\mu+\lambda}\|F\|_{L^4}^4.\end{aligned} \tag{3.25}$$

Integrating (3.25) over $(0,T)$, and choosing δ suitably small, one may arrive at

$$\int_0^T \|P(\rho)\|_{L^4}^4\mathrm{d}t \le CM^{2\gamma} + C\int_0^T \|F\|_{L^4}^4\mathrm{d}t. \tag{3.26}$$

Therefore, collecting (3.21)–(3.23) and (3.26) shows that

$$\int_0^T\int |\nabla u|^4\mathrm{d}x\mathrm{d}t \le CM^{2\gamma} + CM^{\frac32}A_1^{\frac12}(T)A_2(T) + CM^{\frac{\gamma+3}{2}}A_1(T)A_2^{\frac12}(T). \tag{3.27}$$

Combining (3.27) and (3.20) leads to

$$\begin{aligned}A_1(T) &\le C + CM_1 + CM^{\gamma} + CM^{2\gamma} + CM^{\frac32}A_1^{\frac12}(T)A_2(T) + CM^{\frac{\gamma+3}{2}}A_1(T)A_2^{\frac12}(T)\\ &\le \frac{K_1}{2} + CM^{\frac32}K_1^{\frac12}K_2 + CM^{\frac{\gamma+3}{2}}K_1K_2^{\frac12},\end{aligned} \tag{3.28}$$

where

$$K_1 \ge 2(C + CM_1 + CM^{\gamma} + CM^{2\gamma}).$$

On the other hand, combining (3.5) and (3.27) gives

$$\begin{aligned}A_2(T) &\le M_2 + CM^{2\gamma} + CM^{\frac32}A_1^{\frac12}(T)A_2(T) + CM^{\frac{\gamma+3}{2}}A_1(T)A_2^{\frac12}(T)\\ &\le \frac{K_2}{2} + CM^{\frac32}K_1^{\frac12}K_2 + CM^{\frac{\gamma+3}{2}}K_1K_2^{\frac12},\end{aligned} \tag{3.29}$$

where

$$K_2 \geq 2(M_2 + CM^{2\gamma}).$$

Hence if $0 < M \leq \bar{M} := \min\Big\{1, \big(\frac{K_1^{\frac{1}{2}}}{4CK_2}\big)^{\frac{2}{3}}, \big(\frac{1}{4CK_1^{\frac{1}{2}}}\big)^{\frac{2}{3}}, \big(\frac{1}{4CK_2^{\frac{1}{2}}}\big)^{\frac{2}{\gamma+3}}, \big(\frac{K_2^{\frac{1}{2}}}{4CK_1}\big)^{\frac{2}{\gamma+3}}\Big\}$, (3.19) holds. □

We now proceed to derive a uniform (in time) upper bound for the density.

Lemma 3.4 If (ρ, u) is a smooth solution of (1.1), (1.3)–(1.4) as in Lemma 3.3, then

$$\sup_{t\in[0,T]} \|\rho\|_{L^\infty} \leq \frac{7M}{4}, \tag{3.30}$$

provided $1 < \gamma < \frac{6}{5}$ and M appropriate small.

Proof Rewrite the equation of the mass conservation $(1.1)_1$ as

$$D_t\rho = g(\rho) + b'(t), \tag{3.31}$$

where

$$D_t\rho := \rho_t + u\cdot\nabla\rho,\ g(\rho) := -\frac{a\rho^{\gamma+1}}{2\mu+\lambda},\ b(t) := -\frac{1}{2\mu+\lambda}\int_0^t \rho F \mathrm{d}s. \tag{3.32}$$

For all $0 \leq t_1 \leq t_2 \leq T$, one deduces from Lemma 2.1, (3.19), (3.3) and (2.8) that

$$\begin{aligned}
|b(t_2) - b(t_1)| &\leq \frac{CM}{2\mu+\lambda}\int_{t_1}^{t_2} \|F(\cdot,t)\|_{L^\infty}\mathrm{d}s \\
&\leq \frac{aM^{\gamma+1}}{2\mu+\lambda}(t_2-t_1) + \frac{C}{M^{\frac{5}{3}\gamma-1}}\int_0^T \|F(\cdot,t)\|_{L^\infty}^{8/3}\mathrm{d}s \\
&\leq \frac{aM^{\gamma+1}}{2\mu+\lambda}(t_2-t_1) + \frac{C}{M^{\frac{5}{3}\gamma-1}}\int_0^T \|F(\cdot,t)\|_{L^2}^{2/3}\|\nabla F(\cdot,t)\|_{L^6}^2\mathrm{d}s \\
&\leq \frac{aM^{\gamma+1}}{2\mu+\lambda}(t_2-t_1) + \frac{C}{M^{\frac{5}{3}\gamma-3}}\sup_{t\in[0,T]}(\|\nabla u\|_{L^2}^{2/3} + \|P(\rho)\|_{L^2}^{2/3})\int_0^T \|\nabla\dot{u}\|_{L^2}^2\mathrm{d}t \\
&\leq \frac{aM^{\gamma+1}}{2\mu+\lambda}(t_2-t_1) + \frac{C}{M^{\frac{5}{3}\gamma-3}}A_1^{\frac{1}{3}}(T)A_2(T) + \frac{C}{M^{\frac{4}{3}\gamma-3}}A_2(T) \\
&\leq \frac{aM^{\gamma+1}}{2\mu+\lambda}(t_2-t_1) + \frac{C}{M^{\frac{5}{3}\gamma-3}}K_1^{\frac{1}{3}}K_2 + \frac{C}{M^{\frac{4}{3}\gamma-3}}K_2 \\
&\leq \frac{aM^{\gamma+1}}{2\mu+\lambda}(t_2-t_1) + \frac{C_1}{M^{\frac{5}{3}\gamma-3}} + \frac{C_2}{M^{\frac{4}{3}\gamma-3}}.
\end{aligned} \tag{3.33}$$

Therefore, one can choose N_1 and N_0 in (2.3) as

$$N_1 = \frac{aM^{\gamma+1}}{2\mu+\lambda},\ N_0 = \frac{C_1}{M^{\frac{5}{3}\gamma-3}} + \frac{C_2}{M^{\frac{4}{3}\gamma-3}}.$$

Note that

$$g(\zeta) \leq -\frac{a\zeta^{\gamma+1}}{2\mu+\lambda} \leq -N_1 = -\frac{aM^{\gamma+1}}{2\mu+\lambda} \text{ for all } \zeta \geq M.$$

So one can set $\bar{\zeta} = M$ in (2.4). Lemma 2.2 and (3.32) thus yield that

$$\sup_{t\in[0,T]} \|\rho\|_{L^\infty} \leq \max\{\rho_0, M\} + N_0 \leq M + \frac{C_1}{M^{\frac{5}{3}\gamma-3}} + \frac{C_2}{M^{\frac{4}{3}\gamma-3}} \leq \frac{7M}{4}, \tag{3.34}$$

provided

$$\max\left\{\left(\frac{8C_1}{3}\right)^{\frac{1}{\frac{5}{3}\gamma-2}},\left(\frac{8C_2}{3}\right)^{\frac{1}{\frac{4}{3}\gamma-2}}\right\}\le M\le\bar{M},\quad \text{if } 1<\gamma<\frac{6}{5}, \tag{3.35}$$

which completes the proof of this lemma. □

Holding these lemmas on hand, we can deal with the higher order estimates of the solutions which are needed to guarantee the extension of the local classical solution to be a global one. Since the proofs of these lemmas are similar to those in [18], we give out the proofs in Appendix of the paper.

Hereafter, we will always assume that γ, M satisfy (3.35) and the constant C may depend on

$$T,\ \|\rho_0^{\frac{1}{2}}g\|_{L^2},\ \|\nabla g\|_{L^2},\ \|\nabla u_0\|_{H^2},\ \|\rho_0\|_{H^3},\ \|P(\rho_0)\|_{H^3},$$

besides μ, λ, C_0, a, γ, M_1 and M_2.

Lemma 3.5 The following estimates hold

$$\sup_{t\in[0,T]}\int\rho|u_t|^2\mathrm{d}x+\int_0^T\int|\nabla u_t|^2\mathrm{d}x\mathrm{d}t\le C, \tag{3.36}$$

$$\sup_{t\in[0,T]}(\|\rho\|_{H^2}+\|P(\rho)\|_{H^2})\le C. \tag{3.37}$$

Lemma 3.6 The following estimates hold:

$$\sup_{t\in[0,T]}(\|\rho_t\|_{H^1}+\|P_t\|_{H^1})+\int_0^T(\|\rho_{tt}\|_{H^1}^2+\|P_{tt}\|_{H^1}^2)\mathrm{d}t\le C, \tag{3.38}$$

$$\sup_{t\in[0,T]}\int|\nabla u_t|^2\mathrm{d}x+\int_0^T\int\rho u_{tt}^2\mathrm{d}x\mathrm{d}t\le C. \tag{3.39}$$

Lemma 3.7 It holds that

$$\sup_{t\in[0,T]}(\|\rho\|_{H^3}+\|P(\rho)\|_{H^3})\le C, \tag{3.40}$$

$$\sup_{t\in[0,T]}(\|\nabla u_t\|_{L^2}+\|\nabla u\|_{H^2})+\int_0^T(\|\nabla u\|_{H^3}^2+\|\nabla u_t\|_{H^1}^2)\mathrm{d}t\le C. \tag{3.41}$$

Lemma 3.8 For any $\tau\in(0,T)$, there exists some positive constant $C(\tau)$ such that

$$\sup_{t\in[\tau,T]}(\|\nabla u_t\|_{H^1}+\|\nabla^4 u\|_{L^2})+\int_\tau^T\int|\nabla u_{tt}|^2\mathrm{d}x\mathrm{d}t\le C(\tau). \tag{3.42}$$

4 Proof of Theorem 1.1

With all the a priori estimates in Section 3, we now prove the main result of this paper.

Proof of Theorem 1.1 By virtue of Lemma 2.3, there exists a $T_*>0$ such that the Cauchy problem (1.1), (1.3)–(1.4) has a unique classical solution (ρ,u) on $(0,T_*]$. We will use the a priori estimates, Proposition 3.1 and Lemmas 3.7 and 3.8, to extend the local classical solution (ρ,u) to all the time.

First, since

$$A_1(0) \le M_1,\ A_2(0) \le M_2,\ \rho_0 \le 2M,$$

there exists a $T_1 \in (0, T_*]$ such that (3.1) holds for $T = T_1$.

Set

$$T^* = \sup\{T|\ \ (3.1)\ \ \text{holds}\}. \tag{4.1}$$

Then $T^* \ge T_1 > 0$. Hence, for any $0 < \tau < T \le T^*$ with T finite, it follows from Lemmas 3.7 and 3.8 that

$$\nabla u_t,\ \nabla^3 u \in C([\tau, T]; L^2 \cap L^4),\ \nabla u,\ \nabla^2 u \in C([\tau, T]; L^2 \cap C(\overline{\mathbb{R}^3})), \tag{4.2}$$

where we have used the standard embedding

$$L^\infty(\tau, T; H^1) \cap H^1(\tau, T; H^{-1}) \hookrightarrow C([\tau, T]; L^q) \text{ for any } q \in [2, 6).$$

Due to (3.36), (3.39) and (3.42), one can get

$$\begin{aligned}
&\int_\tau^T \|(\rho|u_t|^2)_t\|_{L^1}\mathrm{d}t \\
\le &\int_\tau^T (\|\rho_t|u_t|^2\|_{L^1} + 2\|\rho u_t \cdot u_{tt}\|_{L^1})\mathrm{d}t \\
\le &\, C\int_\tau^T (\|\rho|\mathrm{div}u||u_t|^2\|_{L^1} + \||u||\nabla\rho||u_t|^2\|_{L^1} + \|\rho^{1/2}u_t\|_{L^2}\|\|\rho^{1/2}u_{tt}\|_{L^2})\mathrm{d}t \\
\le &\, C\int_\tau^T (\|\rho|u_t|^2\|_{L^1}\|\nabla u\|_{L^\infty} + \|u\|_{L^6}\|\nabla\rho\|_{L^2}\|u_t\|_{L^6}^2 + \|\rho^{1/2}u_{tt}\|_{L^2})\mathrm{d}t \le C,
\end{aligned}$$

which yields

$$\rho^{1/2}u_t \in C([\tau, T]; L^2).$$

This, together with (4.2), gives

$$\rho^{1/2}\dot{u},\ \nabla\dot{u} \in C([\tau, T]; L^2). \tag{4.3}$$

Next, we claim that

$$T^* = \infty. \tag{4.4}$$

Otherwise, $T^* < \infty$. Then by Proposition 3.1, (3.2) holds for $T = T^*$. It follows from Lemmas 3.7, 3.8 and (4.3) that $(\rho(x, T^*), u(x, T^*))$ satisfies (1.11) and (1.12) with $g(x) = \dot{u}(x, T^*)$, $x \in \mathbb{R}^3$. Lemma 2.3 implies that there exists $T^{**} > T^*$, such that (3.1) holds for $T = T^{**}$, which contradicts (4.1). Hence, (4.4) holds. Lemmas 2.3, 3.7–3.8 and (4.2) thus show that (ρ, u) is in fact the unique classical solution defined on $(0, T]$ for any $0 < T < T^* = \infty$.

The proof of (1.16) is similar to that in [18]. □

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Appendix The Proofs for Higher Derivatives

In this appendix, we first give out some basic estimates for the solution (ρ, u) before giving out the proof of Lemmas 3.5–3.8.

Lemma A.1 The following estimates hold

$$\sup_{t\in[0,T]}(\|\nabla\rho\|_{L^2\cap L^6}+\|\nabla u\|_{H^1})+\int_0^T\|\nabla u\|_{L^\infty}\mathrm{d}t\le C. \tag{A.1}$$

Proof For $2\le p\le 6$, $|\nabla\rho|^p$ satisfies

$$\begin{aligned}&(|\nabla\rho|^p)_t+\mathrm{div}(|\nabla\rho|^p u)+(p-1)|\nabla\rho|^p\mathrm{div}u\\&+p|\nabla\rho|^{p-2}(\nabla\rho)^t\nabla u(\nabla\rho)+p\rho|\nabla\rho|^{p-2}\nabla\rho\cdot\nabla\mathrm{div}u=0.\end{aligned} \tag{A.2}$$

Thus,

$$\begin{aligned}\partial_t\|\nabla\rho\|_{L^p} &\le C(1+\|u\|_{L^\infty})\|\nabla\rho\|_{L^p}+C\|\nabla^2 u\|_{L^p}\\ &\le C(1+\|u\|_{L^\infty})\|\nabla\rho\|_{L^p}+C\|\rho\dot{u}\|_{L^p},\end{aligned}\tag{A.3}$$

due to

$$\|\nabla^2 u\|_{L^p}\le C(\|\rho\dot{u}\|_{L^p}+\|\nabla P(\rho)\|_{L^p}),\tag{A.4}$$

which follows from the standard L^p-estimate for the following elliptic system:

$$\mu\triangle u+(\mu+\lambda)\nabla\mathrm{div}u=\rho\dot{u}+\nabla P(\rho).$$

It follows from Lemma 2.5 and (A.4) that

$$\begin{aligned}\|\nabla u\|_{L^\infty} &\le C(\|\mathrm{div}u\|_{L^\infty}+\|w\|_{L^\infty})\log(e+\|\nabla^2 u\|_{L^6})+C\|\nabla u\|_{L^2}+C\\ &\le C(\|\mathrm{div}u\|_{L^\infty}+\|w\|_{L^\infty})\log(e+\|\dot{u}\|_{L^6}+\|\nabla P(\rho)\|_{L^6})+C\\ &\le C(\|\mathrm{div}u\|_{L^\infty}+\|w\|_{L^\infty})\log(e+\|\dot{u}\|_{L^6})\\ &\quad+C(\|\mathrm{div}u\|_{L^\infty}+\|w\|_{L^\infty})\log(e+\|\rho\|_{L^6})+C.\end{aligned}\tag{A.5}$$

Set

$$f(t):=e+\|\nabla\rho\|_{L^6},\ g(t):=1+(\|\mathrm{div}u\|_{L^\infty}+\|w\|_{L^\infty})\log(e+\|\dot{u}\|_{L^6})+\|\nabla\dot{u}\|_{L^2}.$$

Combining (A.5) with (A.3) and setting $p=6$ in (A.3), one gets

$$f'(t)\le Cg(t)f(t)+Cg(t)f(t)\ln f(t)+Cg(t),$$

which yields

$$(\ln f(t))'\le Cg(t)+Cg(t)\ln f(t),\tag{A.6}$$

due to $f(t)>1$. Note that (2.5), Lemma 2.1 and Lemma 3.4 imply

$$\begin{aligned}\int_0^T g(t)\mathrm{d}t &\le \int_0^T(\|\mathrm{div}u\|_{L^\infty}^2+\|w\|_{L^\infty}^2)\mathrm{d}t+C\\ &\le \int_0^T\left(\frac{1}{2\mu+\lambda}\|F\|_{L^\infty}^2+\frac{1}{2\mu+\lambda}\|P(\rho)-P(\tilde{\rho})\|_{L^\infty}^2+\|w\|_{L^\infty}^2\right)\mathrm{d}t+C\\ &\le \int_0^T(\|F\|_{L^\infty}^2+\|w\|_{L^\infty}^2)\mathrm{d}t+C\\ &\le \int_0^T(\|F\|_{L^2}^2+\|\nabla F\|_{L^6}^2+\|w\|_{L^2}^2+\|\nabla w\|_{L^6}^2)\mathrm{d}t+C\\ &\le C\int_0^T\|\nabla\dot{u}\|_{L^2}^2\mathrm{d}t+C\\ &\le C,\end{aligned}\tag{A.7}$$

which, together with (A.6) and Gronwall's inequality, shows that

$$\sup_{0\le t\le T}f(t)\le C.\tag{A.8}$$

Consequently,

$$\sup_{0\le t\le T}\|\nabla\rho\|_{L^6}\le C. \tag{A.9}$$

As a consequence of (A.5), (A.7) and (A.9), one obtain

$$\int_0^T\|\nabla u\|_{L^\infty}\mathrm{d}t\le C. \tag{A.10}$$

Next, taking $p=2$ in (A.3), one gets by using (A.10) and Gronwall's inequality that

$$\sup_{0\le t\le T}\|\nabla\rho\|_{L^2}\le C,$$

which, together with (A.4), (A.9) and (A.10), gives (A.1). The proof of this lemma is completed. □

Proof of Lemma 3.5 Estimate (3.36) follows directly from the following simple facts:

$$\begin{aligned}\int\rho|u_t|^2\mathrm{d}x&\le\int\rho|\dot u|^2\mathrm{d}x+\int\rho|u\cdot\nabla u|^2\mathrm{d}x\\&\le C+C\|\sqrt{\rho}u\|_{L^2}\|u\|_{L^6}\|\nabla u\|_{L^6}^2\\&\le C+C\|\sqrt{\rho}u\|_{L^2}\|u\|_{L^6}(\|\rho\dot u\|_{L^2}+\|P(\rho)\|_{L^6})^2\\&\le C,\end{aligned} \tag{A.11}$$

and

$$\begin{aligned}\|\nabla u_t\|_{L^2}^2&\le\|\nabla\dot u\|_{L^2}^2+\|\nabla(u\cdot\nabla u)\|_{L^2}^2\\&\le\|\nabla\dot u\|_{L^2}^2+C(\|u\|_{L^\infty}^2\|\nabla^2u\|_{L^2}^2+\|u\|_{L^4}^4)\\&\le\|\nabla\dot u\|_{L^2}^2+C\|u\|_{L^\infty}^2\|\nabla^2u\|_{L^2}^2+C\|\nabla^2u\|_{L^2}^2(\|\rho\dot u\|_{L^2}+\|P(\rho)\|_{L^6})^3\\&\le\|\nabla\dot u\|_{L^2}^2+C\|\nabla u\|_{L^2}^{1/2}\|\nabla u\|_{L^6}^{3/2}\|\nabla^2u\|_{L^2}^2+C\\&\le\|\nabla\dot u\|_{L^2}^2+C.\end{aligned} \tag{A.12}$$

Next, we prove (3.37). Note that P satisfies

$$P_t+u\cdot\nabla P+\gamma P\mathrm{div}u=0, \tag{A.13}$$

which, together with $(1.1)_1$ and a simple computation, yields that

$$\begin{aligned}&\frac{\mathrm{d}}{\mathrm{d}t}(\|\nabla^2P\|_{L^2}^2+\|\nabla^2\rho\|_{L^2}^2)\\&\le C(1+\|\nabla u\|_{L^\infty})(\|\nabla^2P\|_{L^2}^2+\|\nabla^2\rho\|_{L^2}^2)+C\|F\|_{H^2}^2+C\|w\|_{H^2}^2+C,\end{aligned} \tag{A.14}$$

where we have used the following simple fact:

$$\begin{aligned}\|\nabla u\|_{H^m}&\le C(\|\mathrm{div}u\|_{H^m}+\|w\|_{H^m})\\&\le C(\|F\|_{H^m}+\|P(\rho)\|_{H^m}+\|w\|_{H^m})\ \ \text{for } m=1,2.\end{aligned} \tag{A.15}$$

Noticing that F and w satisfy (2.6), we get by the standard L^2-estimate for elliptic system, together with (A.1), that

$$\begin{aligned}\|F\|_{H^2}+\|w\|_{H^2}&\le C(\|F\|_{L^2}+\|\nabla(\rho\dot u)\|_{L^2}+\|w\|_{L^2}+\|\rho\dot u\|_{L^2})\\&\le C(1+\|F\|_{L^2}+\|\nabla(\rho\dot u)\|_{L^2}+\|\nabla\dot u\|_{L^2})\\&\le C(1+\|\nabla\rho\|_{L^3}\|\dot u\|_{L^6}+\|\nabla\dot u\|_{L^2})\\&\le C(1+\|\nabla\dot u\|_{L^2}),\end{aligned} \tag{A.16}$$

which, together with (A.14), Lemma A.1, and Gronwall's inequality, gives directly

$$\sup_{t\in[0,T]}(\|\nabla^2 P\|_{L^2}+\|\nabla^2\rho\|_{L^2})\le C.$$

Thus the proof of this lemma is completed. □

Proof of Lemma 3.6 We first prove (3.38). One deduce from (A.13) and (A.1) that

$$\|P_t\|_{L^2}\le C\|u\|_{L^\infty}\|\nabla P\|_{L^2}+\|\nabla u\|_{L^2}\le C. \tag{A.17}$$

Differentiating (A.13) yields

$$\nabla P_t+u\cdot\nabla\nabla P+\nabla u\cdot\nabla P+\gamma\nabla P\mathrm{div}u+\gamma P\nabla\mathrm{div}u=0.$$

Hence, by (A.1) and (3.36), one gets

$$\|\nabla P_t\|_{L^2}\le C(\|u\|_{L^\infty}\|\nabla^2 P\|_{L^2}+\|\nabla u\|_{L^3}\|\nabla P\|_{L^6}+\|\nabla^2 u\|_{L^2})\le C. \tag{A.18}$$

The combination of (A.17) with (A.18) implies

$$\sup_{0\le t\le T}\|P_t\|_{H^1}\le C. \tag{A.19}$$

Note that P_{tt} satisfies

$$P_{tt}+\gamma P_t\mathrm{div}u+\gamma P\mathrm{div}u_t+u_t\cdot\nabla P+u\cdot\nabla P_t=0. \tag{A.20}$$

Thus, one gets from (A.20), (A.19), (A.1) and (3.36) that

$$\begin{aligned}\int_0^T\|P_{tt}\|_{L^2}^2\mathrm{d}t&\le C\int_0^T(\|P_t\|_{L^6}\|\nabla u\|_{L^3}+\|\nabla u_t\|_{L^2}+\|u_t\|_{L^6}\|\nabla P\|_{L^3}+\|\nabla P_t\|_{L^2})^2\mathrm{d}t\\&\le C.\end{aligned} \tag{A.21}$$

One can hand ρ_t and ρ_{tt} similarly. Thus (3.38) holds.

Next, we prove (3.39). Differentiating $(1.1)_2$ with respect to t, then multiplying the resulting equation by u_{tt}, one gets after integration by parts that

$$\begin{aligned}&\frac{\mathrm{d}}{\mathrm{d}t}\frac12\int\left(\mu|\nabla u_t|^2+(\mu+\lambda)(\mathrm{div}u_t)^2\right)\mathrm{d}x+\int\rho u_{tt}^2\mathrm{d}x\\
=&\frac{\mathrm{d}}{\mathrm{d}t}\left(-\frac12\int\rho_t|u_t|^2\mathrm{d}x-\int\rho_t u\cdot\nabla u\cdot u_t\mathrm{d}x+\int P_t\mathrm{div}u_t\mathrm{d}x\right)\\
&+\frac12\int\rho_{tt}|u_t|^2\mathrm{d}x+\int(\rho_t u\cdot\nabla u)_t\cdot u_t\mathrm{d}x-\int\rho u_t\cdot\nabla u\cdot u_{tt}\mathrm{d}x\\
&-\int\rho u\cdot\nabla u_t\cdot u_{tt}\mathrm{d}x-\int P_{tt}\mathrm{div}u_t\mathrm{d}x\\
:=&\frac{\mathrm{d}}{\mathrm{d}t}I_0+\sum_{i=1}^5 I_i.\end{aligned} \tag{A.22}$$

It follows $(1.1)_1$, (A.1), (3.38) and (3.36) that

$$|I_0|=\left|-\frac12\int\rho_t|u_t|^2\mathrm{d}x-\int\rho_t u\cdot\nabla u\cdot u_t\mathrm{d}x+\int P_t\mathrm{div}u_t\mathrm{d}x\right|$$

$$
\begin{aligned}
&\le C\left|\int \operatorname{div}(\rho u)|u_t|^2\mathrm{d}x\right| + C\|\rho_t\|_{L^3}\|u\cdot\nabla u\|_{L^2}\|u_t\|_{L^6} + C\|P_t\|_{L^2}\|\nabla u_t\|_{L^2}\\
&\le C\int \rho|u||u_t||\nabla u_t|\mathrm{d}x + C\|\nabla u_t\|_{L^2}\\
&\le C\|u\|_{L^6}\|\rho^{1/2}u_t\|_{L^2}^{1/2}\|u_t\|_{L^6}^{1/2}\|\nabla u_t\|_{L^2} + C\|\nabla u_t\|_{L^2}\\
&\le \delta\|\nabla u_t\|_{L^2}^2 + C(\delta),
\end{aligned}
\tag{A.23}
$$

$$
\begin{aligned}
2|I_1| &= \left|\int \rho_{tt}|u_t|^2\mathrm{d}x\right| = \left|\int (\rho_t u + \rho u_t)\cdot\nabla|u_t|^2\mathrm{d}x\right|\\
&\le C\left(\|\rho_t\|_{L^3}\|u\|_{L^\infty} + \|\rho^{1/2}u_t\|_{L^2}^{1/2}\|u_t\|_{L^6}^{1/2}\right)\|u_t\|_{L^6}\|\nabla u_t\|_{L^2}\\
&\le C\|\rho_t\|_{L^3}\|u\|_{L^\infty}\|\nabla u_t\|_{L^2}^2 + \|\rho^{1/2}u_t\|_{L^2}^{1/2}\|\nabla u_t\|_{L^2}^{5/2}\\
&\le C\left(\|\nabla u_t\|_{L^2}^2 + \|\nabla u_t\|_{L^2}^{5/2}\right)\\
&\le C\|\nabla u_t\|_{L^2}^4 + C,
\end{aligned}
\tag{A.24}
$$

and

$$
\begin{aligned}
|I_2| &= \left|\int (\rho_t u\cdot\nabla u)_t\cdot u_t\mathrm{d}x\right|\\
&= \left|\int (\rho_{tt}u\cdot\nabla u\cdot u_t + \rho_t u_t\cdot\nabla u\cdot u_t + \rho_t u\cdot\nabla u_t\cdot u_t)\mathrm{d}x\right|\\
&\le C(\|\rho_{tt}\|_{L^2}\|u\cdot\nabla u\|_{L^3}\|u_t\|_{L^6} + \|\rho_t\|_{L^2}\||u_t|^2\|_{L^3}\|\nabla u\|_{L^6}\\
&\quad +\|\rho_t\|_{L^3}\|u\|_{L^\infty}\|\nabla u_t\|_{L^2}\|u_t\|_{L^6})\\
&\le C(\|\rho_{tt}\|_{L^2}^2 + \|\nabla u_t\|_{L^2}^2).
\end{aligned}
\tag{A.25}
$$

Cauchy's inequality gives

$$
\begin{aligned}
|I_3| + |I_4| &= \left|\int \rho u_t\cdot\nabla u\cdot u_{tt}\mathrm{d}x\right| + \left|\int \rho u\cdot\nabla u_t\cdot u_{tt}\mathrm{d}x\right|\\
&\le C\|\rho^{1/2}u_{tt}\|_{L^2}(\|u_t\|_{L^6}\|\nabla u\|_{L^3} + \|u\|_{L^\infty}\|\nabla u_t\|_{L^2})\\
&\le \delta\|\rho^{1/2}u_{tt}\|_{L^2}^2 + C(\delta)\|\nabla u_t\|_{L^2}^2,
\end{aligned}
\tag{A.26}
$$

and

$$
|I_5| = \left|\int P_{tt}\operatorname{div}u_t\mathrm{d}x\right| \le C\|P_{tt}\|_{L^2}\|\operatorname{div}u_t\|_{L^2} \le C\|P_{tt}\|_{L^2}^2 + \|\nabla u_t\|_{L^2}^2. \tag{A.27}
$$

Due to the regularity of the local solution (2.5), $t\nabla u_t \in C([0,T_*];L^2)$. Thus

$$
\|\nabla u_t(\cdot,T_*/2)\|_{L^2} \le \frac{2}{T_*}\|t\nabla u_t\|_{L^\infty(0,T_*;L^2)} \le C, \tag{A.28}
$$

where C may also depend on $\|\nabla g\|_{L^2}$.

Collecting all estimates (A.23)–(A.28), one deduces from (A.22), (3.38), (3.36) and Gronwall's inequality that

$$
\sup_{T_*/2\le t\le T}\|\nabla u_t\|_{L^2} + \int_{T_*/2}^{T}\int \rho u_{tt}^2\mathrm{d}x\mathrm{d}t \le C. \tag{A.29}
$$

On the other hand, (2.5) gives

$$\sup_{0\le t\le T_*/2}\|\nabla u_t\|_{L^2}+\int_0^{T_*/2}\int\rho u_{tt}^2\mathrm{d}x\mathrm{d}t\le C. \tag{A.30}$$

The combination of (A.29) with (A.30) gives (3.39). This completes the proof of this lemma. □

Proof of Lemma 3.7 It follows from (3.39) and (A.1) that

$$\begin{aligned}\|\nabla(\rho\dot u)\|_{L^2}&\le\||\nabla\rho||u_t|\|_{L^2}+\|\rho\nabla u_t\|_{L^2}+\|\rho|u||\nabla u|\|_{L^2}\|\rho|\nabla u|^2\|_{L^2}+\|\rho|u||\nabla^2u|\|_{L^2}\\&\le\|\nabla\rho\|_{L^3}\|u_t\|_{L^6}+C(\|\nabla u_t\|_{L^2}+\|\nabla\rho\|_{L^3}\|u\|_{L^\infty}\|\nabla u\|_{L^6}\\&\quad+\|\nabla u\|_{L^3}\|\nabla u\|_{L^6}+\|u\|_{L^\infty}\|\nabla^2u\|_{L^2})\\&\le C,\end{aligned}\tag{A.31}$$

thus

$$\sup_{0\le t\le T}\|\rho\dot u\|_{H^1}\le C. \tag{A.32}$$

The standard H^1-estimate for elliptic system gives

$$\begin{aligned}\|\nabla^2u\|_{H^1}&\le C\|\mu\triangle u+(\mu+\lambda)\nabla\mathrm{div}u\|_{H^1}=C\|\rho\dot u+\nabla P\|_{H^1}\\&\le C(\|\rho\dot u\|_{H^1}+\|\nabla P\|_{H^1})\le C,\end{aligned}\tag{A.33}$$

due to $(1.1)_2$, (A.32) and (3.37). As a consequence of (A.1) and (A.33), one has

$$\sup_{0\le t\le T}\|\nabla u\|_{H^2}\le C. \tag{A.34}$$

Therefore, the standard L^2-estimate for elliptic system, (A.1), and Lemma 3.6 gives that

$$\begin{aligned}\|\nabla^2u_t\|_{L^2}&\le C\|\mu\triangle u_t+(\mu+\lambda)\nabla\mathrm{div}u_t\|_{L^2}\\&=C\|\rho_tu_t+\rho u_{tt}+\rho_tu\cdot\nabla u+\rho u_t\cdot\nabla u+\rho u\cdot\nabla u_t+\nabla P_t\|_{L^2}\\&\le C(\|\rho u_{tt}\|_{L^2}+\|\rho_t\|_{L^3}\|u_t\|_{L^6}+\|\rho_t\|_{L^3}\|u\|_{L^\infty}\|\nabla u\|_{L^6}\\&\quad+\|u_t\|_{L^6}\|\nabla u\|_{L^3}+\|u\|_{L^\infty}\|\nabla u_t\|_{L^2}+\|\nabla P_t\|_{L^2})\\&\le C\|\rho u_{tt}\|_{L^2}+C,\end{aligned}\tag{A.35}$$

which, together with (3.39), implies

$$\int_0^T\|\nabla u_t\|_{H^1}^2\mathrm{d}t\le C. \tag{A.36}$$

Applying the standard H^2-estimate for elliptic system again leads to

$$\begin{aligned}\|\nabla^2u\|_{H^2}&\le C\|\mu\triangle u+(\mu+\lambda)\nabla\mathrm{div}u\|_{H^2}\le C(\|\rho\dot u\|_{H^2}+\|\nabla P\|_{H^2})\\&\le C(1+\|\nabla u_t\|_{H^1}+\|\nabla^3P\|_{L^2}),\end{aligned}\tag{A.37}$$

where one has used (A.32) and the following simple facts:

$$\begin{aligned}\|\nabla^2(\rho u_t)\|_{L^2}&\le C(\||\nabla^2\rho||u_t|\|_{L^2}+\||\nabla\rho||\nabla u_t|\|_{L^2}+\|\nabla^2u\|_{L^2})\\&\le C(\|\nabla^2\rho\|_{L^2}\|\nabla u_t\|_{H^1}+\||\nabla\rho\|_{L^3}\|\nabla u_t\|_{L^6}+\|\nabla^2u_t\|_{L^2})\\&\le C+C\|\nabla u_t\|_{H^1},\end{aligned}\tag{A.38}$$

and

$$\begin{aligned}\|\nabla^2(\rho u\cdot\nabla u)\|_{L^2} &\le C(\|\nabla^2(\rho u)||\nabla u|\|_{L^2}+\||\nabla(\rho u)||\nabla^2 u|\|_{L^2}+\|\nabla^3 u\|_{L^2})\\ &\le C(1+\|\nabla^2(\rho u)\|_{L^2}\|\nabla u\|_{H^2}+\|\nabla(\rho u)\|_{L^3}\|\nabla^2 u\|_{L^6})\\ &\le C(1+\|\nabla^2\rho\|_{L^2}\|u\|_{L^\infty}+\||\nabla\rho\|_{L^6}\|\nabla u\|_{L^3}+\|\nabla^2 u\|_{L^2})\\ &\le C,\end{aligned}\tag{A.39}$$

due to (3.37) and (A.34). By using (A.34), (A.37) and (3.37), one may get

$$\begin{aligned}(\|\nabla^3 P\|_{L^2}^2)_t &\le C(\||\nabla^3 u||\nabla P|\|_{L^2}+\||\nabla^2 u||\nabla^2 P|\|_{L^2}+\||\nabla u||\nabla^3 P|\|_{L^2}+\|\nabla^4 u\|_{L^2})\|\nabla^3 P\|_{L^2}\\ &\le C(\|\nabla^3 u\|_{L^2}\|\nabla P\|_{H^2}+\|\nabla^2 u\|_{L^3}\|\nabla^2 P\|_{L^6}+\|\nabla u\|_{L^\infty}\|\nabla^3 P\|_{L^2})\|\nabla^3 P\|_{L^2}\\ &\quad +C(1+\|\nabla^2 u_t\|_{L^2}+\|\nabla^3 P\|_{L^2})\|\nabla^3 P\|_{L^2}\\ &\le C+C\|\nabla^2 u_t\|_{H^1}^2+C\|\nabla^3 P\|_{L^2}^2,\end{aligned}\tag{A.40}$$

which, together with Gronwall's inequality and (A.36), yields that

$$\sup_{0\le t\le T}\|\nabla^3 P\|_{L^2}\le C.\tag{A.41}$$

Collecting all these estimates (A.34)–(A.36) and (3.37) shows that

$$\sup_{0\le t\le T}\|P(\rho)\|_{H^3}+\int_0^T\|\nabla u\|_{H^3}^2\mathrm{d}t\le C.\tag{A.42}$$

It is easy to check similar arguments work for ρ arguments work for ρ by using (A.42). Hence,

$$\sup_{0\le t\le T}\|\rho\|_{H^3}\le C.\tag{A.43}$$

Combining (A.42) and (A.43) shows (3.40). Estimate (3.41) thus follows from (3.39), (A.34), (A.36) and (A.41). Hence the proof of this lemma is completed. □

Proof of Lemma 3.8 Differentiate $(1.1)_2$ with respect to t to get

$$\begin{aligned}&\rho u_{ttt}+\rho u\cdot\nabla u_{tt}-\mu\triangle u_{tt}-(\mu+\lambda)\nabla\mathrm{div}u_{tt}\\ &=\mathrm{div}(\rho u)_t u_t+2\mathrm{div}(\rho u)u_{tt}-2(\rho u)_t\cdot\nabla u_t-(\rho_{tt}u+2\rho_t u_t)\cdot\nabla u-\rho u_{tt}\cdot\nabla u-\nabla P_{tt}.\end{aligned}\tag{A.44}$$

Multiplying (A.44) by u_{tt} and then integrating the resulting equation over $\mathbb{R}^3$, one gets after integration by parts that

$$\begin{aligned}&\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\int\rho|u_{tt}|^2\mathrm{d}x+\int\left(\mu|\nabla u_{tt}|^2+(\mu+\lambda)(\mathrm{div}u_{tt})^2\right)\mathrm{d}x\\ =&-4\int u_{tt}^i\rho u\cdot\nabla u_{tt}^i\mathrm{d}x-\int(\rho u)_t\cdot(\nabla(u_t\cdot u_{tt})+2\nabla u_t\cdot u_{tt})\,\mathrm{d}x\\ &-\int(\rho_{tt}u+2\rho_t u_t)\cdot\nabla u\cdot u_{tt}\mathrm{d}x-\int\rho u_{tt}\cdot\nabla u\cdot u_{tt}\mathrm{d}x+\int P_{tt}\mathrm{div}u_{tt}\mathrm{d}x\\ :=&\sum_{i=1}^5 J_i.\end{aligned}\tag{A.45}$$

We now estimate each J_i $(i=1,\cdots,5)$ as follows:

Hölder's inequality gives

$$|J_1| \le C\|\rho^{1/2}u_{tt}\|_{L^2}\|\nabla u_{tt}\|_{L^2}\|u\|_{L^\infty} \le \delta\|\nabla u_{tt}\|_{L^2}^2 + C(\delta)\|\rho^{1/2}u_{tt}\|_{L^2}^2. \tag{A.46}$$

It follows from (3.36), (3.38), (3.39) and (A.1) that

$$\begin{aligned}|J_2| &\le C(\|\rho u_t\|_{L^3} + \|\rho_t u\|_{L^3})(\|u_{tt}\|_{L^6}\|\nabla u_t\|_{L^2} + \|\nabla u_{tt}\|_{L^2}\|u_t\|_{L^6})\\ &\le C(\|\rho^{1/2}u_t\|_{L^2}^{1/2}\|u_t\|_{L^6}^{1/2} + \|\rho_t\|_{L^6}\|\rho\|_{L^6})\|\nabla u_{tt}\|_{L^2}\\ &\le C\|\nabla u_{tt}\|_{L^2}^2 + C,\end{aligned} \tag{A.47}$$

$$\begin{aligned}|J_3| &\le C(\|\rho_{tt}\|_{L^3}\|u\|_{L^\infty}\|\nabla u\|_{L^3} + \|\rho_t\|_{L^6}\|u_t\|_{L^6}\|\nabla u\|_{L^2})\|u_{tt}\|_{L^6})\\ &\le \delta\|\nabla u_{tt}\|_{L^2}^2 + C(\delta)\|\rho_{tt}\|_{L^2}^2,\end{aligned} \tag{A.48}$$

and

$$\begin{aligned}|J_4| + |J_5| &\le C\|\rho u_{tt}\|_{L^2}\|\nabla u\|_{L^3}\|u_{tt}\|_{L^6} + \|P_{tt}\|_{L^2}\|\nabla u_{tt}\|_{L^2}\\ &\le \delta\|\nabla u_{tt}\|_{L^2}^2 + C(\delta)\|\rho^{1/2}u_{tt}\|_{L^2}^2 + C(\delta)\|P_{tt}\|_{L^2}^2.\end{aligned} \tag{A.49}$$

For any $\tau \in (0, T_*)$, since $t^{1/2}\sqrt{\rho}u_{tt} \in L^\infty(0, T_*; L^2)$ by (2.5), there exists some $t_0 \in (\tau/2, \tau)$ such that

$$\int \rho|u_{tt}|^2 \mathrm{d}x(t_0) \le \frac{1}{t_0}\|t^{1/2}\sqrt{\rho}u_{tt}\|_{L^\infty(0,T_*;L^2)}^2 \le C(\tau). \tag{A.50}$$

Substituting (A.46)–(A.49) into (A.45) and choosing δ suitably small, one obtains by using (3.38), (A.50) and Gronwall's inequality that

$$\sup_{t_0\le t\le T}\int \rho|u_{tt}|^2\mathrm{d}x + \int_{t_0}^T |\nabla u_{tt}|^2 \mathrm{d}x\mathrm{d}t \le C(\tau), \tag{A.51}$$

which, together with (A.35) and (3.39), yields that

$$\sup_{\tau\le t\le T}\|\nabla u_t\|_{H^1} + \int_\tau^T |\nabla u_{tt}|^2\mathrm{d}x\mathrm{d}t \le C(\tau), \tag{A.52}$$

due to $t_0 < \tau$. Now, (3.42) follows from (A.37), (A.52) and (3.40). We finish the proof of this lemma. □