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Research Article

Rate of Convergence of a New Type Kantorovich Variant of Bleimann-Butzer-Hahn Operators

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A new type Kantorovich variant of Bleimann-Butzer-Hahn operator J_n is introduced. Furthermore, the approximation properties of the operators J_n are studied. An estimate on the rate of convergence of the operators J_n for functions of bounded variation is obtained.

1. Introduction

In 1980, Bleimann et al. [1] introduced a sequence of positive linear Bernstein-type operators L_n (abbreviated in the following by BBH operators) defined on the infinite interval $I = [0, \infty)$ by

$$L_n(f,x) = \frac{1}{(1+x)^n}\sum_{k=0}^{n}\binom{n}{k}x^k f\left(\frac{k}{n+1-k}\right), \quad x \in I,\ n \in \mathbb{N}, \tag{1.1}$$

where $\mathbb{N}$ denotes the set of natural numbers.

Bleimann et al. [1] proved that $L_n(f,x) \to f(x)$ as $n \to \infty$ for $f \in C_b(I)$ (the space of all bounded continuous functions on I) and give an estimate on the rate of convergence of $L_n(f,x) \to f(x)$ measured with the second modulus of continuity of f.

In the present paper, we introduce a new type of Kantorovich variant of BBH operator J_n, also defined on I by

$$J_n(f,x) = \sum_{k=0}^{n}\binom{n}{k}p_x^k(1-p_x)^{n-k}\frac{\int_{I_k} f(t)dt}{\int_{I_k} dt}, \tag{1.2}$$

where $p_x = x/(1+x)$ $(x \geq 0)$, $I_k = [k/(n+2-k), (k+1)/(n+1-k)]$, and dt is Lebesgue measure.

The operator (1.2) is different from another type of Kantorovich variant of BBH operator K_n:

$$K_n(f,x) = \frac{n+2}{(1+x)^n} \sum_{k=0}^{n} \binom{n}{k} x^k \int_{k/(n+2-k)}^{(k+1)/(n+1-k)} \frac{f(t)}{(1+t)^2} dt, \tag{1.3}$$

which was first considered by Abel and Ivan in [2]. The integrand function $f(t)/(1+t)^2$ in the operator (1.3) has been replaced with new integrand function $f(t)$ in the operator (1.2). In this paper we will study the approximation properties of J_n for the functions of bounded variation. The rate of convergence for functions of bounded variation was investigated by many authors such as Bojanić and Vuilleumier [3], Chêng [4], Guo and Khan [5], Zeng and Piriou [6], Gupta et al. [7], involving several different operators.

Throughout this paper the class of function Φ is defined as follows:

$$\Phi = \{f \mid f \text{ is of bounded variation on every finite subinterval of } I = [0,\infty)\}. \tag{1.4}$$

Our main result can be stated as follows.

Theorem 1.1. *Let $f \in \Phi$ and let $V_a^b(f)$ be the total variation of f on interval $[a,b]$. Then, for n sufficiently large, one has*

$$\left| J_n(f,x) - \frac{f(x+) - f(x-)}{2} \right| \leq \frac{5(1+x)}{2\sqrt{nx}} |f(x+) - f(x-)| + \frac{9(1+x)^2}{nx} \sum_{k=1}^{n} V_{x-x/\sqrt{k}}^{x+x/\sqrt{k}}(g_x) + O\left(\frac{1}{n}\right), \tag{1.5}$$

where

$$g_x(t) = \begin{cases} f(t) - f(x+), & x < t < \infty, \\ 0, & x = t, \\ f(t) - f(x-), & 0 \leq t < x. \end{cases} \tag{1.6}$$

2. Some Lemmas

In order to prove Theorem 1.1, we need the following lemmas for preparation. Lemma 2.1 is the well-known Berry-Esséen bound for the classical central limit theorem of probability theory. Its proof and further discussion can be founded in Feller [8, page 515].

Lemma 2.1. *Let $\{\xi\}_{i=1}^{\infty}$ be a sequence of independent and identically distributed random variables. And $0 < D\xi_1 < \infty$, $\beta_3 = E|\xi_1 - E\xi_1|^3 < +\infty$, then, there holds*

$$\max_{y \in R} \left| P\left(\frac{1}{b_1\sqrt{n}} \sum_{k=1}^{n} (\xi_k - a_1) \leq y \right) - \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{y} e^{-t^2/2} dt \right| < \frac{c}{\sqrt{n}} \frac{\beta_3}{{b_1}^3}, \tag{2.1}$$

where $a_1 = E\xi_1$, ${b_1}^2 = D\xi_1 = E(\xi_1 - E\xi_1)^2$, $1/\sqrt{2\pi} \leq c \leq 0.82$.

In addition, let $\{\xi\}_{i=1}^{n}$ be the random variables with two-point distribution

$$P_{\xi_i} = \begin{cases} x, & \xi_i = 1 \\ 1 - x, & \xi_i = 0, \end{cases} \tag{2.2}$$

where $i = 1, 2, \ldots, n$. Then we can easily obtain that

$$a_1 = E\xi_1 = x, \quad {b_1}^2 = D\xi_1 = x(1-x), \quad \beta_3 = E|\xi_1 - E\xi_1|^3 \le x(1-x)\left(2x^2 - 2x + 1\right). \tag{2.3}$$

Let $\eta_n = \sum_{i=1}^{n} \xi_i$, then we also have

$$P(\eta_n = k) = \binom{n}{k} x^k (1-x)^{n-k}, \quad k = 0, 1, \ldots, n. \tag{2.4}$$

On the other hand, $J_n(f, x)$ can be written by following integral form:

$$J_n(f, x) = \sum_{k=0}^{n} p_{n,k}\left(\frac{x}{1+x}\right) \frac{\int_{I_k} f(t)dt}{\int_{I_k} dt} = \int_0^{\infty} f(t) H_n(x, t) dt, \tag{2.5}$$

where

$$H_n(x, t) = \sum_{k=0}^{n} p_{n,k}\left(\frac{x}{1+x}\right) \chi_k(t) \frac{1}{\int_{I_k} dt}, \quad \chi_k(t) = \begin{cases} 1, & t \in I_k, \\ 0, & t \notin I_k, \end{cases} \tag{2.6}$$

$I_k = [k/(n+2-k), (k+1)/(n+1-k)]$, $k = 0, 1, 2, \ldots, n$. It is easy to verify that $\int_0^{\infty} H_n(x, u) du = 1$.

Lemma 2.2. *If $x \in (0, \infty)$ is fixed and n is sufficiently large, then*

(a) *for $0 \le y < x$, there holds*

$$\int_0^y H_n(x, t) dt \le \frac{1}{(x-y)^2} \frac{2x(1+x)^2}{n+1}, \tag{2.7}$$

(b) *for $x < z < \infty$, there holds*

$$\int_z^{\infty} H_n(x, t) dt \le \frac{1}{(z-x)^2} \frac{2x(1+x)^2}{n+1}. \tag{2.8}$$

Proof. We first prove (a). Since $0 \le y < x$, $t \in [0, y]$, then $(x-t)/(x-y) \ge 1$. Hence, we have

$$\int_0^y H_n(x, t) dt \le \int_0^y \frac{(x-t)^2}{(x-y)^2} H_n(x, t) dt \le \frac{1}{(x-y)^2} J_n\left((x-t)^2, x\right). \tag{2.9}$$

Direct calculation gives

$$J_n\left((x-t)^2,x\right)=\frac{x(1+x)^2}{n+1}+\frac{(1+x)^4}{3(n+1)(n+2)}+\frac{(1+x)^4(4x+1)}{3(n+1)(n+2)(n+3)}+o\left(n^{-4}\right). \tag{2.10}$$

Hence $\int_0^y H_n(x,t)dt \le (1/(x-y)^2)(2x(1+x)^2/(n+1))$, for n *sufficiently large*.

The proof of (b) is similar. □

Lemma 2.3 (see [9, Theorem 1] or, cf. [10]). *For every $x\in(0,1)$, there holds*

$$p_{n,k}(x)=C_n^k x^k(1-x)^{n-k}\le\frac{1}{\sqrt{2enx(1-x)}}. \tag{2.11}$$

3. Proof of Theorem 1.1

Let $f\in\Phi$, and $x\in I$, Bojanic-Cheng decomposition yields

$$f(t)=\frac{f(x+)-f(x-)}{2}+g_x(t)+\frac{f(x+)-f(x-)}{2}\operatorname{sgn}(t-x)+\delta_x(t)\left[f(x)-\frac{f(x+)-f(x-)}{2}\right], \tag{3.1}$$

where $g_x(t)$ is defined as in (1.6) and

$$\delta_x(t)=\begin{cases}1, & t=x,\\ 0, & t\neq x.\end{cases} \tag{3.2}$$

Obviously, $J_n(\delta_x(t),x)=0$. Thus it follows from (3.1) that

$$\left|J_n(f,x)-\frac{f(x+)-f(x-)}{2}\right|\le\left|J_n(g_x,x)\right|+\left|J_n(\operatorname{sgn}(t-x),x)\frac{f(x+)-f(x-)}{2}\right|. \tag{3.3}$$

First of all, we estimate $|J_n(\operatorname{sgn}(t-x),x)|$

$$J_n(\operatorname{sgn}(t-x),x)=\sum_{k=0}^{n}\frac{(n+1-k)(n+2-k)}{n+2}p_{n,k}\left(\frac{x}{1+x}\right)\int_{I_k}\operatorname{sgn}(t-x)dt, \tag{3.4}$$

where $I_k=[k/(n+2-k),(k+1)/(n+1-k)]$.

Assuming that $x \in [k'/(n+2-k'), (k'+1)/(n+1-k')]$, for some k' $(0 \le k' \le n)$, then we have

$$\begin{aligned}
J_n(\operatorname{sgn}(t-x), x) &= \sum_{k/(n+2-k)>x} p_{n,k}\left(\frac{x}{x+1}\right) - \sum_{(k+1)/(n+1-k)<x} p_{n,k}\left(\frac{x}{x+1}\right) \\
&\quad + \frac{(n+1-k')(n+2-k')}{n+2} p_{n,k'}\left(\frac{x}{1+x}\right)\left(\int_x^{(k'+1)/(n+1-k)} dt - \int_{k'/(n+2-k)}^{x} dt\right) \\
&= 1 - 2\sum_{k/(n+2-k)\le x} p_{n,k}\left(\frac{x}{x+1}\right) \\
&\quad + 2\frac{(n+1-k')(n+2-k')}{n+2} p_{n,k'}\left(\frac{x}{1+x}\right)\int_x^{(k'+1)/(n+1-k)} dt.
\end{aligned} \tag{3.5}$$

Thus

$$\left|J_n(\operatorname{sgn}(t-x), x)\right| \le \left|1 - 2\sum_{k/(n+2-k)\le x} p_{n,k}\left(\frac{x}{x+1}\right)\right| + 2p_{n,k'}\left(\frac{x}{1+x}\right). \tag{3.6}$$

By Lemma 2.3 combining some direct computations, we can easily obtain

$$2p_{n,k'}\left(\frac{x}{1+x}\right) \le \frac{2}{\sqrt{2en(x/(1+x))\cdot(1/(1+x))}} \le \frac{1+x}{\sqrt{nx}}. \tag{3.7}$$

Set $y = x/(1+x) < 1$, then by (2.4) and using Lemma 2.1, we have

$$\begin{aligned}
&\left|1 - 2\sum_{k/(n+2-k)\le x} p_{n,k}\left(\frac{x}{x+1}\right)\right| \\
&\quad = \left|1 - 2\sum_{k\le(n+2)y} p_{n,k}(y)\right| = 2\left|\frac{1}{2} - P(\eta_n \le (n+2)y)\right| \\
&\quad = 2\left|\frac{1}{2} - P\left(\frac{\eta_n - ny}{\sqrt{ny(1-y)}} \le \frac{2y}{\sqrt{ny(1-y)}}\right)\right| \\
&\quad = 2\left|P\left(\frac{\eta_n - ny}{\sqrt{ny(1-y)}} \le \frac{2y}{\sqrt{ny(1-y)}}\right) - \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{(2y-1)/\sqrt{ny(1-y)}} e^{-t^2/2} dt\right. \\
&\qquad \left. + \frac{1}{\sqrt{2\pi}}\int_0^{(2y-1)/\sqrt{ny(1-y)}} e^{-t^2/2} dt\right|
\end{aligned}$$

$$\begin{aligned}
&\leq 2\left|P\left(\frac{\eta_n - ny}{\sqrt{ny(1-y)}} \leq \frac{2y}{\sqrt{ny(1-y)}}\right) - \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{(2y-1)/\sqrt{ny(1-y)}} e^{-t^2/2}dt\right| \\
&\quad + \frac{2}{\sqrt{2\pi}}\int_{0}^{2y/\sqrt{ny(1-y)}} e^{-t^2/2}dt \\
&\leq \frac{2c\beta_3}{\sqrt{n}{b_1}^3} + \frac{2}{\sqrt{2\pi}}\int_{0}^{2y/\sqrt{ny(1-y)}} e^{-t^2/2}dt \\
&\leq \frac{2\times 0.82\times y(1-y)(2y^2-2y+1)}{y(1-y)\sqrt{ny(1-y)}} + \frac{2}{\sqrt{2\pi}}\frac{2y}{\sqrt{ny(1-y)}} \\
&\leq \frac{4}{\sqrt{ny(1-y)}} = \frac{4(1+x)}{\sqrt{nx}}.
\end{aligned} \tag{3.8}$$

Thus, by (3.7), (3.8) we have

$$\left|J_n(\operatorname{sgn}(t-x),x)\right| \leq \frac{4(1+x)}{\sqrt{nx}} + \frac{1+x}{\sqrt{nx}} = \frac{5(1+x)}{\sqrt{nx}}. \tag{3.9}$$

Finally, we estimate $J_n(g_x,x)$.

First, interval $I=[0,\infty)$ can be decomposed into four parts as

$$D_1 = \left[0, x - \frac{x}{\sqrt{n}}\right], \qquad D_2 = \left[x - \frac{x}{\sqrt{n}}, x + \frac{x}{\sqrt{n}}\right], \qquad D_3 = \left[x + \frac{x}{\sqrt{n}}, 2x\right], \qquad D_4 = [2x, +\infty]. \tag{3.10}$$

So $J_n(g_x,x)$ can be divided into four parts

$$J_n(g_x,x) = \int_0^{+\infty} g_x(t)H_n(x,t)dt = \Delta_{1,n}(g_x) + \Delta_{2,n}(g_x) + \Delta_{3,n}(g_x) + \Delta_{4,n}(g_x), \tag{3.11}$$

where $\Delta_{j,n}(g_x) = \int_{D_j} g_x(t)H_n(x,t)dt$.

Noticing $g_x(x)=0$ and for $t\in D_2$, we have $g_x(t)=g_x(t)-g_x(x)$.

Thus

$$\left|\Delta_{2,n}(g_x)\right| \leq \int_{x-x/\sqrt{n}}^{x+x/\sqrt{n}} \left|g_x(t)-g_x(x)\right| H_n(x,t)dt \leq V_{x-x/\sqrt{n}}^{x+x/\sqrt{n}}(g_x) \leq \frac{1}{n}\sum_{k=1}^{n} V_{x-x/\sqrt{k}}^{x+x/\sqrt{k}}(g_x). \tag{3.12}$$

Next, let $y = x - x/\sqrt{n}$, $\lambda_n(x,t) = \int_0^t H_n(x,u)du$.

Now, we recall the Lebesgue-Stieltjes integral representation, and by using partial Lebesgue-Stieltjes integration, we get

$$\begin{aligned}\left|\Delta_{1,n}(g_x)\right| &= \left|\int_0^y g_x(t)d_t\lambda_n(x,t)\right| \\ &= \left|g_x(y)\lambda_n(x,y) - \int_0^y \lambda_n(x,t)d_t g_x(t)\right| \\ &= \left|(g_x(y)-g_x(x))\lambda_n(x,y) - \int_0^y \lambda_n(x,t)d_t(g_x(t)-g_x(x))\right| \\ &\le V_y^x(g_x)\lambda_n(x,y) + \int_0^y \lambda_n(x,t)d_t(-V_t^x(g_x)).\end{aligned} \tag{3.13}$$

An application of (a) in Lemma 2.2 yields

$$\left|\Delta_{1,n}(g_x)\right| \le V_y^x(g_x)\frac{2x(1+x)^2}{(x-y)^2(n+1)} + \frac{2x(1+x)^2}{(n+1)}\int_0^y \frac{1}{(x-t)^2}d_t(-V_t^x(g_x)). \tag{3.14}$$

Furthermore, since

$$\int_0^y \frac{1}{(x-t)^2}d_t(-V_t^x(g_x)) = \frac{-V_y^x(g_x)}{(x-y)^2} + \frac{V_0^x(g_x)}{x^2} + 2\int_0^y \frac{V_t^x(g_x)}{(x-t)^3}dt, \tag{3.15}$$

we have

$$\left|\Delta_{1,n}(g_x)\right| \le \frac{2x(1+x)^2}{n+1}\left[\frac{V_0^x(g_x)}{x^2} + 2\int_0^{x-x/\sqrt{n}} \frac{V_t^x(g_x)}{(x-t)^3}dt\right]. \tag{3.16}$$

Putting $t = x - x/\sqrt{u}$ in the last integral, we have

$$2\int_0^{x-x/\sqrt{n}} \frac{V_t^x(g_x)}{(x-t)^3}dt = \frac{1}{x^2}\int_1^n V_{x-x/\sqrt{u}}^x(g_x)du \le \frac{1}{x^2}\sum_{k=1}^n V_{x-x/\sqrt{k}}^x(g_x). \tag{3.17}$$

It follows from (3.16) and (3.17) that

$$\left|\Delta_{1,n}(g_x)\right| \le \frac{2x(1+x)^2}{(n+1)x^2}\left(V_0^x(g_x) + \sum_{k=1}^n V_{x-x/\sqrt{k}}^x(g_x)\right) \le \frac{4(1+x)^2}{(n+1)x}\sum_{k=1}^n V_{x-x/\sqrt{k}}^x(g_x). \tag{3.18}$$

By a similar method and using Lemma 2.2(b), we obtain

$$\left|\Delta_{3,n}(g_x)\right| \leq \frac{8(1+x)^2}{(n+1)x}\sum_{k=1}^{n} V_x^{x+x/\sqrt{k}}(g_x). \tag{3.19}$$

Now, the remainder of our work is to estimate $\Delta_{4,n}(g_x)$.

For $f(x)$ satisfying the growth condition $f(t) = 0(t^r)$ for some positive integer r as $t \to +\infty$, we obviously have

$$\left|\Delta_{4,n}(g_x)\right| \leq \sum_{k/(n+2-k)>2x} p_{n,k}\left(\frac{x}{1+x}\right)\frac{\int_{\mathrm{I}_k}\left|g_x(t)\right|dt}{\int_{\mathrm{I}_k}dt}. \tag{3.20}$$

Thus, for n sufficiently large, there exists a $M > 0$, such that the following inequalities hold:

$$\begin{aligned}\left|\Delta_{4,n}(g_x)\right| &\leq M\sum_{k/(n+2-k)>2x} p_{n,k}\left(\frac{x}{1+x}\right)\frac{\int_{\mathrm{I}_k}t^r dt}{\int_{\mathrm{I}_k}dt}\\ &= M\sum_{k/(n+2-k)>2(y/(1-y))} p_{n,k}(y)\frac{\int_{\mathrm{I}_k}t^r dt}{\int_{\mathrm{I}_k}dt}\\ &\leq M\sum_{k/(n+2-k)>2(y/(1-y))} p_{n,k}(y)\left(\frac{k+1}{n+1-k}\right)^r,\end{aligned} \tag{3.21}$$

where $y = x/(1+x)$. By the definition of the Stirling numbers $S(r,s)$ of the second kind, we readily have

$$a^r = \sum_{s=1}^{r} S(r,s)a(a-1)\cdots(a-s+1), \quad r \in \mathbb{N}, \tag{3.22}$$

where the Stirling numbers $S(r,s)$ satisfy

$$S(n,0) = \begin{cases} 1 & (n=0),\\ 0 & (n \in \mathbb{N}).\end{cases} \tag{3.23}$$

Thus we can write

$$\sum_{k/(n+2-k)>2(y/(1-y))}\left(\frac{k+1}{n+1-k}\right)^r p_{n,k}(y) = \sum_{s=1}^{r} S(r,s)A_s, \tag{3.24}$$

where

$$\begin{aligned}A_s &= \sum_{k/(n+2-k)>2(y/(1-y))} \frac{(k+1)k\cdots(k-s+2)}{(n+1-k)^r} p_{n,k}(y) \\ &= \sum_{k/(n+2-k)>2(y/(1-y))} \frac{1}{(n+1-k)^r} \cdot \frac{n!(k+1)}{(k-s+1)!(n-k)!} y^k (1-y)^{n-k}.\end{aligned} \tag{3.25}$$

From $k/(n+2-k) > 2x$, $x/(1+x) = y$, we can easily find $k > (2n+4)y/(1+y)$. For a fixed $x > 0$, when $n > 2r + r/x$, we have $(k+1)/(k+1-s) < 2$. Thus there holds

$$A_s \le 2 \sum_{k>(2n+4)y/(1+y)} \frac{1}{(n+1-k)^r} \cdot \frac{n!}{(k-s)!(n-k)!} y^k (1-y)^{n-k}. \tag{3.26}$$

Now using the similar method as that in the proof of Lemma 4 of [11], we deduce that

$$A_s \le \frac{24r!n!y^{s-1}(1+y)^{r-s+2}}{(n+r-s)!(n+r-s+2)}, \quad \text{for } n > 2r + \frac{r}{x}. \tag{3.27}$$

From (3.21), (3.24), and (3.27), we obtain

$$\begin{aligned}\left|\Delta_{4,n}(g_x)\right| &\le M \sum_{k/(n+2-k)>2(y/(1-y))} p_{n,k}(y) \left(\frac{k+1}{n+1-k}\right)^r \\ &= M \sum_{s=1}^{r} S(r,s) A_s = O\left(\frac{1}{n}\right).\end{aligned} \tag{3.28}$$

Finally, by combining (3.12), (3.18), (3.19), and (3.28), we deduce that

$$\begin{aligned}\left|J_n(g_x(t),x)\right| &\le \left|\Delta_{1,n}(g_x)\right| + \left|\Delta_{2,n}(g_x)\right| + \left|\Delta_{3,n}(g_x)\right| + \left|\Delta_{4,n}(g_x)\right| \\ &\le \frac{4(1+x)^2}{(n+1)x} \sum_{k=1}^{n} V_{x-x/\sqrt{k}}^{x}(g_x) + \frac{1}{n} V_{x-x/\sqrt{k}}^{x+x/\sqrt{k}}(g_x) + \frac{8(1+x)^2}{(n+1)x} \sum_{k=1}^{n} V_{x}^{x+x/\sqrt{k}}(g_x) + O\left(\frac{1}{n}\right) \\ &\le \frac{1}{n} V_{x-x/\sqrt{k}}^{x+x/\sqrt{k}}(g_x) + \frac{8(1+x)^2}{(n+1)x} \sum_{k=1}^{n} V_{x-x/\sqrt{k}}^{x+x/\sqrt{k}}(g_x) + O\left(\frac{1}{n}\right) \\ &\le \frac{9(1+x)^2}{nx} \sum_{k=1}^{n} V_{x-x/\sqrt{k}}^{x+x/\sqrt{k}}(g_x) + O\left(\frac{1}{n}\right).\end{aligned} \tag{3.29}$$

Theorem 1.1 now follows from (3.3), (3.9), and (3.29).

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