Global existence theory for the two-dimensional derivative Ginzburg-Landau equation

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Abstract The generalized derivative Ginzburg-Landau equation in two spatial dimensions is discussed. The existence and uniqueness of global solution are obtained by Galerkin method and by *a priori* estimates on the solution in H^1-norm and H^2-norm.

Keywords: Ginzburg-Landau equation, global existence.

THE generalized Ginzburg-Landau equation arises in many nonlinear physical phenomena which include Rayleigh-Benard convection, Taylor-Couette flow in fluid mechanics, drift dissipative wave in plasma physics as well as the turbulent flow in chemical reaction.

Doering *et al.*[1] presented the existence and uniqueness of global solution for G-L equation withtout derivative term ∇u. Duan *et al.*[2] obtained a sufficient condition for the global existence to the generalized G-L equation in one spatial dimension, and Guo and Gao[3] showed the existence of global attractor which has finite Hausdorff and fractal dimension.

In this note, we consider the following complex derivative Ginzburg-Landau equation in two spatial dimensions:

$$\frac{\partial u}{\partial t} = \rho u + (1 + i\upsilon)\Delta u - (1 + i\mu)|u|^{2\sigma}u + \alpha\lambda_1 \circ \nabla(|u|^2 u) + \beta(\lambda_2 \circ \nabla u)|u|^2, \text{ in } \Omega \times \mathrm{R}^+, \tag{1}$$

with initial condition

$$u(x, 0) = u_0(x), \text{ in } \Omega, \tag{2}$$

and periodic boundary condition

$$\Omega = (0, L_1) \times (0, L_1) \text{ and } u \text{ is } \Omega\text{-periodic}, \tag{3}$$

where u is the unknown complex value function, $\sigma > 0$, $\rho > 0$, υ, μ, α and β are real constants, λ_1 and λ_2 are real vectors.

We now state our main results.

Theorem. Assume that $u_0 \in H^2(\Omega)$ and $\exists\ \delta > 0$ such that

$$3 \leqslant \sigma \leqslant \frac{1}{\sqrt{1 + \left(\frac{\mu - \upsilon\delta}{1 + \delta}\right)^2} - 1}.$$

Then there exists a unique global strong solution $u(t)$ of problems (1)—(3) such that

$$u(t) \in L^\infty(0, T; H^2(\Omega)),$$
$$\frac{\partial u}{\partial t} \in L^\infty(0, T; L^2(\Omega))$$

for all $T > 0$.

Introducing the approximate solution sequence $\{U_m\}$ (denoted by $\{u\}$ for the sake of convenience) by Galerkin method we can prove the theorem. How to obtain *a priori* estimates $\|u\|_{H^1}$ and $\|u\|_{H^2}$ plays a key role in the proof. The proof is based on the following two lemmas.

Lemma 1. Assume that $\sigma \geqslant 3$. Then $\forall \varepsilon > 0$, for the solution $u(t)$ we have

$$\frac{1}{2(1+\sigma)}\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega |u|^{2\sigma+2} \leqslant C(\varepsilon)\|u\|_{2\sigma+2}^{2(2\sigma+2)} + \varepsilon\|\Delta u\|^2 + \varepsilon\|\nabla u\|^4 - \frac{1}{4}\int_\Omega |u|^{2\sigma-2}((1+2\sigma)|\nabla|u|^2|^2 - 2\upsilon\sigma\nabla|u|^2 \circ i(u\nabla u^* - u^*\nabla u) + |u\nabla u^* - u^*\nabla u|^2) + C_1,$$

where C_1 depends on the data and T, $\forall T > 0$.

Proof. Take the inner product in H of (1) with $|u|^{2\sigma}u$, and then take the real part of the resulting identity. We find that

$$\frac{1}{2(1+\sigma)}\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega |u|^{2\sigma+2} = \rho\int_\Omega |u|^{2\sigma+2} - \int_\Omega |\nabla u|^2|u|^{2\sigma}$$
$$-\frac{\sigma}{2}\int_\Omega |u|^{2\sigma-2}|\nabla|u|^2|^2 + \frac{1}{2}\upsilon\sigma\int_\Omega |u|^{2\sigma-2}\nabla|u|^2\circ \mathrm{i}(u\nabla u^* - u^*\nabla u)$$
$$-\|(|u|^{2\sigma}u)\|^2 + \mathrm{Re}\,\alpha((\lambda_1\circ\nabla(|u|^2u)),\ (|u|^{2\sigma}u))$$
$$+\mathrm{Re}\beta((\lambda_2\circ\nabla u)|u|^2,\ (|u|^{2\sigma}u)).$$

Due to

$$|u|^2|\nabla u|^2 = \frac{1}{4}|\nabla|u|^2|^2 + \frac{1}{4}|u\nabla u^* - u^*\nabla u|^2,$$

we see that

$$-\int_\Omega |\nabla u|^2|u|^{2\sigma} - \frac{\sigma}{2}\int_\Omega |u|^{2\sigma-2}|\nabla|u|^2|^2$$
$$+\frac{1}{2}\upsilon\sigma\int_\Omega |u|^{2\sigma-2}\nabla|u|^2\circ\mathrm{i}(u\nabla u^* - u^*\nabla u)$$
$$= -\frac{1}{4}\int_\Omega |u|^{2\sigma-2}((1+2\sigma)|\nabla|u|^2|^2$$
$$-2\upsilon\sigma\nabla|u|^2\circ\mathrm{i}(u\nabla u^* - u^*\nabla u) + |u\nabla u^* - u^*\nabla u|^2)\mathrm{d}x.$$
$$|\mathrm{Re}\beta((\lambda_2\circ\nabla u)|u|^2,\ (|u|^{2\sigma}u))|$$
$$\leqslant \frac{1}{2}\|\beta(\lambda_2\circ\nabla u)|u|^2\|^2 + \frac{1}{2}\|(|u|^{2\sigma}u)\|^2$$
$$\leqslant \frac{1}{2}|\beta\lambda_2|^2\int_\Omega |\nabla u|^2|u|^4 + \frac{1}{2}\|(|u|^{2\sigma}u)\|^2$$
$$\leqslant C\|\nabla u\|_4^2\|u\|_8^4 + \frac{1}{2}\|(|u|^{2\sigma}u)\|^2.$$

By Sobolev estimate we have (notice $\sigma\geqslant 3$, $0\leqslant t\leqslant T$ and $\forall 0<\gamma\leqslant 1$),

$$|\mathrm{Re}\beta((\lambda_2\circ\nabla u)|u|^2,\ (|u|^{2\sigma}u))|$$
$$\leqslant \gamma\|u\|_{H^2}^2 + \gamma\|u\|_{H^1}^4 + C(\gamma)\|u\|_q^{2q} + C + \frac{1}{2}\|(|u|^{2\sigma}u)\|^2$$
$$\leqslant \gamma C\|\Delta u\|^2 + 8\gamma\|\nabla u\|^4 + C(\gamma)\|u\|_{2\sigma+2}^{2(2\sigma+2)} + \frac{1}{2}\|(|u|^{2\sigma}u)\|^2 + C.$$

Analogously, we also have

$$|\mathrm{Re}\alpha(\lambda_1\circ\nabla(|u|^2u),\quad (|u|^{2\sigma}u))|$$
$$\leqslant \gamma C\|\Delta u\|^2 + 8\gamma\|\nabla u\|^4 + C(\gamma)\|u\|_{2\sigma+2}^{2(2\sigma+2)} + \frac{1}{2}\|(|u|^{2\sigma}u)\|^2 + C.$$

These inequalities yield Lemma 1 by choosing γ suitably.

Lemma 2. Assume that $u_0\in H^1(\Omega)$ and $\exists\,\delta>0$ such that

$$3\leqslant\sigma\leqslant\frac{1}{\sqrt{1+\left(\dfrac{\mu-\upsilon\delta}{1+\delta}\right)^2}-1}.$$

Then we have

$$\|u(t)\|_{H^1}\leqslant C_2,\quad \forall 0\leqslant t\leqslant T,\quad T>0,$$

where C_2 depends on the data, δ and T.

Proof. Taking the inner product in H of (1) with Δu, and then taking the real part of the resulting identity, we find that

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|\nabla u\|^2 = \rho\|\nabla u\|^2 - \|\Delta u\|^2 + \mathrm{Re}(1+\mathrm{i}\mu)\int_\Omega |u|^{2\sigma}u\Delta u^*$$
$$-\alpha\mathrm{Re}\int_\Omega(\lambda_1\circ\nabla(|u|^2u))\Delta u^* - \beta\mathrm{Re}\int_\Omega(\lambda_2\circ\nabla u)|u|^2\Delta u^*,$$

where

$$\mathrm{Re}(1+\mathrm{i}\mu)\int_\Omega |u|^{2\sigma} u \Delta u^*$$
$$= -\frac{1}{4}\int_\Omega |u|^{2\sigma-2}((1+2\sigma)|\nabla|u|^2|^2 - 2\mu\sigma\nabla|u|^2 \circ \mathrm{i}(u^*\nabla u - u\nabla u^*)$$
$$+ |u^*\nabla u - u\nabla u^*|^2).$$

And by Sobolev estimate we have

$$\left|-\alpha\mathrm{Re}\int_\Omega(\lambda_1 \circ \nabla(|u|^2))\Delta u^* \mathrm{d}x\right| + \left|-\beta\mathrm{Re}\int_\Omega(\lambda_2 \circ \nabla u)\Delta u^* \mathrm{d}x\right|$$
$$\leqslant \varepsilon \|\Delta u\|^2 + \varepsilon\|\nabla u\|^4 + c(\varepsilon)\|u\|_{2\sigma+2}^{2(2\sigma+2)} + C, \quad \forall \varepsilon > 0.$$

In combination with Lemma 1 we arrive at

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\left(\|\nabla u\|^2 + \frac{\delta}{1+\sigma}\|u\|_{2\sigma+2}^{2\sigma+2}\right) \leqslant \rho\|\nabla u\|^2 + C(\varepsilon)(1+\delta)\|u\|_{2\sigma+2}^{2(2\sigma+2)}$$
$$+ (\varepsilon + \varepsilon\delta - 1)\|\Delta u\|^2 + \varepsilon(1+\delta)\|\nabla u\|^4 + C\delta + C$$
$$-\frac{1}{4}\int_\Omega |u|^{2\sigma-2}((1+2\sigma)(1+\delta)|\nabla|u|^2|^2 + 2\sigma(\upsilon\delta - \mu)\nabla|u|^2 \circ \mathrm{i}(u^*\nabla u - u\nabla u^*)$$
$$+ (1+\delta)|u^*\nabla u - u\nabla u^*|^2).$$

We note that

$$\sigma \leqslant \frac{1}{\sqrt{1+\left(\frac{\mu - \upsilon\delta}{1+\delta}\right)^2} - 1}$$

implies that the matrix

$$\begin{pmatrix} (1+2\sigma)(1+\delta) & \sigma(\upsilon\delta - \mu) \\ \sigma(\upsilon\delta - \mu) & 1+\delta \end{pmatrix}$$

is nonnegative definite. Thus, for ε sufficiently small we obtain

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\|\nabla u\|^2 + \frac{\delta}{1+\sigma}\|u\|_{2\sigma+2}^{2\sigma+2}\right) \leqslant C\|\nabla u\|^4 + C\|u\|_{2\sigma+2}^{2(2\sigma+2)} + C$$
$$\leqslant K\left(\|\nabla u\|^2 + \frac{\delta}{1+\sigma}\|u\|_{2\sigma+2}^{2\sigma+2}\right)^2 + C,$$

where $k > 0$ and depends on the data. Then by Gronwall lemma we have

$$\int_0^t (\|\nabla u\|^2 + \|u\|_{2\sigma+2}^{2\sigma+2})\mathrm{d}t \leqslant C, \quad \forall 0 \leqslant t \leqslant T.$$

The above concludes Lemma 2.

By Lemma 2 we can obtain *a priori* estimate on the solution in H^2-norm. Now, let us go back to Galerkin approximate solution sequence $\{U_m\}$. Because we have all necessary estimates and convergence for $\{U_m\}$, we can complete the proof of the Theorem.

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