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ABOUT A CONDITION FOR BLOW UP OF SOLUTIONS OF CAUCHY PROBLEM FOR A WAVE EQUATION *

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Abstract: *For the nonlinear wave equation in* $\mathbf{R}^N \times R^+$ $(N \geqslant 2)$: $\frac{\partial^2 u(x,t)}{\partial t^2} - \frac{\partial}{\partial x_i}\left(a_{ij}(x)\frac{\partial}{\partial x_j}u\right) = |u|^{p-1}\cdot u$, *in* 1980 *Kato proved the solution of Cauchy problem may blow up in finite time if* $1 < p \leqslant \frac{N+1}{N-1}$. *In the present work his result allowing* $1 < p \leqslant \frac{N+3}{N-1}$ *is improved by using different estimates.*

Key words: condition for blow up; wave equation; Cauchy problem
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Consider the Cauchy problem for the wave equation in $\mathbf{R}^N \times \mathbf{R}^+$ $(N \geqslant 2)$:

$$\begin{cases} \dfrac{\partial^2 u(x,t)}{\partial t^2} - \dfrac{\partial}{\partial x_i}\left(a_{ij}(x)\dfrac{\partial}{\partial x_j}u\right) = |u|^{p-1}\cdot u & ((x,t) \in \mathbf{R}^N \times (0,T)), \\ u(x,0) = g(x) \quad (x \in \mathbf{R}^N), \\ u_t(x,0) = h(x) \quad (x \in \mathbf{R}^N), \end{cases} \tag{1}$$

where $u(x,t)$ is nontrivial solution with finite speed of propagation and is supported on a forward cone $\{(x,t)\cdot t \geqslant 0, |x| \leqslant t+d\}$. In 1980 Kato[1] proved the solution may blow up if $1 < p \leqslant \frac{N+1}{N-1}$ by using estimate for $\int_{\mathbf{R}^N} u(x,t)\mathrm{d}x$ under hypotheses $\left(\frac{\mathrm{d}}{\mathrm{d}t}\int_{\mathbf{R}^N} u(x,t)\mathrm{d}x\right)\Big|_{t=0} > 0$ or $\left(\int_{\mathbf{R}^N} u(x,t)\mathrm{d}x\right)\Big|_{t=0} \neq 0$. In this paper, we will improve his result allowing $K_p < \frac{N+3}{N-1}$, by using different estimates for $\int_{\mathbf{R}^N} u^2(x,t)\mathrm{d}x$.

Theorem Assume that

(H1) $1 < p < \frac{N+3}{N-1}$,

(H2) $a_{ij}(x) \in C^2(\mathbf{R}^N)$ and are elliptic,

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(H3) $g(x), h(x) \in C_0^\infty(\mathbf{R}^N)$, $\operatorname{supp}\{g(x), h(x)\} \subseteq \{|x| \leqslant d\}$,

(H4) $\int_{\mathbf{R}^N} g(x)h(x)\mathrm{d}x \geqslant 0$ and $g(x) \not\equiv 0$, which imply

$$\left(\frac{\mathrm{d}}{\mathrm{d}t}\int_{\mathbf{R}^N} u^2(x,t)\mathrm{d}x\right)\bigg|_{t=0} \geqslant 0 \text{ and } \left(\int_{\mathbf{R}^N} u^2(x,t)\mathrm{d}x\right)\bigg|_{t=0} > 0,$$

(H5) $I \triangleq \frac{2}{p+1}\int_{\mathbf{R}^N} |g(x)|^{p+1}\mathrm{d}x - \int_{\mathbf{R}^N} a_{ij}(x)\, D_i g(x) D_j g(x)\mathrm{d}x - \int_{\mathbf{R}^N} |h(x)|^2 \mathrm{d}x \geqslant 0.$

Then $T < \infty$, i. e. $u(x, t)$ may blow up in finite time.

Proof The (H2) ~ (H4) implies the existence of a unique classical solution to problem (1). We will estimate

$$w(t) \triangleq \int_{\mathbf{R}^N} u^2(x,t)\mathrm{d}x$$

by using method similar to Ref. [2]. First, multiplying Eq. (1) by $u(x, t)$ and integrating over R^N, we have

$$\frac{1}{2}w''(t) = \frac{p-1}{p+1}\int_{\mathbf{R}^N} |u|^{p+1}\mathrm{d}x + \frac{2}{p+1}\int_{\mathbf{R}^N} |u|^{p+1}\mathrm{d}x + \int_{\mathbf{R}^N} |u_t|^2\mathrm{d}x - \int_{\mathbf{R}^N} a_{ij} D_i u D_j u \mathrm{d}x. \tag{2}$$

Next, multiplying Eq. (1) by u_t and integrating over $\mathbf{R}^N \times [0,t]$, we have $\left(\text{Notice } a_{ij}D_iuD_ju_t = \frac{1}{2}(a_{ij}D_iuD_ju)_t\right)$

$$\int_{\mathbf{R}^N} |u_t|^2\mathrm{d}x = \frac{2}{p+1}\int_{\mathbf{R}^N} |u|^{p+1}\mathrm{d}x - \int_{\mathbf{R}^N} a_{ij} D_i u D_j u\mathrm{d}x + \int_{\mathbf{R}^N} a_{ij} D_i g D_j g \mathrm{d}x + \int_{\mathbf{R}^N} h^2(x)\,\mathrm{d}x - \frac{2}{p+1}\int_{\mathbf{R}^N} |g(x)|^{p+1}\mathrm{d}x,$$

i. e.,
$$\int_{\mathbf{R}^N} |u_t|^2\mathrm{d}x = \frac{2}{p+1}\int_{\mathbf{R}^N} |u|^{p+1}\mathrm{d}x - \int_{\mathbf{R}^N} a_{ij} D_i u D_j u\mathrm{d}x - I. \tag{3}$$

Eqs. (2) and (3) yield

$$\frac{1}{2}w''(t) = \frac{p-1}{p+1}\int_{\mathbf{R}^N} |u|^{p+1}\mathrm{d}x + \int_{\mathbf{R}^N} |u_t|^2\mathrm{d}x + \int_{\mathbf{R}^N} |u_t|^2\mathrm{d}x + I.$$

By (H5) we obtain

$$\frac{1}{2}w''(t) \geqslant \frac{p-1}{p+1}\int_{\mathbf{R}^N} |u|^{p+1}\mathrm{d}x. \tag{4}$$

Thus $w''(t) \geqslant 0$ and $w'(t)$ is monotone nondecreasing. Therefore $w'(t) \geqslant w'(0) \geqslant 0$ by (H4) and $w(t)$ is also monotone nondecreasing.

Thus by (H4), we have

$$w(t) \geqslant w(0) > 0. \tag{5}$$

Now, by finite speed of propagation and by (H3), we have

$$w(t) = \int_{\mathbf{R}^N} |u|^2 dx = \int_{|x| \leqslant t+d} |u|^2 dx \leqslant$$

$$\left\{ \int_{|x| \leqslant t+d} |u|^{p+1} dx \right\}^{\frac{2}{p+1}} \cdot \left\{ \int_{|x| \leqslant t+d} 1 \cdot dx \right\}^{\frac{p-1}{p+1}},$$

or

$$w(t)^{\frac{p+1}{2}} \leqslant c_1 (t+d)^{N(p-1)/2} \cdot \int_{\mathbf{R}^N} |u|^{p+1} dx.$$

Combining this with Eq. (4), we obtain

$$w''(t) \geqslant c_2 (t+d)^{-N(p-1)/2} \cdot w(t)^{\frac{p+1}{2}},$$

or

$$w''(t) \geqslant c_0 \frac{p+3}{p-1} (t+d)^{-N(p-1)/2} \cdot w(t)^{\frac{p+1}{2}}. \tag{6}$$

Thus $w''(t) > 0$ by (5). Therefore there exists a positive constant ν such that $w''(t) > \nu$. This yields

$$w(t) \geqslant \frac{1}{2} \nu t^2 + w'(0) \cdot t + w(0),$$

or

$$w(t) \geqslant \mu (t+d), \tag{7}$$

where μ is a positive constant.

Again by Eq. (6) we have

$$2w' \left[w'' - c_0 \frac{p+3}{p-1} (t+d)^{-N(p-1)/2} \cdot w^{\frac{p+1}{2}} \right] +$$

$$\frac{4c_0}{p+1} \cdot \frac{N(p-1)}{2} \cdot (t+d)^{\frac{-N(p-1)}{2} - 1} \cdot w^{\frac{p+3}{2}} > 0,$$

i. e.

$$\frac{d}{dt} \left\{ (w'(t))^2 - \frac{4c_0}{p+1} (t+d)^{-N(p-1)/2} \cdot w(t)^{\frac{p+3}{2}} \right\} > 0. \tag{8}$$

Because $w'(t_0) > 0$ for arbitrary fixed to $t_0 > 0$ we can take a sufficiently small $c_0 > 0$ in Eq. (6) such that

$$[w'(t_0)]^2 - \frac{4c_0}{p+1} (t_0+d)^{-N(p-1)/2} \cdot w(t_0)^{\frac{p+3}{2}} > 0.$$

Thus Eq. (8) yields

$$[w'(t)]^2 - \frac{4c_0}{p+1} (t+d)^{-N(p-1)/2} \cdot w(t)^{\frac{p+3}{2}} > 0 \qquad (\text{for } t \geqslant t_0),$$

or

$$w'(t) \geqslant c_3 (t+d)^{-N(p-1)/4} \cdot w(t)^{\frac{p+3}{4}} =$$

$$c_3 (t+d)^{-N(p-1)/4} \cdot w(t)^{\frac{p-1}{4}(1-\theta)} \cdot w(t)^{\frac{p-1}{4}\theta + 1}, \tag{9}$$

where $\theta \in (0,1)$ is sufficiently small constant such that $0 < [N - (1 - \theta)] \cdot \frac{p - 1}{4} < 1$. $\left(\text{Notice } 0 < p - 1 < \frac{4}{N - 1} \text{ by (H1)}\right)$. *Thus Eqs*. (7) and (9) yield

$$w'(t) \geqslant c_5(t + d)^{\frac{-N(p-1)}{4} + \frac{p-1}{4}(1-\theta)} \cdot w(t)^{\frac{p-1}{4}\theta + 1}.$$

Denote $\alpha = \frac{N(p - 1)}{4} - \frac{p - 1}{4}(1 - \theta), \beta = \frac{p - 1}{4}\theta + 1$, then $0 < \alpha < 1$ and $\beta > 1$. Therefore

$$w'(t) \geqslant c_5(t + d)^{-\alpha} \cdot w(t)^{\beta}.$$

This differential inequality implies that for some time to $T_0 < +\infty$:

$$w(t) = \int_{\mathbf{R}^N} u^2(x,t)\mathrm{d}x \to +\infty \qquad (\text{when } t \to T_0^-).$$

Thus, the solution $u(x, t)$ of Eq. (1) may blow up in finite time.

References:

[1] Kato T. Blow up of solutions of some nonlinear hyperbolic equations[J]. *Comm Pure Appl Math*, 1980, 33(4): 501 ~ 505.

[2] Cao Zhenchao, Wang Bixiang, Guo Boling. Global existence theory for the two dimensional derivative G-L equation[J]. *Chinese Science Bulletin*, 1998, **43**(5): 393 ~ 395.