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Dimers on two types of lattices on the Klein bottle

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Abstract
The problem of enumerating close-packed dimers, or perfect matchings, on two types of lattices (the so-called 8.8.4 and 8.8.6 lattices) embedded on the Klein bottle is considered, and we obtain the explicit expression of the number of close-packed dimers and entropy. Our results imply that 8.8.4 lattices have the same entropy under three different boundary conditions (cylindrical, toroidal and Klein bottle) and 8.8.6 lattices have the same property.

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1. Introduction

A central problem in statistical physics and combinatorial mathematics is the enumeration of close-packed dimers, often referred to as perfect matchings in the mathematical literature or Kekulé structures in quantum chemistry, on lattices [19]. In 1961, Kasteleyn [4, 5] found a formula for $m \times n$ quadratic lattice graph with both free and toroidal boundary conditions. Temperley and Fisher [17] used a different method and arrived at the same result at almost exactly the same time. They also considered the entropy of the quadratic lattice graph. Sachs and Zeritz [15] studied the problem involving the enumeration of close-packed dimers of another type of quadratic lattice with different boundary conditions and a different entropy was obtained. This fact showed that the entropy of the quadratic lattice is strongly dependent on the boundary conditions.

The exact solution of the dimer problem was obtained for many other lattices such as 8.8.4 lattice, hexagonal lattice, triangular lattice, kagome lattice, 3.12.12 lattice, union Jack lattice, etc with the toroidal boundary condition [2, 4, 6, 13, 19]. Also, this problem has been extended to the cylindrical condition [10, 13]. Wu and Wang [20] obtained the exact result on the

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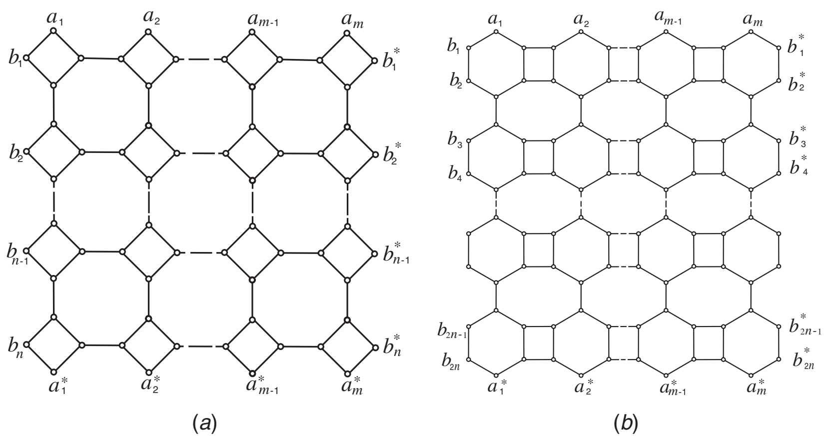

Figure 1. (*a*) $G_1(m, n)$; (*b*) $G_2(m, n)$.

enumeration of close-packed dimers on a finite kagome lattice with general asymmetric dimer weights under the cylindrical boundary condition. The result by Wu and Wang implies that the kagome lattices with the cylindrical and toroidal boundary conditions have the same entropy. This phenomenon also occurred for some other lattices with cylindrical and toroidal boundary conditions [3]. In view of the connection with conformal field theory, where the boundary conditions play a crucial role, there has been considerable renewed interest in considering lattice models on nonorientable surfaces. Lu and Wu [11, 12] have obtained the generating function for simple quadratic lattices embedded on a Klein bottle. In this paper, two types of lattice with the Klein bottle boundary condition are considered.

Two bulk lattices, denoted by $G_1(m, n)$ and $G_2(m, n)$, are illustrated in figures 1(*a*) and 1(*b*), respectively, where $G_2(m, n)$ is a finite subgraph of an edge-to-edge tiling of the plane with two types of vertices—8.8.6 and 8.8.4 vertices—and $G_1(m, n)$ is a finite subgraph of 8.8.4 tiling in the Euclidean plane which has been used to describe phase transitions in the layered hydrogen-bonded $SnCl_2{\cdot}2H_2O$ crystal [17] in physical systems [1, 14, 16]. The 8.8.6 lattice $G_2(m, n)$, whose fundamental part is a hexagon, is composed of mn hexagons. Similarly, the 8.8.4 lattice $G_1(m, n)$, whose fundamental part is a quadrangle, is composed of mn quadrangles.

If we add edges (a_t, a_t^*) for $1 \leqslant t \leqslant m$ in $G_1(m, n)$ (resp. $G_2(m, n)$), we obtain a graph with the cylindrical boundary condition, denoted by $G_1^c(m, n)$ (resp. $G_2^c(m, n)$). Adding edges (a_t, a_t^*), (b_j, b_j^*) for $1 \leqslant t \leqslant m$, $1 \leqslant j \leqslant n$ in $G_1(m, n)$, 8.8.4 lattice with the toroidal boundary condition, denoted by $G_1^t(m, n)$, can be obtained. Salinas and Nagle [16] and Wu [19] showed that the entropy of $G_1^t(m, n)$ is

$$\lim_{m,n\to\infty} \frac{2}{4mn} \log[w_{G_1^t(m,n)}] = \frac{1}{2\pi} \int_0^{\frac{\pi}{2}} \log \frac{5 + \sqrt{25 - 16\cos^2 x}}{2}\, \mathrm{d}x \approx 0.3770.$$

Yan *et al* [21] showed that the entropy of $G_1^c(2m, n)$ is 0.3770 too.

Similarly, by adding edges (a_t, a_t^*), (b_j, b_j^*) for $1 \leqslant t \leqslant m$, $1 \leqslant j \leqslant 2n$ in $G_2(m, n)$, 8.8.6 lattice with the toroidal boundary condition, denoted by $G_2^t(m, n)$, can be obtained. Yan *et al* [21] showed that both $G_2^c(2m, n)$ and $G_2^t(2m, n)$ have the same entropy, i.e.

$$\lim_{m,n\to\infty} \frac{2}{12mn} \log w_{G_2^c(2m,n)} = \lim_{m,n\to\infty} \frac{2}{12mn} \log w_{G_2^t(2m,n)}$$
$$= \frac{2}{3\pi} \int_0^{\frac{\pi}{2}} \log(\cos x + \sqrt{4 + \cos^2 x})\, \mathrm{d}x \approx 0.3344.$$

In this paper, the close-packed dimers of the 8.8.4 lattice $G_1^k(m,n)$ and 8.8.6 lattice $G_2^k(m,n)$ with the Klein bottle boundary condition, where $G_1^k(m,n)$ (resp. $G_2^k(m,n)$) is obtained from $G_1(m,n)$ (resp. $G_2(m,n)$) by adding edges (a_t, a^*_{m+1-t})) (b_j, b_j^*) for $1 \leqslant t \leqslant m$, $1 \leqslant j \leqslant n$ (resp. $1 \leqslant j \leqslant 2n$), are considered. We also obtain the entropies for those lattices. In section 2, we introduce the method of Tesler, and we enumerate the close-packed dimers and entropies in section 3.

2. Tesler's method and crossing orientation

In this section, we review the method of Tesler [18]. It can also be found in [8, 9]. Given an undirected graph $G = (V(G), E(G))$ with a vertex set $V(G) = \{1, 2, ..., 2p\}$, we allow each edge $\{u, v\}$ to have a weight $w_{\{u,v\}}$. In unweighted graphs, set weight to 1 for all edges. Let G^e be an arbitrary orientation of G. Denote the arc of G^e by (u, v) if the direction of it is from u to v. The skew adjacency matrix of G^e, denoted by $A(G^e)$, is defined as follows:

$$A(G^e) = (a_{u,v})_{2p\times 2p},$$

where

$$a_{u,v} = \begin{cases} w_{\{u,v\}} & \text{if } (u,v) \text{ is an arc of } G^e, \\ -w_{\{u,v\}} & \text{if } (v,u) \text{ is an arc of } G^e, \\ 0 & \text{otherwise.} \end{cases}$$

Let $PM = \{\{u_1, u_1'\}, \dots, \{u_p, u_p'\}\}$ range over the partitions of 1, ..., $2p$ into p sets of size 2 and define the signed weight of PM as

$$w_{PM} = \operatorname{sign}\begin{pmatrix} 1 & 2 & \cdots & 2p-1 & 2p \\ u_1 & u_1' & \cdots & u_p & u_p' \end{pmatrix} \cdot a_{u_1,u_1'} \cdots a_{u_p,u_p'},$$

where the sign is of the permutation expressed in two-line notation. The Pfaffian of A is defined as

$$\operatorname{Pf} A = \sum_{PM} w_{PM}.$$

Theorem 1 (Cayley's theorem [7]). *Let $A = (a_{u,v})_{2p\times 2p}$ be a skew symmetric matrix of order of $2p$. Then the determinant of A,* $\det(A) = (\operatorname{Pf} A)^2$.

When PM is a partition that is not a dimer covering, $w_{PM} = 0$, so the nonzero terms of $\operatorname{Pf} A$ correspond to the dimer coverings of G. We call w_{PM} the signed weight of the dimer coverings PM and define the sign of PM to be the sign of w_{PM}. Generally speaking, the terms in the Pfaffian do not possess the same sign; the evaluation of the Pfaffian does not necessarily produce the desired dimer covering number. In order to obtain all dimer configurations correctly counted, we use the method of Tesler.

Any compact boundaryless two-dimensional surface S can be represented in the plane by a plane model [18]. Draw a $2l$-sided polygon P and form l pairs of sides p_j, p_j', $j = 1, \dots, l$. Paste together p_j and p_j'. Any S can be represented by a suitable polygon and pastings. Now take an embedding of a graph G on this surface and draw it within this plane model of the surface. Edges wholly contained inside the polygon P do not cross and are called 0-edges. The edges that go through sides p_j, p_j' of P are called j-edges. We say a face of a planar graph is clockwise odd when it has an odd number of edges pointing along its boundary when traversed clockwise.

To $G_1^k(m,n)$ and $G_2^k(m,n)$ lattices, the graphs embedded on the Klein bottle, we draw its planar subgraph containing all vertices in a four-polygon and label the vertices by $1, 2, \dots,$ shown in figures 2 and 3.

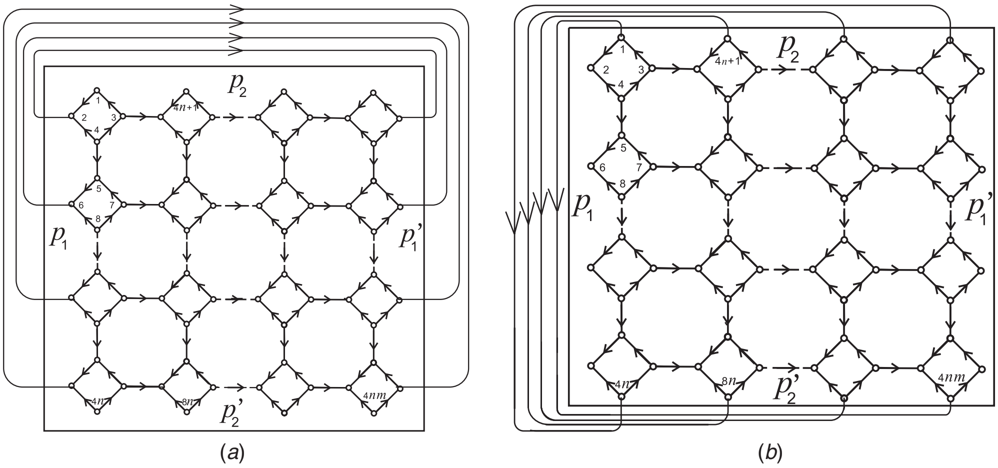

Figure 2. A crossing orientation of $G_1^k(m,n)$. (*a*) The orientation of the subgraph consisted of 0-edges and 1-edges; (*b*) the orientation of the subgraph consisted of 0-edges and 2-edges.

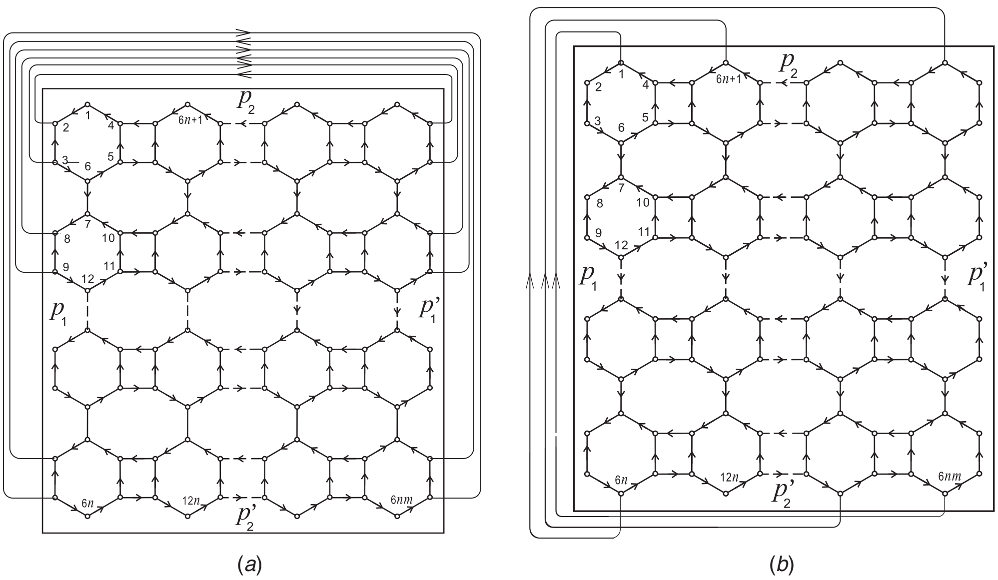

Figure 3. A crossing orientation of $G_2^k(m,n)$ when n is odd, m is even. (*a*) The orientation of the subgraph consisted of 0-edges and 1-edges; (*b*) the orientation of the subgraph consisted of 0-edges and 2-edges.

Crossing orientation rule [18]. *Orient the subgraph of 0-edges so that all its faces are clockwise odd. Orient each j-edge e* ($j > 0$) *as follows. Ignoring all other non-0-edges, there is a face formed by e and certain 0-edges along the boundary of the subgraph of 0-edges. Orient e so that this face is clockwise odd.*

Orient the edges of $G_1^k(m,n)$ as shown in figure 2. Let the directions of the edges in $G_2^k(m,n)$ be the same as shown in figure 3 (reversing the 2-edges when n is even, reversing the 1-edges when m is odd). It can be easily checked that those are crossing orientations.

Introduce two new variables: x_1 and x_2. Multiply the weights of all j-edges by x_j $(j \neq 0)$, and let $X(x_1, x_2)$ be the x-adjacency matrix, where the element of matrix $X(x_1, x_2)$ in row u, column v is

$$X_{u,v} = \begin{cases} 1 & \text{if } (u, v) \text{ is a 0-edge;} \\ -1 & \text{if } (v, u) \text{ is a 0-edge;} \\ x_j & \text{if } (u, v) \text{ is a } j\text{-edge } (j \neq 0); \\ -x_j & \text{if } (v, u) \text{ is a } j\text{-edge } (j \neq 0); \\ 0 & \text{otherwise.} \end{cases}$$

Let $x_1 = \pm 1, x_2 = \pm \mathrm{i}$, and set the weight to be 1 for all 0-edges. Then the number of close-packed dimers is given in [9, 18] as

$$w_G = |\mathrm{Re}(\mathrm{Pf}\, X(1, \mathrm{i}))| + |\mathrm{Im}(\mathrm{Pf}\, X(-1, \mathrm{i}))|. \tag{1}$$

3. The number of close-packed dimers

To obtain the numbers of close-packed dimers of $G_1^k(m,n)$ and $G_2^k(m,n)$, we first introduce some notation. Let B^{-1} be the inverse matrix of B, B^{T} be the transpose of B and $\delta_{t,j}$ be the Kronecker delta equal to 1 if $t = j$ and 0 otherwise. Let I_m denote the $m \times m$ identity matrix. Set

$$R = \begin{bmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & 0 & \cdots \\ \vdots & \vdots & \ddots & \ddots & \ddots \\ 0 & 0 & \cdots & 0 & 1 \\ 0 & 0 & \cdots & 0 & 0 \end{bmatrix}_{n\times n}, \quad K_1 = \begin{bmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & 0 & \ddots \\ \vdots & \vdots & \vdots & \ddots & \ddots \\ 0 & 0 & 0 & \cdots & 1 \\ 1 & 0 & 0 & \cdots & 0 \end{bmatrix}_{m\times m},$$

$$K_2 = \begin{bmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & 0 & \ddots \\ \vdots & \vdots & \ddots & \ddots & \ddots \\ 0 & 0 & \cdots & 0 & 1 \\ -1 & 0 & 0 & \cdots & 0 \end{bmatrix}_{m\times m}, \quad K_3 = \begin{bmatrix} & & & 1 \\ & & 1 & \\ & \vdots & & \\ 1 & & & \end{bmatrix}_{m\times m}.$$

Let U and V be the $m \times m$ matrices with elements

$$U_{t,j} = \sqrt{\frac{1}{m}}\, \mathrm{e}^{\mathrm{i}\frac{2jt\pi}{m}}; \quad V_{t,j} = \sqrt{\frac{1}{m}}\, \mathrm{e}^{\mathrm{i}\frac{(2j-1)t\pi}{m}}; \quad t, j = 1, 2, \ldots, m,$$

respectively. It is not difficult to check that the elements of the U^{-1} and V^{-1} ($m \times m$ matrices) are

$$(U^{-1})_{t,j} = \sqrt{\frac{1}{m}}\mathrm{e}^{-\mathrm{i}\frac{2tj\pi}{m}}; \quad (V^{-1})_{t,j} = \sqrt{\frac{1}{m}}\mathrm{e}^{-\mathrm{i}\frac{(2t-1)j\pi}{m}}.$$

The elements of the $m \times m$ matrices $U^{-1}K_1U$, $U^{-1}(-K_1^{\mathrm{T}})U$ and $U^{-1}K_3U$ are

$$(U^{-1}K_1U)_{t,j} = \mathrm{e}^{\mathrm{i}\theta_t}\delta_{t,j}, \quad \left(U^{-1}\left(-K_1^{\mathrm{T}}\right)U\right)_{t,j} = -\mathrm{e}^{-\mathrm{i}\theta_t}\delta_{t,j}, \quad (U^{-1}K_3U)_{t,j} = \mathrm{e}^{-\mathrm{i}\theta_t}\delta_{t+j,n+1} \tag{2}$$

and the elements of the $m \times m$ matrices $V^{-1}K_2V$, $V^{-1}\big(-K_2^{\mathrm{T}}\big)V$ and $V^{-1}K_3V$ are

$$(V^{-1}K_2V)_{t,j} = \mathrm{e}^{\mathrm{i}\phi_t}\delta_{t,j}, \quad \big(V^{-1}\big(-K_2^{\mathrm{T}}\big)V\big)_{t,j} = -\mathrm{e}^{-\mathrm{i}\phi_t}\delta_{t,j}, \quad (V^{-1}K_3V)_{t,j} = -\mathrm{e}^{-\mathrm{i}\phi_t}\delta_{t+j,n}, \tag{3}$$

where $\theta_t = \frac{2t\pi}{m}$ and $\phi_t = \frac{(2t-1)\pi}{m}$ for $t, j = 1, 2, \ldots, m$.

3.1. *The lattice* $G_1^k(m, n)$

For an even value of m, consider the labelling of vertices of $G_1^k(m, n)$ shown in figure 2. The x-adjacency matrices $X(-1, \mathrm{i})$ and $X(1, \mathrm{i})$ can be written in terms of a linear combination of direct products of the smaller ones:

$$\begin{aligned} X(-1, \mathrm{i}) &= A \otimes I_m + B \otimes K_1 - B^{\mathrm{T}} \otimes K_1^{\mathrm{T}} + C \otimes K_3, \\ X(1, \mathrm{i}) &= A \otimes I_m + B \otimes K_2 - B^{\mathrm{T}} \otimes K_2^{\mathrm{T}} + C \otimes K_3, \end{aligned}$$

where

$$A = \begin{bmatrix} 0 & 1 & -1 & 0 \\ -1 & 0 & 0 & -1 \\ 1 & 0 & 0 & -1 \\ 0 & 1 & 1 & 0 \end{bmatrix} \otimes I_n + \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix} \otimes R - \begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \otimes R^{\mathrm{T}},$$

$$B = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \otimes I_n, \quad C = \begin{bmatrix} 0 & 0 & \cdots & 0 & \mathrm{i} \\ 0 & 0 & \cdots & 0 & 0 \\ \vdots & \vdots & \ldots & \vdots & \vdots \\ 0 & 0 & \cdots & 0 & 0 \\ -\mathrm{i} & 0 & \cdots & 0 & 0 \end{bmatrix}_{4n\times 4n}.$$

We first evaluate the determinant of $X(\pm 1, \mathrm{i})$.

Let

$$D_1(x) = \begin{bmatrix} 0 & 1 & -1 & 0 \\ -1 & 0 & -\mathrm{e}^{-\mathrm{i}x} & -1 \\ 1 & \mathrm{e}^{\mathrm{i}x} & 0 & -1 \\ 0 & 1 & 1 & 0 \end{bmatrix} \quad \text{and} \quad D_2(x) = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ x & 0 & 0 & 0 \end{bmatrix}.$$

Note that K_1 commutes with $-K_1^{\mathrm{T}}$ and K_2 commutes with $-K_2^{\mathrm{T}}$. As a result, $X(-1, \mathrm{i})$ and $X(1, \mathrm{i})$ can be diagonalized by applying a common similarity transformation, respectively. By (2), we have

$$\begin{aligned} &(I_{4n} \otimes U)^{-1}X(-1, \mathrm{i})(I_{4n} \otimes U) \\ &\quad = (I_{4n} \otimes U)^{-1}(A \otimes I_m + B \otimes K_1 - B^{\mathrm{T}} \otimes K_1^{\mathrm{T}} + C \otimes K_3)(I_{4n} \otimes U) \\ &\quad = A \otimes (U^{-1}I_mU) + B \otimes (U^{-1}K_1U) - B^{\mathrm{T}} \otimes (U^{-1}(-K_1^{\mathrm{T}})U) + C \otimes (U^{-1}K_3U) \\ &\quad = \begin{bmatrix} A_1' & & & & & & C_1' & 0 \\ & A_2' & & & & C_2' & & \\ & & \ddots & & \iddots & & & \vdots \\ & & & A_{m/2}' + C_{m/2}' & & & & \\ & & \iddots & & \ddots & & & \vdots \\ & C_{m-2}' & & & & A_{m-2}' & & \\ C_{m-1}' & & & & & & A_{m-1}' & 0 \\ 0 & & \cdots & & \cdots & & 0 & A_m' + C_m' \end{bmatrix}, \end{aligned}$$

where for $t = 1, 2, \ldots, m$,

$$A'_t = A + \mathrm{e}^{\mathrm{i}\theta_t} B - \mathrm{e}^{-\mathrm{i}\theta_t} B^{\mathrm{T}} \tag{4}$$

$$= D_1(\theta_t) \otimes I_n + D_2(1) \otimes R - (D_2(1))^{\mathrm{T}} \otimes R^{\mathrm{T}}, \tag{5}$$

$$C'_t = \mathrm{e}^{-\mathrm{i}\theta_t} C, \tag{6}$$

A similar calculation using (3) gives

$$(I_{4n} \otimes V)^{-1} X(1, \mathrm{i})(I_{4n} \otimes V) = \begin{bmatrix} A''_1 & & & & C''_1 \\ & A''_2 & & C''_2 & \\ & & \vdots & & \\ & C''_{m-1} & & A''_{m-1} & \\ C''_m & & & & A''_m \end{bmatrix},$$

where for $t = 1, 2, \ldots, m$,

$$\begin{aligned} A''_t &= A + \mathrm{e}^{\mathrm{i}\phi_t} B - \mathrm{e}^{-\mathrm{i}\phi_t} B^{\mathrm{T}} \\ &= D_1(\phi_t) \otimes I_n + D_2(1) \otimes R - (D_2(1))^{\mathrm{T}} \otimes R^{\mathrm{T}}, \\ C''_t &= -\mathrm{e}^{-\mathrm{i}\phi_t} C. \end{aligned}$$

Interchanging rows and columns, those matrices can be changed into a block-diagonal form having the same determinants:

$$\det X(-1, \mathrm{i}) = \begin{vmatrix} A'_1 & C'_1 & & & & & & & & \\ C'_{m-1} & A'_{m-1} & & & & & & & & \\ & & A'_2 & C'_2 & & & & & & \\ & & C'_{m-2} & A'_{m-2} & & & & & & \\ & & & & \ddots & & & & & \\ & & & & & A'_{m/2-1} & C'_{m/2-1} & & \\ & & & & & C'_{m/2+1} & A'_{m/2+1} & & \\ & & & & & & & A'_{m/2} + C'_{m/2} & 0 \\ & & & & & & & 0 & A'_m + C'_m \end{vmatrix},$$

$$\det X(1, \mathrm{i}) = \begin{vmatrix} A''_1 & C''_1 & & & & & \\ C''_m & A''_m & & & & & \\ & & A''_2 & C''_2 & & & \\ & & C''_{m-1} & A''_{m-1} & & & \\ & & & & \ddots & & \\ & & & & & A''_{m/2} & C''_{m/2} \\ & & & & & C''_{m/2+1} & A''_{m/2+1} \end{vmatrix}.$$

Let Y be a subset of the row/column index set of P. For convenience, let P^Y denote the determinant of the matrix obtained from P by deleting all rows and columns whose indices are in Y. Note that $\theta_t = -\theta_{m-t}$. Expanding the determinant $L = \begin{vmatrix} A'_t & C'_t \\ C'_{m-t} & A'_{m-t} \end{vmatrix}$ along the first row and then expanding the resulting determinants along the first column, we have

$$L = \begin{vmatrix} D_1(\theta_t)\otimes I_n + D_2(1)\otimes R - (D_2(1))^{\mathrm{T}}\otimes R^{\mathrm{T}} & \mathrm{e}^{-\mathrm{i}\theta_t}C \\ \mathrm{e}^{-\mathrm{i}\theta_{m-t}}C & D_1(\theta_{m-t})\otimes I_n + D_2(1)\otimes R - (D_2(1))^{\mathrm{T}}\otimes R^{\mathrm{T}} \end{vmatrix}$$

$$= \begin{vmatrix} D_1(\theta_t) & D_2(1) & & & & & & & & & & D_2(\mathrm{i}\mathrm{e}^{-\mathrm{i}\theta_t})^{\mathrm{T}} \\ -D_2(1)^{\mathrm{T}} & D_1(\theta_t) & D_2(1) & & & & & & & & & \\ & \ddots & \ddots & \ddots & & & & & & & & \\ & & -D_2(1)^{\mathrm{T}} & D_1(\theta_t) & D_2(1) & & & & & & & \\ & & & -D_2(1)^{\mathrm{T}} & D_1(\theta_t) & -D_2(\mathrm{i}\mathrm{e}^{-\mathrm{i}\theta_t}) & & & & & & \\ & & & & D_2(\mathrm{i}\mathrm{e}^{\mathrm{i}\theta_t})^{\mathrm{T}} & D_1(-\theta_t) & D_2(1) & & & & & \\ & & & & & -D_2(1)^{\mathrm{T}} & D_1(-\theta_t) & D_2(1) & & & & \\ & & & & & & \ddots & \ddots & \ddots & & & \\ & & & & & & & -D_2(1)^{\mathrm{T}} & D_1(-\theta_t) & D_2(1) & & \\ -D_2(\mathrm{i}\mathrm{e}^{\mathrm{i}\theta_t}) & & & & & & & & -D_2(1)^{\mathrm{T}} & D_1(-\theta_t) & & \end{vmatrix}$$

$$= 4L^{\{1,2,3,4\}} + 2\mathrm{i}\sin\theta_t L^{\{1,2,3,4,5\}} - L^{\{1,8n\}} + 2(2\cos\theta_t)^{2n}. \tag{7}$$

Now we turn to calculate $L^{\{1,2,3,4\}}, L^{\{1,2,3,4,5\}}, L^{\{1,8n\}}$. By the Laplace expansion theorem, we obtain several expansions. First, an expansion by rows $1, 2, \ldots, 4(n-1)$:

$$L^{\{1,2,3,4\}} = |A'_t|^{\{1,2,3,4\}}|A'_{m-t}| - |A'_t|^{\{1,2,3,4,4n\}}|A'_{m-t}|^{\{1\}}; \tag{8}$$

then an expansion by rows $1, 2, \ldots, 4n-5$:

$$L^{\{1,2,3,4,5\}} = |A'_t|^{\{1,2,3,4,5\}}|A'_{m-t}| - |A'_t|^{\{1,2,3,4,5,4n\}}|A'_{m-t}|^{\{1\}}; \tag{9}$$

and then an expansion by rows $1, 2, \ldots, 4n-1$:

$$L^{\{1,8n\}} = |A'_t|^{\{1\}}|A'_{m-t}|^{\{4n\}} - |A'_t|^{\{1,4n\}}|A'_{m-t}|^{\{1,4n\}}. \tag{10}$$

Let $\alpha_t = -2\mathrm{i}\sin\theta_t$. Recall that A'_t defined in (5) is a $4n\times 4n$ matrix. We consider the following minors of it:

$$F'_n = |A'_t|^{\{1\}},\quad L'_n = |A'_t|,\quad M'_n = |A'_t|^{\{1,4n\}},\quad J'_n = |A'_t|^{\{4n\}},$$
$$F''_n = |A'_{m-t}|^{\{1\}},\quad L''_n = |A'_{m-t}|,\quad M''_n = |A'_{m-t}|^{\{1,4n\}},\quad J''_n = |A'_{m-t}|^{\{4n\}}.$$

Also, set $\Gamma_{j-1} = \Gamma_j^{\{1,2,3,4\}}$, $j = 2, \ldots, n$, $\Gamma_n \in \{F'_n, J'_n, L'_n, M'_n, F''_n, J''_n, L''_n, M''_n\}$.

It can be checked that

$$\begin{aligned} L'_n &= 4L'_{n-1} + \alpha_t F'_{n-1}, \\ F'_n &= F'_{n-1} - \alpha_t L'_{n-1}. \end{aligned} \tag{11}$$

The recursion relation (11) gives

$$\begin{aligned} L'_n &= 5L'_{n-1} - (4+\alpha_t^2)L'_{n-2}, \\ F'_n &= 5F'_{n-1} - (4+\alpha_t^2)F'_{n-2}. \end{aligned} \tag{12}$$

Note that

$$F'_1 = \begin{bmatrix} 0 & -\mathrm{e}^{\mathrm{i}\theta_t} & -1 \\ \mathrm{e}^{-\mathrm{i}\theta_t} & 0 & -1 \\ 1 & 1 & 0 \end{bmatrix} = -\alpha_t,\quad F'_2 = -5\alpha_t,$$

$$L'_1 = \begin{bmatrix} 0 & 1 & -1 & 0 \\ -1 & 0 & -\mathrm{e}^{\mathrm{i}\theta_t} & -1 \\ 1 & \mathrm{e}^{-\mathrm{i}\theta_t} & 0 & -1 \\ 0 & 1 & 1 & 0 \end{bmatrix} = 4,\quad L'_2 = 16 - \alpha_t^2.$$

Making use of the initial conditions, respectively, solving (12), we obtain

$$F_n' = \frac{-5\alpha_t - \alpha_t\sqrt{9-4\alpha_t^2}}{2\sqrt{9-4\alpha_t^2}}\left(\frac{5+\sqrt{9-4\alpha_t^2}}{2}\right)^{n-1} - \frac{-5\alpha_t + \alpha_t\sqrt{9-4\alpha_t^2}}{2\sqrt{9-4\alpha_t^2}}\left(\frac{5-\sqrt{9-4\alpha_t^2}}{2}\right)^{n-1},$$

$$L_n' = \frac{6-\alpha_t^2+2\sqrt{9-4\alpha_t^2}}{\sqrt{9-4\alpha_t^2}}\left(\frac{5+\sqrt{9-4\alpha_t^2}}{2}\right)^{n-1} - \frac{6-\alpha_t^2-2\sqrt{9-4\alpha_t^2}}{\sqrt{9-4\alpha_t^2}}\left(\frac{5-\sqrt{9-4\alpha_t^2}}{2}\right)^{n-1}. \tag{13}$$

Similarly, a recursion relation can be obtained: $\Gamma_n = 5\Gamma_{n-1} - \left(4+\alpha_t^2\right)\Gamma_{n-2}$, where $\Gamma_{j-1} = (\Gamma_j)^{\{1,2,3,4\}}$, $j = 2, \ldots, n$, $\Gamma_n \in \{J_n', M_n', F_n'', J_n'', L_n'', M_n''\}$.

Note that

$$J_1' = \alpha_t, \quad J_2' = 5\alpha_t, \quad M_1' = 1, \quad M_2' = 1-\alpha_t^2, \quad F_1'' = \alpha_t, \quad F_2'' = 5\alpha_t,$$
$$J_1'' = -\alpha_t, \quad J_2'' = -5\alpha_t, \quad L_1'' = 4, \quad L_2'' = 16-\alpha_t^2, \quad M_1'' = 1, \quad M_2'' = 1-\alpha_t^2.$$

We obtain

$$J_n' = \frac{5\alpha_t + \alpha_t\sqrt{9-4\alpha_t^2}}{2\sqrt{9-4\alpha_t^2}}\left(\frac{5+\sqrt{9-4\alpha_t^2}}{2}\right)^{n-1} - \frac{5\alpha_t - \alpha_t\sqrt{9-4\alpha_t^2}}{2\sqrt{9-4\alpha_t^2}}\left(\frac{5-\sqrt{9-4\alpha_t^2}}{2}\right)^{n-1},$$

$$M_n' = \frac{\sqrt{9-4\alpha_t^2}-3-2\alpha_t^2}{2\sqrt{9-4\alpha_t^2}}\left(\frac{5+\sqrt{9-4\alpha_t^2}}{2}\right)^{n-1} + \frac{\sqrt{9-4\alpha_t^2}+3+2\alpha_t^2}{2\sqrt{9-4\alpha_t^2}}\left(\frac{5-\sqrt{9-4\alpha_t^2}}{2}\right)^{n-1},$$

$$F_n'' = J_n', \quad J_n'' = F_n', \quad L_n'' = L_n', \quad M_n'' = M_n'. \tag{14}$$

By combining (7)–(10), (13) and (14), we obtain

$$\det\begin{bmatrix} A_t' & C_t' \\ C_{m-t}' & A_{m-t}' \end{bmatrix} = L = \left(\frac{5+\sqrt{9-4\alpha_t^2}}{2}\right)^{2n} + \left(\frac{5-\sqrt{9-4\alpha_t^2}}{2}\right)^{2n} + 2\left(2\cos\theta_t\right)^{2n}.$$

Similarly calculating as above, we have

$$\det\begin{bmatrix} A_t'' & C_t'' \\ C_{m-t}'' & A_{m-t}'' \end{bmatrix} = \left(\frac{5+\sqrt{9-4\beta_t^2}}{2}\right)^{2n} + \left(\frac{5-\sqrt{9-4\beta_t^2}}{2}\right)^{2n} + 2\left(2\cos\phi_t\right)^{2n},$$

where $\beta_t = -2\mathrm{i}\sin\phi_t$. Note that $\alpha_t = 0$ $(t = m, m/2)$, expanding the determinant along the first row and then expanding the resulting determinants along the first column, we have

$$\det[A_t' + C_t'] = 4L_{n-1}' - M_n' - 4\left(2\cos\theta_t\right)^{n-1}\mathrm{i} = (4^n - 1) - 4\left(2\cos\theta_t\right)^{n-1}\mathrm{i}(t = m/2, m).$$

So

$$\det X(-1,\mathrm{i}) = \det[A_{m/2}' + C_{m/2}'] \cdot \det[A_m' + C_m'] \cdot \prod_{t=1}^{m/2-1} \det\begin{bmatrix} A_t' & C_t' \\ C_{m-t}' & A_{m-t}' \end{bmatrix}$$
$$= \left((4^n-1)^2 + 4^{n+1}\right)\prod_{t=1}^{m/2-1}$$

$$\times\left[\left(\frac{5+\sqrt{9-4\alpha_t^2}}{2}\right)^{2n}+\left(\frac{5-\sqrt{9-4\alpha_t^2}}{2}\right)^{2n}+2\,(2\cos\theta_t)^{2n}\right]$$

$$\det X(1,\mathrm{i})=\prod_{t=1}^{m/2}\det\begin{bmatrix}A''_t & C''_t\\ C''_{m+1-t} & A''_{m+1-t}\end{bmatrix}$$

$$=\prod_{t=1}^{m/2}\left[\left(\frac{5+\sqrt{9-4\beta_t^2}}{2}\right)^{2n}+\left(\frac{5-\sqrt{9-4\beta_t^2}}{2}\right)^{2n}+2\,(2\cos\phi_t)^{2n}\right].$$

Note that $\det X(1,\mathrm{i})\geqslant 0, \det X(-1,\mathrm{i})\geqslant 0$, so $|\mathrm{Re}(\mathrm{Pf}\,X(1,\mathrm{i}))|=\sqrt{\det X(1,\mathrm{i})}$, $\mathrm{Im}(\mathrm{Pf}\,X(-1,\mathrm{i}))=0$, by (1) we have the following.

Theorem 2. *If m is even, then the number of close-packed dimers of $G_1^k(m,n)$ can be expressed as*

$$w_{G_1^k(m,n)}=\prod_{t=1}^{m/2}\left[\left(\frac{5+\sqrt{9+16\sin^2\frac{(2t-1)\pi}{m}}}{2}\right)^{2n}\right.$$
$$\left.+\left(\frac{5-\sqrt{9+16\sin^2\frac{(2t-1)\pi}{m}}}{2}\right)^{2n}+2\left(2\cos\frac{(2t-1)\,\pi}{m}\right)^{2n}\right]^{\frac{1}{2}}.$$

Similarly, when m is odd, then we have

$$\det X(-1,\mathrm{i})=\det[A'_{m/2}+C'_{m/2}]\prod_{t=1}^{\frac{m-1}{2}}\det\begin{bmatrix}A'_t & C'_t\\ C'_{m-t} & A'_{m-t}\end{bmatrix}$$

$$=(2^n+\mathrm{i})^2\prod_{t=1}^{\frac{m-1}{2}}\left[\left(\frac{5+\sqrt{9-4\alpha_t^2}}{2}\right)^{2n}+\left(\frac{5-\sqrt{9-4\alpha_t^2}}{2}\right)^{2n}+2\,(2\cos\theta_t)^{2n}\right],$$

$$\det X(1,\mathrm{i})=\det[A''_m+C''_m]\prod_{t=1}^{\frac{m-1}{2}}\det\begin{bmatrix}A''_t & C''_t\\ C''_{m+1-t} & A''_{m+1-t}\end{bmatrix}$$

$$=(2^n-\mathrm{i})^2\prod_{t=1}^{\frac{m-1}{2}}\left[\left(\frac{5+\sqrt{9-4\beta_t^2}}{2}\right)^{2n}+\left(\frac{5-\sqrt{9-4\beta_t^2}}{2}\right)^{2n}+2\,(2\cos\phi_t)^{2n}\right].$$

By (1) we have the following.

Theorem 3. *If m is odd, then the number of close-packed dimers of $G_1^k(m,n)$ can be expressed as*

$$w_{G_1^k(m,n)}=(2^n+1)\prod_{t=1}^{\frac{m-1}{2}}\left[\left(\frac{5+\sqrt{9+16\sin^2\frac{(2t-1)\pi}{m}}}{2}\right)^{2n}\right.$$
$$\left.+\left(\frac{5-\sqrt{9+16\sin^2\frac{(2t-1)\pi}{m}}}{2}\right)^{2n}+2\left(2\cos\frac{(2t-1)\,\pi}{m}\right)^{2n}\right]^{\frac{1}{2}}.$$

By theorems 2 and 3, we can calculate the entropy of $G_1^k(m,n)$:

$$\lim_{m,n\to\infty}\frac{2}{4mn}\log w_{G_1^k(m,n)} = \lim_{m,n\to\infty}\frac{1}{2mn}\sum_{t=1}^{\frac{m}{2}}\log\left(\frac{5+\sqrt{9+16\sin^2\frac{(2t-1)\pi}{m}}}{2}\right)^n$$
$$= \frac{1}{2\pi}\int_0^{\frac{\pi}{2}}\log\frac{5+\sqrt{9+16\sin^2 x}}{2}\,\mathrm{d}x \approx 0.3770.$$

3.2. *The lattice* $G_2^k(m,n)$

Introduce 6×6 matrices

$$H_1 = \begin{bmatrix} 0 & 1 & 0 & -1 & 0 & 0 \\ -1 & 0 & -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 & 0 & -1 \\ 0 & 0 & -1 & 0 & 1 & 0 \end{bmatrix}, \quad H_2 = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \end{bmatrix},$$
$$H_3 = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}.$$

If m is even, then labelling the vertices of $G_2^k(m,n)$ as shown in figure 3, the x-adjacency matrices of it, $X(-1,\mathrm{i}), X(1,\mathrm{i})$, can be written in terms of a linear combination of direct products of the smaller ones:

$$\begin{aligned} X(-1,\mathrm{i}) &= A\otimes I_m + B\otimes K_1 - B^{\mathrm{T}}\otimes K_1^{\mathrm{T}} + C\otimes K_3, \\ X(1,\mathrm{i}) &= A\otimes I_m + B\otimes K_2 - B^{\mathrm{T}}\otimes K_2^{\mathrm{T}} + C\otimes K_3, \end{aligned}$$

where

$$A = H_1\otimes I_n + H_2\otimes R - H_2^{\mathrm{T}}\otimes R^{\mathrm{T}},$$
$$B = H_3\otimes I_n,$$
$$C = \begin{bmatrix} 0 & 0 & \cdots & 0 & \mathrm{i} \\ 0 & 0 & \cdots & 0 & 0 \\ \vdots & \vdots & \cdots & \vdots & \vdots \\ 0 & 0 & \cdots & 0 & 0 \\ -\mathrm{i} & 0 & \cdots & 0 & 0 \end{bmatrix}_{6n\times 6n}.$$

In the following, we evaluate the determinant of $X(\pm 1,\mathrm{i})$. Note that K_1 commutes with $-K_1^{\mathrm{T}}$ and K_2 commutes with $-K_2^{\mathrm{T}}$. As a result, $X(-1,\mathrm{i})$ and $X(1,\mathrm{i})$ can be block diagonalized by applying a common similarity transformation. By (2) and (3), we have

$$(I_{6n} \otimes U)^{-1} X(-1, \mathrm{i})(I_{6n} \otimes U)$$
$$= \begin{bmatrix} A'_1 & & & & & & C'_1 & 0 \\ & A'_2 & & & & C'_2 & & \\ & & \ddots & & \iddots & & & \vdots \\ & & & A'_{m/2} + C'_{m/2} & & & & \\ & & \iddots & & \ddots & & & \vdots \\ & C'_{m-2} & & & & A'_{m-2} & & \\ C'_{m-1} & & & & & & A'_{m-1} & 0 \\ 0 & & \cdots & & \cdots & & 0 & A'_m + C'_m \end{bmatrix},$$

$$(I_{6n} \otimes V)^{-1} X(1, \mathrm{i})(I_{6n} \otimes V) = \begin{bmatrix} A''_1 & & & & C''_1 \\ & A''_2 & & C''_2 & \\ & & \vdots & & \\ & C''_{m-1} & & A''_{m-1} & \\ C''_m & & & & A''_m \end{bmatrix},$$

where for $t = 1, 2, \ldots, m$,

$$A'_t = A + \mathrm{e}^{\mathrm{i}\theta_t} B - \mathrm{e}^{-\mathrm{i}\theta_t} B^{\mathrm{T}}, \quad C'_t = \mathrm{e}^{-\mathrm{i}\theta_t} C,$$
$$A''_t = A + \mathrm{e}^{\mathrm{i}\phi_t} B - \mathrm{e}^{-\mathrm{i}\phi_t} B^{\mathrm{T}}, \quad C''_t = -\mathrm{e}^{-\mathrm{i}\phi_t} C.$$

Similar to section 3.2, we can obtain

$$\det[A'_t + C'_t] = -2\mathrm{i}(-4)^n \cos\theta_t \quad (t = m, m/2),$$
$$\det\begin{bmatrix} A'_t & C'_t \\ C'_{m-t} & A'_{m-t} \end{bmatrix} = \left(\frac{8 - \alpha_t^2 + \sqrt{\alpha_t^4 - 16\alpha_t^2}}{2}\right)^{2n} + \left(\frac{8 - \alpha_t^2 - \sqrt{\alpha_t^4 - 16\alpha_t^2}}{2}\right)^{2n} + 2 \times 4^n,$$
$$\det\begin{bmatrix} A''_t & C''_t \\ C''_{m-t} & A''_{m-t} \end{bmatrix} = \left(\frac{8 - \beta_t^2 + \sqrt{\beta_t^4 - 16\beta_t^2}}{2}\right)^{2n} + \left(\frac{8 - \beta_t^2 - \sqrt{\beta_t^4 - 16\beta_t^2}}{2}\right)^{2n} + 2 \times 4^n,$$

where $\alpha_t = -2\mathrm{i}\sin\theta_t = -2\mathrm{i}\sin\frac{2t\pi}{m}$, $\beta_t = -2\mathrm{i}\sin\phi_t = -2\mathrm{i}\sin\frac{(2t-1)\pi}{m}$.

Hence,

$$\det X(-1, \mathrm{i}) = \det[A'_{m/2} + C'_{m/2}] \cdot \det[A'_m + C'_m] \cdot \prod_{t=1}^{m/2-1} \det\begin{bmatrix} A'_t & C'_t \\ C'_{m-t} & A'_{m-t} \end{bmatrix}$$
$$= 4^{2n+1} \prod_{t=1}^{m/2-1} \left[\left(\frac{8 - \alpha_t^2 + \sqrt{\alpha_t^4 - 16\alpha_t^2}}{2}\right)^{2n} + \left(\frac{8 - \alpha_t^2 - \sqrt{\alpha_t^4 - 16\alpha_t^2}}{2}\right)^{2n} + 2 \times 4^n\right];$$
$$\det X(1, \mathrm{i}) = \prod_{t=1}^{m/2} \det\begin{bmatrix} A''_t & C''_t \\ C''_{m+1-t} & A''_{m+1-t} \end{bmatrix}$$
$$= \prod_{t=1}^{m/2} \left[\left(\frac{8 - \beta_t^2 + \sqrt{\beta_t^4 - 16\beta_t^2}}{2}\right)^{2n} + \left(\frac{8 - \beta_t^2 - \sqrt{\beta_t^4 - 16\beta_t^2}}{2}\right)^{2n} + 2 \times 4^n\right].$$

Note that $\det X(1, \mathrm{i}) \geqslant 0$, $\det X(-1, \mathrm{i}) \geqslant 0$, so $|\mathrm{Re}(\mathrm{Pf}\, X(1, \mathrm{i}))| = \sqrt{\det X(1, \mathrm{i})}$, $\mathrm{Im}(\mathrm{Pf}\, X(-1, \mathrm{i})) = 0$, by (1) we have the following.

Theorem 4. *If m is even, then the number of close-packed dimers of $G_2^k(m,n)$ can be expressed as*

$$w_{G_2^k(m,n)} = \prod_{t=1}^{m/2}\left[\left(\frac{8-\beta_t^2+\sqrt{\beta_t^4-16\beta_t^2}}{2}\right)^{2n}+\left(\frac{8-\beta_t^2-\sqrt{\beta_t^4-16\beta_t^2}}{2}\right)^{2n}+2\times 4^n\right]^{1/2},$$

where $\beta_t = -2\mathrm{i}\sin\frac{(2t-1)\pi}{m}$.

Similarly, when m is odd, we have

$$\begin{aligned}\det X(-1,\mathrm{i}) &= \det[A'_{m/2}+C'_{m/2}]\prod_{t=1}^{\frac{m-1}{2}}\det\begin{bmatrix}A'_t & C'_t\\ C'_{m-t} & A'_{m-t}\end{bmatrix}\\ &= (2\times(-4)^n\mathrm{i})\prod_{t=1}^{\frac{m-1}{2}}\\ &\quad\times\left[\left(\frac{8-\alpha_t^2+\sqrt{\alpha_t^4-16\alpha_t^2}}{2}\right)^{2n}+\left(\frac{8-\alpha_t^2-\sqrt{\alpha_t^4-16\alpha_t^2}}{2}\right)^{2n}+2\times 4^n\right],\end{aligned}$$

$$\begin{aligned}\det X(1,\mathrm{i}) &= \det[A''_m+C''_m]\prod_{t=1}^{\frac{m-1}{2}}\det\begin{bmatrix}A''_t & C''_t\\ C''_{m+1-t} & A''_{m+1-t}\end{bmatrix}\\ &= (-2\times(-4)^n\mathrm{i})\prod_{t=1}^{\frac{m-1}{2}}\\ &\quad\times\left[\left(\frac{8-\beta_t^2+\sqrt{\beta_t^4-16\beta_t^2}}{2}\right)^{2n}+\left(\frac{8-\beta_t^2-\sqrt{\beta_t^4-16\beta_t^2}}{2}\right)^{2n}+2\times 4^n\right].\end{aligned}$$

By (1) we have the following.

Theorem 5. *If m is odd, then the number of close-packed dimers of $G_2^k(m,n)$ can be expressed as*

$$\begin{aligned}w_{G_2^k(m,n)} &= 2^{n+1}\prod_{t=1}^{\frac{m-1}{2}}\left[\left(\frac{8-\beta_t^2+\sqrt{\beta_t^4-16\beta_t^2}}{2}\right)^{2n}+\left(\frac{8-\beta_t^2-\sqrt{\beta_t^4-16\beta_t^2}}{2}\right)^{2n}\right.\\ &\qquad\left.+\,2\times 4^n\right]^{1/2},\end{aligned}$$

where $\beta_t = -2\mathrm{i}\sin\frac{(2t-1)\pi}{m}$.

By theorems 4 and 5, the entropy of $G_2^k(m,n)$ can be obtained:

$$\begin{aligned}\lim_{m,n\to\infty}\frac{2}{6mn}\log w_{G_2^k(m,n)} &= \lim_{m,n\to\infty}\frac{1}{3mn}\sum_{t=1}^{\frac{m}{2}}\log\left(\frac{8-\beta_t^2+\sqrt{\beta_t^4-16\beta_t^2}}{2}\right)^n\\ &= \frac{1}{3\pi}\int_0^{\frac{\pi}{2}}\log\frac{8+4\sin^2x+\sqrt{16\sin^4x+64\sin^2x}}{2}\mathrm{d}x\approx 0.3344.\end{aligned}$$

4. Concluding remarks

Fisher and Lebowitz [3] gave examples suggesting that the thermodynamic limit of the free energy (including the entropy) is independent of boundary conditions in statistical mechanics. Kasteleyn [4] discussed the related problem of quadratic lattices with free and toroidal boundary conditions. In this paper, we computed the entropies of the 8.8.6 and 8.8.4 lattices with a Klein bottle boundary condition. Comparing with the results by Salinas and Nagle [16], Wu [19] and Yan *et al* [21], we can see that the 8.8.4 lattices have the same entropy with three different boundary conditions (cylindrical, toroidal and Klein bottle). Also 8.8.6 lattices have the same property.

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