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On the location of zeros of the Homfly polynomial

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Abstract. Wu, Wang, Chang and Shrock initiated the study of zeros of the Jones polynomial since it was the special case of partition functions of the Potts model in physics. The Homfly polynomial is the generalization of the Jones polynomial. Let L be an oriented link, and $P_L(v,z)$ be its Homfly polynomial. In this paper, we study zeros of $P_L(v,z)$ with z fixed. We prove the so-called unit-circle theorem for a family of generalized Jaeger's links $\{D_n(G)|n=1,2,\ldots\}$ which states that $|v|=1$ is the limit of zeros of Homfly polynomials of generalized Jaeger's links $\{D_n(G)|n=1,2,\ldots\}$ if G is bridgeless. Similar to the result of the Jones polynomial, we also prove that zeros of Homfly polynomials are dense in the whole complex plane.

Keywords: rigorous results in statistical mechanics, classical phase transitions (theory), exact results

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Contents

1. Introduction

The study of zeros in physics originated from two very well-known papers [21, 37] on phase transitions of Ising model [5] by Lee and Yang. Zeros of the Jones polynomial [16, 17] are interesting since it is the special case of partition functions of the Potts model, which generalizes the Ising model in physics; see [18, 34]. In [36] and [7], Wu, Wang, Chang and Shrock initiated the study of zeros of the Jones polynomial; they studied some link families and obtained some results by numerical experiments and theoretical analysis at the same time. Since then many works have been carried out, see [6, 9, 10, 12, 14, 15, 24]. The distribution of zeros of the Alexander polynomial [1] has also been studied; see [23, 25]. The Homfly polynomial [8, 26] is the generalization of both the Jones and the Alexander polynomials. The Homfly polynomial can also be generated by the q-state spin-conserving model and IRF models [35]. Let L be an oriented link. We denote by $P_L(v, z)$ the Homfly polynomial of L. In this paper, we shall concentrate on the study of zeros of $P_L(v, z)$, with z fixed, of generalized Jaeger's links.

In [11], Jaeger associated an oriented link to a plane graph by replacing each edge of the graph by an oriented clasp as shown in figure 1. In figure 2, we provide an example of Jaeger's links. Given a connected plane graph G, we cover each edge e of G by an oriented vertical integer tangle $[2n_e]$ as shown in figure 3, and obtain an oriented link. We call it the generalized Jaeger's link, denoted by $D_{\mathbf{n}}(G)$, since Jaeger's link is the case of $n_e = 1$ for each edge e. Note that generalized Jaeger's links have $s(G) + 1$ components, where $s(G)$ is the nullity of G. Thus, given a connected plane graph G with edge set $E(G) = \{e_1, e_2, \ldots, e_m\}$, we define a family of generalized Jaeger's links $\{D_{\mathbf{n}}(G)|\mathbf{n} = (n_1, n_2, \ldots, n_m); n_i = \pm 1, \pm 2, \ldots; i = 1, 2, \ldots, m\}$.

The unit-circle phenomenon also appears in zeros of the chromatic polynomial [4], which is the zero-temperature partition function of the q-state Potts antiferromagnet. In [15], the present authors defined multiple crossing twisted links and proved that the limits of the zeros of their Jones polynomials are the unit circle and some isolated limits. Motivated by results on the chromatic polynomial and the characteristic polynomial in [30] and [38] respectively, the present authors, together with Dong and Tay, proved that zeros

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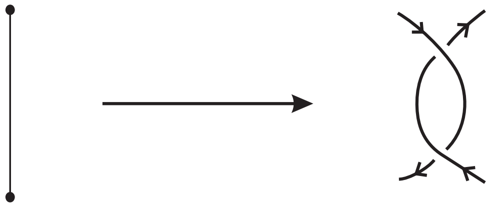

Figure 1. Replacing an edge by an oriented clasp.

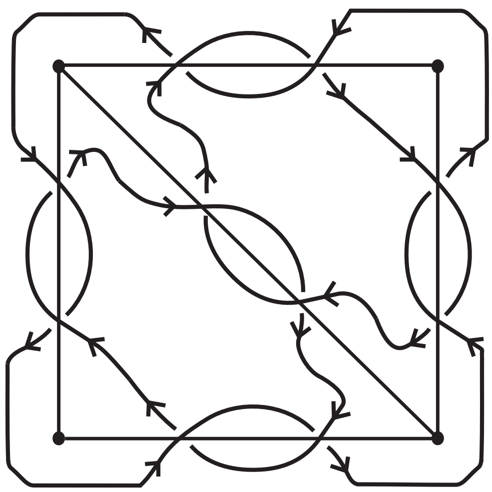

Figure 2. An example of Jaeger's links.

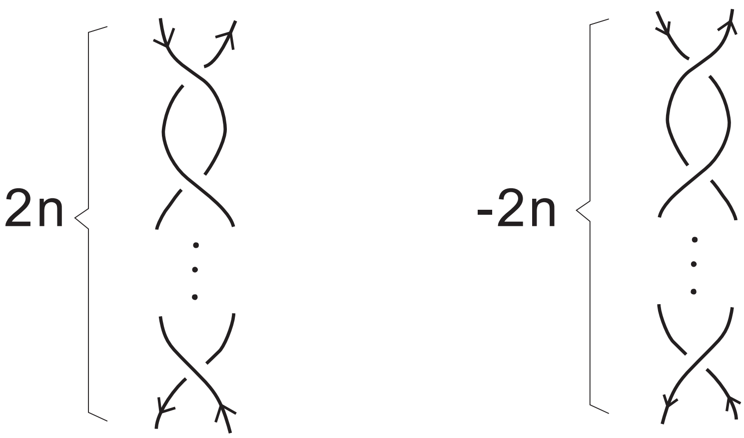

Figure 3. Vertical integer tangles $[2n]$ with $n > 0$ and $2n$ half twists (left) and $n < 0$ and $-2n$ half twists (right).

of Jones polynomials are dense in the whole complex plane [14]. In this paper, we use mathematical analytic results to study similar problems to those in [15] and [14]. Let G be a connected plane graph. Let $D_n(G)$ be the generalized Jaeger's link obtained from G by covering each edge with the vertical integer tangle $[2n]$. Two results are obtained, one is the so-called unit-circle theorem which states that $|v| = 1$ is the limit of zeros of

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Homfly polynomials of $\{D_n(G)|n = 1, 2, \ldots\}$ when G is bridgeless, and the other is that zeros of Homfly polynomials of $\{D_n(S_m)|n = \pm 1, \pm 2, \ldots; m = 1, 2, \ldots\}$ are dense in the whole complex plane, where S_m is the graph which consists of two vertices connected by m parallel edges.

2. Homfly and chain polynomials

The main purpose of this section is to convert the Homfly polynomial of $D_{\mathbf{n}}(G)$ to the chain polynomial [27] of G. Let $G = (V, E)$ be a graph. We denote by $k(G)$ the number of connected components of G. We use $r(G) = |V| - k(G)$ and $s(G) = |E| - |V| + k(G)$ to denote the rank and nullity (i.e. cyclomatic number) of the graph G, respectively.

A weighted graph is a graph G together with a function w mapping E into some commutative ring R with unity 1. If e is an edge of G then $w(e)$ is called the weight of the edge e. The dichromatic polynomial for weighted graphs was introduced by Traldi in [31], which is one of the edge-weighted versions of the celebrated Tutte polynomial in graph theory [33].

Definition 2.1. The dichromatic polynomial $Q_G(t, z)$ of a weighted graph G is defined as

$$Q_G(t,z) = \sum_{F \subset E} \left(\prod_{f \in F} w(f) \right) t^{k\langle F \rangle} z^{s\langle F \rangle},$$

where $k\langle F \rangle$ and $s\langle F \rangle$ are the number of connected components and the nullity of the spanning subgraph $\langle F \rangle$, induced by the edge subset F of G, respectively.

Theorem 2.2 ([13]). *Let G be a connected plane graph with edge set $E = \{e_1, e_2, \ldots, e_m\}$. Let $D_{\mathbf{n}}(G)$ be the associated generalized Jaeger's links. Let the weight of e_i be $\frac{v^{-1}-v}{z}\frac{v^{2n_i}}{1-v^{2n_i}}$. Then the Homfly polynomial of $D_{\mathbf{n}}(G)$*

$$P_{D_{\mathbf{n}}(G)}(v,z) = \left(\frac{z}{v^{-1}-v} \right)^{m+1} \left(\prod_{i=1}^{m} (1 - v^{2n_i}) \right) Q_G \left(\frac{v^{-1}-v}{z}, \frac{v^{-1}-v}{z} \right).$$

Remark 2.3. From theorem 2.2, we can see that the degree of v of $P_{D_{\mathbf{n}}(G)}(v,z)$ is relevant to $\mathbf{n}$, while the degree of z is independent of $\mathbf{n}$. Furthermore, if G is bridgeless, then the highest and the lowest degrees of $P_{D_{\mathbf{n}}(G)}(v,z)$, with respect to z, are $s(G)$ and $-s(G)$, respectively.

Let e be an edge of the graph G, we shall use $G - e$ and G/e to denote the graphs obtained from G by deleting the edge e and contracting the edge e (i.e. deleting the edge e first and then identifying its two end vertices), respectively. When e is a loop, $G - e = G/e$. The following recursive relations hold, which may be taken as an alternative definition of the polynomial, see [32].

(i) If G is an edgeless graph with $n \geq 1$ vertices, then

$$Q_G = t^n. \tag{1}$$

(ii) Otherwise, let e be an edge of G. If e is a loop of G, then

$$Q_G = (1 + w(e)z)Q_{G-e}. \tag{2}$$

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If e is not a loop of G, then

$$Q_G = Q_{G-e} + w(e)Q_{G/e}. \tag{3}$$

To study the chromatic polynomial [28] for the homeomorphism class of graphs, Read and Whitehead Jr introduced a multilinear polynomial of a graph in 1999, the chain polynomial [27], which is associated with a graph whose edges have been labeled with elements of a commutative ring with unity 1.

Let G be a labeled graph. We usually identify the edges with their labels for convenience.

Definition 2.4. The chain polynomial $Ch[G]$ of a labeled graph G is defined as

$$Ch[G] = \sum_{Y \subset E} F_{G-Y}(1-w) \prod_{a \in Y} a,$$

where the sum is over all subsets of $E = E(G)$, $F_{G-Y}(1-w)$ denotes the flow polynomial in $\lambda = 1-w$ of $G-Y$, the graph obtained from G by deleting the edges in Y.

The chain polynomial of a labeled graph can be defined by the following recursive rules [31].

(i) If G is edgeless, then

$$Ch[G] = 1. \tag{4}$$

(ii) Otherwise, let a be an edge of G.

(a) If the edge a is a loop of G, then

$$Ch[G] = (a-w)Ch[G-a]. \tag{5}$$

(b) If the edge a is not a loop, then

$$Ch[G] = (a-1)Ch[G-a] + Ch[G/a]. \tag{6}$$

Comparing (1)–(3) with (4)–(6), we obtain the following lemma.

Lemma 2.5. *Let $G = (V, E)$ be a connected graph, and G^w and G^l be the associated weighted graph and labeled graph, respectively. If, in $Ch[G^l]$, we let $w = 1 - tz$ and $a = 1 + \frac{t}{w(a)}$ for each edge a, then*

$$Q_{G^w}(t, z) = t^{|V|-|E|} \left(\prod_{a \in E} w(a) \right) Ch[G^l].$$

Proof. By induction of the number of edges of G. It is a routine exercise and hence we omit the details here. □

We shall not add the superscripts w and l when they are clear in the context. Combining theorem 2.2 and lemma 2.5, we obtain the following theorem.

Theorem 2.6. *Let G be a connected labeled plane graph with edge set $E = \{a_1, a_2, \ldots, a_m\}$. Let $D_{\mathbf{n}}(G)$ be the associated generalized Jaeger's link. In $Ch[G]$, if we let $w = 1 - (\frac{v^{-1}-v}{z})^2$ and $a_i = \frac{1}{v^{2n_i}}$, then*

$$P_{D_{\mathbf{n}}(G)}(v, z) = \left(\frac{z}{v^{-1}-v} \right)^{s(G)} \left(\prod_{i=1}^{m} v^{2n_i} \right) Ch[G].$$

Proof. Let $\delta = \frac{v^{-1}-v}{z}$ and recall that $w(a_i) = \delta\frac{v^{2n_i}}{1-v^{2n_i}}$ in theorem 2.2. Then, by lemma 2.5, we have

$$Q_G(\delta,\delta)|_{w(a_i)=\delta\frac{v^{2n_i}}{1-v^{2n_i}}} = \delta^{|V|-m}\left(\prod_{i=1}^{m}\delta\frac{v^{2n_i}}{1-v^{2n_i}}\right)Ch[G]|_{w=1-\delta^2;a_i=\frac{1}{v^{2n_i}}}.$$

By theorem 2.2, we have

$$\begin{aligned}P_{D_{\mathbf{n}}(G)}(v,z) &= \delta^{-m-1}\left(\prod_{i=1}^{m}(1-v^{2n_i})\right)Q_G(\delta,\delta)|_{w(a_i)=\delta\frac{v^{2n_i}}{1-v^{2n_i}}}\\ &= \delta^{-s(G)}\left(\prod_{i=1}^{m}v^{2n_i}\right)Ch[G]|_{w=1-\delta^2;a_i=\frac{1}{v^{2n_i}}}.\end{aligned}$$

This completes the proof of theorem 2.6. □

3. Two analytic results

In this section we review two analytic results on zeros of polynomials, which will be used in the next two sections.

Definition 3.1. Suppose that $\{f_n(x)|n=1,2,\ldots\}$ is a family of polynomials. A complex number z is said to be the limit of zeros of $\{f_n(x)|n=1,2,\ldots\}$ if either $f_n(z)=0$ for all sufficiently large n or z is a limit point of the set $\Re(\{f_n(x)\})$, where $\Re(\{f_n(x)\})$ is the union of the zeros of the $f_n(x)$s.

Beraha–Kahane–Weiss's theorem

In [2], the authors proved the Beraha–Kahane–Weiss's theorem. If $\{f_n(x)|n=1,2,\ldots\}$ is a family of polynomials such that

$$f_n(x) = \alpha_1(x)\lambda_1(x)^n + \alpha_2(x)\lambda_2(x)^n + \cdots + \alpha_l(x)\lambda_l(x)^n,$$

where the $\alpha_i(x)$ and $\lambda_i(x)$ are fixed nonzero polynomials, such that no pair $i \neq j$ has $\lambda_i(x) \equiv \omega\lambda_j(x)$ for some complex number ω of unit modulus, then z is a limit of zeros of $\{f_n(x)|n=1,2,\ldots\}$ if and only if

(1) two or more of the $\lambda_i(z)$ are of equal modulus, and strictly greater in modulus than the others; or
(2) for some j, the modulus of $\lambda_j(z)$ is strictly greater than those of the others, and $\alpha_j(z)=0$.

This theorem can also be found in section 3 of [7] and [4]. We call the limits of zeros in (2) of Beraha–Kahane–Weiss's theorem the isolated limits.

The following Sokal's lemma [30] will be used to prove one of main results of the paper.

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Sokal's lemma

Let $F_1(z), F_2(z), G(z)$ be analytic functions on a disc $|z| < R$ satisfying $|G(0)| \leq 1$ and $G(z)$ not constant. Then, for each $\epsilon > 0$, there exists $s_0 < \infty$ such that for all integers $s \geq s_0$ the equation

$$|1 + F_1(z)G(z)^s| = |1 + F_2(z)G(z)^s|$$

has a solution in the disc $|z| < \epsilon$.

4. The unit-circle theorem

Let G be a connected plane graph. Let $D_n(G)$ be the generalized Jaeger's link obtained from G by covering each edge with the vertical integer tangle $[2n]$. In this section we shall study zeros of Homfly polynomials (with z fixed) of the link subfamily $\{D_n(G)|n = 1, 2, \ldots\}$.

Theorem 4.1. *Let G be a connected plane graph with edge set $E = \{e_1, e_2, \ldots, e_m\}$. Then*

$$P_{D_n(G)}(v, z) = \left(\frac{z}{v^{-1} - v}\right)^{s(G)} \sum_{i=0}^{m} v^{2in} \sum_{Y \subset E, |Y|=i} F_{\langle Y \rangle}\left(\left(\frac{v^{-1} - v}{z}\right)^2\right), \qquad (7)$$

where $\langle Y \rangle$ is the spanning subgraph of G induced by the edge subset Y.

Proof. Let $\delta = \frac{v^{-1}-v}{z}$. By theorem 2.6, we have

$$\begin{aligned} P_{D_n(G)}(v, z) &= \delta^{-s(G)} v^{2mn} Ch[G]|_{w=1-\delta^2; a=\frac{1}{v^{2n}}} \\ &= \delta^{-s(G)} v^{2mn} \sum_{Y \subset E} F_{G-Y}(\delta^2)(v^{-2})^{n|Y|} \\ &= \delta^{-s(G)} \sum_{j=0}^{m} v^{2(m-j)n} \sum_{Y \subset E, |Y|=j} F_{G-Y}(\delta^2) \\ &= \delta^{-s(G)} \sum_{i=0}^{m} v^{2in} \sum_{Y \subset E, |Y|=i} F_{\langle Y \rangle}(\delta^2). \end{aligned}$$

□

Remark 4.2. Note that in equation (7), for each $i = 0, 1, \ldots, m$, the summation $\sum_{Y \subset E, |Y|=i} F_{\langle Y \rangle}((\frac{v^{-1}-v}{z})^2)$ is a Laurent polynomial in v and z in general and is independent of n.

Before analyzing zeros of Homfly polynomials (with z fixed) of the link subfamily $\{D_n(G)|n = 1, 2, \ldots\}$, we first give a review of properties of the flow polynomial $F_G(\lambda)$ of the graph G [29].

(P1) If G has no edges, then $F_G(\lambda) = 1$.

(P2) If G has a bridge, then $F_G(\lambda) = 0$.

(P3) If e is a loop of G, then $F_G(\lambda) = (\lambda - 1)F_{G-e}(\lambda)$; if e is not a loop of G, then $F_G(\lambda) = F_{G/e}(\lambda) - F_{G-e}(\lambda)$.

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(P4) If G is bridgeless, then $F_G(\lambda)$ is a polynomial of degree $s(G)$, with coefficient 1.

(P5) If G consists of two graphs H and K which are either disjoint or have a single vertex in common, then $F_G(\lambda) = F_H(\lambda)F_K(\lambda)$.

Theorem 4.3. *Let G be a connected bridgeless plane graph. Then, for any fixed nonzero z, the limits of zeros of Homfly polynomials of $\{D_n(G)|n = 1, 2, \ldots\}$ are the unit circle $|v| = 1$ and some isolated limits.*

Proof. By theorem 4.1, we have

$$\begin{aligned}
P_{D_n(G)}(v,z) &= \sum_{i=0}^{m} v^{2in} \left(\frac{z}{v^{-1}-v}\right)^{s(G)} \sum_{Y\subset E,|Y|=i} F_{\langle Y\rangle}\left(\left(\frac{v^{-1}-v}{z}\right)^2\right) \\
&= \sum_{i=0}^{m} v^{2in} \sum_{Y\subset E,|Y|=i} \left(\frac{z}{v^{-1}-v}\right)^{s(G)} F_{\langle Y\rangle}\left(\left(\frac{v^{-1}-v}{z}\right)^2\right) \\
&= \sum_{i=0}^{m} v^{2in}\alpha_i(v,z).
\end{aligned}$$

By (P2) and (P4), $F_{\langle Y\rangle}(\lambda)$ is either 0 or a polynomial of degree $s(\langle Y\rangle)$. Note that $s(\langle Y\rangle) \le s(G)$ for every edge subset Y of G; this is because the cycle space of $\langle Y\rangle$ is a subspace of the cycle space of G and $s(\langle Y\rangle)$ and $s(G)$ are ranks of the corresponding cycle spaces; for details, see chapter 12 of [3]. Thus, if we view $\alpha_i(v,z)$ as a polynomial in $\frac{v^{-1}-v}{z}$, then its highest degree is no more than $s(G)$ and its lowest degree is no less than $-s(G)$.

Now we let

$$\tilde{P}_{D_n(G)}(v,z) = \left(\frac{v-v^3}{z}\right)^{s(G)} P_{D_n(G)}(v,z) = \sum_{i=0}^{m} v^{2in}\tilde{\alpha}_i(v,z).$$

Bearing in mind that we have supposed z is a fixed nonzero number, then $\tilde{\alpha}_i(v,z)$ is a polynomial in v in the common sense.

Now we apply Beraha–Kahane–Weiss's theorem to the family $\{\tilde{P}_{D_n(G)}(v,z)|n = 1, 2, \ldots\}$. In order to apply Beraha–Kahane–Weiss's theorem, we shall show that

$$\begin{aligned}
\tilde{\alpha}_0(v,z) &= \left(\frac{v-v^3}{z}\right)^{s(G)} \left(\frac{z}{v^{-1}-v}\right)^{s(G)} F_{\langle\emptyset\rangle}\left(\left(\frac{v^{-1}-v}{z}\right)^2\right) \\
&= v^{2s(G)} \qquad \text{(by (P1))}
\end{aligned}$$

and

$$\begin{aligned}
\tilde{\alpha}_m(v,z) &= \left(\frac{v-v^3}{z}\right)^{s(G)} \left(\frac{z}{v^{-1}-v}\right)^{s(G)} F_G\left(\left(\frac{v^{-1}-v}{z}\right)^2\right) \\
&= v^{2s(G)} F_G\left(\left(\frac{v^{-1}-v}{z}\right)^2\right)
\end{aligned}$$

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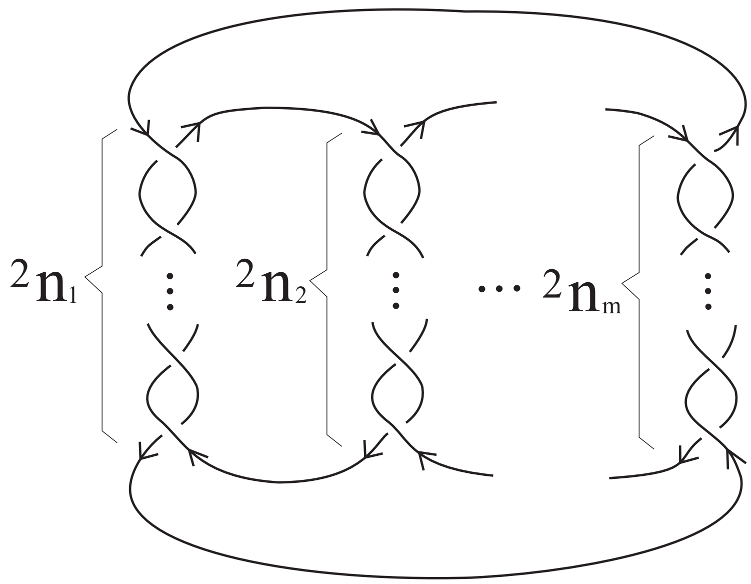

Figure 4. The oriented pretzel link diagram $\tilde{P}(2n_1, 2n_2, \ldots, 2n_m)$.

are both nonzero polynomials. It is clear that $\alpha_0(v, z)$ is nonzero and $\alpha_m(v, z)$ are nonzero since G is bridgeless.

Applying (1) of Beraha–Kahane–Weiss's theorem to the family $\{\tilde{P}_{D_n(G)}(v, z)|n = 1, 2, \ldots\}$, we immediately get the limits $|v| = 1$. Applying (2) of Beraha–Kahane–Weiss's theorem, we obtain $v = 0$ and zeros of $v^{2s(G)}F_G((\frac{v^{-1}-v}{z})^2)$ outside $|v| = 1$.

Note that, for any fixed nonzero z, zeros of $P_{D_n(G)}(v, z)$ are zeros of $\tilde{P}_{D_n(G)}(v, z)$, minus 0, ± 1 with multiplicity $s(G)$. Since ± 1 are on the unit circle $|v| = 1$, thus they must be limits of $\{P_{D_n(G)}(v, z)|n = 1, 2, \ldots\}$. This completes the proof of theorem 4.3. □

Remark 4.4. The conclusion of theorem 4.3 also holds for $\{D_{\mathbf{n}}(G)|\mathbf{n} = (n, \ldots, n, 1, \ldots, 1); n = 1, 2, \ldots\}$ and $\{D_{-n}(G)|n = 1, 2, \ldots\}$.

5. Density in the plane

For $m \geq 3$, an oriented pretzel link $P(2n_1, 2n_2, \ldots, 2n_m)$ is an oriented link that has the oriented pretzel link diagram $\tilde{P}(2n_1, 2n_2, \ldots, 2n_m)$ as shown in figure 4, where $n_1, n_2, \ldots, n_m$ are all nonzero integers. If n_i is positive, the crossings are in the sense shown; if n_i is negative, the crossings are in the opposite sense.

In [20], Landvoy studied the computation of the Jones polynomial of the pretzel link $P(k_1, k_2, \ldots, k_m)$, where k_i is nonzero integer for each $i = 1, 2, \ldots, m$. He first obtained a recursive relation of Kauffman bracket polynomials of pretzel links, then introduced a notation to describe the orientation of pretzel links and calculated the writhe using his notation, and finally provided a Maple program by combining the two. In [19], Kim and Lee calculated the Conway (hence, Alexander) polynomial of pretzel links using a computation tree. Now we use theorem 2.2 to compute the Homfly polynomial of the subfamily $P(2n_1, 2n_2, \ldots, 2n_m)$.

Theorem 5.1. *Let $P(2n_1, 2n_2, \ldots, 2n_m)$ be the oriented pretzel link. Then*

$$P_{P(2n_1,2n_2,\ldots,2n_m)}(v,z) = \left(\frac{z}{v^{-1}-v}\right)^{m+1}\left(c\prod_{i=1}^{m}(1-v^{2n_i}) + \prod_{i=1}^{m}(1+cv^{2n_i})\right),$$

where $c = (\frac{v^{-1}-v}{z})^2 - 1$.

Proof. Let S_m be the graph which consists of two vertices connected by m parallel edges $e_1, e_2, \ldots, e_m$. Note that $P(2n_1, 2n_2, \ldots, 2n_m)$ can be obtained from S_m by replacing the edge e_i by the vertical integer tangle $[2n_i]$ for each $i = 1, 2, \ldots, m$. Now we compute the dichromatic polynomial of S_m. According to definition 2.1, we have

$$\begin{aligned}
Q_{S_m}\left(\frac{v^{-1}-v}{z}, \frac{v^{-1}-v}{z}\right) &= \sum_{F\subset E(S_m)}\left(\prod_{f\in F} w(f)\right)\left(\frac{v^{-1}-v}{z}\right)^{k\langle F\rangle + s\langle F\rangle} \\
&= \sum_{F\subset E(S_m)}\left(\prod_{f\in F} w(f)\right)\left(\frac{v^{-1}-v}{z}\right)^{|F|-2+2k\langle F\rangle}.
\end{aligned}$$

Note that $k\langle\emptyset\rangle = 2$ and $k\langle F\rangle = 1$ for any nonempty subset F. Hence,

$$\begin{aligned}
Q_{S_m}\left(\frac{v^{-1}-v}{z}, \frac{v^{-1}-v}{z}\right) &= \left(\frac{v^{-1}-v}{z}\right)^2 + \sum_{F\neq\emptyset}\left(\prod_{f\in F} w(f)\right)\left(\frac{v^{-1}-v}{z}\right)^{|F|} \\
&= \left(\frac{v^{-1}-v}{z}\right)^2 + \sum_{F\neq\emptyset}\left(\prod_{f\in F}\frac{v^{-1}-v}{z} w(f)\right) \\
&= \left(\frac{v^{-1}-v}{z}\right)^2 - 1 + \sum_{F}\left(\prod_{f\in F}\frac{v^{-1}-v}{z} w(f)\right) \\
&= \left(\frac{v^{-1}-v}{z}\right)^2 - 1 + \prod_{e\in E(S_m)}\left(1 + \frac{v^{-1}-v}{z} w(e)\right).
\end{aligned}$$

Applying theorem 2.2, we have

$$\begin{aligned}
P_{P(2n_1,2n_2,\ldots,2n_m)}(v,z) &= \left(\frac{z}{v^{-1}-v}\right)^{m+1}\left(\prod_{i=1}^{m}(1-v^{2n_i})\right) \\
&\quad\times\left(\left(\frac{v^{-1}-v}{z}\right)^2 - 1 + \prod_{i=1}^{m}\left(1 + \left(\frac{v^{-1}-v}{z}\right)^2\frac{v^{2n_i}}{1-v^{2n_i}}\right)\right) \\
&= \left(\frac{z}{v^{-1}-v}\right)^{m+1}\left(c\prod_{i=1}^{m}(1-v^{2n_i}) + \prod_{i=1}^{m}(1+cv^{2n_i})\right).
\end{aligned}$$

This completes the proof of theorem 5.1. □

Remark 5.2. It is not difficult to obtain that

$$Ch[S_m] = \frac{1}{1-w}\left(\prod_{i=1}^{m}(e_i - w) - w\prod_{i=1}^{m}(e_i - 1)\right).$$

We can also apply theorem 2.6 and the chain polynomial of S_m to obtain theorem 5.1.

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Theorem 5.3. *Let S_m be the graph which consists of two vertices connected by m parallel edges. For any fixed nonzero z, zeros of Homfly polynomials of $\{D_n(S_m)|n = \pm 1, \pm 2, \ldots; m = 1, 2, \ldots\}$ are dense in the whole complex plane.*

Proof. Note that $D_n(S_m) = P(\overbrace{2n, 2n, \ldots, 2n}^{m})$. By theorem 5.1, we have

$$P_{D_n(S_m)}(v,z) = \left(\frac{z}{v^{-1}-v}\right)^{m+1} \left[c(1-v^{2n})^m + (1+cv^{2n})^m\right],$$

where $c = (\frac{v^{-1}-v}{z})^2 - 1$. Thus

$$P_{D_n(S_m)}(v,z) = \left(\frac{z}{v^{-1}-v}\right)^{m+1} \left(\frac{1}{v^2}\right)^{m+1} \times \left[(z^{-2}(1-v^2)^2 - v^2)(v^2 - v^{2n+2})^m + v^2(v^2 + (z^{-2}(1-v^2)^2 - v^2)v^{2n})^m\right].$$

By Beraha–Kahane–Weiss's theorem, for any nonzero integer n, points satisfying the equation $|v^2 - v^{2n+2}| = |v^2 + (z^{-2}(1-v^2)^2 - v^2)v^{2n}|$ and, hence, the equation

$$|1 - v^{2n}| = \left|1 + \left(\left(\frac{v^{-1}-v}{z}\right)^2 - 1\right) v^{2n}\right| \tag{8}$$

are limits of zeros of Homfly polynomials of $\{D_n(S_m)|m = 1, 2, \ldots\}$. Let v_0 be any fixed complex number with $|v_0| \le 1$ and $v_0 \ne 0$. Setting $x = v - v_0$, equation (8) becomes

$$|1 - (x+v_0)^{2n}| = \left|1 + \left(\left(\frac{1+(x+v_0)^2}{z(x+v_0)}\right)^2 - 1\right)(x+v_0)^{2n}\right|. \tag{9}$$

By Sokal's lemma ($F_1(x) = -1, F_2(x) = (\frac{1+(x+v_0)^2}{z(x+v_0)})^2 - 1, G(x) = (x+v_0)^2$), for any sufficiently small $\epsilon > 0$, there exists n_0 such that for any $n \ge n_0$, equation (9) has a zero x satisfying $|x| < \epsilon/2$, i.e. equation (8) has a zero $v = x + v_0$ satisfying $|v - v_0| < \epsilon/2$. For the special case that $v_0 = 0$, by the above result, there exists n_0 such that for any $n \ge n_0$, equation (8) has a zero v satisfying $|v - \epsilon/4| < \epsilon/4$, implying that $|v| < \epsilon/2$.

Therefore, for any fixed x with $|x| \le 1$ in the complex plane, there is a v satisfying equation (8) for some n such that $|v - x| < \epsilon/2$. By the definition of limits of zeros, there is a zero t of the Jones polynomial of $D_n(S_m)$ for some large m such that $|t - v| < \epsilon/2$. Thus $|t - x| < \epsilon$, which means that given a fixed nonzero z, zeros of Homfly polynomials of $\{D_n(S_m)|n = 1, 2, \ldots; m = 1, 2, \ldots\}$ are dense inside the unit circle $|x| = 1$. By the well-known property [22]

$$P_{L^{\mathrm{M}}}(v,z) = P_L(-v^{-1}, z),$$

where L^{M} is the mirror image of L, zeros of the Homfly polynomial of $\{D_n(S_m)|n = -1, -2, \ldots; m = 1, 2, \ldots\}$ are also dense outside the unit circle $|x| = 1$, which completes the proof of theorem 5.3. □

6. Concluding remarks

In this paper, we chose the generalized Jaeger's links as our case study and obtained the unit-circle theorem and density-in-the-plane theorem on zeros of their Homfly polynomials with z fixed. Zeros of the Homfly polynomial with v fixed deserve study. In particular, the case of $v = 1$ of the Conway polynomial (hence, Alexander polynomial) is especially worth studying. However, as remark 2.3 shows, generalized Jaeger's links are not appropriate examples for this study.

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