# INTERPOLATION THEOREM FOR A CONTINUOUS FUNCTION ON ORIENTATIONS OF A SIMPLE GRAPH 

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#### Abstract

Let $G$ be a simple graph. A function $f$ from the set of orientations of $G$ to the set of non-negative integers is called a continuous function on orientations of $G$ if, for any two orientations $O_{1}$ and $O_{2}$ of $G,\left|f\left(O_{1}\right)-f\left(O_{2}\right)\right| \leqslant 1$ whenever $O_{1}$ and $O_{2}$ differ in the orientation of exactly one edge of $G$.

We show that any continuous function on orientations of a simple graph $G$ has the interpolation property as follows:

If there are two orientations $O_{1}$ and $O_{2}$ of $G$ with $f\left(O_{1}\right)=p$ and $f\left(O_{2}\right)=q$, where $p<q$, then for any integer $k$ such that $p<k<q$, there are at least $m$ orientations $O$ of $G$ satisfying $f(O)=k$, where $m$ equals the number of edges of $G$.

It follows that some useful invariants of digraphs including the connectivity, the arcconnectivity and the absorption number, etc., have the above interpolation property on the set of all orientations of $G$.


## 1. Introduction

A variety of research has been clevoted to the orientations of a graph. For example, it is well known that every graph without self loops admits an acyclic orientation; Stanley [22] studied the set of acyclic orientations of a simple graph $G$ and counted the number of acyclic orientations of $G$ by using the chromatic polynomial of $G$; Robbins [16] proved that a nontrivial graph $G$ admits a strongly connected orientation if and only if $G$ is 2 edge-connected; Chvátal and Thomassen [5] further showed that every 2 edge-connected graph of radius $r$ admits an orientation of radius at most $r^{2}+r$; Gerards [7] established an orientation theorem characterizing the class of graphs in which the edges can be oriented in such a way that going along any circuit in the graph, the number of forward edges minus the number of backward edges is equal

[^0]to $0,-1$ or 1 . Roberts and $\mathrm{Xu}[17-20]$ considered optimizing the orientations of grid graphs with respect to various measures; Donald and Elwin [6] investigated the structure of the set of strongly connected orientations of a graph $G$ and showed that any two strongly connected orientations of $G$ can be connected by a sequence of operations called simple transformations.

The above indicates one half of the background for our present work. The other half comes from a number of research related to the interpolation property for some invariants of spanning subgraphs of a given graph. In 1980, at the fourth International Conference on Graph Theory and Applications held in Kalamazoo, G. Chartrand asked [see [3], p. 610]: If a graph $G$ contains spanning trees having $n$ and $m$ end-vertices, with $m<n$, does $G$ contain a spanning tree with $k$ end-vertices for every integer $k$ with $m<k<n$ ? This problem piqued the interest of many graph theorists. It was first affirmatively settled by Schuster [21] in 1983. In 1984 and 1985, Cai [2] and Lin [13] gave different proofs (Lin's is the shortest). Several different generalizations also appeared in Schuster [21], Liu [14], Barefoot [1], and Zhang and Chen [23]. Zhang and Guo [24] further considered similar problem for directed graphs and got the corresponding interpolation theorems. Harary et al. [8, 10, 11] and Lewinter [12] obtained interpolation theorems for more invariants of spanning trees. Recently, Harary and Plantholt [9] classified many known interpolation theorems for spanning trees in $[8,10,11,12,21]$, obtained interpolation results for new invariants and generalized to other families of spanning subgraphs. More recently, S. Zhou [25] used the same idea as in Lin [13] to give a short proof of interpolation theorems for many invariants on spanning subgraphs with equal size.
In this paper, we shall consider the set of all orientations of a simple graph $G$ and establish a general interpolation theorem. It follows that some useful invariants of digraphs (including the connectivity, the arc-connectivity, the absorption number and some other invariants introduced in this paper) have the interpolation property on the set of all orientations of a simple graph $G$.

Throughout the paper, $G=(V(G), E(G))$ is always assumed to be a simple graph with the vertex set $V(G)$ and the edge set $E(G)$. An orientation of $G$ is the digraph obtained from $G$ by assigning a direction to each edge of $G$. The essential concepts in this paper are introduced in the following two definitions.

Definition 1.1 (Graph of orientations of $G$ ). For any two distinct orientations $O_{1}$ and $O_{2}$ of $G$, we say that $O_{1}$ and $O_{2}$ are adjacent if they differ in the orientation of exactly one edge of the underlying graph $G$. This adjacency relation determines a simple graph $\hat{G}$ with the vertex set $V(\hat{G})$ representing all the orientations of $G$. We call $\hat{G}$ the graph of orientations of $G$.

Definition 1.2 (Continuous functions on $G$ ). Let $f$ be a function from the vertex set $V(G)$ to the set of non-negative integers. We say that $f$ is a continuous function on $G$ if, $|f(u)-f(v)| \leqslant 1$ for any two adjacent vertices $u, v$, of $G$. This definition was motivated by Lovácz [15].

For simplicity, a continuous function on the graph of orientations of $G$ will be called a continuous functio on orientations of $G$. For other general graph theoretical terminology, the reader is refered to the book of Chartrand and Lesniak [4].

## 2. Main results

Theorem. Any continuous function $f$ on orientations of a simple graph $G$ has the following interpolation property:

If there are two orientations $O_{1}$ and $O_{2}$ of $G$ with $f\left(O_{1}\right)=p$ and $f\left(O_{2}\right)=q$, where $p<q$, then for any integer $k$ such that $p<k<q$, there are at least $m$ orientations $O$ of $G$ satisfying $f(O)=k$, where $m$ equals the number of edges of $G$.

Proof. Let $\hat{G}$ be the graph of orientations of $G$. We first show that $\hat{G}$ is isomorphic to the $m$-cube $I^{m}$ where $m=|E(G)|$. (Recall that the $m$-cube $I^{m}$ is the graph whose vertices are the $m$-dimensional vectors of 0 's and 1 's, two vertices being adjacent if and only if they differ in exactly one coordinate.) In fact, for any edge of $G$, it can be assigned exactly two distinct directions. We may correspond them to 0 and 1 , respectively, Then a vertex of $\hat{G}$ (i.e., an orientation of $G$ ) corresponds to an $m$-dimensional vector of 0 's and 1 's. It is easily seen that this is a one-to-one correspondence from $V(\hat{G})$ to $V\left(I^{m}\right)$ and preserves adjacency relation. Therefore it gives an isomorphism between the graphs $\hat{G}$ and $I^{m}$.

Notice that $I^{m}$ is $m$-regular. So its connectivity $k\left(I^{m}\right) \leqslant m$. On the other hand, since $I^{m}$ is the product of $m$ paths of length 1 , we may use Menger's Theorem to show $k\left(I^{m}\right) \geqslant m$ by induction on $m$. Thus we have $k\left(I^{m}\right)=m$ and so $k(\hat{G})=m$. Therefore, there are $m$ internally disjoint paths $P$ between $O_{1}$ and $O_{2}$ in $\hat{G}$. Since $f$ is continuous, there must exist at least one $O$ with $f(O)=k$ on every such path $P$, and the theorem follows.

In order to apply the theorem, we recall and introduce some invariants for a digraph $D=(V(D), A(D))$ with the vertex set $V(D)$ and the arc set $A(D)$. (Note that the counterparts of these invariants for undirected graphs are familiar and have been extensively studied.)

Definition 2.1. The connectivity $k_{1}(D)$ of $D$ is defined to be the minimum number of vertices whose removal from $D$ leaves the remaining digraph not strongly connected or reduces $D$ to a single vertex.

Definition 2.2. The arc-connectivity $k_{2}(D)$ of $D$ is defined to be the minimum number of arcs whose removal from $D$ leaves the remaining digraph not strongly connected or reduces $D$ to a single vertex.

Definition 2.3. The directed arboricity $k_{3}(D)$ of $D$ is the minimum number of subsets into which $A(D)$ can be partitioned so that each subset induces a directed forest. (A directed forest is a digraph of which every component is a rooted ditree, where a rooted ditree is a digraph $T$ in which there is a vertex, called the root of $T$, being able to reach any other vertex of $T$ by a clirected path and the underlying undirected graph of $T$ is a tree.)

Definition 2.4. The directed vertex arboricity $k_{4}(D)$ of $D$ is the minimum number of subsets into which $V(D)$ can be partitioned so that each subset induces a directed forest.

Definition 2.5. The directed linear arboricity $k_{5}(D)$ of $D$ is the minimum number of subsets into which $A(D)$ can be partitioned so that each subset induces a directed linear forest. (A directed linear forest is a directed forest of which each component is a directed path.)

Definition 2.6. The directed linear vertex arboricity $k_{6}(D)$ of $D$ is the minimum number of subsets into which $V(D)$ can be partitioned so that each subset induces a directed linear forest.

Definition 2.7. The absorption number $k_{7}(D)$ is the minimum of the cardinalities $|S|$ over all such subsets $S$ of $V(D)$ of which each $S$ satisfies the following: for any $v \in V(D)-S$, there is an arc in $D$ from $v$ to a vertex of $S$.

Now we give the following result on the above invariants.

Corollary. For any simple graph $G$, each of the invariants $k_{i}(i=1,2, \ldots, 7)$ has the interpolation property on the orientations of $G$. That is, if there are two orientations $O_{1}$ and $O_{2}$ of $G$ with $k_{i}\left(O_{1}\right)=p$ and $k_{i}\left(O_{2}\right)=q$, where $p<q$, then for any integer $k$ such that $p<k<q$, there are orientations $O$ of $G$ satisfying $k_{i}(O)=k$. And the number of such $O$ 's is not less than the number of edges of $G$.

Proof. For any given $i=1,2, \ldots, 7$, the function defined by $f(O)=k_{i}(O)$ for each $O \in V(\hat{G})$ is easily seen to be a continuous function on $\hat{G}$. Then the result immediately follows from the Theorem.

Remark. There are other invariants, such as maximum (in-, out-) degree, minimum (in-, out-)degree, and the number of disjoint directed cycles, etc., which can also be included in the corollary.

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