# Chapter 60 Term Structure of Default-Free and Defaultable Securities: Theory and Empirical Evidence

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## 60.1 Introduction

This article provides a survey on term structure models designed for pricing fixed income securities and their derivatives.[1] The past several decades have witnessed a rapid development in the fixed-income markets. A number of new fixed-income instruments have been introduced successfully into the financial market. These include, to mention just a few, strips, debt warrants, put bonds, commercial mortgage-backed securities, payment-in-kind debentures, zero-coupon convertibles, interest rate futures and options, credit default swaps, and swaptions. The size of the fixed-income market has greatly expanded. The total value of the fixed-income assets is about two-thirds of the market value of all outstanding securities.[2] From the investment perspective, it is important to understand how fixed-income securities are priced.

The term structure of interest rates plays a key role in pricing fixed income securities. Not surprisingly, a vast literature has been devoted to understanding the stochastic behavior of term structure of interest rate, the pricing mechanism of fixed-income markets, and the spread between different fixed-income securities. Past research generally focuses on: (i) modeling the term structure of interest rates and yield spreads; (ii) providing empirical evidence; and (iii) applying the theory to the pricing of fixed-income instruments and risk management. As such, our review centers on alternative models of term structure of interest rates, their tractability, empirical performance, and applications.

[1] For a survey on term structure models, see Dai and Singleton (2002b), Dai and Singleton (2003), and Maes (2004).

[2] See The 2008 Statistical Abstract, U.S. Census Bureau.

We begin with the basic definitions and notations in Section 1. We provide clear concepts of term structure of interest rates that are easily misunderstood. Section 2 introduces bond pricing theory within the dynamic term structure model (DTSM) framework. This framework provides a general modeling structure in which most of the popular term structure models are nested. This discussion thus helps understand the primary ingredients to categorize different DTSMs, i.e., the risk-neutral distribution of the state variables and the mapping function between these state variables and instantaneous interest rate.

Sections 3 provides a literature review of the studies on default free bonds. Several widely used continuous-time DTSMs are reviewed here, including affine, quadratic, regime switching, jump-diffusion and stochastic volatility models. We conclude this section with a discussion of empirical performance of these DTSMs, where we discuss some open issues, including the expectation puzzle, the linearity of state variables, the advantages of multifactor and nonlinear models, and their implications for pricing and risk management.

The studies of defaultable bonds are explored in section 4. We review both structural and reduced-form models, with particular attention given to the later. Several important issues in reduced form models are addressed here, including the specification of recovery rates, default intensity, coupon payment, other factors such as liquidity and taxes, and correlated defaults. Since it is convenient to have a closed-form pricing formula, it is important to evaluate the tradeoff between analytical tractability and the model complexity. Major empirical issues are related to uncovering the components of yield spreads and answering the question whether the factors are latent or observable.

Section 5 reviews the studies on two popular interest rate derivatives: interest rate swap and credit default swap. Here we present the pricing formulas of interest rate swap and credit default swap based on risk-neutral pricing theory. Other risk factors, such as counterparty risk and liquidity risk are then introduced into the pricing formula. Following this, we review important empirical work on the determinants of interest rate swap spread and credit default swap spread.

Section 6 concludes the paper by providing a summary of the literature and directions for future research. These include: (i) the economic significance of DTSM specification on pricing and risk management; (ii) the difference of interest rate dynamics in the risk neutral measure and physical measure; (iii) the decomposition of yield spreads; and (iv) the pricing of credit risk with correlated factors.

## 60.2 Definitions and Notations

### 60.2.1 Zero-coupon Bonds

A default-free *zero-coupon bond* (or discount bond) with maturity date $T$ and face value 1 is a claim which has a non-random payoff of 1 for sure at time $T$ and no other payoff before maturity. The price of a zero-coupon bond with maturity date $T$ at time $0 \le t \le T$ is denoted by $D(t,T)$.

### 60.2.2 Term Structure of Interest Rates

Consider a zero-coupon bond with a fixed maturity date $T$. The continuously compounded yield on this bond is

$$r(t,T) = -\frac{1}{T-t}\ln D(t,T) \tag{60.2.1}$$

The zero-coupon *yield curve* or *term structure of interest rates* at time *t* is the function

$$\tau \to r(t,t+\tau):[0,\infty) \to \Re \tag{60.2.2}$$

which maps time to maturity $\tau$ into the yield of the zero-coupon bond with that maturity at time *t*. The price of the zero-coupon bond can be calculated from its yield by

$$D(t,T) = \exp\left[-(T-t)r(t,T)\right] \tag{60.2.3}$$

60.2.3 Instantaneous Interest Rate

The *instantaneous interest rate* at time *t*, $r_t$ is defined as:

$$r_t = \lim_{T\to t}\frac{-\ln D(t,T)}{T-t} \tag{60.2.4}$$

60.2.4 Forward Rate

The *forward rate* at time t, $f_t^{T_1\to T_2}$, is the interest rate between two future time points $T_1$ and $T_2$ which is settled at time *t*. Specifically,

$$f_t^{T_1\to T_2} = \frac{\ln D(t,T_1) - \ln D(t,T_2)}{T_2 - T_1} \tag{60.2.5}$$

Remark: if $T_1 = t$, $f_t^{t\to T_2} = r(t,T_2)$.

60.2.5 Instantaneous Forward Rate

The *instantaneous forward rate* at time *t* with an effective date *T*, $f(t,T)$ is defined as

$$f(t,T) = \lim_{T_2\to T} f_t^{T\to T_2} = \lim_{T_2\to T}\frac{\ln D(t,T) - \ln D(t,T_2)}{T_2 - T} = -\frac{\partial \ln D(t,T)}{\partial T} \tag{60.2.6}$$

## 60.3 Bond Pricing in Dynamic Term Structure Model Framework

### 60.3.1 Spot Rate Approach

Let the instantaneous interest rate $r_t$ be a deterministic function of state variables $Y_t$ and time t, where $Y_t$ is an $K \times 1$ vector,

$$r_t = r(Y_t, t) \tag{60.3.1}$$

and the risk-neutral dynamics of $Y_t$ follow a diffusion process,

$$dY_t = \mu(Y_t, t)dt + \sigma(Y_t, t)dW_t^Q \tag{60.3.2}$$

where $W_t^Q$ is a $K \times 1$ vector of standard and independent Brownian motions under the risk-neutral measure *Q*, $\mu(Y,t)$ (*K* x 1 vector) and $\sigma(Y,t)$ (*K* x *K* matrix) are both deterministic functions of *Y* and *t*.

By risk-neutral pricing theory,[3] the price of a zero-coupon bond with maturity date *T* and face value 1 is given by

$$D(t,T) = E_t^Q\left[\exp\left(-\int_t^T r_s ds\right)\right] \tag{60.3.3}$$

where $E_t^Q$ represents the conditional expectation under risk-neutral measure *Q*. $D(t,T)$ is functionally related to *K* stochastic factors $Y_t$:

$$D(t,T) = D(t,T,Y_t) \tag{60.3.4}$$

By applying the discounted Feynman-Kac theorem[4] to (60.3.3), it can be shown that $D(t,T)$ must satisfy the partial differential equation (PDE),[5]

$$\frac{\partial D}{\partial t} + \mu(Y,t)^T \frac{\partial D}{\partial Y} + \frac{1}{2}\text{Trace}\left[\sigma(Y,t)\sigma(Y,t)^T \frac{\partial^2 D}{\partial Y \partial Y^T}\right] = rD \tag{60.3.5}$$

[3] The Fundamental Theorem of Finance states that under no arbitrage condition, there exists an equivalent martingale measure (risk-neutral) Q under which any security prices scaled by money market account are a martingale process. Such measure is unique if the market is both no arbitrage and complete. See Harrison and Kreps (1979), Duffie (1996), Cochrane (2001).

[4] For a detailed description of the discounted Feynman-Kac theorem, please refer to Shreve (2004).

[5] See also Dai and Singleton (2003).

with the boundary condition $D(T,T)=1$ for all $r_T$. The superscript T represents the transpose of a vector (or matrix).

In order to solve the PDE in (60.3.5) with the boundary condition, we need to specify the diffusion process of state variables $Y_t$ in the risk-neutral measure and the functional form $r(Y_t,t)$ which determines the diffusion process of instantaneous interest rate in the risk-neutral measure. Models involved with such diffusion processes are called the dynamic term structure models (DTSMs).

### 60.3.2 Forward Rate Approach

If we know the initial forward rate curve $f(t,T)$ for all values of $0 \le t \le T \le \bar{T}$, we can recover $D(t,T)$ by

$$D(t,T) = \exp\left(-\int_t^T f(t,v)\,dv\right) \tag{60.3.6}$$

Heath, Jarrow and Morton (1992) propose the following forward rate process[6]

$$df(t,T) = \alpha(t,T)\,dt + \sigma(t,T)\,dW_t \tag{60.3.7}$$

and find that under no arbitrage condition,

(*i*) the forward rate evolves according to the following process

$$df(t,T) = \sigma(t,T)\sigma^*(t,T)\,dt + \sigma(t,T)\,dW_t^Q \tag{60.3.8}$$

where $\sigma^*(t,T) = \int_t^T \sigma(t,v)\,dv$, and

(*ii*) the zero coupon bond price evolves according to the following process

$$dD(t,T) = r_t D(t,T)\,dt - \sigma^*(t,T)D(t,T)\,dW_t^Q \tag{60.3.9}$$

## 60.4 Dynamic Term Structure Models (DTSMs)

In this section, we review the DTSMs commonly used in the pricing of

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[6] Shreve (2004) shows that every DTSM driven by Brownian motion is an HJM model.

default-free bonds. We begin with the affine DTSMs and then the nonlinear DTSMs.

### 60.4.1 Affine DTSMs

Affine DTSMs are characterized by the condition that the yield of zero-coupon bond is an affine (linear plus constant) function of the state variables, i.e.,

$$r(t,T) = A(T-t) + B(T-t)^T Y_t \tag{60.4.1}$$

#### 60.4.1.1 One Factor Affine DTSMs

If $K = 1$, the diffusion process for $r_t$ is given by[7]

$$dr_t = \mu(r_t,t)dt + \sigma(r_t,t)dW_t^Q \tag{60.4.2}$$

which is the one-factor DTSM. Some of the popular one-factor affine models are summarized below.

(1) Vasicek (1977) Model. In this model $r_t$ follows the diffusion process with $\mu(r_t,t) = \kappa(\theta - r_t)$, and $\sigma(r_t,t) = \sigma$. In this case, $D(t,T)$ is given by

$$D(t,T) = \exp\left(-A(T-t) - B(T-t)r_t\right), \ T \geq t \tag{60.4.3}$$

where $A(\tau) = \left(\theta - \dfrac{\sigma^2}{2\kappa^2}\right)\left[\tau - B(\tau)\right] + \dfrac{\sigma^2}{4\kappa}B(\tau)^2$, and $B(\tau) = \dfrac{1-\exp(-\kappa\tau)}{\kappa}$.

The instantaneous forward rate is given by

$$f(t,T) = e^{-\kappa(T-t)}r_t + \left(1 - e^{-\kappa(T-t)}\right)\theta - \frac{\sigma^2}{2\kappa^2}\left(1 - e^{-\kappa(T-t)}\right)^2 \tag{60.4.4}$$

and $v(t,T) = \dfrac{\sigma^2}{2\kappa}\left(1 - e^{-2\kappa(T-t)}\right)$ is the conditional variance of $r_T$.

(2) CIR (Cox, Ingersoll and Ross (1985)) Model. In this model, $r_t$ follows the diffusion process with $\mu(r_t,t) = \kappa(\theta - r_t)$, and $\sigma(r_t,t) = \sigma\sqrt{r_t}$. The zero-coupon

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[7] Throughout this paper, $W_t^Q$ represents the standard Brownian motion under the risk-neutral measure $Q$ and $W_t$ stands for the standard Brownian motion under the physical measure.

bond price $D(t,T)$ is given by

$$D(t,T) = A(T-t)\exp\left(-B(T-t)r_t\right) \tag{60.4.5}$$

where $A(\tau) = \left[\frac{2\gamma e^{(\kappa+\gamma)\tau/2}}{(\kappa+\gamma)\left(e^{\gamma\tau}-1\right)+2\gamma}\right]^{2\kappa\theta/\sigma^2}$ , $B(\tau) = \frac{2\left(e^{\gamma\tau}-1\right)}{(\kappa+\gamma)\left(e^{\gamma\tau}-1\right)+2\gamma}$

and $\gamma = \sqrt{\kappa^2 + 2\sigma^2}$ .

(3) Hull and White (1993) Model. The Hull and White (1993) model is a generalization of the Vasicek (1977) model that considers the time variant properties of $\kappa$ , $\theta$ , and $\sigma$ . In this model, $r_t$ follows the diffusion process with $\mu(r_t,t) = a(t) - b(t)r_t$ , and $\sigma(r_t,t) = \sigma(t)$ , where $a(t)$ , $b(t)$ and $\sigma(t)$ are non-random functions of *t*. $D(t,T)$ is given by

$$D(t,T) = \exp\left(-A(t,T) - B(t,T)r_t\right) \tag{60.4.6}$$

where $A(t,T) = \int_t^T \left[a(s)B(s,T) - \frac{1}{2}\sigma^2(s)B^2(s,T)\right]ds$ , and

$$B(t,T) = \int_t^T \exp\left(-\int_t^s b(u)\,du\right)ds \tag{60.4.7}$$

60.4.1.2 Multi-factor Affine DTSMs

One can develop multi-factor affine DTSMs from the above one-factor examples by simply assuming that $r_t = \delta_0 + \sum_{i=1}^{K} \delta_i Y_{it}$ , with each $Y_i$ following one of the preceding one-factor affine DTSMs.[8] In the following, we first illustrate this type of model by the two-factor model and then generalize it to the multi-factor model.

(1) Canonical Two-Factor Vasicek Model[9]

[8] See Stambaugh (1988), Longstaff and Schwartz (1992), Chen and Scott (1993), Pearson and Sun (1994), Duffie and Singleton (1997) for the multi-factor version of CIR model.
[9] Please refer to Shreve (2004) for a description of the generalized two-factor Vasicek model.

$$\begin{cases} dY_{1t} = -\lambda_1 Y_{1t} dt + dW_{1t}^Q \\ dY_{2t} = -\lambda_{21} Y_{1t} dt - \lambda_2 Y_{2t} dt + dW_{2t}^Q \\ r_t = \delta_0 + \delta_1 Y_{1t} + \delta_2 Y_{2t} \end{cases} \qquad (60.4.8)$$

where $W_{1t}^Q$ and $W_{2t}^Q$ are independent standard Brownian motions under the risk-neutral measure. Given the stochastic processes of the factors, the price of zero-coupon bond $D(t,T)$ is given by

$$D(t,T) = \exp\left(-A(T-t) - B_1(T-t)Y_{1t} - B_2(T-t)Y_{2t}\right) \qquad (60.4.9)$$

where $B_1(\tau)$, $B_2(\tau)$ and $A(\tau)$ satisfy the following ordinary differential equations (ODEs),

$$\begin{cases} B_1'(\tau) = -\lambda_1 B_1(\tau) - \lambda_{21} B_2(\tau) + \delta_1 \\ B_2'(\tau) = -\lambda_2 B_2(\tau) + \delta_2 \\ A'(\tau) = -\frac{1}{2} B_1^2(\tau) - \frac{1}{2} B_2^2(\tau) + \delta_0 \end{cases} \qquad (60.4.10)$$

with the boundary conditions $B_1(0) = B_2(0) = A(0) = 0$.

(2) Canonical Two-Factor CIR Model

$$\begin{cases} dY_{1t} = \left(\mu_1 - \lambda_{11} Y_{1t} - \lambda_{12} Y_{2t}\right) dt + \sqrt{Y_{1t}}\, dW_{1t}^Q \\ dY_{2t} = \left(\mu_2 - \lambda_{21} Y_{1t} - \lambda_{22} Y_{2t}\right) dt + \sqrt{Y_{2t}}\, dW_{2t}^Q \\ r_t = \delta_0 + \delta_1 Y_{1t} + \delta_2 Y_{2t} \end{cases} \qquad (60.4.11)$$

Under some regularity conditions,

$$D(t,T) = \exp\left(-A(T-t) - B_1(T-t)Y_{1t} - B_2(T-t)Y_{2t}\right) \qquad (60.4.12)$$

where $A(\tau)$, $B_1(\tau)$, and $B_2(\tau)$ satisfy the following ODEs,

$$\begin{cases} B_1'(\tau) = -\lambda_{11} B_1(\tau) - \lambda_{21} B_2(\tau) - \frac{1}{2} B_1^2(\tau) + \delta_1 \\ B_2'(\tau) = -\lambda_{12} B_1(\tau) - \lambda_{22} B_2(\tau) - \frac{1}{2} B_2^2(\tau) + \delta_2 \\ A'(\tau) = \mu_1 B_1(\tau) + \mu_2 B_2(\tau) + \delta_0 \end{cases} \qquad (60.4.13)$$

and the boundary conditions $A(0)=0$, $B_1(0)=0$, and $B_2(0)=0$. The superscript $'$ denotes the first-order derivative.

(3) Two-Factor Mixed Model

$$\begin{cases} dY_{1t} = (\mu - \lambda_1 Y_{1t})dt + \sqrt{Y_{1t}}dW_{1t}{}^Q \\ dY_{2t} = -\lambda_2 Y_{2t}dt + \sigma_{21}\sqrt{Y_{1t}}dW_{1t}{}^Q + \sqrt{\alpha + \beta Y_{1t}}dW_{2t}^Q \\ r_t = \delta_0 + \delta_1 Y_{1t} + \delta_2 Y_{2t} \end{cases} \tag{60.4.14}$$

Then,

$$D(t,T) = \exp(-A(T-t) - B_1(T-t)Y_{1t} - B_2(T-t)Y_{2t}) \tag{60.4.15}$$

where $A(\tau)$, $B_1(\tau)$, and $B_2(\tau)$ satisfy the following ODEs,

$$\begin{cases} B_1'(\tau) = -\lambda_1 B_1(\tau) - \frac{1}{2}B_1(\tau)^2 - \sigma_{21}B_1(\tau)B_2(\tau) - (1+\beta)B_2(\tau)^2 + \delta_1 \\ B_2'(\tau) = -\lambda_2 B_2(\tau) + \delta_2 \\ A'(\tau) = \mu B_1(\tau) - \frac{1}{2}\alpha B_2(\tau)^2 + \delta_0 \end{cases} \tag{60.4.16}$$

and the boundary conditions $A(0)=0$, $B_1(0)=0$, and $B_2(0)=0$.

(4) Dai and Singleton (2000) examine the multi-factor affine DTSMs with the following structure:

$$\begin{cases} dY_t = \mathrm{K}(\Theta - Y_t)dt + \Sigma\sqrt{S_t}dW_t{}^Q \\ r(Y_t,t) = \delta_0 + \delta^T Y_t \end{cases} \tag{60.4.17}$$

where $S_t$ is a diagonal matrix with $[S_t]_{ii} = \alpha_i + Y_t^T \beta_i$. Let $\mathbf{B}$ be the *K x K* matrix with the $i^{\text{th}}$ column given by $\beta_i$. By restricting the parameter vector $(\delta_0, \delta, \mathrm{K}, \Theta, \Sigma, \alpha, \mathbf{B})$, they construct admissible affine models that give a unique, well-defined solution for $D(t,T)$ which is equal to $\exp\left(A(T-t) - B(T-t)^T Y_t\right)$ by the following system of ODEs:

$$
\begin{cases}
A'(\tau) = -\Theta^T \mathrm{K}^T B(\tau) + \frac{1}{2}\sum_{i=1}^{K}\left[\Sigma^T B(\tau)\right]_i^2 \alpha_i - \delta_0 \\
B'(\tau) = -\mathrm{K}^T B(\tau) - \frac{1}{2}\sum_{i=1}^{K}\left[\Sigma^T B(\tau)\right]_i^2 \beta_i + \delta
\end{cases}
\tag{60.4.18}
$$

with the initial conditions that $A(0)=0$ and $B(0)=0_{K\times 1}$.

(5) Duffie and Kan (1996) provide sufficient conditions for affine DTSMs that could handle the general correlated affine diffusions:[10]

(i) $\mu(Y,t)$ is an affine function of $Y_t$: $\mu(Y,t)=a+bY_t$, where $a$ is a $K$ x 1 vector and $b$ is a $K$ x $K$ matrix.

(ii) $\sigma(Y,t)\sigma(Y,t)^T$ is an affine function of $Y_t$:

$$
\sigma(Y,t)\sigma(Y,t)^T = h_0 + \sum_{j=1}^{K} h_1^{\,j} Y_{jt} \tag{60.4.19}
$$

where $h_0$ and $h_1^{\,j}$, $j=1,2,...,K$ are $K$ x $K$ matrices.

60.4.2 Quadratic DTSMs

Ahn, Dittmar and Gallant (2002) provide a general specification of quadratic DTSMs. The state variables $Y_t$ are assumed to follow the multivariate Gaussian processes with mean-reverting properties in the risk-neutral measure:

$$
dY_t = \left[\mu - \kappa Y_t\right]dt + \sigma(Y,t)\,dW_t^Q \tag{60.4.20}
$$

where $\mu$ is a $K$ x 1 vector, $\kappa$ and $\sigma$ are $K$ x $K$ matrices, and $W_t^Q$ is a $K$-dimensional vector of the standard Brownian motions that are mutually independent under the risk-neutral measure $Q$.

The instantaneous interest rate $r_t$ is a quadratic function of the state variables,

$$
r(Y,t) = \delta_0 + \delta^T Y_t + Y_t^T \Psi Y_t \tag{60.4.21}
$$

where $\delta_0$ is a constant, $\delta$ is a $K$ x 1 vector, $\Psi$ is a $K$ x $K$ positive semi-definite

[10] See Duffie, Filipovi and Schachermayer (2003) for sufficient and necessary conditions.

matrix, and $\delta_0 - \frac{1}{4}\delta^T \Psi^{-1}\delta \geq 0_{K\times 1}$.

Applying the discounted Feynman-Kac theorem to bond pricing, we obtain:

$$D(t,T) = \exp\left(A(T-t) + B(T-t)^T Y_t + Y_t^T C(T-t) Y_t\right) \tag{60.4.22}$$

where $A(\tau)$, $B(\tau)$, and $C(\tau)$ satisfy the ODEs,

$$\begin{cases} C'(\tau) = 2C(\tau)\sigma\sigma^T C(\tau) - \left(C(\tau)\kappa + \kappa^T C(\tau)\right) - \Psi \\ B'(\tau) = 2C(\tau)\sigma\sigma^T B(\tau) - \kappa^T B(\tau) + 2C(\tau)\mu - \delta \\ A'(\tau) = \text{Trace}\left[\sigma\sigma^T C(\tau)\right] + \frac{1}{2}B(\tau)^T \sigma\sigma^T B(\tau) + B(\tau)^T \mu - \delta_0 \end{cases} \tag{60.4.23}$$

with the initial conditions that $A(0) = 0$, $B(0) = 0_{K\times 1}$, and $C(0) = 0_{K\times K}$.

The yield of zero-coupon bond is a quadratic function of the state variables,

$$r(t,T) = -\frac{A(T-t) + B(T-t)^T Y_t + Y_t^T C(T-t) Y_t}{T-t} \tag{60.4.24}$$

The above is an example of the nonlinear model. Other examples of quadratic DTSMs include:

(1) Beaglehole and Tenney (1991) Model: $\delta_0 = 0$, $\delta = 0_{K\times 1}$, $\Psi$, $\kappa$, and $\sigma$ are diagonal matrix.

(2) Longstaff (1989) Model: $\delta_0 = 0$, $\delta = 0_{K\times 1}$, $\Psi$, $\sigma$ are diagonal matrix, $\kappa = 0_{K\times 1}$, and $\mu \neq 0_{K\times 1}$. The key feature of this model is that the state variables are not mean reverting.

(3) Constantinides (1992) Model: $\delta = 0_{K\times 1}$, $\Psi = I_{K\times K}$, $\sigma$ and $\kappa$ are diagonal matrices.

60.4.3 DTSMs with Jumps

The public announcement of important economic news and the sudden change of monetary policy typically have a jump impact on the interest rates. A number of researchers (see, for example, Das (2002), and Johannes (2004)) find that most classical diffusion processes fail to explain the leptokurtosis of interest rate and suggest the use of jump in DTSMs.

Suppose that $r_t = r(Y_t, t)$ is a function of a jump-diffusion process $Y$ with the risk-neutral dynamics

$$dY_t = \mu(Y_t, t)dt + \sigma(Y_t, t)dW_t^Q + \Delta J_t dZ_t \tag{60.4.25}$$

where $Z_t$ is a Poisson process with risk-neutral intensity $\lambda_t$, and the jump size $\Delta J_t$ follows the distribution $v_t(x) \equiv v(x; Y_t, t)$.

Bas and Das (1996) extend the Vasicek (1977) model to consider the jump behavior of interest rate. Ahn and Thompson (1988) extend the CIR model to the case of state variables following a square-root process with jumps. Duffie, Pan and Singleton (2000) obtain the analytic expressions for $D(t,T)$ with the affine jump-diffusion process. Piazzesi (2001) develops a class of affine-quadratic jump-diffusion models and links the jumps to the resetting of target interest rates by the Federal Reserve Bank.

60.4.4 DTSMs with a Regime Switching

The processes that govern the DTSMs are very likely to change over economic cycles. There is an extensive empirical literature that suggests the regime switching model for DTSMs (see, for example, Sanders and Unal (1988), Gray (1996), Garcia and Perron (1996), Ang and Bekaert (2002)). Suppose that there are (S+1) possible states (regimes) evolved by a conditional Markov chain $s_t : \Omega \to \{0, 1, ..., S\}$ with a $(S+1) \times (S+1)$ transition probability matrix $P_t$ with the property that all rows are sum to one. $P_t^{ij}dt$ is the probability of moving from regime *i* to *j* over the next interval $dt$.

The state variables $Y_t$ in the risk-neutral measure follow the following process

$$dY_t = \mu^j\left(Y_t,t\right)dt + \sigma^j\left(Y_t,t\right)dW_t^Q \tag{60.4.26}$$

where *j* indexes regime *j*. Let $z_t^{\,j} = 1_{s_t=j}, j = 0,1,...,S$ be the regime indicator functions. Then $\mu\left(s_t;Y_t,t\right) = \sum_{j=0}^{S} z_t^{\,j}\mu^j\left(Y_t,t\right)$ , $\sigma\left(s_t;Y_t,t\right) = \sum_{j=0}^{S} z_t^{\,j}\sigma^j\left(Y_t,t\right)$ and $D\left(t,T\right) = \sum_{j=0}^{S} z_t^{\,j}D^j\left(t,T\right)$, where $D^j\left(t,T\right) = D\left(s_t = j;Y_t,t,T\right)$.

Bansal and Zhou (2002) develop a discrete-time regime-switching model where the short interest rate and the market price of risks are subject to discrete regime shifts. Dai and Singleton (2003) propose a DTSM with regime switching that has a closed-form solution for the zero-coupon bond price. The dynamics for each regime *i* in risk-neutral measure is given by

$$\begin{cases} r_t^i \equiv r\left(s_t = i;Y_t,t\right) = \delta_0^i + \delta_Y^T Y_t \\ \mu_t^{\,i} \equiv \mu\left(s_t = i;Y_t,t\right) = \kappa\left(\theta^i - Y_t\right) \\ \sigma_t^{\,i} \equiv \sigma\left(s_t = i;Y_t,t\right) = diag\left(\alpha_k^{\,i} + \beta_k^{\,T}Y_t\right)_{k=1,2,...,K} \end{cases} \tag{60.4.27}$$

where $\delta_0^i$ and $\alpha_k^i$ are constants, $\kappa$ is a constant *K* x *K* matrix, and $\delta_Y$, $\theta^i$ and $\beta_k$ are constant *K* x 1 vectors. With the additional assumption that $P_t$ is state independent, they show that

$$D^i\left(t,T\right) = \exp\left(-A^i\left(T-t\right) - B\left(T-t\right)^T Y_t\right),\ \ 0 \le i \le S \tag{60.4.28}$$

where $A^i\left(\tau\right)$ and $B\left(\tau\right)$ satisfy a set of ODEs.

A characteristic of this model is that regime dependence under the risk-neutral measure enters only through the intercept term $A^i\left(T-t\right)$. The derivative of zero-coupon bond yields with respect to *Y* does not depend on the regime.

In a recent paper, Dai, Singleton and Yang (2007) develop a discrete-time multi-factor DTSM with regime switching that yields a closed-form solution for bond price with the following characteristics: (i) there are two regimes characterized by low (L) and high (H) volatility; (ii) the regime shift probabilities $P_t^{ij}\left(i, j=H,L\right)$ under the physical measure depend on the underlying change of state variables; and (iii) regime-shift risks are priced.

60.4.5 DTSMs with Stochastic Volatility (SV)

The stochastic volatility model introduces an additional factor, i.e., the volatility of $r_t$, in an attempt to explain the instantaneous interest rate dynamics. Examples in this category are:

(1) Longstaff and Schwartz (1992) SV model:

$$\begin{cases} dr_t = \left[\alpha\gamma + \beta\eta - \dfrac{\beta\delta - \alpha\xi}{\beta-\alpha} r_t - \dfrac{\xi-\delta}{\beta-\alpha} V_t\right] dt \\ \qquad + \alpha\sqrt{\dfrac{\beta r_t - V_t}{\alpha\left(\beta-\alpha\right)}} dW_{1t} + \beta\sqrt{\dfrac{V_t - \alpha r_t}{\beta\left(\beta-\alpha\right)}} dW_{2t} \\ dV_t = \left[\alpha^2\gamma + \beta^2\eta - \dfrac{\alpha\beta\left(\delta-\xi\right)}{\beta-\alpha} r_t - \dfrac{\beta\xi - \alpha\delta}{\beta-\alpha} V_t\right] dt \\ \qquad + \alpha^2\sqrt{\dfrac{\beta r_t - V_t}{\alpha\left(\beta-\alpha\right)}} dW_{1t} + \beta^2\sqrt{\dfrac{V_t - \alpha r_t}{\beta\left(\beta-\alpha\right)}} dW_{2t} \end{cases} \tag{60.4.29}$$

where $\alpha$, $\beta$, $\gamma$, $\eta$, $\delta$ and $\xi$ are positive constants and $V_t$ is the instantaneous variance of changes in $r_t$.

(2) Andersen and Lund (1997), and Ball and Torous (1999) SV models:

$$\begin{cases} dr_t = \kappa_1\left(\mu - r_t\right) dt + \sigma_t r_t^{\rho} dW_{1t},\ \rho > 0 \\ d\log\sigma_t^2 = \kappa_2\left(\alpha - \log\sigma_t^2\right) dt + \eta dW_{2t} \end{cases} \tag{60.4.30}$$

(3) Bali (2000) SV model:

$$\begin{cases} dr_t = \kappa_1(\mu - r_t)dt + \sigma_t r_t^{\rho} dW_{1t} \\ d\sigma_t^2 = \kappa_2(\phi - \sigma_t^2)dt + \eta dW_{2t} \\ d\sigma_t = \kappa_3(\varphi - \sigma_t)dt + \iota dW_{3t} \end{cases} \tag{60.4.31}$$

(4) Collin-Dufresne and Goldstein (2002) SV model:[11]

$$\begin{cases} dv_t = k_v(\mu_v - v_t)dt + \sigma_v\sqrt{v_t}dW_{1t}^Q \\ d\theta_t = \left[k_\theta(\mu_\theta - \theta_t) + k_{\theta r}(\mu_r - r_t) + k_{\theta v}(\mu_v - v_t)\right]dt \\ \qquad + \sigma_{\theta v}\sqrt{v_t}dW_{1t}^Q + \sigma_{\theta r}\sqrt{\alpha_r + v_t}dW_{2t}^Q + \sqrt{\sigma_\theta^2 + \beta v_t}dW_{3t}^Q \\ dr_t = \left[k_r(\mu_r - \theta_t) + k_{r\theta}(\mu_\theta - \theta_t) + k_{rv}(\mu_v - v_t)\right]dt \\ \qquad + \sigma_{rv}\sqrt{v_t}dW_{1t}^Q + \sqrt{\alpha_r + v_t}dW_{2t}^Q + \sigma_{r\theta}\sqrt{\sigma_\theta^2 + \beta v_t}dW_{3t}^Q \end{cases} \tag{60.4.32}$$

60.4.6 Other non-affine DTSMs

Besides the quadratic DTSMs, DTSMs with jumps, regime switching, and stochastic volatilities, there are other non-affine DTSMs with $\mu(Y_t,t)$ or $\sigma(Y_t,t)$ not satisfying the conditions of affine DTSMs suggested by Duffie and Kan (1996) and Duffie, Filipovi and Schachermayer (2003). Examples are:

(1) Ahn and Gao (1999) nonlinear model:

$$dr_t = \left(\alpha_1 + \alpha_2 r_t + \alpha_3 r_t^2\right)dt + \sqrt{\alpha_4 + \alpha_5 r_t + \alpha_6 r_t^3}dW_t \tag{60.4.33}$$

(2) Ait-Sahalia (1996) nonlinear model:

$$dr_t = \left(\alpha_{-1} r_t^{-1} + \alpha_0 + \alpha_1 r_t + \alpha_2 r_t^2\right)dt + \sigma r_t^{\rho} dW_t \tag{60.4.34}$$

(3) Black, Derman and Toy (1990), and Black and Karasinski (1991) nonlinear model:[12]

$$d\log r_t = \left(\mu_t - \kappa_t \log r_t\right)dt + \sigma_t dW_t^Q \tag{60.4.35}$$

The other related but different classes of DTSMs include two categories. The

[11] It should be noted that Longstaff and Schwartz (1992) and Collin-Dufresne and Goldstein (2002) SV model are also nested in affine DTSMs since their yields of zero-coupon bonds are also affine to state variables.
[12] Peterson, Stapleton and Subrahmanyam (1998) extend the lognormal model to two factor case.

first class of these models describes the DTSMs with a selection of macroeconomic variables. As Maes (2004) points out, there is a great incentive to investigate the relationship between the dynamics of macroeconomic variables and the term structure since there is strong evidence that term structure predicts movement on macroeconomic activities (see, for example, Estrella and Hardouvelis (1991), Estrella and Mishkin (1996, 1997, 1998)). One important issue involved in these models is to interpret the economic meanings underlying the latent factors in terms of observed and unobserved macroeconomic variables such as inflation and output gaps. Dewachter, Lyrio and Maes (2006) study a continuous-time joint model of macroeconomy and the term structure of interest rates. Ang and Piazzesi (2003), and Dewachter and Lyrio (2006) find that macroeconomic factors clearly affect the short end of term structure. Kozicki and Tinsley (2001, 2002) find that missing factors may be related to the long-run inflation expectation of agents. Dewachter and Lyrio (2006) provide a macroeconomic interpretation for the latent factors in DTSMs: the "level" factor represents the long-run inflation expectation of agents; the "slope" factor captures the temporary business cycle conditions; and the "curvature" factor represents a clear independent monetary policy factor. Moreover, Wu (2006) develops a general equilibrium model of term structure with macroconomic factors.

The second class views the whole term structure as state variables and models their dynamics accordingly. Such high dimensional models are developed by Kennedy (1994) as "Brownian sheets", Goldstein (2000) as "random fields" and Santa-Clara and Sornette (2001) as "stochastic string shocks".

### 60.4.7 Empirical Performance

Because bond pricing is an important issue and there are so many DTSMs, a large number of studies have evaluated different DTSMs and compared their empirical performance in search for a best model. In the following, we summarize major results of empirical term structure studies.

#### 60.4.7.1 Explanation of Expectation Puzzle

The expectation puzzle was documented by Fama (1984), Fama and Bliss (1987), Froot (1989), Campbell and Shiller (1991), and Bekaert, Hodrick and Marshall (1997), which has long posed a challenge for DTSMs.[13] Campbell (1986) introduces a constant risk premium hypothesis to explain the expectations hypothesis. Campbell and Shiller (1991) attribute the expectation puzzle to the time-varying liquidity premium. Backus, Gregory and Zin (1989) show that a model assuming power utility preferences and time-varying expected consumption growth cannot account for this puzzle. Longstaff (2000) tests the expectations hypothesis at the extreme short end of the term structures and finds evidence supporting the hypothesis. Dai and Singleton (2002a) show that a statistical model of stochastic discount factor (SDF) can explain the puzzle. By altering the dependence of risk premia on factors, one can find parameter values for the three factor affine DTSMs that are consistent with the risk premium regressions in Fama and Bliss (1987) and Campbell and Shiller (1991). Using the framework of Campbell and Cochrane (1999), Wachter (2006) proposes a consumption-based model that accounts for many features of the nominal term

[13] Campbell and Shiller (1991) run the regressions $r(t+1,n-1)-r(t,n)=\alpha+(1/n-1)\beta_n(r(t,n)-r(t,1))+e_t$, where $r(t,n)$ is the yield of zero-coupon bond with maturity n at time t. Under the expectation hypothesis, $\beta_n=1$ for all n. However, Campbell and Shiller (1991) show that $\beta_n$ is negative and increasing with n.

structure of interest rates. Bekaert and Hodrick (2001) argue that the past use of the large-sample critical regions instead of the small sample counterparts may have overstated the evidence against the expectations hypothesis. Backus, Foresi, Mozumdar and Wu (2001) find that it is unlikely to explain the expectation puzzle using a one-factor affine DTSM.

#### 60.4.7.2 Linear or Nonlinear Drift of State Variables?

Most of the DTSMs assume that the drifts of state variables are linear (mean reverting). However, empirical findings are inconclusive. Ait-Sahalia (1996) constructs a specification test of DTSMs and rejects the linear drift specification. Stanton (1997) obtains similar results as Ait-Sahalia (1996). Conley, Hansen, Luttmer and Scheinkman (1997) examine the linearity of drift and find that mean reverting is stronger only for large values of interest rate. Ahn and Gao (1999) find nonlinearity in term structure dynamics. Chapman and Pearson (2000) conduct a Monte Carlo simulation of DTSMs with a linear drift and then apply the estimators of Ait-Sahalia (1996) and Stanton (1997) to the simulated data. They find strong mean reversions when interest rate is high. Elerian, Chib and Shephard (2001) and Jones (2003) use a Bayesian approach to show that stronger mean reversion in the extreme levels of interest rate depends critically on the prior distribution. In a survey paper, Chapman and Pearson (2001) examine the interest rate data and find that mean reversion is weak with a wide range of interest rates. The short rate series seems to be a "persistent" time series, i.e., it lingers over long consecutive periods above and below the unconditional long-run mean. Boudoukh, Richardson, Stanton and Whitelaw

(1998) and Balduzzi and Eom (2000) use the nonparametric analysis and find that the drifts in both two- and three-factor DTSMs are nonlinear. Dai and Singleton (2000) empirically test the affine DTSMs and find relatively promising performance of affine DTSMs. But as Ahn, Dittmar and Gallant (2002) point out, the results also suggest that there may be some omitted nonlinearity in the affine DTSMs since the pricing errors of affine DTSMs are sensitive to the magnitude of the slope of the yield curve and highly persistent.

However, there are other empirical studies that support the linear drift. Durham (2003) applies the simulated maximum-likelihood estimator of Durham and Gallant (2002) to compare different DTSMs. The results suggest that simpler drift specifications are preferable to more flexible forms and that the drift function appears to be constant. Li, Pearson and Poteshman (2001) implement a moment-based estimator that accounts for the bias described by Chapman and Pearson (2000) and find no evidence of a nonlinear drift.

In summary, the exact nature of drift for instantaneous interest rate is still inconclusive. Some evidence suggests that mean reversion is stronger in extreme levels of interest rates, while others fail to find strong evidence of nonlinearity.

#### 60.4.7.3 One Factor or Multiple Factors?

While many studies employ single-factor models to describe the interest rate behavior, others suggest using multi-factor models. In a single-factor DTSM, the whole term structure may be inferred from the level of one factor, which is traditionally taken to be the instantaneous interest rate. There are some intuitive reasons to criticize the single-factor DTSMs.[14] First, it implies that changes in term structure and hence bond returns are perfectly correlated across maturities, which can be easily rejected by empirical evidence. Second, it can only accommodate term structures that are monotonically increasing, decreasing or normally humped. An inversely humped or any other shape cannot be generated by single-factor DTSMs. Third, Dewachter and Maes (2000) compare single-factor versus multi-factor DTSMs and find that one factor time-variant parameter models provide a relatively poor fit to the actual term structure observed in the market. Empirically, Brown and Dybvig (1986) find that single-factor DTSMs understate the volatility of long-term yields. Brown and Schaefer (1994) show that the mean reversion coefficient required to explain the cross-maturity patterns at one time is inconsistent with the best fit coefficient.

Empirical research of the term structure models generally suggests that multi-factor DTSMs perform much better than single-factor DTSMs. Dai and Singleton (2000) show a substantial improvement in data fit offered by multi-factor DTSMs. Specifically, the changes in instantaneous interest rate may not only depend on the current level, but also on other factors which may be unobservable or observable. Dai and Singleton (2003) compare different DTSMs and find that: (i) the conditional volatilities of one-factor affine and quadratic DTSMs are affine and these models fail to capture the change of volatility, which suggest the need to use multi-factor DTSMs; (ii) In multi-factor DTSMs, the hump and inverted-hump of

[14] See also Maes (2004).

volatility could be realized by the negative correlation between state variables or the negative correlation between state variables and interest rate; and (iii) the two-factor model performs the best.

A number of studies have attempted to provide economic meanings for the factors included in the multi-factor models. These include Brennan and Schwartz (1979), Richard (1978), Longstaff and Schwartz (1992), Schaefer and Schwartz (1984), Andersen and Lund (1997), Balduzzi, Das, Foresi and Sundaram (1996), and Dewachter, Lyrio and Maes (2006).

#### 60.4.7.4 Affine or Nonlinear DTSMs?

The academic literature has focused on the affine DTSMs which are mainly due to the fact that this class of models yields closed-form bond pricing formulas and can easily handle the cases with multiple factors. However, as Dai and Singleton (2000) and Maes (2004) point out, the affine DTSMs are not able to ensure the positivity of interest rates without having to impose parameter restrictions and without loosing flexibility on the unconditional correlation structure among the state variables. Moreover, the affine DTSMs fail to capture the nonlinearities in the dynamics of interest rates, which are documented by Ait-Sahalia (1999), Boudoukh, Richardson, Stanton and Whitelaw (1998) and Balduzzi and Eom (2000).

Chan, Karolyi, Longstaff and Sanders (1992) compare different DTSMs using GMM. The results show that the models with volatility dependent on risk perform the best. Johannes (2004) examines some classical DTSMs and finds that these models fail to produce the distribution consistent with historical data. He then proposes the jump factor in the DTSM. Hong and Li (2005) propose a nonparametric specification method to test DTSMs. The results show that although significant improvements are achieved by introducing jumps and regime switching into the term structure of interest rate, all models are still rejected, implying that specification errors remain in these DTSMs. Duffee (2002) tests the affine DTSMs and finds that affine DTSMs forecast future yield changes poorly. Affine DTSMs cannot simultaneously match term structure movements and bond return premiums without modifying the dependence of the market price of interest rate risk on interest rate volatility.

#### 60.4.7.5 What do we really care about?

Perhaps there are two more fundamental questions than “how much we know the dynamics of short rates”. The first is do we really care the differences among these models? This question depends on whether different DTSMs have significantly different implications for their applications, such as pricing and risk management (for example the calculation of value-at-risk). Although more complicated models could capture some specific characteristics of underlying variables, the improvement for pricing and hedge may be limited.[15]

Second, we should bear in mind that only the DTSMs in the risk-neutral measure matter for pricing. The dynamics of instantaneous interest rate in the risk-neutral measure is different from that in the physical measure. For example, nonlinearity in the drift based on the physical measure need not imply the nonlinearity in the risk-neutral measure. It is the market price of risk for interest rate that connects the

[15] Bakshi, Cao and Chen (1997) compare different option pricing models and find that the improvement by some complicated models may be limited.

DTSMs in different measures (see, for example, Dai and Singleton (2000), Duffee (2002), Duarte (2004), Ahn and Gao (1999), Cheridito, Filipovi and Kimmel (2007) for different specifications of the market price of risk for interest rate). Therefore, we should be cautious when we attempt to infer the risk-neutral parameter values from the variables in the physical measure. As Dai and Singleton (2003) argue, it seems that having multiple factors in linear models is more important than introducing the nonlinearity into models with a smaller number of factors. Moreover, because of the computational demand of pricing in the presence of nonlinear drifts, attention now continues to focus primarily on DTSMs with a linear drift for state variables.

## 60.5 Models of Defaultable Bonds

Defaultable bonds are bonds whose payoff depends on the occurrence of default event (credit risk). Therefore, modeling the default probability (credit risk) is the key issue for pricing the defaultable bonds. Basically, there are two approaches to model the default: structural and reduced-form approaches.

### 60.5.1 Structural Models

Structural models are pioneered by Black and Scholes (1973) and Merton (1974) that regard the corporate bond as the derivative of firm value. In these models, default occurs at the maturity date of debt provided that the asset value is less than the face value of maturing debt. Default before maturity is not considered in both studies. Black and Cox (1976) propose the first passage time model that defines the default time as the first time the asset value falls below a boundary. Within this framework, default can occur before the maturity of debt. Geske (1977) introduces the coupon payment in the structural model and treats it as a compound option. On each coupon date, if shareholders decide to pay the coupon by selling new equity, the firm stays alive; otherwise, default occurs and bondholders seize firm assets. Leland and Toft (1996) consider the case that the firm continuously issues a constant amount of debt with a fixed maturity that pays continuous coupons. Similar to Geske (1977), the default boundary is endogenous since equity holders can decide whether or not to issue new equity to pay for the debt in case that the firm's payout is not large enough to cover the debt service requirements. Longstaff and Schwartz (1995) introduce interest rate risk described by the Vasicek (1977) model and provide a pricing formula for fixed coupon and floating coupon bonds. Collin-Dufresne and Goldstein (2001) extend the Longstaff and Schwartz (1995) model to account for a stationary leverage ratio. Zhou (1997) extends Merton's approach by modeling the firm's value process as a geometric jump-diffusion process. Anderson and Sundaresan (1996) and Mella-Barral and Perraudin (1997) introduce simplified bargaining games to obtain analytical expressions for the default boundaries in structural models. Duffie and Lando (2001) consider incomplete accounting information in structural models. Related structural models are also studied by Ho and Singer (1982), Titman and Torous (1989), Kim, Ramaswamy and Sundaresan (1993), Leland (1994), Fama and French (1996), Briys and de Varenne (1997), Cathcart and El-Jahel (1998), Goldstein, Ju and Leland (2001), Nielsen, Saá-Requejo and Santa-Clara (2001), Acharya and Carpenter (2002), and Vassalou and Xing (2004).

### 60.5.2 Reduced-form Models

Reduced-form models treat default time as the arrival time of a counting process with the associated intensity process. Jarrow and Turnbull (1995) model the default time as a Poisson process with constant intensity $\lambda$, i.e., the number of events occurring at any time interval $\Delta t$ follows the Poisson distribution with intensity $\lambda\Delta t$,

$$P\left\{N\left(t+\Delta t\right)-N\left(t\right)=n\right\}=\frac{\left(\lambda\Delta t\right)^{n}}{n!}e^{-\lambda\Delta t},n=0,1...,s,\Delta t\geq 0 \tag{60.5.1}$$

where *N(t)* is the number of events until *t*.

Duffie and Huang (1996), Jarrow, Lando and Turnbull (1997), Lando (1998), and Madan and Unal (1998) introduce the doubly stochastic and state dependent default intensity into the Jarrow-Turnbull model, which then becomes the benchmark specification for reduce-form models. The model is formalized as the following. Define the default time $\tau$ as a random variable between [0,*T*], and the probability of no default until time *t* (survival probability), $0\leq t\leq T$, is $S_t=P\left(\tau\geq t\right)$. The unconditional default probability between *t* and $t+\Delta t$ is $P\left(t<\tau\leq t+\Delta t\right)=S_t-S_{t+\Delta t}$. The conditional default probability between *t* and $t+\Delta t$ conditional on no default until *t* is

$$P\left(t<\tau\leq t+\Delta t\mid\tau\geq t\right)=\frac{S_t-S_{t+\Delta t}}{S_t} \tag{60.5.2}$$

The conditional default probability in unit time is defined as

$$\frac{P\left(t<\tau\leq t+\Delta t\mid\tau\geq t\right)}{\Delta t}=\frac{S_t-S_{t+\Delta t}}{S_t\Delta t} \tag{60.5.3}$$

The instantaneous conditional default probability (default intensity) $\lambda_t$ is defined as

$$\lambda_t=\lim_{\Delta t\to 0}\frac{P\left(t<\tau\leq t+\Delta t\mid\tau\geq t\right)}{\Delta t}=-\frac{S_t^{'}}{S_t} \tag{60.5.4}$$

with the initial condition that $S_0 = 1$,

$$S_t = \exp\left(-\int_0^t \lambda_s ds\right) \tag{60.5.5}$$

Then the price of a zero-coupon defaultable bond with face value 1 is given by

$$B(t,T) = E_t^Q\left[\exp\left(-\int_t^\tau r_u du\right)\omega_\tau 1_{\{\tau\le T\}}\right] + E_t^Q\left[\exp\left(-\int_t^T r_u du\right)1_{\{\tau>T\}}\right] \tag{60.5.6}$$

where $E_t^Q\left[\exp\left(-\int_t^\tau r_u du\right)\omega_\tau 1_{\{\tau\le T\}}\right]$ is the present value of the recovery upon default $\omega_\tau = \omega(Y_\tau,\tau)$ at the default arrival time $\tau$ whenever $\tau \le T$, and $E_t^Q\left[\exp\left(-\int_t^T r(u)du\right)1_{\{\tau>T\}}\right]$ is the present value of face value conditional on no default before maturity. Simplifying the pricing formula (see, e.g., Lando (1998)) gives the following result

$$B(t,T) = E_t^Q\left[\int_t^T \lambda_s\omega_s \exp\left(-\int_t^s (r_u + \lambda_u)du\right)ds\right] + E_t^Q\left[\exp\left(-\int_t^T (r_s + \lambda_s)ds\right)\right] \tag{60.5.7}$$

The pricing formula of defaultable bonds depends on three variables: recovery rate, default-free interest rates and the default intensity. Differences in the treatment of three factors differentiate each reduced-form model.

60.5.2.1 Recovery Rate

(1) Fractional Recovery of Par, Payable at Maturity

This recovery formulation refers to the case that in the event of default, a fraction $\omega$ of the face value is recovered but the payment is postponed until the maturity of defaultable bond. This in fact results in the following specification:

$$\omega_\tau = \omega D(\tau,T) \tag{60.5.8}$$

If $\omega$ is constant and state-independent,

$$B(t,T)=E_t^Q\left[\omega\int_t^T \lambda_s \exp\left(-\int_t^T r_u du-\int_t^s \lambda_u du\right)ds\right]+E_t^Q\left[\exp\left(-\int_t^T (r_s+\lambda_s)ds\right)\right] \quad (60.5.9)$$

which becomes

$$\begin{aligned}B(t,T)=\omega&\left[\int_t^T E_t^Q\left[\lambda_s \exp\left(-\int_t^s \lambda_u du\right)\right]E_t^Q\left(\exp\left(-\int_t^T r_u du\right)\right)ds\right]\\&+E_t^Q\left[\exp\left(-\int_t^T r_s ds\right)\right]E_t^Q\left[\exp\left(-\int_t^T \lambda_s ds\right)\right]\end{aligned} \quad (60.5.10)$$

if $r_t$ and $\lambda_t$ are independent. The Jarrow, Lando and Turnbull (1997) model is a special case of this class.

(2) Fractional Recovery of Par, Payment at Default

If a constant and state-independent fraction $\omega$ of the face value is recovered and paid at the time of default,

$$B(t,T)=E_t^Q\left[\omega\int_t^T \lambda_s \exp\left(-\int_t^s (r_u+\lambda_u)du\right)ds\right]+E_t^Q\left[\exp\left(-\int_t^T (r_s+\lambda_s)ds\right)\right] \quad (60.5.11)$$

which becomes

$$\begin{aligned}B(t,T)=\omega&\left[\int_t^T E_t^Q\left[\lambda_s \exp\left(-\int_t^s \lambda_u du\right)\right]E_t^Q\left(\exp\left(-\int_t^s r_u du\right)\right)ds\right]\\&+E_t^Q\left[\exp\left(-\int_t^T r_s ds\right)\right]E_t^Q\left[\exp\left(-\int_t^T \lambda_s ds\right)\right]\end{aligned} \quad (60.5.12)$$

if $r_t$ and $\lambda_t$ are independent. Examples of this class of models are Duffie (1998a), Duffie and Singleton (1999a), Longstaff, Mithal and Neis (2005), and Liu, Shi, Wang and Wu (2007).[16]

Due to the identification problem in obtaining separate estimates for the recovery rate and the default intensity, most empirical studies try to estimate the default intensity process from defaultable bond data using an exogenously given recovery rate. Houweling and Vorst (2005) find that under some specification of $\lambda$ and *r*, the

[16] There is another class of models that specify the recovery as a fraction of market value, see, Lando (1998), and Li (2000b).

value of recovery rate does not substantially affect the results if it lies within a logical interval.

### 60.5.2.2 Dynamics of Interest Rate and Default Intensity

Although many complicated DTSMs (for example, DTSMs with jump, and regime switching) are applicable to model the dynamics of interest rate and default intensity in the pricing of defaultable bonds, affine models are still the most favored framework due to the analytical tractability of these models. Consider the state vector $Y$ that follows an affine-jump diffusion process

$$dY_{jt} = \kappa_j\left(\mu_j - Y_{jt}\right)dt + \sigma_j\sqrt{Y_{jt}}dW_{jt}^Q + \Delta J_{jt}dZ_{jt},\ j = 1,\cdots,\ K \tag{60.5.13}$$

where $W_{jt}^Q$, $j = 1,\cdots,\ K$ are independent standard Brownian motions under the risk-neutral measure. $r_t$ and $\lambda_t$ are typically modeled by making them dependent on a set of common stochastic factors $Y$, which introduces stochasticity and correlation in the process of $r_t$ and $\lambda_t$. For example,

$$\begin{cases} r_t = a_r^0(t) + a_r^1(t)Y_{1t} + \ldots + a_r^K(t)Y_{Kt} \\ \lambda_t = a_\lambda^0(t) + a_\lambda^1(t)Y_{1t} + \ldots + a_\lambda^K(t)Y_{Kt} \end{cases} \tag{60.5.14}$$

Duffie and Singleton (2003) formulate an intensity process as a mean-reverting process with jumps. The default intensity between jump events is given by:

$$\frac{d\lambda_t}{dt} = \kappa\left(\gamma - \lambda_t\right) \tag{60.5.15}$$

Thus, at any time t between two jumps,

$$\lambda_t = \gamma + e^{-\kappa(t-T)}\left(\lambda_T - \gamma\right) \tag{60.5.16}$$

where $T$ is the time of last jump and $\lambda_T$ is the jump intensity at time $T$. Suppose that jumps occur at Poisson arrival time with an intensity $c$ and that jump sizes are exponentially distributed with mean $J$, Duffie and Singleton (2003) show that the conditional survival probability from $t$ to $s$ is:

$$P(\tau > s \mid \tau \geq t) = e^{\alpha(s-t)+\beta(s-t)\lambda_t} \tag{60.5.17}$$

where

$$\begin{cases} \beta(\tau) = -\dfrac{1-e^{-\kappa\tau}}{\kappa} \\ \alpha(\tau) = -\gamma\left(\tau - \dfrac{1-e^{-\kappa\tau}}{\kappa}\right) - \dfrac{c}{J+\kappa}\left[Jt - \ln\left(1+\dfrac{1-e^{-\kappa\tau}}{\kappa}J\right)\right] \end{cases} \tag{60.5.18}$$

Duffie and Singleton (1999a) model $r_t$ and $\lambda_t$ as

$$\begin{cases} r_t = \rho_0 - Y_{0t} + Y_{1t} + Y_{2t} \\ \lambda_t = bY_{0t} + Y_{3t} \end{cases} \tag{60.5.19}$$

where $Y_{0t}$, $Y_{1t}$, $Y_{2t}$ and $Y_{3t}$ are independent CIR (square-root) processes under the risk-neutral measure, and $\rho_0$ and b are constants. The degree of negative correlations between $r_t$ and $\lambda_t$ is controlled by the choice of b.[17]

Liu, Shi, Wang and Wu (2007) model $r_t$ and $\lambda_t$ as:

$$\begin{cases} \lambda_t = \lambda_t^* + \beta(r_t - \bar{r}) \\ dr_t = \kappa_r(\mu_r - r_t)dt + \sigma_r\sqrt{r_t}dW_{1t}^Q \\ d\lambda_t^* = \kappa_\lambda(\mu_\lambda - \lambda_t^*)dt + \sigma_\lambda\sqrt{\lambda_t^*}dW_{2t}^Q \end{cases} \tag{60.5.20}$$

where $W_{1t}^Q$, $W_{2t}^Q$ are two independent standard Brownian motions under the risk-neutral measure. The degree of negative correlations between $r_t$ and $\lambda_t$ is controlled by the choice of $\beta$. On the other hand, Duffie (1998b), Bielecki and Rutkowski (2000, 2004) apply the spread forward rate and price the zero-coupon defautable bond as

$$B(t,T) = \exp\left(-\int_t^T \left(f(t,v) + s(t,v)\right)dv\right) \tag{60.5.21}$$

where $f(t,v)$ is the default-free forward rate, and $s(t,v)$ is the spread forward

[17] Chen, Cheng, Fabozzi and Liu (2006) also propose a pricing model with correlated factors.

rate.

### 60.5.2.3 Coupon

It is quite natural to extend the pricing of zero-coupon defautable bonds to coupon bonds. Assuming that the coupon $C$ is paid continuously and the recovery is a fraction of the par value which is paid at default time, the price of coupon defaultable bond is given by

$$\begin{aligned} B(C,t,T) = {} & E_t^Q\left(C\int_t^T \exp\left(\int_t^s -(r_u+\lambda_u)du\right)ds\right) + E_t^Q\left[\exp\left(-\int_t^T (r_s+\lambda_s)ds\right)\right] \\ & + E_t^Q\left[\omega\int_t^T \lambda_s \exp\left(-\int_t^s (r_u+\lambda_u)du\right)ds\right] \end{aligned} \tag{60.5.22}$$

If $r_t$ and $\lambda_t$ are assumed to be independent, it can be further simplified to

$$\begin{aligned} B(C,t,T) = {} & \left(C\int_t^T E_t^Q\left[\exp\left(-\int_t^s r_u du\right)\right] E_t^Q\left[\exp\left(-\int_t^s \lambda_u du\right)\right] ds\right) \\ & + E_t^Q\left[\exp\left(-\int_t^T r_s ds\right)\right] E_t^Q\left[\exp\left(-\int_t^T \lambda_s ds\right)\right] \\ & + \omega\left[\int_t^T E_t^Q\left[\exp\left(-\int_t^s r_u du\right)\right] E_t^Q\left[\lambda_s \exp\left(-\int_t^s \lambda_u du\right)\right] ds\right] \end{aligned} \tag{60.5.23}$$

The case of discrete coupon payments and the fractional recovery of par value paid at maturity can be derived in a similar way.

### 60.5.2.4 Liquidity and Taxes

Standard term structure models of default risk assume that yield spreads between corporate (defaultable) bonds and government (default-free) bonds are determined by two factors: default risk ($\lambda_t$) and the expected loss ($1-\omega$) in the event of default. However, recent studies have shown that other factors, such as liquidity and taxes can significantly affect corporate bond yield spreads.[18]

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[18] See Elton, Gruber, Agrawal and Mann (2001), Longstaff, Mithal and Neis (2005), and Liu, Shi, Wang and Wu (2007).

Using a reduced-from approach, Longstaff, Mithal and Neis (2005) introduce the liquidity factor into the defaultable bond pricing formula and obtain a closed-form solution for corporate bond price with default and liquidity,

$$\begin{aligned} B(C,t,T) = & E_t^Q\left[C\int_t^T \exp\left(-\int_t^s (r_u+\lambda_u+l_u)\,du\right)ds\right] \\ & +E_t^Q\left[\exp\left(-\int_t^T (r_s+\lambda_s+l_s)\,ds\right)\right] \\ & +E_t^Q\left[\omega\int_t^T \lambda_s \exp\left(-\int_t^s (r_u+\lambda_u+l_u)\,du\right)ds\right] \end{aligned} \tag{60.5.24}$$

where $r_t$, $\lambda_t$ and $l_t$ denote the default free interest rate, default intensity and liquidity intensity at time *t*, respectively. Their dynamics follow

$$\begin{cases} d\lambda_t = \kappa_\lambda(\mu_\lambda - \lambda_t)dt + \sigma_\lambda\sqrt{\lambda_t}\,dW_{1t}^Q \\ dl_t = \sigma_l dW_{2t}^Q \end{cases} \tag{60.5.25}$$

where $W_{1t}^Q$ and $W_{2t}^Q$ are independent standard Brownian motions under the risk-neutral measure.

The tax effect, on the other hand, is much more complicated due to changes in tax rate and differential tax treatments of capital gain (loss) in discount (premium) bonds. For example, for the discount bonds, when there is no default, the difference between the face value and the price is regarded as the capital gain and should be taxed by capital gain tax rate. When there is a default before maturity, the investor expects a capital loss and receives a tax rebate from the government. Moreover, the premium and discount of corporate bonds must be amortized, which makes the pricing more complicated. Liu, Shi, Wang and Wu (2007) deal with these issues by assuming a buy-and-hold strategy of bond and obtain the pricing formula for corporate bond with taxes. The price of discount defaultable coupon bond without amortization is given by.[19]

$$
\begin{aligned}
B(t,t_M) = {} & E_t^Q\left[C(1-\tau_i)\sum_{m=1}^{M}\exp\left(-\int_t^{t_m}(r_s+\lambda_s)ds\right)\right] \\
& +E_t^Q\left[\left(1-\tau_g\left(1-B(t,t_M)\right)\right)\exp\left(-\int_t^{t_M}(r_s+\lambda_s)ds\right)\right] \\
& +E_t^Q\left[\int_t^{t_M}\left(\omega+\tau_g\left(B(t,t_M)-\omega\right)\right)\lambda_s\exp\left(-\int_t^{s}(r_u+\lambda_u)du\right)ds\right]
\end{aligned}
\tag{60.5.26}
$$

where $\tau_i$ is the income tax rate, $\tau_g$ is the capital tax rate, $t_i$, $i=1,...,M$ is the discrete time of coupon payment. After a simple transformation,

$$
\begin{aligned}
B(t,t_M) = {} & \frac{1}{Z}E_t^Q\left[C(1-\tau_i)\sum_{m=1}^{M}\exp\left(-\int_t^{t_m}(r_s+\lambda_s)ds\right)\right] \\
& +E_t^Q\left[(1-\tau_g)\exp\left(-\int_t^{t_M}(r_s+\lambda_s)ds\right)\right] \\
& +E_t^Q\left[\int_t^{t_M}(1-\tau_g)\omega\lambda_s\exp\left(-\int_t^{s}(r_u+\lambda_u)du\right)ds\right]
\end{aligned}
\tag{60.5.27}
$$

and $Z=1-\tau_g E_t^Q\left\{\left[\exp\left(-\int_t^{t_M}(r_s+\lambda_s)ds\right)\right]+\left[\int_t^{t_M}\lambda_s\exp\left(-\int_t^{s}(r_u+\lambda_u)du\right)ds\right]\right\}$

In a recent paper, Lin, Liu and Wu (2007) propose a generalized defaultable bond pricing model with default, liquidity and tax. The price of discount defaultable coupon bond without amortization is given by

$$
\begin{aligned}
B(t,t_M) = {} & \frac{1}{Z}E_t^Q\left[C(1-\tau_i)\sum_{m=1}^{M}\exp\left(-\int_t^{t_m}(r_s+\lambda_s+l_s)ds\right)\right] \\
& +E_t^Q\left[(1-\tau_g)\exp\left(-\int_t^{t_M}(r_s+\lambda_s+l_s)ds\right)\right] \\
& +E_t^Q\left[\int_t^{t_M}(1-\tau_g)\omega\lambda_s\exp\left(-\int_t^{s}(r_u+\lambda_u+l_u)du\right)ds\right]
\end{aligned}
\tag{60.5.28}
$$

and

$$
Z=1-\tau_g E_t^Q\left\{\left[\exp\left(-\int_t^{t_M}(r_s+\lambda_s+l_s)ds\right)\right]+\left[\int_t^{t_M}\lambda_s\exp\left(-\int_t^{s}(r_u+\lambda_u+l_u)du\right)ds\right]\right\}.
$$

Under the similar assumptions of $r_t$, $\lambda_t$ and $l_t$ as in Longstaff, Mithal and Neis

[19] For the case of premium bond and amortization, see Liu, Shi, Wang and Wu (2007).

(2005), Lin, Liu and Wu (2007) derive a closed-form solution for defaultable coupon bond with default, liquidity and tax.

#### 60.5.2.5 Correlated Default

Default correlation is an important issue when dealing with the default or survival probability of more than one firm. Schonbucher (2003) lays out some basic properties to model the correlated defaults. First, the model must be able to produce default correlations of a realistic magnitude. Second, it must keep the number of parameters used to describe the dependence structure under control, without growing dramatically with the number of firms. Third, it should be a dynamic model, capable of modeling the number of defaults as well as the timing of default. Fourth, it should be able to reproduce the periods with default clustering. Finally, the easier calibration and implementation of the model, the better.

Consider two firms A and B that have not defaulted before time t ($0 \le t \le T$), whose default probabilities before $T$ are given by $p_A$ and $p_B$. Then the linear correlation coefficient[20] between the default indicator random variables $1_A = 1_{\{\tau_A \le T\}}$ and $1_B = 1_{\{\tau_B \le T\}}$ is given by[21]

$$\rho(1_A, 1_B) = \frac{p_{AB} - p_A p_B}{\sqrt{p_A(1-p_A)p_B(1-p_B)}} \qquad (60.5.29)$$

There are three different approaches to model default correlation in the literature.[22] We describe these approaches below.

[20] This correlation is based on risk-neutral measure which is different from physical measure used to compute the correlation from empirical default events. Jarrow, Lando and Yu (2005), Yu (2002, 2003) provide a procedure for calculating physical default correlation through risk-neutral densities.

[21] Another measure of default dependence between firms is the linear correlation between default time, $\rho(\tau_A, \tau_B)$.

[22] See also Elizalde (2003).

(1) Conditionally Independent Default (CID) Models

The conditionally independent default approach introduces correlation by making them dependent on a set of common variables $Y$ and on a firm-specific factor $\lambda_{it}^{*}$. Suppose there are *I* firms, and the default intensity for firm *i*, *i* = 1,⋯, *I*, is

$$\lambda_t^i = a_{\lambda_i}^0 + a_{\lambda_i}^1 Y_{1t} + \ldots + a_{\lambda_i}^K Y_{Kt} + \lambda_{it}^{*} \tag{60.5.30}$$

where the firm-specific default factor $\lambda_{it}^{*}$ is independent across firms.

For example, Duffee (1999) considers a CID model as follows:

$$\begin{cases} r_t = a_r^0 + Y_{1t} + Y_{2t} \\ \lambda_t^i = a_{\lambda_i}^0 + a_{\lambda_i}^1 \left( Y_{1t} - \bar{Y}_{1t} \right) + a_{\lambda_i}^2 \left( Y_{2t} - \bar{Y}_{2t} \right) + \lambda_{it}^{*} \\ d\lambda_{it}^{*} = \kappa_i \left( \mu_i - \lambda_{it}^{*} \right) dt + \sigma_i \sqrt{\lambda_{it}^{*}} dW_{it} \end{cases} \tag{60.5.31}$$

In this model, $\lambda^i$ captures the stochasticity of intensities and the coefficients $a_{\lambda_i}^1$ and $a_{\lambda_i}^2$, $i = 1,\ldots,I$, capture the correlation between intensities themselves, and between intensities and interest rates.

The main drawback of the CID models is that they fail to generate high default correlation. However, Yu (2003) argues that it is not a problem of the CID approach itself but a problem of the choice of state variables. Driessen (2005) introduces two more common factors for Duffee's (1999) model and finds that they elevate the default correlation.[23] The risk-neutral default density of firm *i, i=1, 2, ⋯, I* is a function of *K* common factors, a firm-specific factor and two more factors that determine the risk free rates,

[23] There are two possible ways to deal with the high default correlation issue. One way is to introduce joint jumps in the default intensities (Duffie and Singleton (1999b)). The other way is to consider the default-event triggers that cause joint defaults (Duffie and Singleton (1999b), Kijima (2000), and Kijima and Muromachi (2000)).

$$\begin{cases} \lambda_t^i = a_{\lambda_i}^0 + \sum_{j=1}^{K} a_{\lambda_i}^j \left(Y_{jt} - \overline{Y}_{jt}\right) + \left(\lambda_{it}^* - \overline{\lambda}_{it}^*\right) + b_{i1}\left(X_{1t} - \overline{X}_{1t}\right) + b_{i2} X_{2t},\ i = 1, 2, \ldots, N \\ r_t = a_r^0 + a_r^1 X_{1t} + a_r^2 X_{2t} \end{cases} \tag{60.5.32}$$

where $Y_{jt}$, $j=1, 2, \cdots, K$ and $\lambda_{it}^*$ follow the independent square-root process, and the two additional terms $X_{1t}$ and $X_{2t}$ follow a bivariate process,

$$\begin{bmatrix} dX_{1t} \\ dX_{2t} \end{bmatrix} = \begin{bmatrix} \kappa_{11} & 0 \\ \kappa_{21} & \kappa_{22} \end{bmatrix} \begin{bmatrix} \theta_1 - X_{1t} \\ -X_{2t} \end{bmatrix} dt + \begin{bmatrix} \sqrt{X_{1t}} & 0 \\ 0 & \sqrt{1 + \beta X_{1t}} \end{bmatrix} \begin{bmatrix} dW_{1t} \\ dW_{2t} \end{bmatrix} \tag{60.5.33}$$

where $W_{1t}$ and $W_{2t}$ are independent standard Brownian motions under the physical measure. The introduction of two more terms could allow for the correlation between spreads and risk free rates.

(2) Contagion Model

Contagion models account for two more empirical facts: (i) the default of one firm can trigger the default of other related firms; (ii) the default times tend to concentrate in certain periods of time. It includes the propensity model proposed by Jarrow and Yu (2001) and infectious defaults in Davis and Lo (2001).

Jarrow and Yu (2001) extend the CID model to account for counterparty risk, i.e., the risk that the default of one firm may increase the default probability of other related firms. They differentiate the total *I* firms into primary firms (1,⋯,*K*) and secondary firms (*K*+1,⋯,*I*). The default intensities of the primary firms are modeled using CID and do not depend on the default status of any other firm. The default of a primary firm increases the default intensities of secondary firms, but not the converse (asymmetric dependence). Thus, the secondary firms' default intensities are given by

$$\lambda_t^i = \overline{\lambda}_t^i + \sum_{j=1}^{K} a_{i,t}^j 1_{\{\tau_j \le t\}} \tag{60.5.34}$$

for $i = K+1,...,I$ and $j = 1,...,K$.

Davis and Lo (2001) assume that each firm has an initial $\lambda_t^i$ for $i = 1,...,I$. When a default occurs, the default intensity of all remaining firms is increased by a factor $a > 1$, called the enhancement factor, to $a\lambda_t^i$.

(3) Copula

The copula approach takes the marginal probabilities as input and introduces the dependence structure to generate joint probabilities. Specifically, if we want to model the default time, the joint default probabilities are given by

$$\mathbf{F}(t_1,...,t_I) = \mathbf{P}[\tau_1 \le t_1,...,\tau_I \le t_I] = \mathbf{C^d}(F_1(t_1),...,F_I(t_I)) \tag{60.5.35}$$

If we want to model the survival times,

$$\mathbf{S}(t_1,...,t_I) = \mathbf{P}[\tau_1 > t_1,...,\tau_I > t_I] = \mathbf{C^s}(s_1(t_1),...,s_I(t_I)) \tag{60.5.36}$$

where $\mathbf{C^d}$ and $\mathbf{C^s}$ are two different copulas[24].

Examples of copulas are:

(i) Independent Copula. The *I*-dimensional independent copula is given by

$$C(u_1,...,u_I) = \prod_{i=1}^{I} u_i \tag{60.5.37}$$

(ii) Perfect Correlation Copula. The *I*-dimensional perfect correlation copula is given by

$$C(u_1,...,u_I) = \min(u_1,...,u_I) \tag{60.5.38}$$

(iii) Normal Copula. The *I*-dimensional normal copula with correlation matrix $\Sigma$ is given by

[24] For a more detailed description of the copula theory, please refer to Joe (1997), Frees and Valdez (1998), Costinot, Roncalli and Teiletche (2000), and Embrechts, Lindskog and McNeil (2001).

$$\mathbf{C}(u_1,...,u_I)=\Phi_{\Sigma}^{I}\left(\Phi^{-1}(u_1),...,\Phi^{-1}(u_I)\right) \tag{60.5.39}$$

where $\Phi_{\Sigma}^{I}$ represents an *I*-dimensional normal distribution function with a covariance matrix $\Sigma$, and $\Phi^{-1}$ denotes the inverse of the univariate standard normal distribution function.

(iv) *t* Copula. Let X be an random vector distributed as an *I*-dimensional multivariate *t*-student with *v* degrees of freedom, mean vector $\mu$ (for v>1) and covariance matrix $\frac{v}{v-2}\Sigma$ (for v > 2). We could then express X as

$$X=\mu+\frac{\sqrt{v}}{\sqrt{S}}Z \tag{60.5.40}$$

where *S* is a random variable distributed as a $\chi^2$ with *v* degrees of freedom and *Z* is an *I*-dimensional normal random vector that is independent of *S* with zero mean and linear correlation matrix $\Sigma$. The *I*-dimensional *t*-copula of *X* could be expressed as

$$\mathbf{C}(u_1,...,u_I)=t_{v,R}^{I}\left(t_v^{-1}(u_1),...,t_v^{-1}(u_I)\right) \tag{60.5.41}$$

where $t_{v,R}^{I}$ represents the distribution function of $\frac{\sqrt{v}}{\sqrt{S}}Z$, $Z$ is an *I*-dimensional normal random vector which is independent of *S* with mean zero and covariance matrix *R*. $t_v^{-1}$ denotes the inverse of the univariate *t*-student distribution function with *v* degrees of freedom and $R_{ij}=\frac{\Sigma_{ij}}{\sqrt{\Sigma_{ii}\Sigma_{jj}}}$.

(v) Archimedean Copulas. An *I*-dimensional Archimedean copula function is represented by

$$\mathbf{C}(u_1,...,u_I)=\phi^{-1}\left(\phi(u_1)+...+\phi(u_I)\right) \tag{60.5.42}$$

where the function $\phi:[0,1]\rightarrow R^{+}$, called the generator of the copula, is invertible and satisfies the conditions that $\phi'(u)<0$, $\phi''(u)>0$, $\phi(1)=0$, and $\phi(0)=\infty$.

Examples of generator functions are:

Clayton: $\phi(u)=\dfrac{u^{-\theta}-1}{\theta}$, for $\theta\ge 0$

Frank: $\phi(u)=-\ln\dfrac{e^{-\theta u}-1}{e^{-\theta}-1}$, for $\theta\in R\setminus\{0\}$

Gumbel: $\phi(u)=(-\ln u)^{\theta}$, for $\theta\ge 1$

Product: $\phi(u)=-\ln u$

Studies incorporating copulas into the reduced-form approach to account for default dependence include Li (2000a), Schonbucher and Schubert (2001), and Frey and McNeil (2001).

### 60.5.3 Empirical Issues

#### 60.5.3.1 The Components of Yield Spread

Understanding the determinants of corporate bond spreads is important for both academics and practitioners. For academics, valuation of corporate bonds requires a pricing model that incorporates all relevant factors. From the investment perspective, investors need to know the required premia of default, liquidity, and taxes in order to be compensated properly for the risk and tax burden of holding corporate bonds. Furthermore, from the corporate finance perspective, understanding the components of the corporate yield spread aids in capital structure decisions as well as determination of timing and maturity of debt and equity issuances.

A vast literature has been devoted to studies of determinants of corporate bond spreads. This includes, among others, Jones, Mason and Rosenfeld (1984), Duffee (1999), Duffie and Singleton (1999a), Elton, Gruber, Agrawal and Mann (2001), Collin-Dufresne, Goldstein and Martin (2001), Huang and Huang (2003), Eom, Helwege and Huang (2004), De Jong and Driessen (2004), and Ericsson and Renault (2006). These studies reported mixed results for the component of corporate bond yield spreads. Jones, Mason and Rosenfeld (1984) apply Merton model to a sample of firms with simple capital structures and secondary market prices during 1977-1981 period and find that the predicted prices (yields) from the model are too high (low). Ogden (1987) finds that the Merton model underpredicts spreads by 104 basis points on average. Lyden and Saraniti (2000) compare the Merton model and the Lonstaff and Schwartz (1995) model and find that both models underestimate the yield spreads. Huang and Huang (2003) evaluate the performance of most popular term structure models and report estimates of default risk premia substantially below the corporate bond spread. Collin-Dufresne, Goldstein and Martin (2001) analyze the effect of key financial variables suggested by structural models on corporate bond spreads and find that these variables explain only a small portion of variations in spreads. Sarig and Warga (1989), Helwege and Turner (1999) examine the shape of credit term structure, while Duffee (1999) and Brown (2001) test the correlation between interest rates and spreads. Most of these empirical studies conclude that term structure models underpredict yield spreads.

On the other hand, Eom, Helwege and Huang (2004) empirically test five structural models, i.e., the Merton model, Geske (1977) model, Longstaff and Schwartz (1995) model, Leland and Toft (1996) model and Collin-Dufresne, Goldstein and Martin (2001) model. The results show that term structure models can over or under estimate corporate bond spreads and prediction errors are high.

Besides default risk, liquidity and tax are two additional factors that affect corporate bond yield spreads. One of the biggest challenges in term structure models is to estimate the liquidity premium of corporate bond spreads. Empirical estimation of liquidity premium is difficult because liquidity is unobservable and bond prices only reflect the combined effects of liquidity and default risk. Thus, the liquidity and default premia cannot be separately identified from the data on term structure of corporate bond prices alone. Longstaff, Mithal and Neis (2005) overcome this identification problem by using additional information from the credit default swap. They find that the majority of the corporate bond spread is due to default risk and the nondefault component is time varying and strongly related to the measures of bond-specific illiquidity as well as macroeconomic measures of bond market liquidity. Specifically, when the Treasury curve is used as the default free discount function, the average size of default component is 51% for AAA/AA bonds, 56% for A bonds, 71% for BBB bonds and 83% for BB bonds.

Yawitz, Maloney and Ederington (1985) estimate nonlinear models of corporate and municipal bonds with default risk and taxes. Elton, Gruber, Agrawal and Mann (2001), Liu, Shi, Wang and Wu (2007) show that the tax premium accounts for a significant portion of corporate bond spreads. Elton, Gruber, Agrawal and Mann (2001) assume a tax rate equal to the typical statutory state income tax. Liu, Shi, Wang and Wu (2007) propose a pricing model that accounts for stochastic default probability and differential tax treatments for discount and premium bonds. By estimating parameters directly from bond data, they obtain a significantly positive income tax rate of marginal investor after 1986. Empirical evidence shows that taxes explain a substantial portion of observed spreads. Taxes on average account for 60%, 50% and 37% of the observed corporate-Treasury yield spreads for AA, A and BBB bonds, respectively.

Lin, Liu and Wu (2007) further account for stochastic default probability, liquidity, and differential tax treatments for discount and premium bonds in the pricing model. The model provides more precise estimates of the tax and liquidity components of spreads. They find that a substantial portion of the corporate yield spread is due to taxes and liquidity. The liquidity component in the spread is highly correlated with bond-specific and market-wide liquidity measures whereas the tax component is insensitive to these liquidity measures. On average, 51% of corporate yield spread is attributable to the default component, 32% to the tax component, and 17% to the liquidity component. The default component represents 39% of the spread for AAA/AA bonds, 46% for A bonds, 60% for BBB bonds and 73% for BB bonds. The tax component explains 39% of the spread for AAA/AA bonds, 36% for A bonds, 25% for BBB bonds and 16% for BB bonds. The liquidity component accounts for 21% of the spread for AAA/AA bonds, 18% for A bonds, 15% for BBB bonds and 11% for BB bonds.

Berndt, Lookman and Obreja (2006) investigate the source for common variation in U.S. credit markets that is not related to changes in riskfree rates or expected default losses. They extract a latent common component from firm-specific changes in default risk premium, named as “default risk premium (DRP) factor”, and find that its change is priced in the corporate bond market. The DRP factor could explain a maximum 35% of the credit market returns. Moreover, the DRP factor also captures the jump-to-default risk associated with market-wide credit events.

60.5.3.2 State Variables: Latent or Observable

Standard term structure models specify the default intensity as the function of latent variables which are unobservable and follow some diffusion processes. By contrast, some empirical studies specify the default intensity as the function of observable state variables in the reduced-form models. For example, Bakshi, Madan and Zhang (2006) model the aggregate defaultable discount rate $R_t = r_t + (1-\omega)\lambda_t$ as:

$$R_t = \Lambda_0 + \Lambda_r r_t + \Lambda_Y Y_t \tag{60.6.1}$$

where $Y_t$ denotes the firm-specific distress index. Bakshi, Madan and Zhang (2006) consider leverage, book-to-market, profitability, equity-volatility, and distance-to-default to be the firm specific distress variables and show that interest rate risk is of the first-order importance for explaining variations in single-name defaultable bond yields. When applying to low-grade bonds, a credit risk model that takes leverage into consideration reduces absolute yield mispricing by as much as 30%.

Janosi, Jarrow and Yildirim (2002) assume that

$$R_t = a_0 + a_1 r_t + a_2 Z_t \tag{60.6.2}$$

where $Z_t$ is a standard Brownian motion driving the S&P500 index.

Chava and Jarrow (2004) estimate a reduced-form model with accounting and market variables using historical bankruptcy data. In their model, the default correlation can be computed directly from physical intensity rather than those transformed from the risk-neutral intensity estimated from credit spreads. Using this model, one can also estimate an affine model with latent variables from bankruptcy, which better captures the common variations in default rates and may lead to more accurate default correlation estimates.

## 60.6 Interest Rate and Credit Default Swaps

### 60.6.1 Valuation of Interest Rate Swap

An interest rate swap is an agreement between two parties (known as counterparties) where one stream of future interest payments is exchanged for another based on a specified principal amount. Interest rate swaps often exchange a fixed payment for a floating payment that is linked to an interest rate (most often the LIBOR). With the interest rate swap, a company agrees to pay cash flows equal to interest at a predetermined fixed rate on a notional principal for a number of years. In return, it receives interest at a floating rate on the same notional principal for the same period of time. Interest rate swaps are simply the exchange of one set of cash flows (based on interest rate specifications) for another. Swaps are contracts set up between two or more parties, and can be customized in many different ways.

When an interest rate swap is first initiated, it is generally a plain vanilla and its value is zero. However, it could be positive or negative after time goes on. Similar to the pricing of corporate bonds, there are two approaches to value the interest rate swap: the structural approach and the reduced-form approach.[25] Structural models such as Cooper and Mello (1991) and Li (1998) uses Merton's (1974) approach to price the interest rate swap. Models developed more recently adopt the reduced-form approach, which regards the swap as the difference between two bonds and focuses on the rate used to discount the future cash flows of the interest rate swap. Studies using the reduced-form models of interest rate swaps include, among others, Duffie and Huang (1996), Duffie and Singleton (1997), Gupta and Subrahmanyam (2000), Collin-Dufresne and Solnik (2001), Grinblatt (2001), Liu, Longstaff and Mandell (2004), and Li (2006). In what follows, we focus on the literature on the reduced-form approach.

Consider a plain vanilla fixed-for-floating swap with maturity $\tau$ and the nominal principal equal to 1. The floating side is reset semi-annually to the six-month LIBOR rate from six months prior. The fixed side pays a coupon rate $c$ at the reset dates. Let $r_t^L$ be the LIBOR rate set at date $t$ for loans maturing six months later. From the standpoint of the floating-rate payer, an interest rate swap could be regarded as a long position in a fixed rate bond and a short position in a floating-rate bond. The fixed-side coupon rate $c$ is set at initial date $t$ so that the present value of expected net cash flows from the long and short positions is zero at the initial date, that is,

$$0 = \sum_{j=1}^{2\tau} E_t^Q \left[ \exp\left( -\int_t^{t+0.5j} R_s ds \right) \left( c - r_{t+0.5(j-1)}^L \right) \right] \qquad (60.6.3)$$

Then

[25] In practice, there is another approach for valuing the interest rate swap as a series of forward rate agreement (FRAs), see Hull (2006).

$$c=\frac{\sum_{j=1}^{2\tau}E_t^Q\left[\exp\left(-\int_t^{t+0.5j}R_s ds\right)r_{t+0.5(j-1)}^L\right]}{\sum_{j=1}^{2\tau}E_t^Q\left[\exp\left(-\int_t^{t+0.5j}R_s ds\right)_t\right]} \tag{60.6.4}$$

where $R_s$ are the discount rates of the cash flows. With the assumption that risky zero-coupon bonds are priced at the appropriate LIBOR rate in the interbank lending market, Duffie and Singleton (1997) show that

$$c=\frac{1-B_t^\tau}{\sum_{j=1}^{2\tau}B_t^{0.5j}} \tag{60.6.5}$$

with

$$B_t^m=E_t^Q\left[\exp\left(-\int_t^{t+m}R_s ds\right)\right] \tag{60.6.6}$$

Here $B_t^m$ means the present value of 1 dollar payable at time *t+m*. For the floating rate payer, the floating rate swap with a face value of 1 at time $t+\tau$ is equivalent to a floating rate bond which has the value at par at the initial date. Thus, the value of the floating side of the swap is 1 minus the present value of 1 dollar payable at time t+$\tau$ , $B_t^\tau$ .

Duffie and Singleton (1997) suggest that $R_t$ could also be interpreted as a default-adjusted discount rate. If the recovery rate is $\omega_t$ and the default intensity is $\lambda_t$, then

$$R_t=r_t+(1-\omega_t)\lambda_t \tag{60.6.7}$$

If the relative liquidities of interest rate swap and Treasury market are also considered,

$$R_t=r_t+(1-\omega_t)\lambda_t-l_t \tag{60.6.8}$$

where $l_t$ is a convenience yield that accounts for the effect of differences in liquidity and repo specialness between the Treasury and the swap market.[26]

Duffie and Huang (1996) change $R_t$ to account for the counterparties' asymmetric default risks. The economic intuition is clear. Suppose at any given time *t*, the current market value of the swap with no default is $V_t$ for party A, which could be positive or negative. On the other hand, the value is $-V_t$ for party B. If $V_t > 0$, then party A is at risk to the default of party B between *t* and *t+1*. Thus, under the risk-neutral measure, $V_t$ equals the default probability of party B between *t* and *t+1* multiplied by the recovery value, plus the survival probability of party B between *t* and *t+1*, multiplied by the market value given no default, which is the risk-neutral expected present value of receiving $V_{t+1}$ at *t+1*, plus any interest paid to A by B between *t* and *t+1*. If $V_t < 0$, this recursive method is the same, except for the fact that now B is at risk to default of A, so the default probability and recovery rate are those of A. This could be given mathematically by

$$R_{v,t} = r_t + s_t^A 1_{\{v<0\}} + s_t^B 1_{\{v\geq 0\}} \tag{60.6.9}$$

with

$$s_t^i = \left(1 - \omega_t^i\right)\lambda_t^i \tag{60.6.10}$$

Liu, Longstaff and Mandell (2004) use a five-factor affine framework to model the swap spread:

$$\begin{cases} r_t = \delta_0 + Y_{1t} + Y_{2t} + Y_{3t} \\ \lambda_t = \delta_2 + \gamma r_t + Y_{5t} \\ l_t = \delta_1 + Y_{4t} \end{cases} \tag{60.6.11}$$

where $\delta_0$, $\delta_1$, $\delta_2$ and $\gamma$ are constants, $\left[Y_1, Y_2, Y_3, Y_4, Y_5\right]$ are five state variables

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[26] See, for example Grinblatt (2001).

with dynamics in risk-neutral measures following

$$dY_t = -\beta Y_t dt + \Sigma dW_t^Q \tag{60.6.12}$$

Li (2006) proposes a similar reduced form model of interest rate swaps:

$$\begin{cases} r_t = \delta_0 + Y_{1t} + Y_{2t} + Y_{3t} \\ \lambda_t = \delta_1 r_t + Y_{4t} \\ l_t = \delta_2 r_t + Y_{5t} \\ dY_{it} = \kappa_i \left( \mu_i - Y_{it} \right) dt + \sigma_i \sqrt{Y_{it}}\, dW_{it}^Q,\ i = 1,2,3,4 \\ dY_{5t} = \kappa_5 \left( \mu_5 - Y_{5t} \right) dt + \sigma_5 dW_{5t}^Q \end{cases} \tag{60.6.13}$$

where $W_{it}^Q$ are independent standard Brownian motions under the risk-neutral measure.

### 60.6.2 Valuation of Credit Default Swaps

Credit derivatives have emerged as a remarkable and rapidly growing area in global derivatives and risk management practice which have been perhaps the most significant and successful financial innovation of the last decade. The growth of the global credit derivatives market continues to exceed expectations. According to BBA (British Bankers' Association) Credit Derivatives Report 2006, the outstanding notional amount of the market will reach $33 trillion at the end of 2008. Single-name credit default swaps (CDS) represent a substantial proportion of the market. CDS are the most liquid products among the credit derivatives currently traded which make up the bulk of trading volume in credit derivatives markets. Moreover, CDS along with total return swaps and credit spread options are the basic building blocks for more complex structured credit products. The CDS market has supplanted the bond market as the industry gauge for a borrower's credit quality.

Credit default swaps are structured as instruments which make an agreed payoff upon the occurrence of a credit event. That is, in a CDS, the protection seller and the protection buyer enter a contract which requires that the protection seller compensates the protection buyer if a default event occurs before maturity of the contract. If there is no default event before maturity, the protection seller pays nothing. In return, the protection buyer typically pays a constant quarterly fee to the protection seller until default or maturity, whichever comes first. This quarterly payment, usually expressed as a percentage of its notional principal value, is the CDS spread or premium.

60.6.2.1 Credit Event in CDS

As with all other financial markets, the liquidity and efficiency of aligning buyers and sellers depend on consistent, reliable and understandable legal documentation. The International Swaps and Derivatives Association (ISDA) has been a strong force in maintaining the uniformity of documentation of CDS products through the assistance and support of its members, primarily the dealer community. The credit definitions by ISDA allow specification of the following credit events:

(i) Bankruptcy,

(ii) Failure to pay above a nominated threshold (say in excess of US$1 million) after expiration of a specific grace period (say, 2 to 5 business days),

(iii) Obligation default or obligation acceleration,

(iv) Repudiation or moratorium (for sovereign entities), and

(v) Restructuring.

There are significant issues in defining the credit events. This reflects the heterogeneous nature of credit obligations. In general, items 1, 2 and 5 are commonly used as credit events in CDS for firms. Four types of restructuring have been given by ISDA: full restructuring; modified restructuring (only bonds with maturity shorter than 30 months can be delivered); modified-modified restructuring (restructured obligations with maturity shorter than 60 months and other obligations with maturity shorter than 30 months can be delivered); and no restructuring.

The payment following the occurrence of a credit event is either repayment at par against physical delivery of a reference obligation (physical settlement) or the notional principal minus the post default market value of the reference obligation (cash settlement). In practice, physical settlement is the dominant settlement mechanism, though the proportion has dropped a lot (according to BBA Credit Derivative Report 2006). The delivery of obligations in case of physical settlement can be restricted to a specific instrument, though usually the buyer may choose from a list of qualifying obligations, irrespective of currency and maturity as long as they rank *pari passu* with (have the same seniority as) the reference obligation. This latter feature is commonly referred to as the cheapest-to-deliver option. Theoretically, all deliverable obligations should have the same price at default and the delivery option would be worthless. However, in some credit events, e.g., a restructuring, this option is favorable to the buyer, since he can deliver the cheapest bonds to the seller. Counterparties can limit the value of the cheapest-to-deliver option by restricting the range of deliverable obligations, e.g., to non-contingent, interest-paying bonds.

60.6.2.2 Valuation of Credit Default Swap without Liquidity Effect

Let $\rho$ be the premium paid by the buyer of default protection. Assuming that the premium is paid continuously, the present value of the premium leg of a credit-default swap can be written as

$$E_t^Q\left\{\rho\int_t^T \exp\left[-\int_t^s (r_u+\lambda_u)du\right]ds\right\} \tag{60.6.14}$$

If the bond defaults, a bondholder recovers a fraction $\omega$ of the par value and the seller of default protection pays $1-\omega$ of the par value to the buyer. The value of the protection leg of the credit default swap is given by

$$E_t^Q\left\{\int_t^T (1-\omega)\lambda_s \exp\left[-\int_t^s (r_u+\lambda_u)du\right]ds\right\} \tag{60.6.15}$$

Equating the premium leg to the protection leg, we can solve for the CDS premium:

$$\rho = \frac{E_t^Q\left\{(1-\omega)\int_t^T \lambda_s \exp\left[-\int_t^s (r_u+\lambda_u)du\right]ds\right\}}{E_t^Q\left\{\int_t^T \exp\left[-\int_t^s (r_u+\lambda_u)du\right]ds\right\}} \tag{60.6.16}$$

An analytical solution could be obtained if we assume $r_t$ and $\lambda_t$ follow the affine class of diffusion in the risk-neutral measure. For example, Longstaff, Mithal and Neis (2005) assume a CIR processes for $\lambda_t$:

$$d\lambda_t = \kappa_\lambda(\mu_\lambda - \lambda_t)dt + \sigma_\lambda\sqrt{\lambda_t}dW_t^Q \tag{60.6.17}$$

where $W_t^Q$ is a standard Brownian motion under the risk-neutral measure. Other related studies include Duffie (1999), Hull and White (2000, 2001), and Houweling and Vorst (2005).

60.6.2.3 Valuation of Credit Default Swap with Liquidity

The liquidity effect on the CDS is asymmetric since the premium leg and the protection leg are subject to different liquidity risks. The premium leg should be discounted with CDS-specific liquidity factor since it depends on the liquidity of CDS market, while the protection leg has no liquidity problem.[27] The CDS premium is then given by

$$\rho = \frac{E_t^Q\left\{(1-\omega)\int_t^T \lambda_s \exp\left[-\int_t^s (r_u + \lambda_u)\, du\right] ds\right\}}{E_t^Q\left\{\int_t^T \exp\left[-\int_t^s (r_u + \lambda_u + l_u)\, du\right] ds\right\}} \tag{60.6.18}$$

60.6.3 Empirical Issues

Due to the rapid growing swap markets, a number of empirical studies have attempted to identify the possible determinants of swap spreads and related issues on pricing mechanism, market efficiency, and information spillovers among different markets.

60.6.3.1 Determinants of Interest Rate Swap Spread

One of the stylized facts we observe for interest rate swap is that there is a positive spread between the swap rate and the government default-free interest rate, which is termed as the swap spread. Most of the empirical studies of interest rate swap focus on the determinants of interest rate swap spreads, which basically include two components: default and liquidity components. The default component generally involves two types of default risk. First, the counterparties may default on their future

[27] Bühler and Trapp (2007) argue that the protection leg should be discounted with corporate bond specific liquidity factor if the protection is paid in physical settlement. However, we argue that it is unnecessary since it could be introduced easily by adjusting $\omega$.

obligations. This is called counterparty default risk. Second, the underlying floating rate in a swap contract is usually set at the LIBOR rate, which is a default-risky interest rate.[28]

Sun, Sundaresan and Wang (1993) provide the earliest empirical investigation of default risk in swap spread and find evidence of default risk premium in the swap spread.[29] Brown, Harlow and Smith (1994) study US dollar swaps from 1985 to 1991 and find a positive relationship between the LIBOR spread and the swap spread, while Eom, Subrahmanyam and Uno (2000) find a similar evidence in Japan. Minton (1997) uses two proxies for default risk premium (corporate quality spread and aggregate default spread)[30] and finds that default risk is important for interest rate swap spreads. In general, a 100 basis-point increase in the bond spread of BBB bonds results in a 12-15 basis-point increase in the swap spread. Lang, Litzenberger and Liu (1998) argue that the sharing of surplus created by swaps affects swap spreads. Fehle (2003) run VAR regressions of swap spreads on default risk and liquidity proxies using international data. The difference between LIBOR and Treasury-bill rate is employed as a proxy for liquidity, while the level, slope, and volatility of term structure and the difference between the yields on a portfolio of corporate bonds and a corresponding Treasury bond act as proxies for default risk. The results show that swap spreads are sensitive to bond spreads in most currencies and maturities. However, there are no clear patterns across bond spreads from different ratings and across swap maturities.

[28] Li (2006) summarizes six reasons to suggest that the counterparty default risk is not important for swap spreads.
[29] See also Sundaresan (2003).
[30] Corporate quality spread is defined as the difference between the yields on portfolios of Moody's Baa-rated corporate debt and portfolios of Moody's Aaa-rated corporate debt, while aggregate default spread is defined as the difference between the yields on portfolios of Moody's Baa-rated corporate debt and the average of ten and thirty year US Treasury yields that match the maturities of Baa-rated corporate debt.

Most of these studies use a linear regression approach and do not apply the dynamic reduced-form model.

Within the reduced-form framework, Duffie and Singleton (1997) find that both liquidity and default risks are necessary to explain the variation in swap spreads. However, the effect of liquidity factor does not last long while the default risk becomes more important for longer time horizons. He (2000) finds that the liquidity component could explain most variations in swap spreads. Grinblatt (2001) attributes the swap spread to liquidity differences between government securities and short-term Eurodollar borrowing and finds that his model could generate a wide variety of swap spread curves and explain about 35% to 40% of the variations in US swap spreads across time. Li (2006) attributes the liquidity component of swap spreads to the liquidity difference between the Treasury and swap markets and decomposes the swap spreads into default and liquidity components. The parameter estimates show that the default and liquidity components of swap spreads are both negatively related to riskless interest rates. A further analysis reveals that default risk accounts for the levels of swap spreads, while the liquidity component explains most of the volatilities of swap spreads.

#### 60.6.3.2 Determinants of Credit Default Swap Spread

Given the short history of credit derivatives market and the limited data availability, there has been little empirical work in this arena. Most of them focus on the determinants of CDS spreads, spillover between CDS and other financial markets, and their role in price discovery of credit conditions.

(1) Determinants of CDS spreads

Houweling and Vorst (2005) implement a set of reduced-form models on market CDS quotes and corporate bond quotes and find that financial markets may not regard Treasury bonds as the default-free benchmark. Zhu (2006) examines the long-term pricing accuracy in the CDS market relative to the bond market. His study looks into the underlying factors that explain the price differentials and explores the short-term dynamic linkages between the two markets in a time series framework. The panel data study and the VECM analysis both suggest that short-term deviations between the two markets are largely due to the higher responsiveness of CDS spreads to changes in the credit condition.

Zhang, Zhou and Zhu (2005) introduce jump risks of individual firms to explain credit default swap spreads. Using both historical and realized measures as proxies for various aspects of the jump risks, they find evidence that long-run historical volatility, short-run realized volatility, and various jump-risk measures all have statistically significant and economically meaningful effects on credit spreads. More important, the sensitivities of credit spreads to volatility and jump risk depend on the credit grade of the entities and these relationships are nonlinear. Negative jumps tend to have larger effects.

Blanco, Brennan and Marsh (2005) test the theoretical equivalence of credit default swap spreads and credit spreads derived by Duffie (1999). Their empirical evidence strongly supports the parity relation as an equilibrium condition. Moreover, CDS spreads lead credit spreads in the price discovery process.

(2) Spillover between CDS and Other Financial Markets

Longstaff, Mithal and Neis (2003) examine weekly lead-lag relationships between CDS spread changes, corporate bond spread changes, and stock returns of US firms in a VAR framework. They find that both stock and CDS markets lead the corporate bond market which provides support for the hypothesis that information seems to flow first into credit derivatives and stock markets and then into corporate bond markets. However, there is no clear lead pattern of the stock market to the CDS market and vice versa.

Jorion and Zhang (2007) examine the information transfer effect of credit events across the industries and document the intra-industry credit contagion effect in the CDS market. The empirical evidence strongly supports the domination of contagion effects over competition effects for Chapter 11 bankruptcies and competition effects over contagion effects for Chapter 7 bankruptcies.

Acharya and Johnson (2007) quantify insider trading in the CDS market and show that the information flow from the CDS market to the stock market is greater for negative credit news and for entities that subsequently experience adverse shocks. The degree of information flow increases with the number of banks that have ongoing lending (and hence monitoring) relations with a given entity. The results suggest that the CDS market leads the stock market in information transmission.

(3) Price Discovery of Credit Condition

Norden and Weber (2004) analyze the response of stock and CDS markets to rating announcements made by the three major rating agencies during the period

2000-2002. The results show that the CDS market reacts earlier than the stock market with respect to reviews for downgrade by S&P and Moody's.

Hull, Predescu and White (2004) examine the relationship among CDS spreads, bond yields and benchmark risk-free rates used by participants in the derivative market. They show that using swap rates as the risk free benchmark produces better goodness-of-fit compared to using other risk-free rate proxies such as Treasury rates. Their empirical evidence also suggests that the CDS market anticipates credit rating announcements, especially negative rating events.

Tang and Yan (2006) study the effects of liquidity in the CDS market and liquidity spillover from other markets to the CDS market using a large data set. They find substantial liquidity spillover from bond, stock and option markets to the CDS market.

## 60.7 Concluding Remarks

This paper provides a comprehensive survey of term structure models, pricing applications and empirical evidence. Historically, two major considerations shape the development of DTSMs: (i) explaining the stylized facts of term structure; and (ii) the tradeoff between mathematical complexity and analytical tractability.

We begin with a generalized pricing framework by which most of the DTSMs are nested. Based on different specifications on the risk-neutral dynamics of state variable and the mapping function between state variable and short term interest rates, we categorize DTSMs as affine, quadratic, regime switching, jump, stochastic volatility models. We compare the empirical performance of these DTSMs in fitting the interest rate behavior in the physical measure and the price of default-free government bonds.

Empirical findings on DTSMs are not conclusive. Multifactor models seem to perform better than single-factor models. However, there remains serious concern about the applicability of nonlinear DTSMs. Moreover, the economic value of some DTSMs in bond pricing and risk management, and the relationship between the dynamics of term structures under the risk-neutral and physical measures remain open questions.

We also evaluate the usefulness of DTSMs in the pricing of defaultable bonds. In standard term structure models, the yield spread is determined by two factors: the risk of default (modeled by default intensity) and the expected loss in the event of default (modeled by recovery rate). However, most of the empirical evidence has shown that default risk can only explain a portion of credit spreads, and non-default components, such as liquidity and tax, are also important for the credit spread. Lin, Liu and Wu (2007) propose a corporate bond pricing model that incorporates the default probability, liquidity and tax to decompose the corporate bond yield spread into three components. They find that default, liquidity and tax are all important factors for explaining corporate yield spreads. However, there are two caveats. First, empirically Lin, Liu and Wu (2007) follow the approach of Longstaff, Mithal and Neis (2005) by assuming that CDS premium contains no liquidity component. This assumption has been questioned by several studies such as Acharya and Johnson (2007) and Tang and Yan (2007). Second, Lin, Liu and Wu (2007) assume that liquidity intensity and default intensity are independent while in reality they are more likely to be correlated. To accommodate this correlation, we need to obtain a closed-form solution for the corporate bond pricing formula with correlated factors.

Finally, research on the components of swap spreads is inconclusive. Most studies assume that the CDS premium contains no liquidity component, while several recent studies show the existence of liquidity premium in CDS. Other potentially interesting research subjects in this area include the significance of fixed-income derivative markets in affecting information transmission, price discovery, and liquidity in the spot markets. For example, there are two possible effects on corporate bond trading by CDS trading. First, it provides an easier way to trade the credit risk, which makes investors more reluctant to trade corporate bonds and hence decreases the corporate bond liquidity. Second, CDS trading provides a way to hedge the credit risk, which complements corporate bond trading and increases the liquidity of corporate bonds.

The fixed income research will continue to be an exciting field. The recent literature on pricing derivatives using DTSMs shows an enormous potential for new insights using derivatives data in model estimation. It is expected that the fixed-income derivative market will provide important information for researchers to better understand credit risk and liquidity of the underlying market and to develop more sophisticated models of term structure to address the unresolved issues.

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