Mean-Preserving-Spread Risk Aversion and The CAPM

Phelim P. Boyle and Chenghu Ma*

October 27, 2006

Abstract

This paper establishes conditions under which the classical CAPM holds in equilibrium. Our derivation uses simple arguments to clarify and extend results available in the literature. We show that if agents are risk averse in the sense of mean-preserving-spread (MPS) the CAPM will necessarily hold, along with two-fund separation. We derive this result without imposing any distributional assumptions on asset returns. The CAPM holds even when the market contains an infinite number of securities and when investors only hold finite portfolios. Our paper complements the results of Duffie(1988) who provided an abstract derivation of the CAPM under some somewhat more technical assumptions.

In addition we use simple arguments to prove the existence of equilibrium with MPS-risk-averse investors without assuming that the market is complete. Our proof does not require any additional restrictions on the asset returns, except that the co-variance matrix for the returns on the risky securities is non-singular.

Keywords: CAPM equilibrium, two-fund separation, generalized efficient portfolio, MPS-risk-aversion. *JEL Classification*: D50, D81, G10, G11

1 Introduction

This paper provides general conditions for the validity of the classical CAPM as an equilibrium model in economies with a frictionless market. First, we show that, if equilibrium exists, then the asset returns must satisfy the CAPM if all investors are MPS-risk-averse (Theorem1). Second, we prove the existence

of equilibrium in the CAPM without assuming complete markets (Theorem 2). What is remarkable for the existence of equilibrium CAPM lies in the fact that, as illustrated in Section 4 below, generically, the optimal demand correspondences for MPS-risk-averse investors do not exist for an arbitrary given set of security prices. Yet, we manage to prove the existence of an equilibrium CAPM by restricting the prices to be located in a zero-measure set. Precisely, the existence proof is based on the validity of the CAPM when equilibrium exists as is proved in Theorem 1. It is noted that prices satisfying the CAPM constitute a measure-zero set in a suitably defined topological space for security prices.

Given the importance of the CAPM, it is of interest to see to what extent it holds in equilibrium and this topic has been discussed in the literature. It has been known for a long time[1] that if all investors have mean-variance preferences, then the CAPM will necessarily hold. It is also known that mean-variance preferences persist when asset returns are elliptically distributed (see Chamberlain 1983, and Owen and Rabinovitch 1983). It is, therefore, of particular interest to explore if the CAPM holds when preferences are not necessarily in the mean-variance class (but, see Duffie 1988, and the discussions below).

Following the insights of Sharpe-Lintner-Mossin, the key observation which leads to the validity of the CAPM is that (mean-variance) investors optimally choose to hold combinations of two efficient portfolios: the risk free asset and the so-called tangent portfolio. This is known as two-fund separation theorem (see Black 1972 and Tobin 1958, and also Bottazzi, Hens and Löffler 1998 for recent developments). Therefore, to seek conditions for the CAPM, it is sufficient to seek conditions under which the two-fund separation theorem holds. The first effort in this direction was made by Cass and Stiglitz (1970). They used an expected utility framework and derived a parametric specification of expected utility functions which were sufficient for two-fund separation in the sense that, given the utility function, changes in wealth would not change the risky portfolio which the investor would optimally invest (if the optimal solution exists). In contrast to Chamberlain (1983), and Owen and Rabinovitch (1983), Cass and Stiglitz's observation on two-fund separation was made without imposing any distributional restrictions on asset returns. As a result, it is not clear if Cass and Stiglitz's separating risky portfolios remain the same for different utility functions belonging to the parametric class discovered by them.

Further to Cass and Stiglitz (1970), Ross (1978) developed distributional conditions on asset returns to ensure two-fund separation with the separating portfolios being common for all risk averse expected utility investors. Ross showed that two-fund separation holds if and only if asset returns are driven by two common factors with residual returns (to the factors) having zero (conditional) mean conditional on the linear span formed by the factors. It is noted that Chamberlain's class of elliptical distributions, which is sufficient for two-fund separation, belongs to the Ross's class. More recently, Berk (1997) derived joint restrictions on investor's expected utility functions, the aggregate

---

[1]See Sharpe (1964) and Lintner (1965) and Mossin (1966).

endowment and asset returns for two-fund separation and the CAPM, which accommodates both Ross (1978) and Chamberlain (1983) as special cases.

The conditions required for the CAPM in this paper advance the existing literature and lead to a clearer understanding of the extent to which the CAPM holds in equilibrium. In comparison with the above cited work, this paper makes no distributional assumptions on asset returns, and purely focusses on investor's risk preferences over the random payoffs. Essentially, we show that the CAPM holds as long as investors have MPS-risk-averse preferences. Loosely speaking, an investor exhibits MPS-risk-aversion, if for all random payoffs X and $Y = X + \varepsilon$, the investor would prefer X to Y whenever $E[\varepsilon] = 0$ and $\mathrm{Cov}(X, \varepsilon) = 0$. The notion of MPS-risk-aversion is appealing because (a) it captures the investor's psychological aversion towards 'increase in risk' in a natural way. (b) MPS-risk-averse preferences are not restricted to certain classes of expected or non-expected utility functions. Therefore, they are not subject to criticisms such as the well-known Allais Paradox and other deficiencies associated with expected utility functions. It is noted that expected utility functions may violate MPS-risk-aversion — and it could even be argued that this constitutes another drawback of expected utility functions.

An alternative derivation of the CAPM under the MPS-risk-averse behavior assumption was given by Duffie (1988, Theorem 11E). He used the notion of *variance-aversion* which is the same as MPS-risk-aversion introduced here. It is noted that the derivation of the CAPM in Duffie's book does not rely on the two-fund separating properties of the efficient frontier. Duffie's derivation rests on several explicit and implicit technical assumptions on asset returns: first, the existence of a continuous linear pricing rule in the market span; second, the market subspace is assumed to be a closed subset in $\mathbb{L}_2$. Whilst the linearity of the pricing rule is necessarily implied by the no-arbitrage condition, as part of the equilibrium conditions, the continuity assumption on the pricing rule is somewhat arbitrary and rather strong. Moreover, the closed-ness assumption on the market subspace is especially strong when the market subspace is infinite dimensional. For example, it is well known that the set of all bounded random payoffs $\mathbb{L}_\infty$ as a market subspace in $\mathbb{L}_2$ is not closed under the $\mathbb{L}_2$-norm. The market subspace studied in this paper also violates Duffie's closed-ness condition. Notice further that the continuity assumption, together with the closed-ness of the market subspace, implies the linear representation of the pricing rule in $\mathbb{L}_2$. The latter is crucial for Duffie's derivation of the CAPM.

In contrast, the derivation provided in this paper follows the heritage of the traditional approach, as in Sharpe (1964) and Lintner (1965). It is based on the relevance of the mean-variance efficient frontier and the investor's optimal portfolio holdings. This paper deviates from Duffie (1988), but is similar to Chamberlain and Rothschild (1983) and Nielsen (1990), in that we restrict investors to hold portfolios $\theta \in \Theta$ involving only a finite number of securities even though the market contains an infinite number of tradable securities. The role played by the dimensionality of the number of securities is important in this paper. Hence it is useful to give a brief road map of the areas where the dimensionality of the number of security space impacts the analysis as well as an

outline of the relevant results. We first discuss this in connection with the mathematics of the efficient frontier and then its implications for two fund separation and risk decomposition.

To discuss the efficient frontier we need more notation. Let $J = \{1, \cdots, j, \cdots\}$ be the set of risky assets. The set

$$\Theta = \left\{\theta \in \mathbb{R}^{\#J} : \sup\{j \in J : \theta_j \neq 0\} < \infty\right\}$$

which is itself an infinite dimensional vector space, forms a dense, but not closed, subset of Hilbert space $\mathbb{H}_2$. The Hilbert space contains all square-integrable risky portfolios (see section 3 for detail).[2] Elements in $\mathbb{H}_2$ are referred to as 'generalized portfolios' because they may involve compositions of an infinite number of securities.

We shall see that the efficient frontier for Θ, in general, does not exist. This is not surprising because a fully diversified portfolio may involve the composition of an infinite number of securities. The efficient frontier for $\mathbb{H}_2$, as it turns out, is relatively easier to obtain, and is referred to as *the generalized efficient frontier* (hence g.e.f.). The characterization of the g.e.f. is a straightforward extension to Markowitz's efficient frontier with a finite number of securities. In fact, under fairly general conditions, the g.e.f. is well-defined and inherits many of the key properties of the classical efficient frontier (for a finite number of securities). These include the validity of

(a) (generalized) two-fund separation, and

(b) risk decomposition.

For (a), we show that any generalized efficient portfolio must be expressed as a convex combination of two arbitrary generalized efficient portfolios. For (b), we show that, all arbitrary (generalized) portfolios, including all finite portfolios in Θ, must be expressible as a mean-preserving-spread of some generalized efficient portfolios. In the presence of a risk free asset, the generalized efficient portfolios would involve convex combinations of the risk free asset and a so-called generalized tangent portfolio θ_T.

The generalized efficient frontier is also shown to be relevant for portfolio decision making when investors are restricted to hold finite portfolios in Θ. This is the case despite the fact that the efficient frontier formed by portfolios in Θ does not exist. In general, MPS-risk-averse investors would not be satiated with an arbitrary portfolio containing just a finite number of securities. This leads to non-existence or emptiness of the optimal demand correspondence for MPS-risk-averse investors. Nevertheless, in the presence of a risk less asset, if the generalized tangent portfolio θ_T itself were finite (thus located in Θ), the efficient frontier formed by portfolios in Θ (with the risk free asset) would be, generically, well-defined and be given by the tangent ray and the reflection of the tangent ray. In these circumstances, both the mutual fund separation theorem

---

[2]Defined later as $\mathbb{H}_2\left(\Sigma\right)$. Here, we use $\mathbb{H}_2$ to simplify the notation for convenience.

and the risk decomposition theorem hold true even with $\#J = \infty$. In fact, we show that, when the economy involves a finite number of MPS risk averse investors and when the market portfolios contain a finite number of tradable securities, in equilibrium the generalized tangent portfolio θ_T would necessarily be located in Θ, and be given by the market portfolio. It is in this sense we say that the classical mutual fund separation along with risk-decomposition extend to economies with an infinite number of securities $(\#J = \infty)^3$. This in turn implies the validity of the equilibrium CAPM.

The assumption on the finiteness of the market portfolio is not restrictive. In practice, the market portfolio is an index of a finite number of stocks. The class of derivative securities written on the stocks or the index of stocks, which represents a large or even an infinite number of traded financial securities, are not in the composite of the market portfolio. The equilibrium CAPM provides a mechanism to price not only those primitive securities in the composite of the market portfolio, but also securities that are out of the composite of the market portfolio.

The relationship between mean-variance utility functions and the MPS risk-averse preferences has been analyzed by Löffler (1996). Löffler showed that, when the market contains a finite number of risky assets (but no less than three) and when the market span forms a convex cone of $\mathbb{L}_2$, if the MPS risk averse preference is represented by continuously Frechét-differentiable (in the $\mathbb{L}_2$-norm) utility functions, then the preference must admit a mean-variance utility representation.[4] This result is interesting and surprising. Nevertheless, one may still want to treat the mean-variance preferences as a subset of the MPS-risk-averse preferences. First, a binary relationship satisfying the MPS risk aversion property constitutes a partial order, which may not admit an utility representation. Indeed, we do not need to assume the MPS preference to admit an utility representation for most of the analysis in this paper. Second, even if preferences admit utility representations, most of the analysis carried out in this paper does not require the utility function to be differentiable. The latter, according to Löffler (1996), is crucial for the preferences to admit a mean-variance utility representation. Third, in this paper we restrict attention to monotone MPS-risk-averse preferences that are not satiated by future payoffs/cash flows. This is in contrast to the general class of mean-variance utility functions that may violate the non-satiation property. Finally, we do not restrict the market to contain a finite number of securities instead it may contain an infinite number of securities. The assumption of a finite number of securities is crucial for the validity of Löffler (1996)'s representation theorem.

---

[3]The mutual fund separation result along with risk decomposition would hold true for generalized portfolios if investors were allowed to hold portfolios involving an infinite number of securities. These are valid for arbitrary given risky returns that may not necessarily conform to the equilibrium.

[4]Without the Frechét differentiability assumption it is not clear if the MPS-risk-averse preferences admit mean-variance representation. The analysis carried out in this paper does not require the differentiability assumption. In fact, we even do not need to assume the preference to admit an utility representation except for the proof on the existence of an equilibrium.

As a separate contribution, we provide an elementary proof of the existence of market equilibrium for economies with a finite or countably infinite number of tradable securities. By elementary we mean that the proof does not use fixed point arguments, and we consider general MPS-risk-averse investors. The proof is based on the observation that the CAPM holds as long as equilibrium exists in an economy with MPS-risk-averse investors. For the existence proof, we need to assume that, investors' preferences restricted to the set of efficient portfolios (if the efficient frontier is well-defined) admit some continuous utility representations.

The existence of equilibrium in the CAPM framework with mean-variance investors has been well documented. For example, Dana (1999) provides a simple existence proof by assuming a complete market. More recently Hara (2001) contains an existence proof by dropping the market completeness and maintaining the assumption of mean-variance preferences. Nielsen (1989) and Sun and Yang (2003) prove existence under joint restrictions on the mean-variance utility functions and asset returns. These latter proofs do not assume homogeneous beliefs, but rely crucially on the assumption of mean-variance preferences. Our proofs are along the lines of Hara's approach. The distinguishing features of the existence proof provided in this paper are:

- We consider general MPS-risk-averse investors.

- Similar to Hara (2001), we do not assume the market to be complete.

The remainder of the paper is organized as follows: Section 2 describes the model and summarizes the main results. Section 3 derives the generalized efficient portfolios where the number of risky securities can be infinite. Section 4 discusses the MPS-risk-averse investor's optimal portfolio choice and its relevance to the generalized efficient frontier defined for an infinite number of tradable securities. Section 5 includes a formal derivation of the CAPM, along with the two-fund separation theorem. Section 6 concludes the paper. The proof of the existence of equilibrium is outlined in the Appendix.

2 Outline of the Model and Main Results

This section describes the basic framework and summarizes the main results of the paper.

We consider a two-period exchange economy with a frictionless capital market and heterogenous agents. The uncertainty is summarized by a probability state space $(\Omega, \mathbb{P})$ with probability measure $\mathbb{P}$. The topological properties of the state space are otherwise not specified. There exists a countable (finite or infinite) set of non-redundant risky securities $J = \{1, \cdots, j, \cdots\}$ available for trade. Let $\#J$ be the number of risky assets. Security j is associated with a state-contingent random payoff $\delta^j : \Omega \to \mathbb{R}$. We assume that there is a risk free asset, denoted as security 0. The risk free asset has a unit payoff in all future states (i.e., $\delta^0 \equiv 1$).

For the given set of risky assets, let Θ be a set of all admissible portfolio holdings on J.[5] Similar to Chamberlain and Rothschild (1983) and Nielsen (1990), we restrict Θ to consist of portfolios that involve only a finite number of risky assets:

$$\Theta = \left\{ \theta \in \mathbb{R}^{\#J} : \sup \left\{ j \in J : \theta_j \neq 0 \right\} < \infty \right\}. \tag{1}$$

For all $\theta \in \Theta$, θ_j represents the proportion invested in the risky security j with $\theta_0 \equiv 1 - \sum_{j \in J} \theta_j$ being the proportion invested in the risk free asset. Let $\delta^\theta \equiv \delta^0 + \sum_{j \in J} \theta_j \left(\delta^j - \delta^0 \right)$ be the portfolio payoff. Denote by

$$\mathbb{D} \equiv \left\{ \delta : \exists \, \theta \in \Theta \text{ such that } \delta^\theta = \delta \right\} \tag{2}$$

the market span formed by all tradable securities.

With this notation, the exchange economy is summarized by

$$\mathcal{E} \equiv \left((\Omega, \mathbb{P}), \mathbb{D}, \left\{ \succeq^i, \phi^i \right\}_{i \in I} \right) \tag{3}$$

where

- We assume that the payoffs of the J risky securities are all in the Hilbert space $\mathbb{L}_2(\Omega, \mathbb{P})$ — the space of square integrable random variables. Moreover, we assume that the variance-covariance matrix, Σ_0 is positive definite; that is, for all $\theta \in \Theta$, $\theta^\mathsf{T} \Sigma_0 \theta \geq 0$ and $\theta^\mathsf{T} \Sigma_0 \theta = 0 \Leftrightarrow \theta = 0$.

- The number of agents (investors) I is finite. Investor $i \in I$ is concerned only with his or her next period wealth. Investor i's preference over all random payoffs in $\mathbb{D} \subseteq \mathbb{L}_2(\Omega, \mathbb{P})$ is summarized by a preference relation $\succeq^i \subset \mathbb{D} \times \mathbb{D}$. For all i, $\succeq^i$ is monotonic, and displays risk aversion in the sense of MPS-risk-aversion.

- Investor i is endowed with a fixed number of shares $\phi^i \in \Theta$ of securities.

Let p_j be the market price of security j. The (total) return on security j is thus given by $R^j \equiv \frac{\delta^j}{p_j}$. After the market opens at $t = 0$, investors can adjust their positions at the given market prices $p \equiv [p_0, \cdots, p_j, \cdots]$. At price p, agent i's initial wealth is given by $W_0^i \equiv p \bullet \phi^i$. Let θ^i be agent i's portfolio holdings with θ_j^i representing the proportion of i's initial wealth invested in the risky security j. The total number of shares invested in security j by agent i is thus given by $\frac{\theta_j^i W_0^i}{p_j}$ with corresponding market value of $\theta_j^i W_0^i$. The total amount invested in the risk free asset is $W_0^i \left(1 - \theta^\mathsf{T} e \right)$, where $e \equiv [1, 1, \cdots]^\mathsf{T}$ is a column vector with unit elements.

---

[5] Much of the analysis below applies for a general admissible set such as $\Theta = \mathbb{H}_2(\Sigma)$ defined in section 3 below, which may contain portfolios with an infinite number of securities.

Definition 1 *Market equilibrium consists of a price vector p and a portfolio allocation $\left\{\theta^i\right\}_{i\in I} \subseteq \Theta$ such that: (i) $\sum_i \frac{\theta_j^i W_0^i}{p_j} = \phi_j^M \equiv \sum_i \phi_j^i$ for all j; and (ii) for all i, θ^i is the most preferred portfolio amongst all feasible portfolios θ for agent i.*

Let θ^M be the *market portfolio* with $\theta_j^M = \frac{\phi_j^M p_j}{\sum_j \phi_j^M p_j}$ for all $j \in J$. Let R^M be the return of the market portfolio.

We are now in a position to state the first main result of the paper:

Theorem 1 *Given the economy $\mathcal{E}$ with a (possibly infinite) number of tradable securities ($\#J \leq \infty$) and a finite number of MPS-risk-averse investors. If equilibrium exists with $p_j \neq 0$, for all j, then the capital asset pricing model (CAPM) holds: for all θ,*

$$\mu\left[\theta\right] = R_f + \beta^\theta \left(\mu^M - R_f\right), \tag{4}$$

where $R_f = p_0^{-1}$ is one plus the risk free interest rate, $\mu\left[\theta\right] \equiv E\left[R^\theta\right]$ is the expected return for portfolio θ,

$$\beta^\theta \equiv \frac{Cov\left(R^\theta, R^M\right)}{Var\left(R^M\right)} \tag{5}$$

and $\mu^M = E\left[R^M\right]$.

The proof of Theorem 1 is based on the properties of the efficient frontier together with the implications of the choice behavior for MPS-risk-averse investors. We provide the full details in Sections 3, 4 and 5.

The second main result of this paper deals with the existence of equilibrium.[6]

Theorem 2 *Given the economy $\mathcal{E}$ with a (possibly infinite) number of tradable securities ($\#J \leq \infty$) and a finite number of MPS-risk-averse investors. Assuming that investors' preferences (as partial orders) constrained on the set of efficient portfolios admit some well-defined continuous utility representations (whenever the efficient frontier is well-defined). If the payoffs associated with the initial portfolios[7] are non-negative, and if the payoffs for individual securities are non-negative and have a positive definite covariance matrix, then the equilibrium exists and the CAPM holds.*

The proof of Theorem 2 is contained in the Appendix. In the proof, we will make extensive use of the results developed in the paper, including the validity of the CAPM (Theorem 1), the two-fund separation theorem (Proposition 8) and other properties of the efficient frontier.

---

[6]The existence theorem is stated for the case when investors are restricted to hold finite number of securities. The existence proof extends as well to the case when investors were allowed to hold infinite portfolios.

[7]Those are the portfolios with which the investors are endowed.

3 The Efficient Frontier

In this section we provide a unified treatment of efficient portfolios. First, the definition of an efficient frontier adopted here extends the original definition in Markowitz (1954) by allowing an infinite number of tradable securities. Second, the framework is general enough to accommodate cases with or without a risk free asset (even though we assume the existence of a risk free asset). Finally, in defining the efficient frontier, it is not necessary to assume the covariance matrix Σ be non-singular.

Let R_f and R be respectively the risk free return and the return vector for the risky assets. Let $\mathbb{X} \subseteq \mathbb{R}^{\#J}$ be an arbitrary admissible portfolio space[8]. For any given $\theta \in \mathbb{X}$, the resulting portfolio return $R^\theta \equiv R_f + \theta^\mathsf{T} [R - R_f e]$ has mean return and standard deviation given by

$$\mu [\theta] = R_f + \theta^\mathsf{T} [\mu - R_f e] \text{ and } \sigma [\theta] = (\theta^\mathsf{T} \Sigma \theta)^{\frac{1}{2}},$$

where μ and Σ represent the vector of expected returns and the variance-covariance matrix of the risky returns. For pure risky portfolios θ we have $\theta^\mathsf{T} e = 1$ and $\mu [\theta] = \theta^\mathsf{T} \mu$. Similarly, for any given two portfolios θ and θ', the covariance of the portfolio returns is given by

$$\sigma [\theta, \theta'] = \theta^\mathsf{T} \Sigma \theta'. \tag{6}$$

The following definition of an efficient portfolio applies whether or not there is a risk free asset.

Definition 2 *For any given $\mu_0 \in \mathbb{R}$, portfolio $\theta_0 \in \mathbb{X}$ is said to be efficient at μ_0 if*

$$\theta_0 = \arg \min_{\theta \in \mathbb{X}} \{ \sigma [\theta] : \mu [\theta] = \mu_0 \} ; \tag{7}$$

that is, amongst all portfolios with an expected return of μ_0, the efficient portfolio θ_0 has the minimum risk(standard deviation). The curve $\mathcal{I} (\mathbb{X})$ defined below, which is formed by the set of efficient portfolios,

$$\mathcal{I} (\mathbb{X}) \equiv \{ (\mu [\theta], \sigma [\theta]) : \theta \in \mathbb{X} \text{ is efficient} \}$$

is referred to as the mean-variance efficient frontier, or simply the 'efficient frontier', with respect to $\mathbb{X}$.

The next proposition describes a general property of efficient portfolios.

Proposition 1 *Let $\theta_0 \in \mathbb{X}$ be an efficient portfolio with mean μ_0. For all $\theta \in \mathbb{X}$ with $\mu [\theta] = \mu_0$, we have: $R^\theta = R^{\theta_0} + \varepsilon$ with $E [\varepsilon] = 0$ and $Cov(R^{\theta_0}, \varepsilon) = 0$.*

---

[8]The admissible set $\mathbb{X}$ in this section is not restricted so long as the portfolio returns are associated with finite means and variances. For example, the notion of efficient portfolios introduced below are well-defined for $\mathbb{X} = \Theta$ when all portfolios are restricted to contain only a finite number of risky assets. The notion of efficient frontier is also well-defined for $\mathbb{X} = \mathbb{H}_2 (\Sigma)$ that is defined in Section 3.1 below.

Proof. Consider the set of portfolios $\{\alpha\theta + (1-\alpha)\theta_0 : \alpha \in \mathbb{R}\}$ formed by convex combinations of θ_0 and θ. These portfolios all have the same mean return given by μ_0. Since θ_0 is efficient at μ_0 with standard deviation σ_0, $\sigma[\alpha\theta + (1-\alpha)\theta_0]$ must achieve its minimum at $\alpha = 0$; that is,

$$0 = \arg\min_{\alpha} \left\{ \alpha^2\sigma^2[\theta] + 2\alpha(1-\alpha)\sigma[\theta,\theta_0] + (1-\alpha)^2\sigma_0^2 \right\}. \tag{8}$$

The first order condition leads to $\sigma[\theta,\theta_0] = \sigma_0^2$. Let $\varepsilon \equiv R^\theta - R^{\theta_0}$. We have: $E[\varepsilon] = 0$ and $\mathrm{Cov}(R^{\theta_0}, \varepsilon) = \sigma[\theta,\theta_0] - \sigma_0^2 = 0$. ∎

Notice that this proof does not require any assumption on the finiteness of the number of securities, nor does it require any assumption on the non-singularity of the covariance matrix Σ. Similar to the well-known case with a finite number of securities, it can be further shown that the efficient frontier induced by pure (possibly an infinite number of) risky assets, if well-defined, will form a hyperbola (see section 3.1 below). The efficient frontier in the presence of a risk free asset is formed by a tangent ray plus its reflection ray. The efficient portfolio at any given μ_0, if exists, might not be unique unless Σ is non-singular.

From the definition of the efficient frontier, it is also noted that

- Dropping redundant securities does not affect the shape of the efficient frontier;

- For finite portfolios $\mathbb{X} = \Theta$, with any arbitrary given set of non-redundant securities we can always construct a new sequence of portfolios, known as 'normalized portfolios' so that (a) the normalized portfolios are un-correlated with each other; (b) each constitutes a portfolio of a finite number of risky securities together with the risk free asset; and (c) the set of normalized portfolios, together with the risk free asset, generates the same market span as was generated from the original set of tradable securities.[9]

---

[9]Given returns $\{R_j\}_{j=1}^{\infty}$ for an arbitrary collection of non-redundant risky assets with positive definite variance-covariance matrix, the returns $\left\{R_j^*\right\}_{j=1}^{\infty}$ associated with the so-called 'normalized portfolios' are defined by setting

$$
\begin{aligned}
R_1^* &= R_1 \\
R_2^* &= R_2 - \alpha_{2,1}\left(R_1^* - R_f\right) \\
&\cdots \\
R_j^* &= R_j - \alpha_{j,1}\left(R_1^* - R_f\right) - \cdots - \alpha_{j,j-1}\left(R_{j-1}^* - R_f\right) \\
&\cdots
\end{aligned}
$$

with $\alpha_{j,k} = \frac{\mathrm{Cov}(R_j, R_k^*)}{\mathrm{Var}(R_k^*)}, k = 1,2,\cdots,j-1$. The normalized returns are un-correlated to each other, and each constitutes a finite portfolio. The normalized risky returns defined above are non-zero and risky because the risky assets are non-redundant and because the variance-covariance matrix is positive definite. Finally, it is also noted that the return for each risky asset, say R^j, can be expressed as a return of a finite portfolio formed by the 'normalized portfolios'.

It is evident that the normalized portfolios induce the same efficient frontier as the original set of tradable securities. For the rest of this section, without loss of generality, we only deal with normalized portfolios, so that the variance and covariance matrix Σ is expressed as a (infinite dimensional) diagonal matrix with its elements to be given by the variances of the corresponding normalized risky portfolios $\sigma_j^2, j = 1, 2, \cdots$. The infinite-dimensional matrix Σ is understood to be associated with an well-defined inverse matrix Σ^{-1} which is also diagonal with diagonal elements given by $\sigma_j^{-2}, j = 1, 2, \cdots$.

3.1 Generalized efficient portfolios: pure risky assets

First, we introduce some machinery to handle an infinite number of securities. For the given positive definite matrix Σ we consider the following infinite dimensional Hilbert space

$$\mathbb{H}_2(\Sigma) \equiv \left\{ \theta \in \mathbb{R}^{\#J} : \theta^\mathsf{T} \Sigma \theta < \infty \right\} \tag{9}$$

with its inner product given by $\langle \theta, \theta' \rangle_\Sigma \equiv \theta^\mathsf{T} \Sigma \theta'$ and norm given by $\|\theta\|_\Sigma \equiv \sqrt{\langle \theta, \theta \rangle_\Sigma}$. The Hilbert space consists of all weighted square-summable sequences. A pure generalized risky portfolio corresponds to an element in $\mathbb{H}_2(\Sigma)$ with $e^\mathsf{T} \theta = 1$. The inner product of two generalized portfolios gives the covariance of the two (generalized) portfolio returns, while the norm of a generalized portfolio gives the standard deviation of the (generalized) portfolio return.

We note that a generalized risky portfolio may involve holdings of an infinite number of risky assets. The portfolio space Θ that contains portfolios with finite number of risky assets is known to form a dense subset of the Hilbert space $\mathbb{H}_2(\Sigma)$; that is, (a) $\Theta \subset \mathbb{H}_2(\Sigma)$; (b) for all $\theta \in \mathbb{H}_2(\Sigma)$ there exists a sequence $\{\theta_n\}_{n=1}^\infty$ in Θ such that $\|\theta - \theta_n\|_\Sigma \to 0$ as $n \to \infty$. However, the set Θ as a subset of $\mathbb{H}_2(\Sigma)$ is not closed under the $\|\cdot\|_\Sigma$-norm.

We impose the following two conditions on μ and Σ.

A.1 Σ is positive definite and $\Sigma^{-1}\mu$ and $\Sigma^{-1}e \in \mathbb{H}_2(\Sigma)$;

A.2 Non-degeneracy: μ is not proportional to e.

Given A.1 and A.2, as in the case with a finite number of securities, we introduce the notion of *'generalized efficient portfolio' (g.e.p.)* and *'generalized efficient frontier' (g.e.f.)* $\mathcal{I}_g$ formed by the set of generalized pure risky portfolios:

Definition 3 *The 'generalized efficient portfolio' (g.e.p.) at μ_0 is given by*

$$\theta_0 = \left[\Sigma^{-1}\mu, \Sigma^{-1}e\right] A^{-1} [\mu_0, 1]^\mathsf{T} \in \mathbb{H}_2(\Sigma) \tag{10}$$

where $A \equiv [\mu, e]^\mathsf{T} \Sigma^{-1} [\mu, e]$ is a 2×2 positive definite matrix. The frontier formed by g.e.p.s in the μ-σ plane is referred to as the 'generalized efficient frontier' (g.e.f.) and is denoted by $\mathcal{I}_g$.

Notice that, under condition A.1, we have $e^{\mathsf{T}}\theta_0 = \langle \Sigma^{-1}e, \theta_0 \rangle_\Sigma = 1$ and $\mu^{\mathsf{T}}\theta_0 = \langle \Sigma^{-1}\mu, \theta_0 \rangle_\Sigma = \mu_0$. The non-degeneracy condition A.2 is satisfied when there are two risky assets with distinct expected returns. Assumption A.2 implies $\Sigma^{-1}e$ is not proportional to $\Sigma^{-1}\mu$, which in turn implies

$$\langle \Sigma^{-1}e, \Sigma^{-1}\mu \rangle_\Sigma < \left\| \Sigma^{-1}e \right\|_\Sigma \times \left\| \Sigma^{-1}\mu \right\|_\Sigma. \tag{11}$$

This last inequality ensures the matrix A is non-singular with a well-defined inverse A^{-1}.

Remark 1 *The g.e.p. θ_0 will, in general, not be efficient with respect to Θ because it may involve an infinite number of securities for almost all μ and μ_0. There are, however, exceptional cases for which there exist g.e.p.s belonging to Θ. This happens when, there exists a constant number k so that the infinite vector $\Sigma^{-1}(\mu - ke) \neq \emptyset$ contains a finite number of non-zero elements.[10] In this case, we can always find a μ_0 at which the g.e.p. θ_0 involves a finite number of risky assets. So, generically, risky portfolios in Θ would not constitute as g.e.p.s even though the g.e.p.s are well-defined for all arbitrary μ_0 under conditions A.1 and A.2.*

Remark 2 *Although the efficient frontier generated from portfolios in Θ may not exist, all generalized efficient portfolios on $\mathcal{I}_g$ can be arbitrarily approximated by finite portfolios because Θ is dense in $\mathbb{H}_2(\Sigma)$. It is in this sense, we refer to the g.e.f. $\mathcal{I}_g$ as the 'asymptotic efficient frontier' for Θ. In Proposition 2 below, we shall also see that the g.e.f. $\mathcal{I}_g$ constitutes the efficient frontier generated from the enlarged portfolio set $\mathbb{X} = \mathbb{H}_2(\Sigma)$.*

The g.e.f. defined above inherits many of the properties of the original Markowitz's efficient frontier for a finite number of securities. The next proposition summarizes the properties of the g.e.f. $\mathcal{I}_g$. The proofs are standard, and are thus omitted.

Proposition 2 *Under conditions A.1 and A.2, the g.e.f. $\mathcal{I}_g$ generated by generalized risky portfolios (with possibly an infinite number of securities) forms an hyperbola in the μ-σ plane:*

1. For all μ_0, the variance of the generalized efficient portfolio θ_0 is given by

$$\sigma_0^2 = [\mu_0, 1] A^{-1} [\mu_0, 1]^{\mathsf{T}}. \tag{12}$$

2. The (generalized) minimum variance portfolio on the hyperbola is given by

$$\underline{\theta} = \left\| \Sigma^{-1}e \right\|_\Sigma^{-2} \times \Sigma^{-1}e \tag{13}$$

which corresponds to the g.e.p. at

$$\underline{\mu} = \left\| \Sigma^{-1}e \right\|_\Sigma^{-2} \times \langle \Sigma^{-1}e, \Sigma^{-1}\mu \rangle_\Sigma. \tag{14}$$

---

[10]This happens when the normalized risky portfolios are associated with finite distinct expected returns.

3. The g.e.p. θ_0 at μ_0 solves the following quadratic minimization problem with constraints:

$$\theta_0 = \arg \min_{\substack{\theta \in \mathbb{H}_2(\Sigma) \\ s.t. \\ \langle \Sigma^{-1}e, \theta \rangle_\Sigma = 1 \\ \langle \Sigma^{-1}\mu, \theta \rangle_\Sigma = \mu_0}} \|\theta\|_\Sigma^2 . \qquad (15)$$

4. For the g.e.p. θ_0 at arbitrary $\mu_0 \neq \underline{\mu}$, there exists a so-called 'generalized zero-beta portfolio' θ_0' located on $\mathcal{I}_g^-$ such that $\langle \theta_0, \theta_0' \rangle_\Sigma = 0$. Moreover, θ_0' corresponds to the g.e.p. at μ_0', which is the intercept on the μ-axis of the tangent line at (μ_0, σ_0) along the g.e.f. $\mathcal{I}_g$.

5. Generalized Black Separation: Convex combinations of g.e.p.s on $\mathcal{I}_g$ are still generalized efficient; and all g.e.p.s can be expressed as convex combinations of two arbitrary distinct g.e.p.s, particularly for those orthogonal portfolios θ_0 and θ_0'.

In the above proposition, we see from statement 2, that the minimum variance generalized portfolio $\underline{\theta}$ always involves an infinite number of securities. The variance of the minimum variance (generalized) portfolio $\underline{\theta}$, which is given by $\sigma^2[\underline{\theta}] = \|\Sigma^{-1}e\|_\Sigma^{-2} = [e^\intercal \Sigma^{-1}e]^{-1}$, is strictly smaller than the minimum variance generated from any finite number of risky assets in Θ. In fact, $\sigma^2[\underline{\theta}]$ is the asymptotic limit to the minimum variances generated from finite risky portfolios in Θ. This is consistent with the logic of risk diversification.

Statement 3 of the proposition confirms that the g.e.f. $\mathcal{I}_g$ constitutes the efficient frontier for $\mathbb{X} = \mathbb{H}_2(\Sigma)$. Statements 4 and 5 extend the classical mutual fund separation properties of the efficient frontier to the case with an infinite number of securities, keeping in mind that it is valid for the generalized efficient frontier (rather than the efficient frontier for Θ).

Next we consider the situation when there is a risk free asset.

3.2 Generalized efficient frontier with a risk free asset

We proceed to characterize the generalized efficient frontier when the market contains a risk free asset, in addition to an infinite number of risky assets. Similar to the finite dimensional case, the generalized efficient frontier in the presence of a risk free asset is determined by the relative position of the risk free return R_f to that of the minimum variance (generalized) portfolio $\underline{\theta}$ in the μ-σ plane. It is noted that the generalized tangent portfolio θ_T on the hyperbola, which constitutes a generalized efficient risky portfolio, intersects the μ-axis at the given risk free return R_f, may or may not exist.

Even when the generalized tangent portfolio that crosses the μ-axis at R_f exists, it is not necessarily on the top half of the hyperbola (see Figure 2 below). Therefore, the tangent ray, even when it exists, does not necessarily constitute the generalized efficient frontier. This happens when the generalized tangent portfolio belongs to the lower section of the hyperbola.
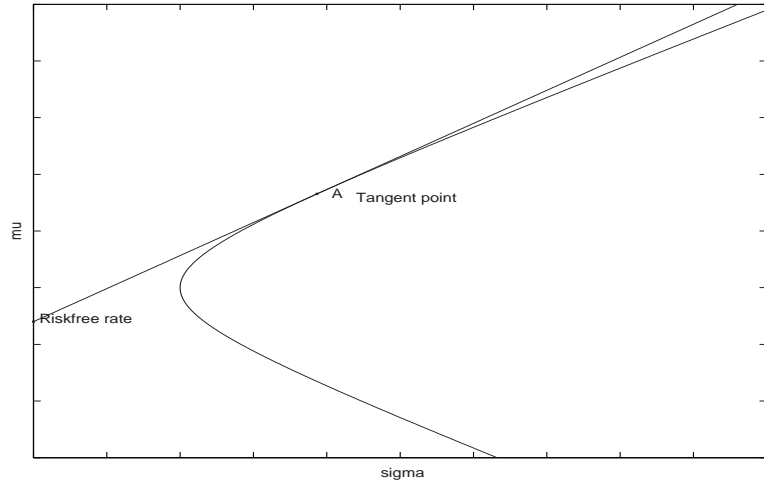
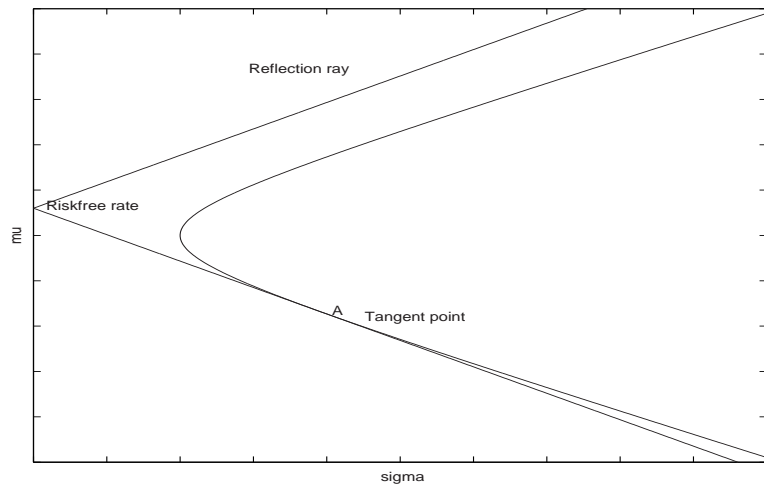Figure 1: Plot of case when $\underline{\mu} > R_f$.

Figure 2: Plot of case when $\underline{\mu} < R_f$.

14

To fully characterize the generalized efficient frontier, we need to distinguish between the following two cases:

Case 1: $\underline{\mu} \neq R_f$. In this case, there is a unique generalized pure risky portfolio, say $\theta_T \in \mathcal{I}_g$, on the hyperbola, which could be either on the top or the bottom of the hyperbola, such that the tangent line at θ_T intersects the μ-axis at R_f. The generalized portfolio risk and expected return are thus bordered by the tangent ray and the reflection of the tangent ray in the μ-σ plane. Given this observation, we have:

Proposition 3 *Suppose A.1 and A.2 hold with $\underline{\mu} \neq R_f$. The generalized tangent portfolio $\theta_T \in \mathbb{H}_2(\Sigma)$ exists uniquely, and is given by*

$$\theta_T = \frac{\Sigma^{-1}\mu - R_f \Sigma^{-1} e}{\langle \Sigma^{-1} e, \Sigma^{-1}\mu \rangle_\Sigma - R_f \|\Sigma^{-1} e\|_\Sigma^2}. \tag{16}$$

Moreover,

(a) *The generalized efficient frontier is formed by the tangent rays on the μ-σ plane, called the generalized efficient rays:*

$$\mu = R_f \pm \frac{\mu[\theta_T] - R_f}{\sigma[\theta_T]}\sigma, \text{ for all } \sigma \geq 0. \tag{17}$$

(b) *For all $\theta \in \Theta$, which includes all individual securities, it must hold true that*

$$\mu[\theta] = R_f + \beta_T^\theta (\mu[\theta_T] - R_f), \tag{18}$$

where $\beta_T^\theta \equiv \frac{\langle \theta_T, \theta \rangle_\Sigma}{\|\theta_T\|_\Sigma^2} \left(= \frac{\sigma[\theta, \theta_T]}{\sigma^2[\theta_T]}\right)$. Moreover, the portfolio return admits the following decomposition:

$$R^\theta - R_f = \beta_T^\theta \left[R^{\theta_T} - R_f \right] + \varepsilon^\theta, \tag{19}$$

where ε^θ has a zero mean and is un-correlated with R^{θ_T}.

Proof. The generalized tangent portfolio θ_T is well-defined when $\underline{\mu} \neq R_f$. By definition, θ_T maximizes the absolute generalized Sharpe-ratio $\frac{|\mu[\theta] - R_f|}{\sigma[\theta]}$ among all generalized risky portfolios. When $\underline{\mu} \neq R_f$, the optimal solution exists unique and is given by the expression (16). Statements (a) and (b) are valid following the same argument as in the finite dimensional case as in Huang and Litzenberger (1988, Chapter 3.18 & 3.19). ∎

Notice that the generalized efficient frontier, which is well-defined when $\underline{\mu} \neq R_f$, would constitute the efficient frontier for portfolios in Θ if, and only if, the tangent generalized portfolio θ_T itself belongs to Θ. This is possible when and only when there exist only a finite number of normalized risky assets having their expected returns to deviate from the risk free rate. If this were the case, the generalized efficient rays would constitute the efficient frontier formed by finite portfolios in Θ (in addition to the risk free asset).
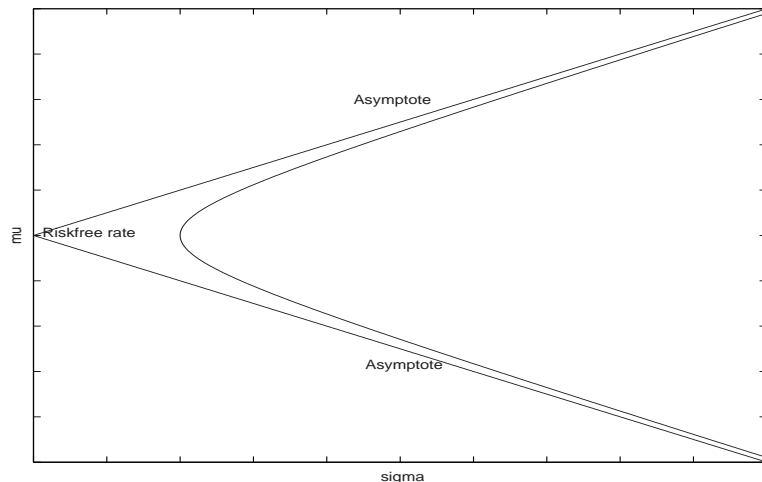
Case 2: $\underline{\mu} = R_f$.

Figure 3: Plot of case when $\underline{\mu} = R_f$.

It is not difficult to see that the generalized efficient portfolio for all $\mu_0 \neq R_f$ does not exist.

Proposition 4 *If $\underline{\mu} = R_f$, the generalized efficient portfolio does not exist at all $\mu_0 \neq R_f$.*

    Proof. When $\underline{\mu} = R_f$, the set of all feasible portfolios in the μ-σ plane is strictly bordered by the two asymptotic rays of the hyperbola, originating from R_f at the μ-axis. For all $\mu_0 \neq R_f$, any generalized portfolio θ which is strictly located within the two asymptotic rays, with mean-return no less than μ_0, can not be efficient since there exists another generalized portfolio to the left of $(\mu[\theta], \sigma[\theta])$. The new portfolio has the same mean and a smaller variance. Therefore, the generalized efficient portfolio at μ_0 does not exist. Any bundle (μ, σ) other than $(R_f, 0)$ located on or outside the asymptotic rays is not achievable. ∎

    When there is a finite number of risky assets, it is well known that the optimal portfolio for mean-variance investors, if it exists, must be located on the Markowitz mean-variance efficient frontier. The existing literature tells us little about the relevance of efficient frontier on portfolio choices made by investors whose preferences are not in the mean-variance class.[11] The difficulties in establishing such relevancy are well known: First, an investor's optimal portfolio may not exist even though the efficient frontier is well defined. This occurs, for example, when the security prices violate the no-arbitrage conditions which

---

[11]See Ross (1978) and Huang and Lizenberger (1988, Chapter 4.2) for results on two-fund separation for investors with risk averse expected utility functions. They prove that, the separating portfolios, if they exist, must be on the efficient frontier. See also Berk (1997) for conditions on asset returns and the expected utility functions that are sufficient for both two-fund separation and the CAPM.

are necessary for the existence of an optimal portfolio for all investors with increasing and continuous utility functions. Second, even if the optimal portfolio exists, the efficient portfolio with mean return corresponding to that of the optimal portfolio may not exist. This occurs, for example, when the mean return of the minimum variance portfolio is equal to the risk free rate. Finally, when the optimal portfolio and efficient portfolio both exist, it is still not obvious if the investor would choose to optimally hold the efficient portfolio because the investor may care about the higher moments beyond the second. In this section we study the optimal choice behavior for *risk averse* investors and explore the relevance of the efficient portfolios in their optimal choices.

We begin by recalling the definition of MPS-risk-aversion:

Definition 4 *An investor is said to be risk averse if $E[X] \succeq X$ for all $X \in \mathbb{D}$; and is said to be 'risk averse in the sense of mean-preserving spread', or simply 'MPS-risk-averse', if*

$$X \succeq X + \varepsilon, \ \text{where } E[\varepsilon] = 0 \ \text{and } Cov(X, \varepsilon) = 0,$$

with strict preference if $\varepsilon \neq 0$.

In this definition, ε is required to have zero mean and zero correlation with X. Therefore, it is natural to take $X + \varepsilon$ as being more risky than X, and thus MPS-risk-aversion captures agent's negative attitudes towards an increase in risk. Since X is a mean preserving spread of $E[X]$, MPS-risk-aversion implies risk aversion.

The notion of MPS risk aversion is the same as the notion of strict variance-averse preferences proposed by Duffie (1988). Moreover, Löffler (1996) showed that if the preference relation $\succeq$ on $\mathbb{D}$ is continuous in the $\mathbb{L}_2$-norm and is represented by a continuously differentiable utility function, and if the market contains a finite (but no less than three) number of tradable securities, then MPS-risk-averse preferences must admit a mean-variance utility representation. The equivalence between MPS-risk-aversion and mean-variance breaks down when $J = 2$. The smoothness assumption on the agent's utility function would appear to be essential for Löffler's result. Here, we distinguish between MPS-risk-averse preferences with mean-variance utility function because a binary relationship satisfying MPS-RA property forms a partial order, which may not necessarily admit an utility representation. Indeed, the analysis below is carried through without assuming the preference relationship to be complete and to admit an utility representation.

The concept of MPS differs also from the notion of second-order stochastic dominance (SSD). Rothschild and Stiglitz (1970) showed that X SSD $X + \varepsilon$ if, and only if, $E[\varepsilon \mid X] = 0$. The latter implies, but is not implied by, $E[\varepsilon] = 0$ and $Cov(X, \varepsilon) = 0$. Therefore, if a random variable Y is second-order stochastically dominated by the random variable X, then Y must be identical (in distribution) to a MPS of X. The converse is, in general, not true. The two partial orders, namely SSD and MPS, are equivalent for certain

class of distributions, for example, when the random variables are normally distributed.[12]

For MPS-risk-averse investors, our next result follows as a corollary to Proposition 1.

Proposition 5 *Let θ^* be an optimal portfolio holding for a MPS-risk-averse investor. Let μ^* be the portfolio mean return for the optimal portfolio θ^*. Then, if the efficient portfolio at μ^* exists, the optimal portfolio must be efficient.*

Proof. Let θ_0 be an efficient portfolio with mean return given by μ^*. By Proposition 1, the return of θ^* must be a mean-preserving spread of the return of θ_0. We must have $R^{\theta^*} = R^{\theta_0}$ since otherwise if $R^{\theta^*} \neq R^{\theta_0}$, $W_0 R^{\theta_0}$ must be preferred to $W_0 R^{\theta^*}$ by the MPS-risk-averse investor at initial wealth $W_0 > 0$, which contradicts the optimality of θ^*. Therefore, θ^* must be efficient at μ^*. ∎

In general, an MPS-risk-averse investor's optimal portfolio holding may not necessarily belong to the efficient frontier if the efficient portfolio at μ^* does not exist. It is obvious that any mean-variance investor with utility function $u(\mu, \sigma)$ which is continuous, monotonically increasing in the first argument, and decreasing in the second argument, must display MPS-risk-aversion. For mean-variance investors, we have a stronger result.

Proposition 6 *For mean-variance investors, their optimal portfolios, if they exist, must be efficient.*

Proof. Suppose to the contrary that there exists θ_0 such that $\mu[\theta_0] = \mu[\theta^*]$ and $\sigma[\theta_0] < \sigma[\theta^*]$. We have: $u\left(W_0 \mu[\theta_0], W_0 \sigma[\theta^0]\right) > u(W_0 \mu[\theta^*], W_0 \sigma[\theta^*])$. This contradicts the optimality of θ^*. ∎

Remark 3 *The existence of an efficient portfolio is a necessary condition for the existence of an optimal solution for mean-variance investors. That is, the existence of an optimal solution with a mean return μ^* implies the existence of an efficient portfolio at μ^*. Conversely, if no efficient portfolio exists at μ_0, then no investor with mean-variance preferences will optimally choose to hold portfolios with a mean return equal to μ_0.*

3.3 Mutual fund separation

It is well known that, with a finite number of securities, efficient frontiers are well-defined and all efficient portfolios can be expressed as a convex combination of two efficient portfolios. The optimal portfolio holdings for MPS-risk-averse investors can be easily characterized because they would have to be located on the efficient frontier.

This section concerns optimal portfolio choices for MPS-risk-averse investors (who prefer more to less) when they face a possibly infinite number of investment opportunities ($\#J = \infty$). Here we deal with an arbitrary given set of asset

---

[12]For normal distributions, $E[\varepsilon \mid X] = 0$ if, and only if, $E[\varepsilon] = 0$ and $\text{Cov}(X, \varepsilon) = 0$.

returns and for these cases we saw that the efficient frontiers are not always well-defined. Therefore, it is not surprising to find violation with respect to mutual fund separation, and to find non-existence of the optimal portfolio holdings (in Θ) for MPS-risk-averse investors. The situation will be dramatically different when the market is restricted to be in equilibrium. In equilibrium, we shall show that the efficient frontier (with a risk less asset) is well-defined, and the mutual fund separation holds (see section 5).

3.3.1 Choices with pure risky portfolios

We consider the case when the market contains purely risky portfolios. As a corollary to Propositions 1 & 2, the following important risk decomposition result is readily obtained:

Proposition 7 *Assume conditions A.1 and A.2 hold. For all $\theta \in \mathbb{H}_2(\Sigma)$, there exists a unique $\theta_0 \in \mathcal{I}_g$ such that R^θ is a MPS of R^{θ_0}.*

Note that, the risk-decomposition theorem holds for all generalized risky portfolios in $\mathbb{H}_2(\Sigma)$, thus in particular for risky portfolios in Θ. That is, the return for each risky portfolio θ in Θ must admit as a MPS to that of a g.e.p. θ_0 on $\mathcal{I}_g$. This implies that MPS-risk-averse investors would tend to hold generalized efficient portfolios if they were allowed to hold an infinite number of securities. In other words, investors would, in general, not be satiated with holding any arbitrary finite number of securities. However there are two exceptions to this result.

(a) when $\#J < \infty$, the g.e.f. $\mathcal{I}_g$ reduces to the Markowitz efficient frontier, and all g.e.p.s become efficient;

(b) $\#J = \infty$ and the (normalized) risky assets contain just a finite number distinct expected returns. There exists **one** (and only one) g.e.p. on $\mathcal{I}_g$ that involves a finite number of securities. Unless investors wish to invest in such a portfolio, their optimal portfolios being restricted in Θ would not exist.

So, in general, when confronted with an infinite number of risky investment opportunities, the MPS-risk-averse investors would rarely be satiated with a finite number of risky assets. This results in non-existence with respect to their optimal portfolio holdings in Θ.

Remark 4 *Their optimal portfolios would, however, be well-defined and be located on the g.e.f. $\mathcal{I}_g$, if they are allowed to hold an infinite number of securities, corresponding to $\mathbb{H}_2(\Sigma)$, keeping in mind that the (generalized) classical Black separation holds for $\mathbb{H}_2(\Sigma)$ following from Proposition 2.*

3.3.2 Portfolio choices with a risk free asset

The same conclusion applies to the case with a risk free asset. Suppose $\underline{\mu} \neq R_f$ so that the generalized tangent portfolio θ_T is well-defined. When $\theta_T \notin \Theta$, all generalized efficient portfolios (except the risk free asset) would not be admissible, yet highly desirable by the MPS-risk-averse investors ending up with the generic non-existence of optimal portfolios in Θ. For the case when $\theta_T \in \Theta$, the tangent ray and its reflection constitute the efficient frontier generated from risky portfolios in Θ and the risk free asset. The optimal portfolio holdings for MPS-risk-averse investors can be easily characterized. We have:

Proposition 8 *Assume A.1 and A.2 hold. Consider an investor with monotonic and MPS-risk-averse preferences.*

(a) *The investor's optimal portfolio, if it exists, must have an expected return no less than the risk free interest rate, $\mu^* \geq R_f$.*

(b) *When $\underline{\mu} \neq R_f$, the optimal portfolio, if it exists must be on the generalized efficient rays and consist of a combination of the risk free asset and the generalized tangent portfolio θ_T.*

(c) *When $\underline{\mu} = R_f$, the optimal portfolio, if not risk free, will not exist.*

Proof. To prove (a), suppose to the contrary, that $\mu^* < R_f$. For any given positive initial wealth W_0, we have: $W_0 R_f \succ W_0 \mu^* \succeq W_0 R^{\theta^*}$. That is, the investor will choose to invest in the risk free asset only. This contradicts the optimality of portfolio θ^*. Therefore, the optimal portfolio, if it exists, must have an expected return no less than R_f .

For (b), with $\underline{\mu} \neq R_f$ and $\mu^* \geq R_f$, from Proposition 3-(a), we know that the efficient portfolio exists at μ^* with a mean return given by μ^*. By Proposition 5, the optimal portfolio is efficient. Hence, Proposition 3-(b) implies that the optimal portfolio can be expressed as a combination of the risk free asset and the generalized tangent portfolio θ_T.

For (c), with $\underline{\mu} = R_f$ and $\mu^* \geq R_f$, by Proposition 4, the efficient portfolio does not exist for all $\mu_0 > R_f$, particularly for μ^* if $\mu^* \neq R_f$. By Proposition 5, the optimal portfolio would either be risk free or inefficient, or would not exist. We can further show that, the optimal portfolio would not exist if it is not risk free. Suppose, to the contrary that, the optimal portfolio θ^* exists and is not risk free. It must be located strictly within the area bordered by the two asymptotic rays in the μ-σ plane. In particular, we can always find portfolios θ' and θ'' in $\Theta, \theta' \neq \theta''$ so that θ^* is strictly located inside the area bordered by the sub-efficient frontier, $\mathcal{I}_0$, formed by the three (risky) portfolios θ^*, θ' and θ''.[13] Since Proposition 8 holds for the case with a finite number of securities, θ^* can be expressed as a mean-preserving-spread of a portfolio on $\mathcal{I}_0$. The latter contradicts to the optimality of θ^* for the MPS-risk-averse investor. ■

---

[13]Since, by assumption, θ^* is not efficient, we may set portfolios θ' and θ'' to be such that $\mu' < \mu^*, \sigma' < \sigma^*, \mu'' > \mu^*$ and $\sigma'' = \sigma^*$.

Finally, if investors are allowed to hold an infinite number of securities, say in $\mathbb{H}_2\left(\Sigma\right)$, their optimal portfolios would exist and be expressed as combinations of the risk free asset and the generalized tangent portfolio θ_T. The classical mutual fund separation theorem holds for $\mathbb{H}_2\left(\Sigma\right)$ though it does not hold for Θ when investors are restricted to hold a finite number of securities. Recall that there would be no more than one generalized efficient risky portfolio that is actually located in Θ. This is true in particular for the generalized tangent portfolio. Since, generically, the generalized tangent portfolio does not belong to Θ, so are those generalized efficient portfolios along the generalized efficient rays. It is in this sense we may say that, generically, the optimal demand correspondence in Θ for MPS risk averse investors does not exist.

4   Equilibrium CAPM

This section builds on our earlier results to derive the CAPM with MPS-risk-averse investors. First, we prove that, the CAPM holds for economies that contain a finite number of securities. Then we extend the result to the case when the number of securities is infinite.

Recall that investors have homogeneous beliefs and so they will perceive the same generalized efficient frontier as derived in the previous sections. Given the two-fund separation theorem (Proposition 8) and Proposition 3-(b), to prove the validity of the CAPM, it is sufficient to show that, (a) the generalized tangent portfolio exists in equilibrium, and (b) the generalized tangent portfolio coincides with the market portfolio in Θ.

The lemma below shows that, in equilibrium, we must have $\underline{\mu} \neq R_f$.

Lemma 1 *For economies with a finite number of tradable securities, existence of equilibrium implies $\underline{\mu} \neq R_f$.*

Proof. Suppose to the contrary that the equilibrium exists with $\underline{\mu} = R_f$. By Proposition 4, with $\underline{\mu} = R_f$, the efficient portfolios do not exist at all $\mu_0 \neq R_f$. By Proposition 8-(c), the optimal portfolio would be either given by the risk free asset, or would not exist. Since all investors investing in the risk free asset will necessarily violate the market clearing conditions for the risky assets, we can thus conclude that, the optimal portfolios do not exist for at least one investor. The latter contradicts the assumption of the existence of equilibrium. Therefore, in equilibrium, we must have $\underline{\mu} \neq R_f$. ■

We can further identify the generalized tangent portfolio to coincide with the market portfolio in equilibrium.

Lemma 2 *Given an economy $\mathcal{E}$ with a finite number of securities. Let $p_j \neq 0$, $j = 0, \cdots, J$, be the equilibrium prices. Then, the equilibrium generalized tangent portfolio must be given by the market portfolio; that is, $\theta_T = \theta^M$.*

Proof. By Lemma 1, in equilibrium, we have $\underline{\mu} \neq R_f$. Let θ^i be investor i's optimal portfolio holding. By Proposition 8, we can write: $\theta^i = \alpha_i \theta_T, \alpha_i \in \mathbb{R}_+$

or $\mathbb{R}_-$ depending on whether $\underline{\mu} > R_f$ or $\underline{\mu} < R_f$, with $1 - \alpha_i$ of the initial wealth invested in the risk free asset. Since, by assumption, the market is in equilibrium, we must have:

(i) $\sum_i (1 - \alpha_i) W_0^i = 0$;

(ii) $\sum_i \frac{\theta_j^i W_0^i}{p_j} = \phi_j^M$, for all $j = 1, \cdots, J$.

Condition (i) is the market clearing condition for the risk free asset, which means that the total borrowing equals the total lending. Condition (ii) is the market clearing condition for the risky assets — the total number of shares held by the investors must equal the total number of shares outstanding. Condition (ii) can be re-written as: $\sum_i \theta_j^i W_0^i = p_j \phi_j^M$ for all j; that is, the total dollar amount invested in security j equals the total market capitalization of the security. With $\theta^i = \alpha_i \theta_T$, condition (ii) further reduces to

$$\left(\sum_i \alpha_i W_0^i \right) \theta_T = [\phi_1^M p_1, \cdots, \phi_J^M p_J]^{\mathsf{T}} \equiv \left(\sum_j \phi_j^M p_j \right) \theta^M.$$

From (i), we have:

$$\sum_i \alpha_i W_0^i = \sum_i W_0^i \equiv \sum_i \sum_j \phi_j^i p_j \equiv \sum_j \phi_j^M p_j.$$

This yields $\theta_T = \theta^M$; that is, the generalized tangent portfolio coincides with the market portfolio. ∎

Remark 5 *Notice that, when $p_j \neq 0$ for all j, the risky returns have a non-singular variance-covariance matrix as long as the covariance matrix of the pay-offs δ is non-singular. This last assumption is to ensure that none of the tradable securities is redundant.*

Remark 6 *The market clearing condition (i) for the risk free asset is actually implied by the market clearing condition (ii) for the risky assets. We have seen that, condition (ii), together with $\theta^i = \alpha_i \theta_T$, implies $\left(\sum_i \alpha_i W_0^i \right) \theta_T = \left(\sum_j \phi_j^M p_j \right) \theta^M$. With $\theta_T^{\mathsf{T}} e = \left(\theta^M \right)^{\mathsf{T}} e = 1$, we have: $\sum_i \alpha_i W_0^i = \sum_j \phi_j^M p_j \equiv \sum_i W_0^i$; that is, condition (i) is valid. Of course, we must have $\theta_T = \theta^M$.*

Remark 7 *We can further show that, if equilibrium exists with $W_0^i > 0$ for all i, then we must have $\mu^M > \underline{\mu} > R_f$ and the market portfolio θ^M must be on the efficient ray.*

Proof of Theorem 1. Lemmas 1 and 2, together with Proposition 3-(b), imply the validity of the CAPM for economies with a finite number of securities. This concludes the proof of Theorem 1 for the case of $\#J < \infty$.

We can further prove the validity of the CAPM for economies with an infinite number of securities ($\#J = \infty$).

Let $\{\theta_i\}_{i \in I}$ be an equilibrium allocation for the infinite economy $\mathcal{E}$. We consider a hypothetic economy, say $\mathcal{E}_0$, which contains both the initial allocation $\{\phi_i\}_{i \in I}$ and the equilibrium allocation $\{\theta_i\}_{i \in I}$ in the market span. In fact, we can construct $\mathcal{E}_0$ by selecting all securities j in its market span whenever there exists an agent i such that either ϕ_i or θ_i involves a non-zero position in j. The constructed economy $\mathcal{E}_0$ has the following characteristics:

- $\mathcal{E}_0$ is finite. This is because the number of agents is assumed to be finite, and because, for all i, the portfolios ϕ_i and $\theta_i \in \Theta$ involve only a finite number of securities.

- The allocation $\{\theta_i\}_{i \in I}$ constitutes an equilibrium allocation for $\mathcal{E}_0$. This is because, given the equilibrium prices for $\mathcal{E}$, the portfolio θ_i, which is feasible and optimal for investor i in economy $\mathcal{E}$, remains feasible and optimal for i in economy $\mathcal{E}_0$. Moreover, by definition, the allocation $\{\theta_i\}_{i \in I}$ satisfies the market clearing conditions for $\mathcal{E}_0$.

- For all $j \in J$ that is tradable in $\mathcal{E}_0$, the CAPM holds for j; moreover, for all i, θ_i must be expressed as a combination of the risk free asset and the market portfolio. This is because, the CAPM and two-fund separation hold for all finite economies.

Now, consider an arbitrary security $j \in J$ which was excluded from the economy $\mathcal{E}_0$ as a tradable security. We modify the economy $\mathcal{E}_0$ constructed above by adding security j as a tradable security. This will not change the equilibrium allocation because $\{\theta_i\}_{i \in I}$ is an equilibrium allocation for the original economy $\mathcal{E}$ and because the number of shares outstanding for security j is zero. Consequentially, the CAPM must hold for security j as well. Since j is arbitrary, we conclude that the CAPM must hold for all $\theta \in \Theta$ that involves a portfolio of any finite number of securities. This completes the proof of Theorem 1 for the case when $\#J = \infty$. ∎

We see that, in equilibrium, adding (non-active) new risky assets does not affect investors' optimal portfolio holdings. But, we must emphasize that, adding new (non-redundant) assets would certainly change the shape of the (generalized) efficient frontier, which would in general be shifted further towards the left. The equilibrium conditions imply that, even though the shape of the efficient frontier formed by the risky portfolios will change with newly added risky securities, it won't change the composition of the generalized tangent portfolio(s) — in equilibrium the market portfolio still coincides with the generalized tangent portfolio for the new efficient frontier(s). This last observation holds true more generally when prices of all securities are determined through the CAPM (see Lemma 8 for proof). Moreover, since, in equilibrium, the generalized tangent portfolio is given by the market portfolio which involves a finite number of securities, the equilibrium efficient frontier for Θ with $\#J = \infty$ (and with a risk less asset) is well-defined and is given by the tangent ray and its reflection ray.

23

This last observation implies the validity of two-fund separation, along with risk-decomposition; that is, in equilibrium, investors optimally hold portfolios that involve combinations of the market portfolio and the risk free asset.[14]

We conclude this section with two additional remarks.

Remark 8 *In comparison to Duffie (1988, Theorem 11E), this proof of the CAPM is elementary and does not involve topological assumptions on the market span. In addition, we do not impose the continuity assumption on the equilibrium pricing rule. It is easy to verify that, the market span generated from an infinite number of tradable securities studied in this paper does not necessarily form a closed set in $\mathbb{L}_2$. For example, consider the sequence of portfolios $\{\theta_n\}_{n=1}^{\infty} \subset \Theta$ defined to be such that $\theta_n(j) = 1$ for all $1 \leq j \leq n$ and $\theta_n(j) = 0$ for $j > n$. Each portfolio involves a finite number of securities, but the sequence does not converge to an element in Θ. Even if the payoffs of all individual risky securities $\{\delta^j\}_{j=1}^{\infty}$ belong to $\mathbb{L}_2$, the payoff $\sum_{j \in J} \delta^j$ (in the limit) induced by the sequence, may not have a finite $\mathbb{L}_2$-norm.*

Remark 9 *Under fairly general conditions, the CAPM remains valid as an equilibrium model if the MPS-risk-averse investors are allowed to invest in an infinite number of securities. This is because, the generalized efficient frontier, which is well-defined under A.1. and A.2, would constitute the efficient frontier formed by infinite portfolios. Since MPS-risk-averse investors would all invest in the risk-free asset and the generalized tangent portfolio, in equilibrium, the market portfolio must coincide with the generalized tangent portfolio. The validity of the CAPM along with two-fund separation theorems follow from the properties of the generalized efficient frontier summarized in Propositions 2 and 3.*

5   Concluding remarks

This paper proves that the CAPM holds for economies with MPS-risk-averse investors. The CAPM model is shown to be valid without imposing any distributional restrictions on asset returns and the number of tradable securities. This approach contrasts with multi-factor models in the literature based on assumptions on the existence of some exogenous factor structure in modelling asset returns. Our results suggest that, so long as investors exhibit MPS-risk-aversion, the relevance of all those factors, that affect asset returns would all be summarized through the return of the market portfolio. This is true, at least, in equilibrium.

We implicitly assume that the investor's endowment is in the market span. For cases with non-spannable endowments such as in Hara (2001), the results on

---

[14]Black's separation theorem for risky portfolios still does not hold in equilibrium when we restrict $\theta \in \Theta$. This is because the efficient portfolios formed by finite risky portfolios are generically not defined (even in equilibrium) except when μ_0 is set at the expected return of the market portfolio, for which the efficient portfolio is given by the market portfolio. Black's separation holds, however, for generalized efficient portfolios in $\mathbb{H}_2(\Sigma)$.

the validity of the CAPM and the existence proofs are still valid. This is because the part of the endowment which is not market spannable must be orthogonal to the market span following the orthogonal decomposition theorem for the Hilbert spaces. Therefore, MPS-risk-averse investors still hold portfolios along the efficient rays. Moreover, the derivation of the CAPM and the existence proof provided in the Appendix below remain valid for the cases with non-spannable endowments.

In future research it would be desirable to consider economies with heterogenous beliefs as in Nielsen(1990) and Sun and Yang (2003) by maintaining the MPS-risk-averse behavior assumption studied in this paper.

It would be useful to extend this analysis to multi-period settings with MPS-risk-averse agents. In particular it would be of interest to explore the implications of MPS-risk-averse behavior assumption on the agents' portfolio trading strategies. Concerning the equilibrium security prices, we conjecture that for i.i.d. economies with MPS-risk-averse agents the CAPM will still constitute an equilibrium model. It remains to be seen to what extent the CAPM or the conditional-CAPM will represent an equilibrium model for non-i.i.d. economies with MPS-risk-averse agents. These topics will be examined in future work.

6   Appendix: Existence of Equilibrium CAPM

This appendix contains the proof of the existence of a market equilibrium satisfying the CAPM. Let Σ_0 be the positive definite variance-covariance matrix for the payoffs of all individual risky securities. Assume that $\phi^i \bullet \delta \geq 0$ for all i, and that $\delta_j \geq (\neq) \, 0$ for all j. Assume further that the payoff, $\delta^M \equiv \phi^M \bullet \delta \geq (\neq) \, 0$, of the market portfolio has a positive standard deviation $\sigma \left[\delta^M \right] > 0$. In addition to the assumption of MPS-risk-aversion, we restrict investors' preference to be represented by some increasing, continuous and strictly quasi-concave utility functions $U_i : \mathbb{D} \to \mathbb{R}^{15}$, $i = 1, \cdots, I$.

For all $d \in \mathbb{D}$, let $\Psi(d)$ be the *price* of the payoff d, that is implicitly determined from the CAPM.[16]

Lemma 3 *If equilibrium exists, then there exists $t > 0$ and $s > 0$ such that, for all $d \in \mathbb{D}$,*

$$\Psi(d) = E\left[\pi d\right] \ \ with \ \pi = t - s\widehat{c} \tag{20}$$

where $\widehat{c} \equiv \frac{\delta^M - E\left[\delta^M\right]}{\sigma\left[\delta^M\right]}$, $t \equiv \Psi(1) > 0$ *and* $s \equiv \frac{\Psi(1)E\left[\delta^M\right] - \Psi\left(\delta^M\right)}{\sigma\left[\delta^M\right]} > 0$.

Proof. By Theorem 1, the CAPM holds in equilibrium. Substitute the return $\frac{d}{\Psi(d)}$ together with $R_f = \frac{1}{\Psi(1)}$ and $R^M = \frac{\delta^M}{\Psi(\delta^M)}$ into the CAPM relation, and solve for $\Psi(d)$. This yields the desired expression for $\Psi(d)$ with t and s

---

[15]A real function f on $\mathbb{D}$ is said to be quasi-concave if for all $A \in \mathbb{R}$, the set $\{d \in \mathbb{D} : f(d) \geq A\}$ is a convex set (if not empty). It is said to be strictly quasi-concave if for all distinct d and $d' \in \{d \in \mathbb{D} : f(d) \geq A\}$, and for all $\alpha \in (0,1)$, $f(\alpha d + (1-\alpha) d') > A$.

[16]Notice that, for all $d \in \mathbb{D}$ the (equilibrium) price $\Psi(d)$ of d is well defined and is the common price of all financing portfolios.

as specified above. We note that s is positive because the equilibrium price of the market portfolio must not be greater than the present value of its expected payoff; otherwise, nobody would hold a positive quantity in the market portfolio, which leads to a violation of the market clearing condition. ∎

Note that π is a discount factor. Therefore, given the payoff on the market portfolio, and given the expressions for t and s, the equilibrium pricing rule is fully determined by $\Psi(1)$ and $\Psi[\delta^M]$, which are the prices of the risk free bond and that of the market portfolio.

Lemma 4 *If π is an equilibrium discount factor, then for all constants $k > 0$ $k\pi$ is also an equilibrium discount factor.*

By Lemmas 3 and 4, we can normalize the equilibrium discount factor in (20) to be such that $t + s = 1$ and write $\pi = \pi(r)$ with

$$\pi(r) = 1 - r - r\widehat{c}, \text{ for some } r \in [0,1]. \tag{21}$$

To prove the existence of equilibrium, we need to show that there exists an r such that, given prices determined by (20) with $\pi = \pi(r)$, the optimal portfolio exists for each investor and satisfies the market clearing conditions.

To ensure the existence of an optimal portfolio, we need to restrict r to be such that the pricing rule (20) satisfies the no-arbitrage conditions. Let $\mathbb{D}^\perp$ be the vector space that is orthogonal to the market span; that is, for all $d \in \mathbb{D}$ and $v \in \mathbb{D}^\perp$, $E[dv] = 0$.

Lemma 5 *If π is an equilibrium discount factor, then for all $v \in \mathbb{D}^\perp$, $\pi + v$ is also an equilibrium discount factor that supports the same equilibrium as π. In particular, there exists an $v \in \mathbb{D}^\perp$ such that $\pi + v > 0$.*

The first statement of the Lemma is obvious. The second part follows from the fundamental theorem of Dybvig and Ross (1987) since in equilibrium the market admits no arbitrage opportunities, and the latter is equivalent to the existence of a positive discount factor.

Given $\pi(\cdot)$ as defined above, let O be the subset of r for which a positive discount factor can be found which would support the same price vector as the one generated by $\pi(r)$; that is,

$$O \equiv \left(r \in [0,1] : \text{ there exists } v \in \mathbb{D}^\perp \text{ such that } \pi(r) + v > 0 \right). \tag{22}$$

Notice that, when $\pi(r)$ is positive, we can set $v = 0$.

Lemma 6 *The set O is an open and convex subset of $[0,1]$; in particular, we can write: $O = [0, r^*)$, for some $r^* \leq 1 \notin O$.*

Proof. First, we see that $0 \in O$ with $\pi(0) = 1$ and with $v = 0$. Second, we have $1 \notin O$. This is because, with $\pi = \pi(1) = -\widehat{c}$, the price for the market portfolio is negative: $\Psi(\delta^M) = -\sigma[\delta^M] < 0$. This violates the no-arbitrage condition since, by assumption, the payoff of the market portfolio, δ^M

is non-negative and does not equal to zero. Since $\pi(r)$ is linear in r, for any arbitrary $r_0 \in O$ with $\pi(r_0) + v_0 > 0$, $v_0 \in \mathbb{D}^\perp$, we have $\pi(r) + v_0 > 0$ for all $r \in \{r \in [0,1) : |r - r_0| < \epsilon\}$, and for all $\epsilon > 0$ that are sufficiently small. In particular, for $r_0 > 0$, and for ϵ sufficiently small, we have: $(r_0 - \epsilon, r_0 + \epsilon) \subset O$. Therefore, $O \subseteq [0,1]$ is open.

For all r_0 and r_1 in O, let v_0 and $v_1 \in \mathbb{D}^\perp$ be such that $\pi(r_0) + v_0 > 0$ and $\pi(r_1) + v_1 > 0$. For all $x \in [0,1]$, we have $v = xv_0 + (1-x)v_1 \in \mathbb{D}^\perp$ since $\mathbb{D}^\perp$ is a vector space. Moreover, we have: $\pi(r) + v = x[\pi(r_0) + v_0] + (1-x)[\pi(r_1) + v_1] > 0$; or, $r = xr_0 + (1-x)r_1 \in O$. Therefore, O is a convex subset of $[0,1]$.

Let $r^* \equiv \sup\{r : r \in O\} > 0$. We have $r^* \leq 1$ and, for all $r < r^*$, $[0,r] \subseteq O$ by the convexity of O. We can further claim that $r^* \notin O$, otherwise, $[0,r^*] = O$ which contradicts the openness of O. Therefore, we must have $O = [0,r^*)$. ∎

For all $r \in O$, let $\theta_T(r)$ and $\theta^M(r)$ respectively be the generalized tangent portfolio and market portfolio. For all $x \in \mathbb{R}^{\#J}$, let $\text{diag}[x]$ be the diagonal matrix with j-th diagonal element given by x_j.

Lemma 7 *For all $0 < r \in O$, the generalized tangent portfolio $\theta_T(r)$ is well-defined and is given by*

$$\theta_T(r) = \frac{diag\left[\Psi_r(\delta)\right] \Sigma_0^{-1} \left(E[\delta] - \Psi_r^{-1}(1)\Psi_r(\delta)\right)}{\Psi_r^{\mathsf{T}}(\delta) \Sigma_0^{-1} \left(E[\delta] - \Psi_r^{-1}(1)\Psi_r(\delta)\right)}. \tag{23}$$

Proof. Since no-arbitrage implies a positive linear pricing rule, the price $\Psi_r(\delta^j)$ for all securities j must be positive and be linear in r. We can further verify that, for all $0 < r \in O$, $\underline{\mu}(r) \neq \Psi_r^{-1}(1) = R_f$. This in turn implies, by Proposition 8, the existence of the efficient rays with the generalized tangent portfolio $\theta_T(r)$ well-defined and given by

$$\theta_T(r) = \frac{\Sigma^{-1}(\mu - R_f e)}{\langle \Sigma^{-1}e, \Sigma^{-1}\mu \rangle_\Sigma - R_f \|\Sigma^{-1}e\|_\Sigma^2}.$$

Notice that

$$\langle \Sigma^{-1}e, \Sigma^{-1}\mu \rangle_\Sigma - R_f \|\Sigma^{-1}e\|_\Sigma^2 = \|\Sigma^{-1}e\|_\Sigma^2 \left(\underline{\mu}(r) - R_f\right) \neq 0$$

for $0 < r \in O$. Let $E[\delta]$ and $\Psi_r(\delta)$ be the expected payoff vector and the positive price vector for the J risky assets. The desired expression for the generalized tangent portfolio is valid because

$$
\begin{aligned}
\Sigma^{-1} &= \text{diag}\left[\Psi_r(\delta)\right] \Sigma_0^{-1} \text{diag}\left[\Psi_r(\delta)\right], \\
E[\delta] - \Psi_r^{-1}(1)\Psi_r(\delta) &= \text{diag}\left[\Psi_r(\delta)\right](\mu - R_f e), \\
\Psi_r^{\mathsf{T}}(\delta) &= e^{\mathsf{T}}\text{diag}\left[\Psi_r(\delta)\right].
\end{aligned}
$$

∎

We can also show that

Lemma 8 *For all $0 < r \in O$, the generalized tangent portfolio is given by the market portfolio:*[17] $\theta_T(r) = \theta^M(r).$

Proof. By definition, the market portfolio is given by

$$\theta^M(r) \equiv \frac{\mathrm{diag}\left[\Psi_r(\delta)\right]\phi^M}{\Psi_r(\delta^M)}.$$

Since both the generalized tangent portfolio and the market portfolio have unit length, and since $\mathrm{diag}[\Psi_r(\delta)]$ is non-singular, it is sufficient to show that the ϕ^M is proportional to $\Sigma_0^{-1}\left(E[\delta] - \Psi_r^{-1}(1)\Psi_r(\delta)\right)$. By the definition of the discount factor, we have:

$$E[\delta] - \Psi_r^{-1}(1)\Psi_r(\delta) = \frac{r}{1-r}E[\widehat{c}\delta] = \frac{r}{1-r}\frac{1}{\sigma[\delta^M]}\Sigma_0\phi^M.$$

This yields

$$\phi^M = \left(r^{-1} - 1\right)\sigma\left[\delta^M\right]\Sigma_0^{-1}\left(E[\delta] - \Psi_r^{-1}(1)\Psi_r(\delta)\right)$$

as desired. ∎

In the light of this observation, by the two-fund separation theorem (Proposition 8), all MPS-risk-averse investors will choose to hold a combination of the market portfolio and the risk free asset. For all $r \in O$, let

$$\alpha^i(r) = \arg\max_{a\in\mathbb{R}_+} U_i\left((1-a)\frac{\Psi_r\left(\phi^i\cdot\delta\right)}{\Psi_r(1)} + a\frac{\Psi_r\left(\phi^i\cdot\delta\right)}{\Psi_r(\delta^M)}\delta^M\right) \qquad (24)$$

be the proportion of i's wealth $W_0^i \equiv \Psi_r\left(\phi^i\cdot\delta\right)$ that is optimally invested in the market portfolio $\theta^M(r)$.

Lemma 9 *For all $r \in O$ and for all i, we have:*

1. $\alpha^i(0) = 0.$

2. *If $\{r_n\}_{n=1}^\infty \subset O$ converges to $r^* \notin O$, then $\lim_{n\to\infty}\alpha^i(r_n) = +\infty.$*

3. $\alpha^i : O \to \mathbb{R}_+$ *is continuous.*

Proof. At $r = 0$, the price of the market portfolio is given by $E\left[\delta^M\right]$. The expected return of the market portfolio is thus given by the risk free interest rate $\left(\Psi_r^{-1}(1) = 1\right)$. This implies that, for all a, the portfolio return $(1-a)\frac{1}{\Psi_r(1)} + a\frac{\delta^M}{\Psi_r(\delta^M)}$ is a mean-preserving spread of the risk free return $\frac{1}{\Psi_r(1)}$. Therefore, all risk averse investors would optimally invest in the risk free asset with zero position in the market portfolio. That is, $\alpha^i(0) = 0$ for all i.

---

[17]Notice that, for arbitrary given prices, the market portfolio does not necessarily coincide with the tangent portfolio. But, with prices generated by the discount factor $\pi(r)$, for all $r \in O$, the two portfolios can be shown to coincide.

Since, by assumption, the utility function is continuous, the real function $f : \mathbb{R}_+ \times [0,1] \to \mathbb{R}$ defined by

$$f(a,r) \equiv U_i \left((1-a) \frac{\Psi_r \left(\phi^i \cdot \delta \right)}{\Psi_r (1)} + a \frac{\Psi_r \left(\phi^i \cdot \delta \right)}{\Psi_r (\delta^M)} \delta^M \right) \qquad (25)$$

is also continuous and strictly quasi-concave in a for any given $r \in [0,1]$. Consider the set-valued function

$$G(r) \equiv \left\{ a \in \mathbb{R}_+ : f(a,r) = \sup_{a \in \mathbb{R}_+} f(a,r) \right\}, r \in [0,1], \qquad (26)$$

which is either single-valued or empty. By Lemma 6, the set of $r \in [0,1]$ for which $G(r)$ is non-empty is given by $O \equiv [0, r^*) \subset [0,1]$. Therefore, for all $r \in O$, we can write $G(r) = \left\{ \alpha^i (r) \right\}$.

To prove the second statement, let $\{r_n\}_{n=1}^\infty \subset O$ converge to $r^* \notin O$. Consider the resulting sequence $\left\{ \alpha^i (r_n) \right\}_{n=1}^\infty \subset \mathbb{R}_+$. To show that $\lim_{n \to \infty} \alpha^i (r_n) = +\infty$, suppose, to the contrary, that $\left\{ \alpha^i (r_n) \right\}_{n=1}^\infty$ has a finite limit point given by $\alpha^* \geq 0$. Let $\{n_k\}_{k=1}^\infty$ be a convergent subsequence that converges to α^*. We have: for all $a \in \mathbb{R}_+$, $f(a, r_n) \leq f \left(\alpha^i (r_n), r_n \right)$ for all n, particularly holds true for the subsequence $\{n_k\}_{k=1}^\infty$. Let $k \to \infty$, by continuity of f, we have: $f(a, r^*) \leq f(\alpha^*, r^*)$, which holds true for all a. Therefore, $\alpha^* \in \arg\max_{a \in \mathbb{R}_+} f(a, r^*) \equiv G(r^*)$. This, however, contradicts the emptiness of $G(r^*)$ at r^*. Therefore, we must have: $\lim_{n \to \infty} \alpha^i (r_n) = +\infty$.

To show that $\alpha^i : O \to \mathbb{R}_+$ is continuous, it is sufficient to show that: for all $\{r_n\}_{n=1}^\infty \subset O$ converging to $r \in O$, the resulting sequence $\left\{ \alpha^i (r_n) \right\}_{n=1}^\infty$ converges to $\alpha^i (r)$. Firstly, we show that $\left\{ \alpha^i (r_n) \right\}_{n=1}^\infty$ must be a bounded sequence. Suppose, without loss of generality, that $\lim_{n \to \infty} \alpha^i (r_n) = +\infty$. Let $\{a_n\}_{n=1}^\infty \subset \mathbb{R}_+$ be an arbitrary sequence that converges to $\alpha^i (r)$. For any arbitrary $a > 0$, let $x_n = \frac{a}{\alpha^i (r_n)}$ for all n. For n sufficiently large, we have $x_n \in (0,1)$. Consider the sequence $\left\{ (1 - x_n) a_n + x_n \alpha^i (r_n) \right\}_{n=1}^\infty$ formed by the convex combinations of a_n and $\alpha^i (r_n)$. The sequence converges to $\alpha^i (r) + a$. By the quasi-concavity of f, and by the optimality of $\alpha^i (r_n)$, we have: $f \left((1 - x_n) a_n + x_n \alpha^i (r_n), r_n \right) \geq f(a_n, r_n)$. Let $n \to \infty$, it yields $f \left(\alpha^i (r) + a, r \right) \geq f \left(\alpha^i (r), r \right)$. This contradicts the unique optimality of $\alpha^i (r)$ for the given $r \in O$.

Now, let $\alpha \geq 0$ be any finite limit point of $\left\{ \alpha^i (r_n) \right\}_{n=1}^\infty$, and let $\{n_k\}_{k=1}^\infty$ be the convergent subsequence. We have for all $a \in \mathbb{R}_+$ and for all k, $f(a, r_{n_k}) \leq f \left(\alpha^i (r_{n_k}), r_{n_k} \right)$. Let $k \to \infty$, by continuity of f, we have: $f(a, r) \leq f(\alpha, r)$, which holds for all a. Also, since $r \in O$, $\sup_{a \in \mathbb{R}_+} f(a, r)$ achieves its maximum uniquely at $\alpha^i (r)$. Therefore, we conclude that $\alpha = \alpha^i (r)$. This implies that $\left\{ \alpha^i (r_n) \right\}_{n=1}^\infty$ has a unique limit point $\alpha^i (r)$; or, equivalently, $\lim_{n \to \infty} \alpha^i (r_n) = \alpha^i (r)$. This ends the proof of the third statement. ∎

As a necessary condition for the existence of a market equilibrium, the mar-

ket clearing condition for the risky assets implies:

$$\sum_i \alpha^i(r) \Psi_r(\phi^i \cdot \delta) = \Psi_r(\delta^M) ; \tag{27}$$

that is, the aggregate investment in all risky assets equals the value of the market portfolio. We have,

Proposition 9 *There exists an $0 < r \in O$ that solves equation (27).*

Proof. Let $\alpha^M(r) = \sum_i \alpha^i(r) \frac{\Psi_r(\phi^i \cdot \delta)}{\Psi_r(\delta^M)}, r \in O$. By Lemma 9, the function $\alpha^M : O \to \mathbb{R}_+$ is continuous, and has the following two properties:
(a) $\alpha^M(0) = 0$, (b) $\lim_{n \to \infty} \alpha^M(r_n) = \infty$ for all $\{r_n\}_{n=1}^{\infty} \subset O$ converging to r^*. By the continuity of α^M, there exists an $r \in (0, r^*)$ such that $\alpha^M(r) = 1$; or, equivalently, $\sum_i \alpha^i(r) \Psi_r(\phi^i \cdot \delta) = \Psi_r(\delta^M)$. ∎

Now, we are ready to prove the main existence theorem.

Proposition 10 *There exists an $r \in (0,1)$ such that $\pi(r)$ constitutes an equilibrium discount factor.*

Proof. Let $0 < r \in O$ be a solution to equation (27). Lemma 8, together with Proposition 8, implies that i's optimal portfolio θ^i in the risky assets, for the given r, is proportional to the market portfolio. We write $\theta^i = \alpha_i(r) \theta^M(r)$. With initial wealth given by $W_0^i \equiv \Psi_r(\phi^i \cdot \delta)$, we have:

$$\sum_i (1 - \alpha_i(r)) W_0^i$$
$$= \Psi_r(\phi^M \cdot \delta) - \sum_i \alpha_i(r) \Psi_r(\phi^i \cdot \delta)$$
$$= \Psi_r(\delta^M) - \sum_i \alpha_i(r) \Psi_r(\phi^i \cdot \delta)$$
$$= 0;$$

that is, the net borrowing in the risk free asset is zero; and for all risky assets j, we have:

$$\sum_i \frac{\theta_j^i W_0^i}{\Psi_r(\delta^j)}$$
$$= \sum_i \frac{\alpha_i(r) \theta_j^M(r) \Psi_r(\phi^i \cdot \delta)}{\Psi_r(\delta^j)}$$
$$= \sum_i \frac{\alpha_i(r) \Psi_r(\phi^i \cdot \delta)}{\Psi_r(\delta^j)} \theta_j^M(r)$$
$$= \frac{\Psi_r(\delta^M)}{\Psi_r(\delta^j)} \frac{\Psi_r(\delta^j) \phi_j^M}{\Psi_r(\delta^M)}$$
$$= \phi_j^M;$$

30

or, in other words, the total number of shares invested in risky asset j equals to the number of shares outstanding for the asset. Therefore, the pricing rule resulting from the discount factor $\pi(r)$ constitutes a market equilibrium. This concludes the proof. ∎

References

[1] Berk J. (1997), "Necessary and Sufficient Conditions for the CAPM", *Journal of Economic Theory 73:* 245-257

[2] Black F. (1972), "Capital Market Equilibrium with Restricted Borrowing", *Journal of Business 45:* 444-455.

[3] Bottazzi J., T. Hens and A. Löffler (1998), "Market Demand Functions in the Capital Asset Pricing Model", *Journal of Economic Theory 79:* 192-206.

[4] Cass D. and J. Stiglitz (1970), "The Structure of Investor Preferences and Asset Returns, and Separability in Portfolio Selection: A Contribution to the Pure Theory of Mutual Funds", *Journal of Economic Theory 2:* 122-160.

[5] Chamberlain G. (1983), "A Characterization of the Distribution that Imply Mean-Variance Utility Functions", *Journal of Economic Theory 29:* 185-201.

[6] Chamberlain G. and M. Rothschild (1983), "Arbitrage, Factor Structure, and Mean-Variance Analysis on Large Asset Markets", *Econometrica 50:* 1281-1304.

[7] Constantinides G. and A. Malliaris (1995), "Portfolio Theory" in *Handbooks in Operations Research and Management Science, Vol. 9:* 1-30.

[8] Dana R. A., "Existence, Uniqueness and Determinacy of Equilibrium in CAPM with a Riskless Asset", *Journal of Mathematical Economics 32:* 167-175.

[9] Diamond P. and J. Stiglitz (1974), "Increases in Risk and Risk Aversion", *Journal of Economic Theory 8:* 337-360.

[10] Duffie D. (1988), *Security Markets: Stochastic Models.* Academic Press.

[11] Dybvig P. and S. Ross (1987), "Arbitrage" in *The New Palgrave: A Dictionary of Economics.* The Macmillan Press Limited.

[12] Epstein L. (1992), "Behavior under Risk: Recent Developments in Theory and Applications" in *Advances in Economic Theory — Sixth World Congress,* J.J. Laffont eds., Cambridge University Press.

[13] Hansen L. and R. Jagannathan (1991), "Implications of Security Market Data for Models of Dynamic Economies", *Journal of Political Economy 99:* 225-262.

[14] Hara C. (2001), "Equilibrium Prices of the Market Portfolio in the CAPM with Incomplete Financial Markets", working paper, University of Cambridge.

[15] Huang C.F. and R. Litzenberger (1988). *Foundations for Financial Economics.* Prentice Hall, Inc.

[16] Lintner J. (1965), "The Valuation of Risky Assets and the Selection of Risky Investment in Stock Portfolios and Capital Budgets", *Review of Economics and Statistics 47:* 13-37.

[17] Litzenberger R. and K. Ramaswamy (1979), "On Distributional Restrictions for Two Fund Separation", *TIMS Studies in the Management Science 11:* 99-107.

[18] Löffler A. (1996), "Variance Aversion Implies μ-σ^2-Criterion", *Journal of Economic Theory 69:* 532-539.

[19] Markowitz (1959). *Portfolio Selection.* New York: John Wiley and Sons, Inc.

[20] Mossin J. (1966), "Equilibrium in a Capital Asset Market", *Econometrica 34:* 768-783.

[21] Nielsen L. (1990), "Existence of Equilibrium in CAPM", *Journal of Economic Theory 52:* 223-231.

[22] Owen J. and R. Rabinovitch (1983), "On the Class of Elliptical Distributions and Their Applications to the Theory of Portfolio Choice", *Journal of Finance 38:* 745-752.

[23] Quiggin J. (1982), "A Theory of Anticipated Utility", *Journal of Economic Behavior and Organization 3:* 323-343.

[24] Sharpe W. (1964), "Capital Asset Prices: A Theory of Capital Market Equilibrium under Conditions of Risk", *Journal of Finance 19:* 425-442.

[25] Ross S. (1971), "Risk and Efficiency", University of Pennsylvania Discussion Paper No.196, 1971.

[26] Ross S. (1978), "Mutual Fund Separation in Financial Theory — The Separating Distributions", *Journal of Economic Theory 17:* 254-286.

[27] Sun N. and Z. Yang (2003), "Existence of Equilibrium and Zero-Beta Pricing Formula in the Capital Asset Pricing Model with Heterogeneous Beliefs", *Annals of Economics and Finance 4:* 51-71.

[28] Tobin J. (1958), "Liquidity Preference and Behavior Towards Risk", *Review of Economic Studies 25:* 65-86.

[29] Yaari M. (1984), "Risk Aversion without Diminishing Marginal Utility", International Center for Economics and Related Disciplines, London.

[30] Yaari M. (1987), "A Dual Theory of Choice under Risk", *Econometrica 55:* 95-115.