

MPS Risk Aversion and MV Analysis in Continuous Time with Lévy Jumps

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Abstract

This paper studies sequential portfolio choices by MPS-risk-averse investors in a continuous time jump-diffusion framework. It is shown that the optimal trading strategies for MPS risk averse investors, if they exist, must be located on a so-called ‘temporal efficient frontier’ (t.e.f.). Analytic and qualitative characterizations of the t.e.f. are provided and are shown to form a hyperbola in the μ - σ plane. This paper also provides insights on (i) dynamic consistency underlying those temporal efficient trading strategies; (ii) mutual fund separation in extending the classical notion of Tobin (1958) and Black (1972) to this continuous-time setting; (iii) risk decomposition in presence of Lévy jumps, and (iv) differences between MPS risk averse investors

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and the expected utility investors concerning their optimal trading behaviour.

1 Introduction

This paper is on sequential portfolio choice, dynamic risk management, and intertemporal mean-variance efficiency in a context of continuous time jump-diffusion framework. The analysis extends the existing literature in two-folds: First, investors are assumed to display mean-preserving-spread (MPS) risk aversion.¹ Second, asset returns follow a jump-diffusion process, and the motion of the stock prices is assumed to be driven by an exogenous Markov state process $\{x_t\}$ that is itself a Lévy process. Our contribution involves an analytic characterization of the so-called temporal efficient frontier in the jump-diffusion framework, along with several insights into optimal trading strategies conducted by MPS risk averse investors.

1.1 MPS Risk Aversion and Portfolio Choice

The notion of MPS-risk-aversion is taken from Boyle and Ma (2002). An investor is said to display MPS-risk-aversion if s/he prefers X to Y whenever Y is identical in distribution to an MPS of X . The MPS risk averse preference differs from risk averse expected utility. MPS risk averse preference as a partial order may not admit a mean-variance utility representation though any mean-variance utility function (that is decreasing in standard deviation) must display MPS-risk-aversion. These are well illustrated in Boyle and Ma (2002).

It is also noted that, mean-preserving-spread as a partial order can capture higher moments of the distribution, in addition to the first two moments.

¹Here, MPS is the abbreviation of ‘mean-preserving-spread’. For arbitrary random payoffs X and Y , Y is said to be a mean-preserving-spread of X if there exists an ε , with $E[\varepsilon] = 0$ and $\text{Cov}(X, \varepsilon) = 0$, such that $Y = X + \varepsilon$. MPS constitutes a partial order on the space of random payoffs.

The MPS risk averse preference is thus readily able to model investors' psychological aversion towards downside risk, in contrast to the classical mean-variance preferences. Moreover, according to Boyle and Ma (2002), MPS-risk-aversion constitutes the key behavioral assumption underlying the classical Markowitz's (1952, 1959) mean-variance analysis, and also for the validity of the CAPM, one of the corner stones for modern finance which has originated from Sharpe (1964) and Lintner (1965), along with the insight from Tobin (1958) and Black (1972) on mutual fund separation.

The notion of MPS risk aversion can be readily extended to continuous time setting. For any arbitrary given trading session $[0, T]$ concerned, let $X_{0,T}$ and $X'_{0,T}$ be the final payoffs resulting from self-financing trading strategies ϕ and ϕ' , respectively. Typical MPS-risk-averse investors would prefer ϕ to ϕ' whenever $X'_{0,T}$ is expressed as an MPS of $X_{0,T}$, provided that both trading strategies have the same initial cost of capital.

In the meantime, we may also extend Markowitz's mean-variance efficiency in continuous time. Consider the set of terminal payoffs generated from self-financing trading strategies on trading session $[0, T]$. A trading strategy with instantaneous expected growth rate μ_0 on $[0, T]$ is said to be temporal efficient at μ_0 if there exists no other self-financing trading strategy that has a smaller risk and yet achieves the target rate μ_0 within the trading session $[0, T]$. Here, the risk is understood to be measured by the standard deviation / variance with respect to the terminal wealth. The curve on (μ, σ) -plane, which is formed by the set of all mean-variance efficient trading strategies, is known as *temporal efficient frontier* (t.e.f.). Almost by definition, all MPS risk averse investors will invest optimally along the t.e.f..

Unlike Markowitz's one-period problem, to construct the temporal efficient trading strategy and to analytically derive the t.e.f. is, by no means, an easy task. This is mainly due to the fact that, the temporal MV-efficient trading strategies involve continuous trading, and investors need to revise its portfolio holdings continuously upon newly arrival of information. In conse-

quence, to analytically characterize the set of efficient trading strategies, we need to fully characterize the revision of the optimal portfolio holdings for the entire trading session. This represents a challenging mathematical problem relative to the original one-period problem studied by Markowitz (1952) and Boyle and Ma (2002). So, this paper represents probably the very original effort in tackling the sequential portfolio choice problem.

Interestingly, as illustrated below in this paper, we may transform the efficient portfolio choice problem into a so-called ‘optimal tracking problem’. The optimal tracking problem is best understood as a problem of controlling a moving object to reach a specific target within a prespecified time interval $[0, T]$. The controller must try to maintain the moving object to move at about a constant pre-specified target speed, while maintaining the time- T location of the object as close to the target location as possible. This problem is solved by applying the standard variational method together with the dynamic programming technique.

It is shown that the optimal trading strategies for MPS risk averse investors, if exist, must be located on the t.e.f.. Analytic and qualitative characterizations of the t.e.f. are provided and are shown to form a hyperbola in the μ - σ plane. This paper also provides insights into

- dynamic consistency underlying those temporal efficient trading strategies;
- mutual fund separation in extending the classical notion of Tobin (1958) and Black (1972) to continuous-time;
- risk decomposition in continuous time with Lévy jumps, and
- differences between MPS risk averse investors and the expected utility investors concerning their optimal trading behavior.

1.2 Managing Risk of Rare Events

The existing literature on risk management and portfolio choice in continuous time has largely been in the context of diffusion. The books by Korn (1997) and Merton (1990) contain an extensive coverage of the literature. The sequential choice problem, or simply Merton's problem, can be traced back to Merton (1971,1973) for his derivation of the optimal risky portfolio by expected utility investors with constant Arrow-Pratt measure of relative risk aversion (RRA). The optimal portfolio is expressed as

$$\theta^* = \frac{1}{\text{RRA}} \times [\sigma\sigma^\top]^{-1} [\mu - r\bar{1}],$$

where $\sigma\sigma^\top$ is the instantaneous variance-covariance matrix for the risky assets, and $\mu - r\bar{1}$ is the excess instantaneous mean return vector for the risky assets. The formula suggests that, for EU investors with constant RRA, the optimal portfolio is proportional to the instantaneous tangent portfolio that is located on the so-called 'local mean-variance efficient frontier'. This is in analoguing to Markowitz's original finding (for mean-variance investors) in static setting.

Research on mean-variance analysis and portfolio choice in continuous time includes Richardson (1989), Duffie and Richardson (1991), Schweizer (1992), and more recently Bielecki, Jin, Pliska and Zhou (2005). Treatments in all these papers are within the pure diffusion framework. Mean-variance analysis in continuous time jump-diffusion framework, which remains largely an untouched research territory, constitutes the aim of this paper.

The diffusion specification on the state process, along with the security price processes, fails to capture the presence of rare events, the possibility of some sudden changes in the economic conditions, shifts in economic policies, and other possible changes in the economic environment, which all leads to dramatic changes in security prices. Empirical research conducted by Press (1967), Jarrow and Rosenfeld (1984), Ball and Torous (1985), and recent

studies by Bates (1996), Eraker, Johannes and Polson (2003) and Eraker (2004), all confirmed the existence of significant jump risk in security prices.

The jump risk can be best modelled by introducing the Lévy jump in characterizing the motion of the state process along with the security price process (see Merton 1976, Naik and Lee 1990, Ma 1992, 2005, 2006). By assuming the state process to follow a Lévy process, we are able to simultaneously model both the Brownian motion, corresponding to continuous local risk, and the ‘jump’ events summarized through a so-called Lévy measure. The Lévy measure summarizes both the frequency and the jump size distribution.

To hedge against jump risk and the risk of rare events has long been recognized as a challenging task. Naik and Lee (1990) are among the first to point out that the market becomes incomplete in presence of jump risk. Ma (1992) goes further to assert that it is essentially impossible to fully hedge against the jump risk using options trading unless there is no Brownian risk, and when the jump size is constant. The latter corresponds to the case studied by Cox and Ross (1976). In fact, even with portfolios involving arbitrary large but finite number of assets, as is to be illustrated below, efficient portfolio in presence of jump risk could deviate dramatically from what has been prescribed above in the diffusion case, without mentioning the possibility of obtaining a fully hedged portfolio. This is largely due to the fact that, the temporal efficient trading strategy is to hedge against not only the local instantaneous risk associated with the Brownian motion, but also the rare jump risk, in addition to the risk on how investors perceive future investment opportunities. The latter is known as ‘shadow risk’ as it reflects risk associated with the shadow prices.

While it is hard to fully hedge against different sources of risk simultaneously, the relevance of temporal mean-variance efficient trading strategies as a useful venue towards portfolio risk management can be readily established. Following the same logic in Boyle and Ma (2002), we can show (see section

2.3 below) that, to investors who display MPS-risk-aversion, their optimal trading strategies, if they exist, must be temporal mean-variance efficient. Indeed, what we wish to accomplish in this paper is exactly to construct analytically the temporal mean-variance efficient frontier in presence of jump risk.

1.3 EU vs MPS RA Investors: How Differently They Trade?

As a relevant issue, we explore the difference between MPS risk averse investors and expected utility investors concerning their trading behavior. We shall restrict our discussion to the case in absence of jump risks — this corresponds to the case in which investors from either groups would invest in the same composition of risky portfolio. Yet, they are found to follow very different trading patterns concerning the portfolio weights assigned to the separating portfolios. Such differences can be also best reflected through the instantaneous mean return and local volatility with respect to the balance of the investment accounts respectively managed by investors from the two groups.

Various observations concerning mutual fund separation and risk decomposition can be made as well in this general jump-diffusion framework. Extension of Black's (1972) two mutual fund separation theorem within the temporal framework is found sensible to the existence of jump risk. In general, investors with different target rates may end up adopting very different trading strategies; in particular, the optimal trading strategy by an MPS risk averse investor may not be expressed as convex combinations of two arbitrary efficient trading strategies. Moreover, in general, it no longer holds true that, "if ϕ and ϕ' are temporal efficient trading strategy (say with targeting rate μ and μ'), then for all constant $\alpha \in (0, 1)$, $\alpha\phi + (1 - \alpha)\phi'$ is efficient". Conditions for the validity of statement of this sort will be explored in this paper.

1.4 Organization of The Paper

The rest of the paper is organized as follows: In Section 2, we provide a setup of the model. It includes an introduction of the state process in summarizing uncertainty of the economy, and specifications on the return process for all tradable securities. Formal definition of temporal efficient trading strategy, along with some discussions on the behavior assumption underlying MPS-risk-averse investors and on the formulation of the corresponding sequential portfolio choice problem, are also covered in this section. In Section 3, we transform the efficient portfolio choice problem into an optimal tracking problem, and show how the problem can be solved with the dynamic programming technique. Section 3 also contains some useful insights into the qualitative properties of the t.e.f. and its relationship with the local instantaneous mean-variance efficient frontier. Sections 4 and 5 concern implications on the robustness of mutual fund separations and risk diversification in continuous time, respectively. Section 6 is an in-depth discussion on the difference between MPS risk averse investors and the expected utility investors concerning their optimal trading behavior. Some useful remarks are provided in Section 7.

2 Setup of the Model

We take as primitive a trading session $[0, T]$. The uncertainty is summarized by a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ with increasing and right continuous information filtration. The market contains a fixed number of securities, indexed by $j \in \{1, 2, \dots, J\}$, in addition to a risk-free saving account with instantaneous risk free interest rates $\{r_t\}$. A typical task for investors is to choose a trading strategy within the trading session to maximize its temporal utility. The trading strategy is associated with a portfolio holding of the tradable securities at each spot market t contingent on realizations of state of nature ω in Ω .

The following are some additional descriptions with respect to each component of the market environment; namely, the information structure, the market structure, and the behavior assumptions underlying the economic agents, namely, the investors.

2.1 Jump-Diffusion State Process and Returns

Throughout this paper, we make the following specifications on the information structure along with the return process for all tradable securities. These specifications are largely taken from Ma (1992, 2006), and are known to be sufficiently flexible for carrying out most applications in continuous time finance.

- The nature of uncertainty in this economy is assumed to be summarized by a probability space $(\Omega, \mathcal{F}, \mathbb{P})$. The information filtration $\mathcal{F} = \{\mathcal{F}_t\}_{t \in [0, T]}$ is generated from a n -dimensional Markov jump-diffusion Lévy process $\{x_t\}$ that is governed by the following stochastic differential equation:

$$dx_t = b(t, x_t) dt + a(t, x_t) dB_t + \int_{\mathbb{R}^k} l(t, x_t, u) v(dt, du) \quad (1)$$

with initial state x_0 , and with coefficients b, a and l to be deterministic continuous functions of dimension $n \times 1, n \times m$ and $n \times 1$, respectively. Here, $\{B_t\}_{t \in [0, T]}$ is a m -dimensional standard Brownian motion; and $v(\cdot, \cdot)$ on $[0, T] \times \mathbb{R}^k$ is a random Poisson measure of a k -dimensional Lévy process. The corresponding Lévy measure of the Lévy process is denoted $v(du)$. The process $\{x_t\}$ is understood as the state process as driving force for the evolution of the economy, particularly for the motion of stock prices described below.

- Security j is associated with a life-long cumulative dividend stream $\{D_t^j\}_{t \in [0, T]}$ and a price process $\{S_t^j\}_{t \in [0, T]}$. The payoffs and security

prices are all adapted to the state process $\{x_t\}$. The total instantaneous return for each of the tradable security j admits the following decomposition

$$\frac{dX_t^j}{S_t^j} = \mu^j(t, x_t) dt + \sigma^j(t, x_t) dB_t + \int \gamma^j(t, x_t, u) v(dt, du) \quad (2)$$

with $dX_t^j \equiv dS_t^j + dD_t^j$. We assume that $\gamma^j > -1$.

- The interest rates $\{r_t\}$ are also assumed to be adapted to the state process $\{x_t\}$. We write $r_t \equiv r(t, x_t)$.

Under these specifications, the cash balance $\{X_t^\phi\}$ of the investment account resulting from self-financing trading strategy ϕ evolves according to

$$\frac{dX_t^\phi}{X_t^\phi} = [r_t + \theta_t^\top (\mu_t - r_t \mathbf{1})] dt + \theta_t^\top \sigma_t dB_t + \int \theta_t^\top \gamma_t(u) v(dt, du) \quad (3)$$

with

$$\theta_t \triangleq [\theta_t^1, \dots, \theta_t^J]^\top, \mu_t \triangleq [\mu_t^1, \dots, \mu_t^J]^\top \quad (4)$$

$$\sigma_t \triangleq [\sigma_t^1, \dots, \sigma_t^J]^\top, \gamma_t \triangleq [\gamma_t^1, \dots, \gamma_t^J]^\top. \quad (5)$$

Here, θ_t^j is the proportion invested in security j with $\theta_t^j \triangleq \frac{\phi_t^j S_t^j}{\phi_t \cdot S_t}$ for all $j \neq 0$. The cash position deposited in the saving account is given by $(1 - \sum_{j \neq 0} \theta_t^j) X_t^\phi$.

Remark 1 *It is noted that, the cash balance of the investment account remains positive throughout the entire trading session so long as $\theta_t^\top \gamma_t > -1$ for all $t \in (0, T)$, \mathbb{P} -a.s.. In other words, for trading strategies to satisfy the restriction $\theta^\top \gamma > -1$, the resulting cash balance $\{X_t^\phi\}$ must be positive and*

evolve according to the following SDE:

$$d \ln X_t^\phi = \left(r_t + \theta_t^\top [\mu_t - r_t \mathbf{1}] - \frac{1}{2} \theta_t^\top \sigma_t \sigma_t^\top \theta_t \right) dt + \theta_t^\top \sigma_t dB_t + \int \ln [1 + \theta_t^\top \gamma_t(u)] v(dt, du).$$

A trivial trading strategy resulting in positive cash balance is to set $\theta \equiv \emptyset$; that is, to invest in the risk free saving account. For the special case where there exists no jump risk, the resulting cash balance $\{X_t^\phi\}$ must be positive \mathbb{P} -a.s. for any self-financing trading strategy ϕ . Since, in this paper, we do not impose explicitly restrictions on borrowing, we do not need to impose positive cash balance as a constraint.²

2.2 Temporal MV-Efficiency

The concept of temporal MV-efficiency is a straightforward extension to the notion of Markowitz's static MV-efficiency.

Definition 2 *Within an arbitrarily given trading session $[0, T]$, for all $\mu_0 \in \mathbb{R}$, a self-financing trading strategy $\phi(\mu_0)$ is said to be temporal MV-efficient, or simply MV-efficient, at μ_0 if*

$$\phi(\mu_0) = \arg \min_{\phi \in \Phi_T^0} \left\{ \sigma_0 \left[R_{0,T}^\phi \right] : \mathbb{E}_0 \left[R_{0,T}^\phi \right] = e^{\mu_0 T} \right\}; \quad (6)$$

that is, among all self-financing trading strategies with the target rate of return μ_0 , the corresponding MV-efficient trading strategy $\phi(\mu_0)$ is the one that involves the minimum risk in reaching the target rate μ_0 .

²A different problem can be formulated here is when negative cash balance is not permitted, for example, as part of the liquidity restrictions from financial regulation. In this circumstance, we must impose explicitly the non-negative constraint as part of the restriction to feasible trading strategies even though investor can tolerate negative payoffs psychologically.

The notion of temporal MV-efficiency extends to any arbitrary sub-trading sessions, say $(t, T]$. Since an MV-efficient trading strategy involves continuous trading, investors need to revise their portfolio holdings continuously upon new arrival of information. As a result, to analytically characterize the efficient trading strategy represents a more challenging mathematical problem relative to the original one-period problem studied by Markowitz (1952) and Boyle and Ma (2002). One may thus wonder if it is possible to construct a dynamic consistent efficient trading strategy so that the efficient trading strategy designed at time $t = 0$ for the entire trading session $[0, T]$ would be optimally carried out at all future contingencies; in particular, if the efficient trading strategy contingent on the sub-trading session $[t, T]$ would remain efficient for the sub-session.

One may also wonder how it is possible to analytically characterize the temporal efficient frontier, along with the evolution of the temporal efficient frontier in time. All these will be tackled in the following sections.

2.3 MPS-Risk-Aversion and Temporal MV-efficiency

The notion of mean-preserving-spread (MPS) is taken from Boyle and Ma (2002), which can be readily extended to the context of sequential choice in continuous time. For any arbitrary trading session $[0, T]$, let X and X' be the end of session payoffs resulting from self-financing trading strategies ϕ and ϕ' , respectively. X' is said to be a *mean-preserving-spread* of X if $X' = X + \varepsilon$ with $\mathbb{E}_0[\varepsilon] = 0$ and $\text{Cov}_0(X, \varepsilon) = 0$.

We consider investors who care only about the terminal wealth at the end of trading session $[0, T]$, and rank different self-financing trading strategies by applying the mean-preserving-spread criterion with respect to the resulting end of session returns or terminal payoffs. Or, more explicitly, let $X_{0,T}^\phi$ and $X_{0,T}^{\phi'}$ be the final payoffs respectively resulting from self-financing trading strategies ϕ and ϕ' . MPS-risk-averse investors would prefer ϕ to ϕ' whenever $X_{0,T}^{\phi'}$ is expressed as an MPS of $X_{0,T}^\phi$.

The following observation is on the MV-efficient trading strategy and its relevance for MPS risk averse investors' choices:

Proposition 3 *Let $\phi_0 \in \Phi_T^0$ be an efficient self-financing trading strategy at μ_0 . Then, for all self-financing trading strategies $\phi \in \Phi_T^0$ with expected growth rate μ_0 , $X_{0,T}^\phi$ must be expressed as an MPS of $X_{0,T}^{\phi_0}$. Moreover, for MPS risk averse investors, its optimal trading strategy ϕ^* , if exists, must be MV-efficient.*

Proof. The total return, denoted $R_{0,T}^\phi$, associated with trading strategy ϕ can be interpreted as the end of session payoff starting with unit initial cash position. Consider the set of trading strategies

$$\{\alpha\phi + (1 - \alpha)\phi_0 : \alpha \in \mathbb{R}\}$$

formed by convex combinations of trading strategies ϕ_0 and ϕ . These trading strategies remain self-financing, and to have their total return be given by $\alpha R_{0,T}^\phi + (1 - \alpha) R_{0,T}^{\phi_0}$. The expected total return is $e^{\mu_0 T}$. Since ϕ_0 is efficient at μ_0 , the total risk $\sigma_0 [\alpha R_{0,T}^\phi + (1 - \alpha) R_{0,T}^{\phi_0}]$ must achieve its minimum at $\alpha = 0$. The first order condition leads to $\text{Cov}_0 (R_{0,T}^\phi, R_{0,T}^{\phi_0}) = \sigma_0^2 [R_{0,T}^\phi]$. Let $\varepsilon_T \equiv R_{0,T}^\phi - R_{0,T}^{\phi_0}$. We have: $\mathbb{E}_0 [\varepsilon_T] = 0$ and $\text{Cov}_0 (R_{0,T}^{\phi_0}, \varepsilon_T) = 0$.

To prove the second part of the proposition, suppose to the contrary that there exists a self-financing and budget feasible trading strategy ϕ to be such that $\mu_T^\phi = \mu_T^*$ and $\sigma_T^\phi < \sigma_T^*$, where μ_T^* and σ_T^* are respectively the expected final payoff and standard deviation of the final payoff, X_T^* , generated from the optimal trading strategy ϕ^* . Let $W_0 > 0$ be the initial cash position of the investment account. We may write $X_T^* = W_0 R_{0,T}^*$ and $X_T^\phi = W_0 R_{0,T}^\phi$. Consider the set of trading strategies $\{\alpha\phi + (1 - \alpha)\phi^* : \alpha \in \mathbb{R}\}$ formed by the (extended) convex combinations of ϕ and ϕ^* , all are self-financing and budget feasible, and are with the same expected total return. Let $\phi_0 \neq \phi^*$

be the trading strategy in the set that minimizes the total risk

$$\sigma_0 \left[\alpha X_T^\phi + (1 - \alpha) X_T^* \right] = W_0 \sigma_0 \left[\alpha R_{0,T}^\phi + (1 - \alpha) R_{0,T}^* \right].$$

By the first part of the proposition, $X_T^* = W_0 R_{0,T}^*$ must be a mean-preserving-spread of $X_T^{\phi_0} = W_0 R_{0,T}^{\phi_0}$. In consequence, the MPS-risk-averse investor would prefer ϕ_0 to ϕ^* . This contradicts the optimality of ϕ^* for MPS-risk-averse investors. ■

The quest for the relevance of Markowitz's efficient frontier has been controversial ever since its original launch in the early 1950's. The attack on mean-variance analysis has mainly concerned the mean-variance behavior assumptions of the investors. One may ask: How can we assume that investors only care about mean and variance towards their portfolio decision making, while there are substantial evidence in suggesting that investors care about downside risk more than the risk measured by the standard deviation or variance? It is noted that the measure of downside risk in general involves higher order moments than mean and variance. Indeed, people do care about higher moments in describing their preferences.

The attack does not apply to MPS risk averse investors. This is because, as pointed out by Boyle and Ma (2002), MPS risk aversion as a partial order may not admit a mean-variance utility representation. Also, MPS risk averse investors may indeed care about downside risk and higher moments of the distribution. Yet, what has suggested by Proposition 3 is that the MPS risk averse investors' optimal portfolio, if it exists, must be located on the t.e.f. So, it is exactly in this sense that we say that Proposition 3 establishes the relevance of t.e.f. for choices by MPS risk averse investors.

3 MV Efficiency: An Optimal Tracking Problem

The objective of this section is to analytically characterize the MV-efficient trading strategy along with the evolution of the MV-efficient frontier in continuous-time. The stock returns are understood to be driven by state process $\{x_t\}$. The cash balance of the investment account evolves according to equation (3):

$$\frac{dX_t^\phi}{X_t^\phi} = [r_t + \theta_t^\top (\mu_t - r_t \bar{1})] dt + \theta_t^\top \sigma_t dB_t + \int \theta_t^\top \gamma_t(u) v(dt, du) \quad (7)$$

with initial cash position $X_0^\phi = W_0 > 0$ and with $\theta_t^j \equiv \frac{\phi_t^j S_t^j}{X_t^\phi}, j \neq 0$. When the investment account is with unit initial cash position, say $W_0 = 1$, X_t^ϕ corresponds to the total return resulting from the trading strategy within the trading session $(0, t]$, for which we may write $X_t^\phi = R_{0,t}^\phi$.

To understand the evolution of the efficient trading strategy within a pre-specified trading session $(0, T]$, we propose the following so-called *optimal tracking problem*:

Problem 4 (Optimal Tracking) *Consider an investor whose objective is to achieve a pre-specified expected growth rate μ_0 for the cash balance of the managed investment account for a pre-specified trading session $[0, T]$ and all its subsessions, and he wishes to achieve the target rate by taking the minimum risk measured by the standard deviation of the end of session wealth.*

The problem described above is referred to as a problem on ‘tracking a target rate’. The question is: How would the optimal tracking strategy evolve in time, if it exists? One may imagine that, to track the target rate, at each future spot market, say at time $t \in (0, T)$, the investor may wish to revise his portfolio holding upon newly arrival of information, by maintaining the

same target rate μ_0 set, and by choosing a trading strategy to minimize the risk for the sub- trading session $[t, T]$ starting from t .

In the following paragraphs, we will formulate the sequential choice problem as an optimal tracking problem; in particular, we will show that the solution to the optimal tracking problem must constitute a temporal MV-efficient trading strategy.

Let ϕ be an arbitrary tracking strategy with $\mathbb{E}_t [X_T^\phi] = e^{\mu_0(T-t)} X_t^\phi$ and with $\sigma_t^2 [X_T^\phi] = \mathbb{E}_t \left[\left(X_T^\phi \right)^2 \right] - \mathbb{E}_t^2 [X_T^\phi]$. Since the target rate is fixed in advance, the time- t optimization problem for the optimal tracking problem is reduced to the following minimization problem under constraint:

$$\mathfrak{J}(t, x, X) \triangleq \min_{\substack{\phi \in \Phi_t^0 \\ \mathbb{E}_t [X_T^\phi] = e^{\mu_0(T-t)} X}} \mathbb{E}_t \left[\left(X_T^\phi \right)^2 \right] \quad (8)$$

for all $t \in [0, T]$, where $X_t^\phi = X$ is the starting cash position of the investment account, and $x_t = x$ is the time- t state for the state variable.

The following observation on the relationship between the optimal tracking strategy and the MV-efficient trading strategy can be readily established:

Proposition 5 *For all $t \in (0, T)$ the optimal tracking strategy ϕ^* on $(t, T]$, if exists, must constitute an MV-efficient trading strategy for the sub-session $(t, T]$. Moreover, it corresponds to the efficient trading strategy with expected growth rate to be given by μ_0 .*

Proof. Let ϕ^* be the optimal tracking strategy on $(0, T]$ with end of session payoff X_T^* . The time- t variance $\sigma_t^2 [X_T^\phi | X_t^\phi = X]$ is minimized at ϕ^* with the minimum variance to be given by $\sigma_t^2 [X_T^*] = \mathfrak{J}(t, x_t, X) - e^{2\mu_0(T-t)} X^2$. With $\{X_t^\phi\}$ satisfying equation (3) which is of homogeneous of degree one to the initial cash balance, we may write $X_T^\phi = X_t^\phi R_{t,T}^\phi$ with $R_{t,t}^\phi = 1$, and write $\mathfrak{J}(t, x, X) = \mathfrak{J}(t, x, 1) X^2$. The time- t optimal tracking strategy ϕ^*

must be given by

$$\phi^* = \arg \min_{\phi \in \Phi_T^0} \sigma_t \left[R_{t,T}^\phi \right] \quad (9)$$

s.t.
 $\mathbb{E}_t [R_{t,T}^\phi] = e^{\mu_0(T-t)}$

for all $t \in [0, T]$, with $\left\{ R_{t,s}^\phi \right\}_{s \in [t, T]}$ solving the SDE (3) under initial condition $R_{t,t}^\phi = 1$. Therefore, by definition of MV-efficiency, restriction of ϕ^* on $(t, T]$ constitutes an efficient trading strategy at μ_0 for the trading session $(t, T]$.

■

So, to find the MV-efficient trading strategy we need to solve the optimal tracking problem (8).

3.1 Dynamic Consistency

The following lemma shows that the optimal tracking strategy displays dynamic consistency:

Lemma 6 *The optimal tracking strategy ϕ^* , if it exists, must display dynamic consistency; that is, for the given optimal tracking strategy ϕ^* on $(0, T]$, the restriction of ϕ^* on $(t, T]$ must constitute an optimal tracking strategy for the sub-session $(t, T]$.*

Proof. Let θ^* be the corresponding portfolio holding to the optimal tracking strategy ϕ^* . Suppose to the contrary that there exists a tracking strategy ϕ with resulting portfolio holdings θ so that

$$\mathbb{E}_t \left[\left(X_T^\phi \right)^2 \mid X_t^\phi = X \right] \leq \mathbb{E}_t \left[\left(X_T^* \right)^2 \mid X_t^* = X \right]$$

for all X , \mathbb{P} -a.s., on the sub-session $(t, T]$. Now, we construct a new trading strategy, say $\widehat{\phi}$, that trades according to the optimal tracking strategy θ^* on the session $(0, t]$ and according to θ on $(t, T]$ with $X_t^\phi = X_t^*$. This newly designed trading strategy is, by construction, self-financing. It also constitutes

a tracking strategy on $(0, T]$:

$$\begin{aligned}
& \mathbb{E}_0 \left[X_T^{\widehat{\phi}} \mid X_0^{\widehat{\phi}} = W_0 \right] \\
&= \mathbb{E}_0 \left[\mathbb{E}_t \left[X_T^{\widehat{\phi}} \mid X_t^{\widehat{\phi}} = X_t^* \right] \mid X_0^{\widehat{\phi}} = W_0 \right] \\
&= \mathbb{E}_0 \left[\mathbb{E}_t \left[X_T^{\phi} \mid X_t^{\phi} = X_t^* \right] \mid X_0^{\widehat{\phi}} = W_0 \right] \\
&= \mathbb{E}_0 \left[e^{\mu_0(T-t)} X_t^* \mid X_0^* = W_0 \right] \\
&= e^{\mu_0(T-t)} e^{\mu_0 t} W_0 \\
&= e^{\mu_0 T} W_0.
\end{aligned}$$

Taking the conditional expectations on both sides of the above inequality to obtain

$$\mathbb{E}_0 \left[\left(X_T^{\widehat{\phi}} \right)^2 \right] < \mathbb{E}_0 \left[\left(X_T^* \right)^2 \right].$$

This contradicts the optimality of ϕ^* . ■

Dynamic consistency enables us to work backward in constructing the optimal tracking strategy. This in turn results in the following recursive equation for the value function. We have:

Lemma 7 *For all $t \in (0, T)$ and $\Delta > 0$ that is sufficiently small, it must hold true that*

$$\mathfrak{J}(t, x, X) = \min_{\substack{\phi \in \Phi_{t+\Delta}^0 \\ \text{s.t.} \\ \mathbb{E}_t[X_{t+\Delta}^{\phi}] = e^{\mu_0 \Delta} X}} \mathbb{E}_t \left[\mathfrak{J} \left(t + \Delta, x_{t+\Delta}, X_{t+\Delta}^{\phi} \right) \right] \quad (10)$$

in particular, the maximum is achieved at the optimal tracking strategy ϕ^ with $\{\mathfrak{J}(t, x_t, X_t^*)\}_{t \in [0, T]}$ to form a martingale on $[0, T]$.*

Proof. Let ϕ^* be the optimal tracking strategy on $(0, T]$ with end of session payoff X_T^* . We have, by Lemma 6, for all $t \in [0, T]$,

$$\begin{aligned}
& \mathfrak{J}(t, x_t, X_t^*) \\
&= \mathbb{E}_t [(X_T^*)^2 \mid X_t^*] \\
&= \mathbb{E}_t [\mathbb{E}_{t+\Delta} [(X_T^*)^2 \mid X_{t+\Delta}^*] \mid X_t^*] \\
&= \mathbb{E}_t [\mathfrak{J}(t + \Delta, x_{t+\Delta}, X_{t+\Delta}^*)];
\end{aligned}$$

that is, $\{\mathfrak{J}(t, x_t, X_t^*)\}_{t \in [0, T]}$ forms a martingale on $[0, T]$. We can further verify that trading according to ϕ^* on $(t, t + \Delta]$ constitutes a tracking strategy for the sub-session $(t, t + \Delta]$. We have:

$$\begin{aligned}
& e^{\mu_0(T-t)} X \\
&= \mathbb{E}_t [X_T^* \mid X_t^* = X] \\
&= \mathbb{E}_t [\mathbb{E}_{t+\Delta} [X_T^* \mid X_{t+\Delta}^*] \mid X_t^* = X] \\
&= \mathbb{E}_t [e^{\mu_0(T-t-\Delta)} X_{t+\Delta}^* \mid X_t^* = X] \\
&= e^{\mu_0(T-t-\Delta)} \mathbb{E}_t [X_{t+\Delta}^* \mid X_t^* = X].
\end{aligned}$$

This yields

$$\mathbb{E}_t [X_{t+\Delta}^* \mid X_t^* = X] = e^{\mu_0 \Delta} X.$$

Now, consider an arbitrary tracking strategy ϕ that achieves the same target rate on $(t, t + \Delta]$; that is,

$$\mathbb{E}_t [X_{t+\Delta}^\phi \mid X_t^\phi = X] = e^{\mu_0 \Delta} X.$$

We extend the trading strategy on $(t, t + \Delta]$ to the entire session $(t, T]$ by setting $\theta = \theta^*$ on $(t + \Delta, T]$. With starting balance $X_{t+\Delta}^\phi$, which is available at the beginning of the sub-trading session $(t + \Delta, T]$, to be fully invested according to θ^* , this newly defined trading strategy, denoted $\widehat{\phi}$, is by construction self-financing. We can further verify that it forms a tracking strategy on

$(t, T]$; that is,

$$\begin{aligned}
& \mathbb{E}_t \left[X_T^{\hat{\phi}} \mid X_t^\phi = X \right] \\
&= \mathbb{E}_t \left[\mathbb{E}_{t+\Delta} \left[X_T^* \mid X_{t+\Delta}^\phi \right] \mid X_t^\phi = X \right] \\
&= \mathbb{E}_t \left[e^{\mu_0(T-t-\Delta)} X_{t+\Delta}^\phi \mid X_t^\phi = X \right] \\
&= e^{\mu_0(T-t-\Delta)} e^{\mu_0\Delta} X \\
&= e^{\mu_0(T-t)} X.
\end{aligned}$$

With these, by the optimality of ϕ^* , we obtain

$$\begin{aligned}
& \mathfrak{J}(t, x, X) \\
&\leq \mathbb{E}_t \left[\left(X_T^{\hat{\phi}} \right)^2 \mid X_t^{\hat{\phi}} = X \right] \\
&= \mathbb{E}_t \left[\mathbb{E}_{t+\Delta} \left[\left(X_T^* \right)^2 \mid X_{t+\Delta}^\phi \right] \mid X_t^\phi = X \right] \\
&= \mathbb{E}_t \left[\mathfrak{J}(t + \Delta, x_{t+\Delta}, X_{t+\Delta}^\phi) \mid X_t^\phi = X \right].
\end{aligned}$$

We have already learnt (Re: Lemma 6) that the inequality holds with equality when it trades according to ϕ^* on $(t, t + \Delta]$, which itself constitutes a tracking strategy on $(t, t + \Delta]$. These enable us to conclude the validity of (10). ■

3.2 HJB Equation: MPS-RA Investor

With above observation on the dynamic consistency for the corresponding sequential choice problem induced by the optimal tracking problem, we can readily derive the following HJB equation for the optimal tracking problem. We have:

Proposition 8 *The HJB equation for the optimal tracking problem takes the*

form

$$0 = \min_{\substack{\theta \in \mathbb{R}^J \\ \text{s.t.} \\ \theta^\top (\mu + \int \gamma(u) v(du) - r\bar{1}) = \mu_0 - r}} \mathcal{A}^{(\theta)} \mathfrak{J}(t, x, X) \quad (11)$$

where

$$\begin{aligned} & \mathcal{A}^{(\theta)} \mathfrak{J}(t, x, X) \\ = & \mathfrak{J}_t + \mathfrak{J}_x b + X \mathfrak{J}_X (r + \theta^\top [\mu - r\bar{1}]) \\ & + \frac{1}{2} tr([a^\top, X \sigma^\top \theta] H_{\mathfrak{J}} [a^\top, X \sigma^\top \theta]^\top) \\ & + \int (\mathfrak{J}(t, x + l, X + X \theta^\top \gamma) - \mathfrak{J}) v(du) \end{aligned}$$

is the infinitesimal generator induced by the joint jump-diffusion process $\{x_t, X_t^\phi\}$.

Proof. For any arbitrarily given tracking strategy ϕ , it must hold true that

$$\lim_{\Delta \rightarrow 0^+} \frac{\mathbb{E}_t [e^{-\mu_0 \Delta} X_{t+\Delta}^\phi] - X}{\Delta} = 0.$$

This, by Itô's lemma, yields

$$\mu_0 - r = \theta^\top \left(\mu - r\bar{1} + \int \gamma(u) v(du) \right). \quad (12)$$

Moreover, by Lemma 7, for tracking strategy ϕ , it must hold true that

$$\begin{aligned} 0 & \leq \lim_{\Delta \rightarrow 0^+} \frac{\mathbb{E}_t \left[\mathfrak{J}(t + \Delta, x_{t+\Delta}, X_{t+\Delta}^\phi) \right] - \mathfrak{J}(t, x, X)}{\Delta} \\ & = \mathcal{A}^{(\theta)} \mathfrak{J}(t, x, X) \end{aligned}$$

and that, it holds with equality at the optimal tracking strategy. So, we conclude that $\mathcal{A}^{(\theta)} \mathfrak{J}(t, x, X)$ achieves its minimum (zero) with the optimal tracking strategy, which is summarized by the risky portfolio $\theta^*(t, x, X)$,

among all $\theta \in \mathbb{R}^J$ that are constrained by equation (12). ■

With the HJB equation (11) we transform the sequential optimal tracking problem into a static optimization problem with constraint. This enables us to fully analytically characterize the optimal portfolio holding as feedback to the evolution of the state process. We now turn to next section to work out the technical details.

3.3 Analytic Solution

By Proposition 5, the optimal tracking strategy at μ_0 must constitute an MV-efficient trading strategy with the same target rate μ_0 . The HJB equation derived in Proposition 8 for the optimal tracking portfolio enables us to analytically characterize the efficient trading strategy in continuous-time.

With $\mathfrak{J}(t, x, X) = J(t, x) X^2$ and $J(t, x) \equiv \mathfrak{J}(t, x, 1) > 0$, the static optimization problem associated with the HJB equation can be further simplified into the following quadratic optimization problem:

$$\theta^* = \arg \min_{\substack{\theta \in \mathbb{R}^J \\ \text{s.t.} \\ \theta^\top \lambda = \mu_0 - r}} \theta^\top c + \frac{1}{2} \theta^\top \Sigma \theta \quad (13)$$

where

$\lambda \triangleq \mu - r\bar{1} + \int \gamma v(du)$ is the vector of excess instantaneous returns for all risky assets (in presence of jump risk) over the risk free interest rates.

$\Sigma \triangleq \sigma \sigma^\top + J^{-1} \int J(t, x+l) \gamma \gamma^\top v(du)$ is the *shadow instantaneous variance-covariance matrix* (in presence of jump risk) derived from the returns of risky assets. The contribution from the jump risk is indexed by $J(t, x+l)/J(t, x)$. The weights depend on how the total risk of the investment account reacts to the jumps in the state variables.

$c \triangleq J^{-1} \{ \sigma a^\top J_x + \int [J(t, x+l) - J] \gamma v(du) \}$ corresponds to the instantaneous covariances of the risky returns with the depreciation rate of the

total risk associated with the investment account. The total risk is measured by J , which is, in general, driven by the state variables x .

Remark 9 *It is noted that the total risk J of the investment account varies with the level of target rate μ_0 . In consequence, both Σ and c are specific to the target rate μ_0 as well. But, there is an exception to this, which corresponds to the case where asset returns are not driven by the state variables. In this case, we will have $J = J(t)$ with $c = 0$ and $\Sigma = \Sigma_0$, where*

$$\Sigma_0 \triangleq \sigma\sigma^\top + \int \gamma\gamma^\top v(du) \quad (14)$$

is the objective- rather than shadow- instantaneous variance-covariance matrix for the returns of risky assets. This special case will be studied in some detail in Example 11 below.

The quadratic optimization problem formulated above for the local efficient portfolio holding resembles the optimization problem solved by Markowitz (1952) in deriving the mean-variance efficient frontier in static setting except that, here c may not be equal to zero, and that we made use of shadow instantaneous variance-covariance matrix Σ .

The optimal solution can be easily found by applying the standard Lagrangian method. Suppose Σ is positive definite, the efficient portfolio θ^* is unique and is given by

$$\theta^* = -\Sigma^{-1}c + (\mu_0 - r + c^\top\Sigma^{-1}\lambda) (\lambda^\top\Sigma^{-1}\lambda)^{-1} \Sigma^{-1}\lambda. \quad (15)$$

Let $\underline{\theta} \triangleq -\Sigma^{-1}c$. The risky portfolio $\underline{\theta}$ constitutes the so-called *instantaneous minimum shadow variance portfolio* since it is the risky portfolio that minimizes the shadow risk in absence of tracking constraint. The second term in the efficient risky portfolio θ^* is proportional to $\Sigma^{-1}\lambda$, and, in turn,

proportional to the instantaneous shadow tangent portfolio

$$\theta_m \triangleq (\bar{\Gamma}^\top \Sigma^{-1} \lambda)^{-1} \Sigma^{-1} \lambda \quad (16)$$

which is obtained by incorporating the jump risk. So, to obtain the efficient portfolio, investors need to do the following:

- (a) to hedge against the systematic risk with the minimum (shadow) variance portfolio $\underline{\theta}$;
- (b) to invest the remaining cash balance into a combination of the instantaneous (shadow) tangent portfolio θ_m and the risk free saving account. The weight invested in the shadow tangent portfolio varies with the target rate μ_0 . If $\bar{\Gamma}^\top \Sigma^{-1} \lambda > 0$, the position invested in the tangent portfolio would increase with the target rate μ_0 .

It remains to determine the risk associated with the efficient trading strategy. With

$$\sigma_t^2 [X_T^*] \equiv [J(t, x_t) - e^{2\mu_0(T-t)}] X_t^{*2}, t \in [0, T] \quad (17)$$

the risk that is associated with the efficient trading strategy ϕ^* (with expected growth rate μ_0) is fully summarized by the J -function. With efficient portfolio $\theta^*(t, x)$ given by (15), we can readily derive the functional equation for J by substituting $\mathfrak{J}(t, x, X) = J(t, x) X^2$ and the expression for $\theta^*(t, x)$ into the HJB equation (11). This results in the following so-called *generalized Riccarti equation* for the J -function:

$$\frac{\mathcal{A}J}{J} = -2\mu_0 + c^\top \Sigma^{-1} c - (\mu_0 - r + c^\top \Sigma^{-1} \lambda)^2 (\lambda^\top \Sigma^{-1} \lambda)^{-1} \quad (18)$$

with terminal condition $J(T, x) \equiv 1$. Here, \mathcal{A} is the infinitesimal generator induced by the state process $\{x_t\}$.

So, in summarizing the above derivations, we obtain the following characterization theorem for the efficient trading strategy:

Proposition 10 *Let $J \in \mathbb{C}^{1,2}((0, T) \times \mathbb{R}^n)$ solve the generalized Riccati equation (18). The efficient trading strategy ϕ^* is with its time- t portfolio $\theta^*(t, x)$ to be given by (15), and is with its risk exposure to be determined by equation (17). Moreover, the cash balance $\{X_t^*\}$ resulting from ϕ^* is governed by SDE*

$$\frac{dX_t^*}{X_t^*} = \mu_0 dt + \theta_t^{*\top} \left[\sigma_t dB_t + \int \gamma_t(u) \tilde{v}(dt, du) \right]. \quad (19)$$

The following is an illustrative example of the formation of MV-efficient trading strategies, along with the evolution of risk and cash balance resulting from such trading strategies:

Example 11 *We consider the case where the coefficients of asset returns are state-independent; that is,*

$$r = r(t), \mu = \mu(t), \sigma = \sigma(t), \gamma = \gamma(t, u) \quad (20)$$

in addition to the assumption that the state process $\{x_t\}$ contains no jump risk ($l \equiv \emptyset$) and that the asset returns contain no systematic instantaneous risk ($\sigma a^\top = \emptyset$). Under these specifications, we have $c = \emptyset, \Sigma = \Sigma_0$. We will also have $J = J(t)$; that is, the total risk of the investment account is deterministic and state-independent. With $J = J(t)$ the efficient portfolio holding admits a simple expression:

$$\theta^* = (\mu_0 - r) (\lambda^\top \Sigma_0^{-1} \lambda)^{-1} \Sigma_0^{-1} \lambda. \quad (21)$$

We can further characterize the total risk involved in the managed investment account. The generalized Riccati equation for the total risk J of the investment account is reduced to an ordinary linear differential equation:

$$\frac{dJ(t)}{dt} = -\xi_0(t) J(t) \quad \text{with } J(T) = 1 \quad (22)$$

where

$$\xi_0(t) \triangleq 2\mu_0 + (\lambda^\top \Sigma_0^{-1} \lambda)^{-1} (\mu_0 - r)^2.$$

The solution is given by $J(t) = e^{\int_t^T \xi_0(s) ds}$. It is noted that $\ln J$, as function of μ_0 , forms a hyperbola. This is analogue to the classical static Markowitz's efficient frontier. Finally, the cash balance $\{X_t^*\}$ resulting from the efficient trading strategy must evolve in time according to the following SDE:

$$\begin{aligned} \frac{dX^*}{X^*} &= \mu_0 dt + (\mu_0 - r) (\lambda^\top \Sigma_0^{-1} \lambda)^{-1} \\ &\quad \times \lambda^\top \Sigma_0^{-1} \left[\sigma dB_t + \int \gamma \tilde{v}(dt, du) \right]. \end{aligned} \quad (23)$$

4 Temporal v.s. Local Mutual Fund Separation

Here, we wish to distinguish between local instantaneous MV-efficiency and temporal MV-efficiency. The former is defined locally at any specific time and state, say at (t, x) , and refers to the Markowitz's efficient frontier derived from the time- t instantaneous variance-covariance matrix Σ_0 and excess instantaneous mean return λ , in addition to the risk free interest rate r . The latter is, on the other hand, defined over trading strategies on an entire trading session $(0, T]$. The concept of temporal MV-efficiency is relevant to MPS risk averse investors' trading in continuous time since we have already learnt that, MPS risk averse investors only invest according to some MV-efficient trading strategies. The relationships between local and temporal MV-efficiency can be readily deduced from expression (15) for the local portfolio holding $\theta^*(t, x)$ induced by the temporal MV-efficient trading strategy $\phi(\mu_0)$ with target rate μ_0 . We have the following observations:

- (a) As indicated in Figure 1, the portfolio holding $\theta^*(t, x)$ induced by a temporal efficient trading strategy, in general, is not located on the

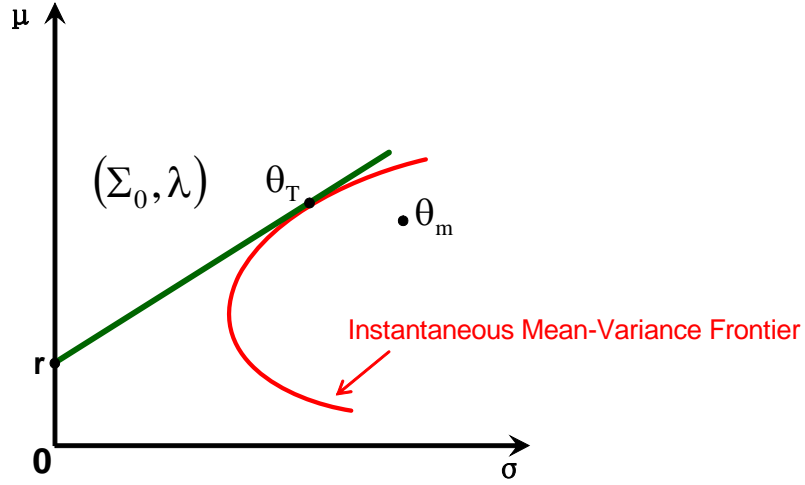


Figure 1: Instantaneous MV Frontier

instantaneous Markowitz's MV-efficient frontier derived from (Σ_0, λ, r) ; instead, for the given shadow instantaneous variance-covariance matrix Σ , $\theta^*(t, x)$ given by (15) admits the following alternative expression:

$$\theta^* = \arg \min_{\substack{\theta \in \mathbb{R}^J \\ \text{s.t.} \\ r + [\theta - \underline{\theta}]^\top \lambda = \mu_0 - \underline{\theta}^\top \lambda}} \frac{1}{2} (\theta - \underline{\theta})^\top \Sigma (\theta - \underline{\theta})$$

that is, as depicted in Figure 2, the distorted portfolio $\theta^* - \underline{\theta}$ constitutes a local shadow efficient portfolio induced by (Σ, λ, r) with shadow variance-covariance matrix Σ (rather than Σ_0), and be proportional to the local instantaneous shadow tangent portfolio θ_m . We have θ^* to admit the following expression

$$\theta^* = \underline{\theta} + (\mu_0 - r - \underline{\lambda}) \lambda_m^{-1} \theta_m \quad (24)$$

where $\underline{\lambda}$ and λ_m are the excess instantaneous return for the risky portfolio $\underline{\theta}$ and θ_m , respectively. It is noted that the temporal efficient

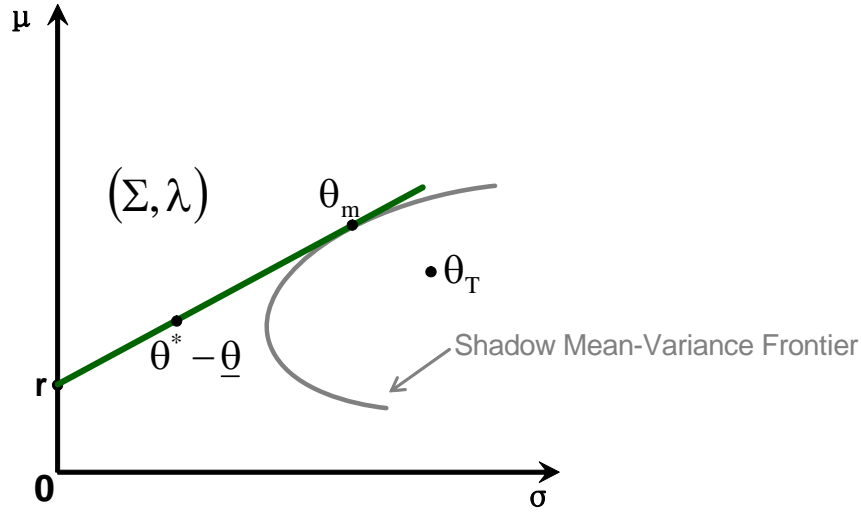


Figure 2: Shadow MV Frontier

portfolio θ^* would fall into the local Markowitz's mean-variance efficient frontier induced by (Σ_0, λ, r) when $c = \emptyset$ and $\Sigma = \Sigma_0$. This corresponds to the case when the market involves no systematic jump risk ($l = \emptyset$) and when asset returns contain no systematic risk ($\sigma a^\top = \emptyset$).

- (b) Concerning mutual fund separation in a local spot market, expression (15) leads to a three-fund separation for the local portfolio holding θ^* induced by temporal efficient trading strategies. The local portfolio holding can be decomposed into $\underline{\theta}$, plus a risky portfolio that is proportional to the instantaneous shadow tangent portfolio θ_m , in addition to the risk free saving account. But, in contrast to the classical Markowitz's efficient frontier, both $\underline{\theta}$ and θ_m derived above are specific to the target rate μ_0 . In other words, different target rates would correspond to different shadow efficient frontiers with different minimum-variance portfolios $\underline{\theta}$ and instantaneous shadow tangent portfolios θ_m . This is because the shadow instantaneous variance-covariance matrix

Σ itself varies with the target rate μ_0 .

(c) Let

$$\theta^0 \triangleq \underline{\theta} + (c^\top \Sigma^{-1} \lambda - r) (\lambda^\top \Sigma^{-1} \lambda)^{-1} \Sigma^{-1} \lambda \quad (25)$$

$$\theta^1 \triangleq \theta^0 + (\lambda^\top \Sigma^{-1} \lambda)^{-1} \Sigma^{-1} \lambda \quad (26)$$

From expression (15) we obtain the following decomposition for the efficient risky portfolio:

$$\theta^* = (1 - \mu_0) \theta^0 + \mu_0 \theta^1. \quad (27)$$

Let ϕ^0 and ϕ^1 correspond to the trading strategies generated from risky portfolios θ^0 and θ^1 , respectively. Then, concerning temporal mutual fund separation for a given trading session $(0, T]$, we have:

$$\phi(\mu_0) = (1 - \mu_0) \phi^0 + \mu_0 \phi^1 \quad (28)$$

that is, the efficient trading strategy admits a decomposition into a convex combination of two trading strategies ϕ^0 and ϕ^1 . Notice that, in this above decomposition, the trading strategies ϕ^0 and ϕ^1 are, in general, not efficient and are target rate specific (unless $c = \emptyset$ and $\Sigma = \Sigma_0$).

For the special case when $c = \emptyset$ and $\Sigma = \Sigma_0$, as is the case when the market involves no systematic jump risk ($l = \emptyset$) and when asset returns contain no systematic risk ($\sigma a^\top = \emptyset$), the separating trading strategies ϕ^0 and ϕ^1 would be invariant with the target rate, and correspond to the efficient trading strategies $\phi(0)$ and $\phi(1)$ at $\mu_0 = 0$ and $\mu_0 = 1$, respectively. In this case, we obtain a mutual fund separation that extends the classical Black's separation theorem to continuous time;

that is, for all $\mu_0 \in \mathbb{R}$,

$$\phi(\mu_0) = (1 - \mu_0)\phi(0) + \mu_0\phi(1). \quad (29)$$

This can be re-stated in the language of Black: First, all (extended) convex combinations of efficient trading strategies are efficient. Second, any efficient trading strategy can be expressed as a unique linear combination of two arbitrary, but fixed, efficient trading strategies. This separation theorem in continuous time goes deeper than the classical Black's separation theorem because the issue of how investors revise their portfolio weights on each of the separating portfolios can not be addressed in the static setting.

- (d) Under the same specifications as in (c) towards temporal mutual fund separation, all myopic MPS-risk-averse investors, who care about final payoff at the end of a specific, but arbitrary, trading session $(0, T]$, would optimally invest in two mutual funds, each corresponding to some arbitrary, but distinct, efficient trading strategies, say, for example, $\phi(0)$ and $\phi(1)$, on $(0, T]$. It is understood that, once an investor sets a target rate, and decides on the specific mutual funds to invest for a given trading session, the separating theorem suggests that there is no need to revise the weight put into each of the separating funds during the entire trading session. This extends Tobin's two-fund separation into continuous-time. But, unlike Tobin's separating funds, which differ from each other largely from the compositions of the separating portfolios (say, one involving purely risky assets, and the other involving the risk free bond only), the separating funds in this continuous-time setting differ from each other concerning the corresponding trading strategies. In each spot market, each separating fund has a combination of the minimum-variance portfolio $\underline{\theta}$, the instantaneous shadow tangent portfolio θ_m , along with the risk free saving account. The two

separating funds differ from each other on the weights allocated to each of these portfolios, and in the ways how the weights are to be adjusted as time evolves within the trading session.

In summary, it is the local instantaneous shadow efficient frontier, which is target rate specific, that is relevant in characterizing the composition of the temporal efficient portfolio in each spot market. A three fund separation is valid with respect to efficient local portfolio holdings, but each of the separating risky portfolios $\underline{\theta}$ and θ_m is target rate specific. Similar observations hold true regarding temporal mutual fund separations. Generally speaking, each efficient trading strategy can be decomposed into an extended convex combination of two separating trading strategies ϕ^0 and ϕ^1 , both are target rate specific unless the state variables have no jump risk ($l = \emptyset$) and when asset returns contain no systematic risk ($\sigma a^\top = \emptyset$).

5 Risk-Return Relationship

We start with some useful notations. For any arbitrary risky portfolio $\theta \in \mathbb{R}^J$, the corresponding instantaneous excess return, systematic factor risks, and shadow portfolio risk are respectively denoted by

$$\lambda^\theta \triangleq \theta^\top \lambda, c^\theta \triangleq \theta^\top c, \widehat{\sigma}[\theta] \triangleq \sqrt{\theta^\top \Sigma \theta}. \quad (30)$$

For the instantaneous shadow tangent portfolio θ_m , we denote

$$\lambda_m \triangleq \theta_m^\top \lambda, c_m \triangleq \theta_m^\top c, \widehat{\sigma}_m \triangleq \sqrt{\theta_m^\top \Sigma \theta_m}. \quad (31)$$

Moreover, for all θ , $\widehat{\sigma}[\theta_m, \theta] \triangleq \theta^\top \Sigma \theta_m$ is the shadow covariance between the two portfolios.

To derive the risk-return relationship, we multiply Σ on both sides of equation (16) to obtain

$$\lambda = (\bar{\Gamma}^\top \Sigma^{-1} \lambda) \Sigma \theta_m. \quad (32)$$

With this, we see that, for all risky portfolio θ ,

$$\lambda^\theta = (\bar{\Gamma}^\top \Sigma^{-1} \lambda) \hat{\sigma} [\theta_m, \theta] \quad \text{and} \quad \lambda_m = (\bar{\Gamma}^\top \Sigma^{-1} \lambda) \hat{\sigma}_m^2. \quad (33)$$

This, in turn, implies the following risk-return relationship that admits the standard linear- β representation:

$$\lambda^\theta = \hat{\beta}_m^\theta \lambda_m \quad \text{with} \quad \hat{\beta}_m^\theta \triangleq \hat{\sigma} [\theta_m, \theta] / \hat{\sigma}_m^2. \quad (34)$$

It is noted that $\hat{\beta}_m^\theta$ is computed using shadow variance and covariance matrix Σ . For those pure risky portfolios ($\bar{\Gamma}^\top \theta = 1$), which include all individual securities j , the above linear- β model can be re-stated in terms of the total instantaneous returns:

$$\begin{aligned} & \mu_t^\theta + \int \gamma_t^\theta(u) v(du) \\ &= r_t + \hat{\beta}_{m,t}^\theta \left(\mu_{m,t} + \int \gamma_{m,t}(u) v(du) - r_t \right). \end{aligned} \quad (35)$$

With $\Sigma = \Sigma_0 + \int \left(\frac{J_\pm}{J} - 1 \right) \gamma \gamma^\top v(du)$, the shadow variance and covariance matrix is expressed as the total instantaneous variance-covariance matrix, plus a term that represents the deviation of the shadow variance-covariance matrix from the objective variance-covariance matrix. Σ is reduced to the objective instantaneous variance-covariance matrix Σ_0 when the state process $\{x_t\}$ contains no jump risks ($l = \emptyset$). So, the newly derived linear- β model (35) can be regarded as a two-factor model with total risk premium of the risky portfolio, that is measured by $\hat{\beta}_m^\theta$, to admit the following decomposition:

$$\hat{\beta}_m^\theta = \frac{\theta_m^\top \Sigma_0 \theta + \int \left(\frac{J_\pm}{J} - 1 \right) \theta_m^\top \gamma \gamma^\top \theta v(du)}{\theta_m^\top \Sigma_0 \theta_m + \int \left(\frac{J_\pm}{J} - 1 \right) \theta_m^\top \gamma \gamma^\top \theta_m v(du)} \quad (36)$$

where the first term is interpreted as the risk premium for hedging against

the total instantaneous risk (Brownian + jump) associated with the risky portfolio, and the second term is an additional risk premium to compensate for the systematic jump risk involved in the ‘shadow’ risk of the managed investment account.

Alternatively, with $\Sigma_0 = \sigma\sigma^\top + \int \gamma\gamma^\top v(du)$, we may write

$$\widehat{\beta}_m^\theta = \frac{\theta_m^\top \sigma \sigma^\top \theta + J^{-1} \int J_+ \theta_m^\top \gamma \gamma^\top \theta v(du)}{\theta_m^\top \sigma \sigma^\top \theta_m + J^{-1} \int J_+ \theta_m^\top \gamma \gamma^\top \theta_m v(du)} \quad (37)$$

in which the first component represents part of the risk premium for hedging against Brownian uncertainty involved in the risky portfolio, while the second component represents part of the risk premium for hedging against the jump risks involved in the portfolio and the systematic jump risk that affects the overall volatility (J) of the investment account.

When state variables involve no systematic jump risk, we have $J_+ = J$ and $\widehat{\beta}_m^\theta = \beta_m^\theta \triangleq \frac{\theta_m^\top \Sigma_0 \theta}{\theta_m^\top \Sigma_0 \theta_m}$. There is no need to hedge against the systematic jump risk. In this case, the above shadow linear- β model is reduced to the classical linear- β model with β as a measure of risk premium for hedging against portfolio risk using the instantaneous tangent portfolio θ_m .

5.1 Risk Decomposition

With the validity of the shadow linear- β model (35) derived in the previous section, we obtain as follows a so-called ‘orthogonal risky decomposition’ in continuous-time: For each risky portfolio θ , we may write

$$\theta = \frac{\lambda^\theta}{\lambda_m} \theta_m + \left(\theta - \frac{\lambda^\theta}{\lambda_m} \theta_m \right). \quad (38)$$

Let $\theta^e \equiv \theta - \frac{\lambda^\theta}{\lambda_m} \theta_m$. Then, it is easy to see that θ^e has zero excess instantaneous return; moreover, with the validity of the shadow linear- β model, we can readily verify that θ^e is orthogonal to the shadow instantaneous tangent

portfolio θ_m in the sense that $\widehat{\sigma}[\theta^e, \theta_m] = 0$. We have

$$\begin{aligned}\widehat{\sigma}[\theta^e, \theta_m] &= \widehat{\sigma}[\theta, \theta_m] - \frac{\lambda^\theta}{\lambda_m} \widehat{\sigma}_m^2 \\ &= -\frac{\widehat{\sigma}_m^2}{\lambda_m} \left(\lambda^\theta - \widehat{\beta}_m^\theta \lambda_m \right) = 0.\end{aligned}$$

Particularly, for temporal MV efficient portfolio θ^* with target rate μ_0 , which is given by (15), the orthogonal decomposition takes the following form

$$\theta^* = \left(\underline{\theta} - \widehat{\beta}_m^\theta \theta_m \right) + \left(\frac{\mu_0 - r}{\lambda_m} \right) \theta_m. \quad (39)$$

For the validity of this decomposition, notice that, the right hand side is a linear combination of $\underline{\theta}$ and θ_m with total excess instantaneous return to be given by $\mu_0 - r$, which is the same as that of θ^* ; and that, all temporal efficient portfolio is a linear combination of $\underline{\theta}$ and θ_m . Notice also that, the first term on the right hand side of equation (39) is the residual risky portfolio induced by the minimum variance portfolio $\underline{\theta}$. It has zero excess instantaneous return, and is orthogonal to the shadow instantaneous tangent portfolio θ_m under the shadow variance-covariance matrix Σ . These enable us to conclude the validity of ‘shadow orthogonal decomposition’ (39).

From shadow orthogonal decomposition (39), we see that the temporal efficient portfolio θ^* is in general not located on the shadow instantaneous efficient frontier unless $\underline{\theta} - \widehat{\beta}_m^\theta \theta_m = \emptyset$, or, equivalently, when the minimum variance portfolio $\underline{\theta}$ is itself proportional to the instantaneous tangent portfolio. This occurs when $c = \emptyset$, for which we have $\underline{\theta} = \emptyset$.

5.2 Temporal v.s. Instantaneous Efficient Frontier

With the help of ‘orthogonal decomposition’ for temporal efficient risk portfolios θ^* , we can readily plot on the μ - σ plane, the so-called *temporal efficient*

frontier \mathcal{I}_t . The time- t temporal efficient frontier is defined by setting

$$\mathcal{I}_t \triangleq \left\{ \begin{array}{l} (\mu_0, \sigma_0) : \mu_0 \in \mathbb{R}, \sigma_0 = \sqrt{\theta^{*\top} \Sigma_0 \theta^*} \text{ and} \\ \theta^* \text{ is temporal efficient at } t \text{ with target rate } \mu_0 \end{array} \right\}. \quad (40)$$

We proceed to characterize the temporal efficient frontier \mathcal{I}_t induced by those temporal efficient trading strategies. For all θ and $\theta' \in \mathbb{R}^J$, we denote $\sigma[\theta, \theta'] \triangleq \theta^\top \Sigma_0 \theta'$ and $\sigma[\theta] \triangleq \sqrt{\sigma[\theta, \theta]}$; particularly for $\underline{\theta}$ and θ_m we denote $\underline{\sigma} = \sigma[\underline{\theta}]$, $\sigma_m = \sigma[\theta_m]$ and $\beta_m^\theta = \frac{\sigma[\underline{\theta}, \theta_m]}{\sigma_m^2}$. With these notions, we can readily compute the portfolio variance for the temporal efficient portfolio θ^* given by equation (39). Setting $\sigma_0 = \sigma[\theta^*]$, we obtain

$$\begin{aligned} \sigma_0^2 &= \sigma^2 \left[\underline{\theta} + \left(\frac{\mu_0 - r}{\lambda_m} - \widehat{\beta}_m^\theta \right) \theta_m \right] \\ &= \underline{\sigma}^2 + 2 \left(\frac{\mu_0 - r - \lambda^\theta}{\lambda_m} \right) \beta_m^\theta \sigma_m^2 + \left(\frac{\mu_0 - r - \lambda^\theta}{\lambda_m} \right)^2 \sigma_m^2 \\ &= \underline{\sigma}^2 - (\beta_m^\theta \sigma_m)^2 + \left(\frac{\mu_0 - r - \lambda^\theta}{\lambda_m} + \beta_m^\theta \right)^2 \sigma_m^2. \end{aligned}$$

This yields a hyperbola on the μ - σ plane

$$\mu_0 = r + \lambda^\theta - \beta_m^\theta \lambda_m \pm \left| \frac{\lambda_m}{\sigma_m} \right| \sqrt{\sigma_0^2 - \underline{\sigma}^2 + (\beta_m^\theta \sigma_m)^2} \quad (41)$$

for all $\sigma_0 \geq \sqrt{\underline{\sigma}^2 - (\beta_m^\theta \sigma_m)^2}$, with

$$\begin{aligned} \lambda^\theta &= -c^\top \Sigma^{-1} \lambda \\ \underline{\sigma}^2 &= c^\top \Sigma^{-1} \Sigma_0 \Sigma^{-1} c \\ \beta_m^\theta \lambda_m &= -(\lambda^\top \Sigma^{-1} \Sigma_0 \Sigma^{-1} \lambda)^{-1} (c^\top \Sigma^{-1} \Sigma_0 \Sigma^{-1} \lambda) (\lambda \Sigma^{-1} \lambda) \\ \beta_m^\theta \sigma_m &= -(\lambda^\top \Sigma^{-1} \Sigma_0 \Sigma^{-1} \lambda)^{-1/2} (c^\top \Sigma^{-1} \Sigma_0 \Sigma^{-1} \lambda) \\ \left| \frac{\lambda_m}{\sigma_m} \right| &= (\lambda^\top \Sigma^{-1} \Sigma_0 \Sigma^{-1} \lambda)^{-1/2} (\lambda \Sigma^{-1} \lambda). \end{aligned}$$

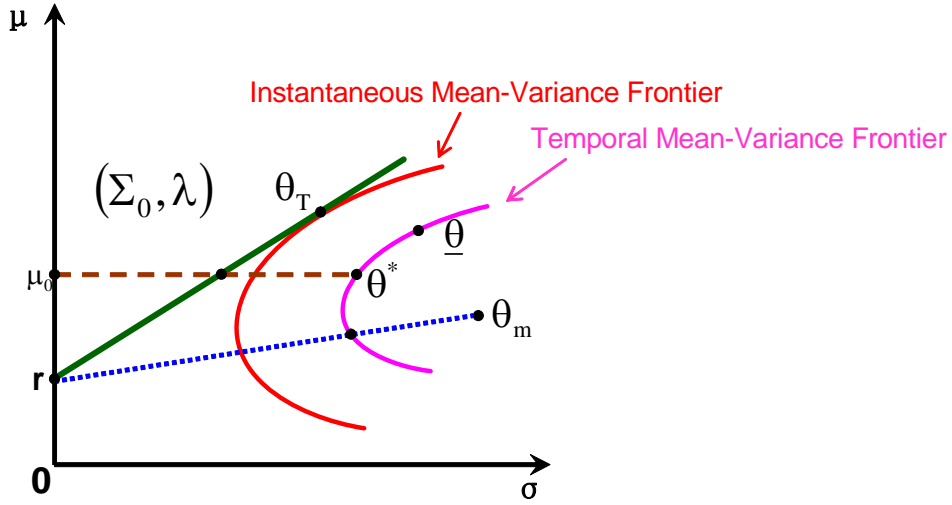


Figure 3: Temporal MV Frontier

It is noted that, the temporal efficient frontier differs from the local instantaneous Markowitz's efficient frontier derived from (Σ_0, λ, r) . The latter is known to be composed of two efficient rays that intersect the μ -axis at r with tangent portfolio proportional to $\Sigma_0^{-1}\lambda$. Here, the minimum shadow variance portfolio $\underline{\theta} = -\Sigma^{-1}c$, which is in general not located on the instantaneous efficient rays, is always on the temporal efficient frontier \mathcal{I}_t . In fact, it corresponds to the temporal efficient portfolio at $(\mu_0, \sigma_0) = (r + \lambda^{\underline{\theta}}, \underline{\sigma})$. In contrast, the shadow instantaneous tangent portfolio θ_m , which is always on the instantaneous shadow efficient frontier, in general, does not belong to the instantaneous efficient rays, nor does it belong to the temporal efficient frontier \mathcal{I}_t .

For the special case of $c = \emptyset$, the temporal efficient frontier, which is composed of two efficient rays that also intersects the μ -axis at r , has its slope $\frac{\lambda_m}{\sigma_m}$ to be different from the corresponding slope of the instantaneous efficient rays. The temporal efficient frontier coincides with the instantaneous efficient

frontier when the state process contains no systematic jump risk ($l = \emptyset$).

In conclusion, the temporal and instantaneous efficient frontiers, in general, differ from each other. While the instantaneous efficient frontier is composed of two efficient rays, the temporal efficient frontier, in general, forms a hyperbola on the μ - σ plane. The temporal and instantaneous efficient frontiers coincide to each other when asset returns involve no systematic risk ($c = \emptyset$) and when the state process contains no systematic jump risk ($l = \emptyset$).

6 EU v.s. MPS-Risk-Averse Investors

With the above analytic characterizations of the optimal portfolio choices by investors displaying MPS risk aversion, we may readily dig into the difference between expected utility investors and MPS-risk-averse investors concerning their optimal trading behaviors. We start by asking the following general question:

Problem 12 *How is it possible to distinguish between expected utility investors and investors who display MPS-risk-aversion in their trading behaviors?*

To answer this question, we restrict our discussion to the special case of Example 11, where asset returns and state variables contain no jump risks ($\gamma = \emptyset$ and $l = \emptyset$) and where asset returns are state-independent ($\sigma a^\top = \emptyset$). This case is of particular interesting because, as to be illustrated below, it becomes less obvious on how to distinguish between these two groups of investors concerning their trading behavior. Following the standard treatment in literature such as Merton (1971), we consider expected utility functions with constant relative risk aversion (i.e., $\text{RRA}_u = 1 - \alpha$) and assume $\alpha < 1$.

It is well-known that (see, Merton 1971), expected utility investors with constant RRA would optimally invest in a common instantaneous tangent

portfolio that is given by

$$\theta_m = \frac{[\sigma\sigma^\top]^{-1} [\mu - r\bar{1}]}{\bar{1}^\top [\sigma\sigma^\top]^{-1} [\mu - r\bar{1}]}.$$
 (42)

We have also learned from Example 11 that MPS-risk-averse investors would invest in the same risky portfolio. So, it is impossible to distinguish these two groups of investors by merely looking into the compositions of their risky portfolios. This, however, does not mean that these two groups of investors would trade in the same fashion. In fact, as illustrated below, investors from these two groups act very differently. We look into the corresponding portfolio weights on the risky portfolio θ_m within the trading session.

Let θ^α be the optimal risky portfolio for the expected utility investor with relative risk aversion to be given by $1 - \alpha$. Let θ^* be the local portfolio corresponding to an MV-efficient trading strategy with target rate μ_0 . We write

$$\begin{aligned}\theta^\alpha &= k_\alpha \times [\sigma\sigma^\top]^{-1} [\mu - r\bar{1}] \\ \theta^* &= k^* \times [\sigma\sigma^\top]^{-1} [\mu - r\bar{1}]\end{aligned}$$

where

$$k_\alpha \triangleq \frac{1}{1 - \alpha} \text{ and } k^* \triangleq \frac{\mu_0 - r}{[\mu - r\bar{1}]^\top [\sigma\sigma^\top]^{-1} [\mu - r\bar{1}]}.$$

Both θ^α and θ^* are proportional to a common risky portfolio which is itself proportional to the instantaneous tangent portfolio θ_m . We see that, for expected utility investors, the portfolio weight k_α is constant and invariante to changes in the market environment (which is summarized by the risk free interest rates r , the drifts μ and volatilities σ of the risky assets), while for MPS risk-averse investors, the corresponding portfolio weight k^* is very sensitive to such changes.

The difference between the trading behavior for investors from these two groups can alternatively be reflected from the cash balance dynamics for the investment accounts managed by these investors. To investment accounts managed by those MPS-risk-averse investors, the resulting cash balances must evolve in time according to

$$\frac{dX^*}{X^*} = \mu_0 dt + \frac{(\mu_0 - r) [\mu - r\bar{\mathbf{1}}]^\top [\sigma\sigma^\top]^{-1} \sigma}{[\mu - r\bar{\mathbf{1}}]^\top [\sigma\sigma^\top]^{-1} [\mu - r\bar{\mathbf{1}}]} dB_t \quad (43)$$

for some constant μ_0 . Similarly, the cash balance of an investment account managed by an expected utility investor must evolve according to

$$\frac{dX^{(\alpha)}}{X^{(\alpha)}} = r dt + \frac{[\mu - r\bar{\mathbf{1}}]^\top [\sigma\sigma^\top]^{-1}}{1 - \alpha} ([\mu - r\bar{\mathbf{1}}] dt + \sigma dB_t). \quad (44)$$

The former is associated with a constant expected growth rate μ_0 , while the latter is with time-varying instantaneous growth rate which is greater than the risk free interest rate r .

It is also interesting to notice that the instantaneous volatilities of the two managed investment accounts tends to move in the opposite directions. With

$$\sigma^2 [\ln X^*] = \frac{(\mu_0 - r)^2}{[\mu - r\bar{\mathbf{1}}]^\top [\sigma\sigma^\top]^{-1} [\mu - r\bar{\mathbf{1}}]} \quad (45)$$

$$\sigma^2 [\ln X^{(\alpha)}] = \frac{[\mu - r\bar{\mathbf{1}}]^\top [\sigma\sigma^\top]^{-1} [\mu - r\bar{\mathbf{1}}]}{(1 - \alpha)^2} \quad (46)$$

we see that, roughly speaking, at times when the interest rates are relatively stable (i.e. r does not change much), the account managed by the expected utility investors are experiencing high (and increasing) volatilities, and the volatilities for the accounts managed by the MPS-risk-averse investors has to be low (and declining). Or, in other words, for a given target rate μ_0 and a risk aversion coefficient α , the two volatilities tend to move in opposite

directions. This is true particularly when the interest rates are stable.

As a separate, but relevant, observation, we have

$$\sigma [\ln X^{(\alpha)}] \geq \sigma [\ln X^*] \Leftrightarrow \mu^\alpha \geq \mu_0 \quad (47)$$

whenever $\mu_0 > r$. Here,

$$\mu^\alpha = r + \frac{[\mu - r\bar{1}]^\top [\sigma\sigma^\top]^{-1} [\mu - r\bar{1}]}{1 - \alpha}$$

is the instantaneous portfolio return induced by the optimal portfolio θ^α . We may thus say that, at times when the accounts managed by those expected utility investors become more (less) volatile relative to those managed by some other MPS risk averse investors, it has to be the case when members in the former group are expecting higher (lower) rewards (in terms of instantaneous expected returns) than the fixed target rates (μ_0) set by those MPS risk averse investors.

Therefore, in conclusion, even for the special case when investors from either group would optimally choose to invest in the same local risky portfolio θ_m , their trading behavior could be very different, referring to the portfolio weight allocated to the common risky portfolio. The difference in the trading behavior of investors from these two groups is also reflected in the dynamics for the cash balance of the investment account respectively managed by investors from these two groups. Such difference in trading reflects their psychological differences in attitudes towards risk.

7 Concluding Remarks

The temporal efficient frontier (t.e.f.) in presence of Lévy jumps is shown to inherit much of the mathematical properties of the classical Markowitz's static MV efficient frontier, and it resembles the local instantaneous efficient frontier corresponding to Merton (1971) and Bielecki et al (2005). All form

hyperbola in the μ - σ plane.

The t.e.f. is found not to coincide with the local instantaneous frontier — the continuous time analogue of Markowitz’s mean-variance frontier. This observation is potentially useful in understanding the existence of documented financial anomalies in empirical finance — MPS risk averse investors may not wish to invest along the local instantaneous Markowitz’s mean-variance frontier, but instead hold portfolios on the t.e.f.. The optimal portfolio on the t.e.f. could well fall strictly within the instantaneous local Markowitz’s efficient frontier.

Our observations on mutual fund separation are also profound and interesting. In contrast to the classical two-fund separation along the line of Black (1972) and Tobin (1958), our study shows that MPS-risk-averse investors’ optimal trading strategy is target rate specific. Precisely, investors with different target rates may end up investing into different managed mutual funds, each involving a specific set of separating portfolios. Our theoretic findings are, nevertheless, much in line with the real world phenomena on the existence of various types of mutual funds offered by different financial institutes, each aiming to attract demand from some specific groups of investors — a picture that is in sharp contrast to the theoretical prediction made by Black (1972) and Tobin (1958).

Finally, our study sheds light on the difference between expected utility and MPS-risk-averse investors concerning their trading behavior in sequential time frame. Even though these two groups of investors may end up holding a common risky portfolio in each spot market, the differences between their trading behaviors are most reflected through the portfolio weights assigned to each of the separating portfolios within the time frame and across states. Precisely, the portfolio weights corresponding to investors respectively from the two groups are associated with recognizable different time patterns. We showed that such difference in trading behavior would be also reflected from the time patterns of the instantaneous returns and the volatilities of the funds

respectively managed by investors from these two groups.

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