Aggregation in Incomplete Market with General Utility Functions

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Abstract

This paper tackles the "aggregation problem" for stochastic economies with possibly incomplete market. An "aggregation theorem" is proved towards an analytic construction of the representative agent's utility function. This is done within a general time-state setup with general utility functions and without restrictions on the initial resource allocations. Welfare implications, concerning the social welfare loss resulting from market incompleteness, are readily reflected from the constructed representative agent's utility function.

Key Words: Aggregation, constrained Pareto optimal, incomplete market

JEL Classification numbers: D52, G11, G12.

1 Introduction

Modern equilibrium asset pricing theory has been largely built upon the assumption on the existence of a representative agent (Lucas 1978, Breeden 1979, CIR 1985, Epstein-Zin 1989). The quest for the existence of such an agent in a multi-agents environment is known as "aggregation problem" in literature. When aggregation holds, the aggregated demand and the equilibrium security prices can be easily characterized. It will be largely determined by the fundamental of the macro economy together with the psychological sentiment of the investment community that is summarized by the representative agent's utility function.

The aggregation problem represents one of the most challenging problem in modern economic theory. The proof of the aggregation theorem attracted attention from academia ever since Arrow-Debreu (1954) launched their proofs of the

welfare theorems as a foundation of modern finance theory. For complete market stochastic economy of Arrow-Debreu (1954), a well-defined representative agent's utility function can be constructed (Nigishi 1960).

When market is incomplete, equilibrium allocations may no longer be Pareto efficient (Hart 1975, Magill and Quinzii 1986), and marginal rates of substitution at an arbitrary equilibrium allocation may no longer equal to each other among the agents. In consequence, the utility function defined as the maximum of weighted individual utility functions, known as "social welfare function", no longer constitutes the representative agent's utility function. The existence of the representative agent along with the construction of representative agent's utility function (if exists) has thus become an open question, which has been silently waiting for answer for more than five decades. The difficulty and its great importance associated with the aggregation problem is best reflected from the following quote taking from Rubinstein (1974):

> "The chief difficulties befouling the analysis of securities market equilibrium is the problem of aggregation. ......" (Mark Rubinstein 1974).

The difficulty, as to be illustrated in this paper, is a conceptual one. The existing aggregation results for incomplete market are largely derived under certain restrictions on individual agents' utility functions. These include Gorman (1953), Rubinstein (1974), Milne (1979), and Detemple and Gottardi (1998). The first three papers, see also Hara (2008) for a dynamic extension, consider utility functions falling into the HARA class with certain homogeneity structure in preferences. These aggregation results rely heavily on the HARA utility specification because utility functions in the HARA class admit a linear risk tolerance representation (in consumption), which in turn ensure a mutual fund separating property to hold. Aggregation is thus obtained as a direct consequence of mutual fund separation. Detemple and Gottardi (1998) alike assume agents' having "almost identically homothetic" utility functions. Under such utility specifications the market becomes "effective complete" in the sense that the equilibrium allocation fulfills the Pareto optimality condition. In this case, the social welfare function constructed by Nigishi (1960) constitutes representative agent's utility function.

The aggregation problem to be tackled in this paper is for incomplete markets, particularly applies for economies with its equilibrium allocations falling outside the Pareto efficient set. The aggregation theorem obtained in this paper does not rely on any particular specification of agents' utility functions (except for monotonicity, concavity and smoothness), and without imposing any restrictions on the initial resource allocation.

The proof of the aggregation theorem is built up on an earlier result on the "constrained Pareto optimality" of equilibrium allocations when the market is incomplete. According to Grossman (1977), even though an equilibrium allocation in incomplete market might fail to be Pareto efficient, yet the equilibrium allocation must be "constrained Pareto efficient" in the sense that no Pareto improvement can be constructed by restricting the net proceeds of the

re-allocations belong to the market span. The latter is equivalent to say that, no Pareto improvement can be obtained by trading within the existing market place.

The key breakthrough leading to our proof of the aggregation theorem is a newly proved "characterization theorem" (Theorem 3) for the set of constrained Pareto efficient allocations. The characterization theorem extends or builds upon the existing results in literature (see Ohlson 1987). Indeed, the characterization theorem itself represents an interesting and important discovery because it provides an ideal venue to explore the welfare implications of stochastic economies with possibly incomplete markets!

The remaining of the paper is organized as follows: Section 2 contains some preliminaries and set-up of the model. Section 3 provides an characterization theorem for the set of constrained Pareto efficient allocations. Section 4 is on the construction of representative agent's utility function when the equilibrium allocation might fail to be Pareto efficient. Section 5 contains several concluding remarks.

2 Setup of the Stochastic Economy

Consider a stochastic financial market economy of finite periods T and finite state space Ω. A set of agents $\mathcal{I} = \{1, \cdots, I\}$ have a homogeneous information structure that is summarized by a sequence of increasing partitional information filtration $\{\mathcal{F}_t\}_{t=0}^T$. Let $m = \Sigma_{t=1}^T \# (\mathcal{F}_t)$, $\# (\mathcal{F}_t)$ be the number of (base) disjoint events in $\mathcal{F}_t$.

2.1 The Agents

Each agent $i \in \mathcal{I}$ in the economy is characterized by a bundle $\{\succeq_i, e^i\}$ in which

- $\succeq^i$ is agent i's preference relation on $\mathbb{R}^{m+1}$;

- $e^i \in \mathbb{R}^{m+1}$ is agent i's endowment.[1]

We assume that

Condition U $\succeq^i, i \in \mathcal{I}$ be represented by a utility function $u^i : \mathbb{R}^{m+1} \mapsto \mathbb{R}$, where u^i is continuously differentiable, strictly increasing and concave.

Let
$$\mathbb{A} = \underbrace{\mathbb{R}^{m+1} \times \cdots \times \mathbb{R}^{m+1}}_{I}$$

be the allocation space among all society members — the agents. Given the aggregate endowment $e = \sum_{i \in \mathcal{I}} e^i$, an allocation $\mathbf{a} \in \mathbb{A}$ is called *feasible* if

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[1]Here, the endowments are in the form of cashes across times and states. We can introduce endowments in terms of shares in the tradable securities as well. The analysis below indeed carries through to this general case. Our assuming no endowment in shares is for the sake of notational simplicity.

$\sum_{i \in \mathcal{I}} a^i = e$. An feasible allocation $\mathbf{a}$ is said to *dominate* feasible allocation $\mathbf{b}$ if (a) for each i, $a^i \succeq^i b^i$, and (b) if there exists, at least, one agent i such that $a^i \succ^i b^i$. A non-dominated feasible allocation is *Pareto efficient*. The set of all Pareto efficient allocations is denoted *PE*. Recall also the following well documented result towards the construction of the set of Pareto efficient allocations *PE*.

Theorem 1 *Suppose condition U are satisfied. An allocation $\mathbf{c}^* \in \mathbb{A}$ belongs to* PE *if and only if there exist $\{\lambda_i\}_{i \in \mathcal{I}} \gg 0$ such that*

$$\mathbf{c}^* = \arg \max \left\{ \sum_{i \in \mathcal{I}} \lambda_i u^i(c^i) : \sum_{i \in \mathcal{I}} c^i = e \right\}; \tag{1}$$

in particular, let $MRS^i(c^i) = \frac{1}{\partial u^i(c^i)/\partial c_0^i} \nabla u^i(c^i)$ be the marginal rate of substitution for i; then, at $\mathbf{c}^*$ it must hold true that

$$MRS^1(c^{1*}) = \cdots = MRS^I(c^{I*}). \tag{2}$$

According to Theorem 1, the set *PE* can be fully characterized as outcomes of "social-welfare maximization". Here, the welfare function refers to an arbitrary positive linear combination of (all) agents' utility functions.

2.2   The Market

Let $p = \{p_t\}$ be a $\mathcal{F}_t$-adapted $\mathbb{R}^J$-valued price process for the J tradable securities. A portfolio at time t corresponds to a position, measured in unit, on each of the tradable securities at the time. Let $\phi_t \in \mathbb{R}^J$ be the time-t portfolio. A trading strategy, or trading plan, is thus an $\mathcal{F}_t$-adapted process $\phi = \{\phi_t\}_{t \geq 0}$. The set of all trading strategies is denoted $\Phi = \mathbb{R}^{(m+1) \times J}$.

For any given trading strategy ϕ, the initial cash investment is thus given by $\phi_0 \cdot p_0 = p_0^\phi$. The resulting future cash flows are thus given by the following flow budget constraints

$$d_t^\phi = \phi_{t-1} \cdot (p_t + \delta_t) - \phi_t \cdot p_t$$

for $t = 1, \cdots, T$, with $d_0^\phi = -p_0^\phi$. The *market span*, denoted $\mathbb{D}(p, \delta)$, consists of all possible cash streams that can be generated by trading; that is,

$$\mathbb{D}(p, \delta) = \left\{ d \in \mathbb{R}^{m+1} : \exists \phi \in \Phi \text{ such that } d^\phi = d \right\}.$$

The market span $\mathbb{D}(p, \delta)$ forms a vector subspace of $\mathbb{R}^{m+1}$. Its *orthogonal complement* is denoted $\mathbb{D}^\perp(p, \delta)$, or simply $\mathbb{D}^\perp$, also forms a vector subspace of $\mathbb{R}^{m+1}$. Recall the following well-known "orthogonal decomposition theorem":

Lemma 2 *For all $x \in \mathbb{R}^{m+1}$, there exist unique $x_{\mathbb{D}} \in \mathbb{D}$ and $x_{\mathbb{D}^\perp} \in \mathbb{D}^\perp$ such that $x = x_{\mathbb{D}} + x_{\mathbb{D}^\perp}$; moreover, it holds true that*[2]

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[2] Here, $x \circ d$ is an inner product defined on $\mathbb{R}^{m+1}$; while $p \cdot y$ is the inner product on $\mathbb{R}^J$.

1. $x \circ x_{\mathbb{D}} \geq 0$ with strict inequality if $x \notin \mathbb{D}^{\perp}$;

2. $x \circ x_{\mathbb{D}^{\perp}} \geq 0$ with strict inequality if $x \notin \mathbb{D}$.

Let $\mathbb{M}(p, \delta)$, or simply $\mathbb{M}$, be the set of all future cash flows contained in the market span. So, for each $d \in \mathbb{D}$, we may write $d = (d_0, d_{-0}) \in \mathbb{R} \times \mathbb{M}$ and write $\mathbb{D} \subseteq \mathbb{R} \times \mathbb{M}$. The market at (p, δ) is *complete* if $\mathbb{M} = \mathbb{R}^m$; otherwise, it is *incomplete*. So, in a complete market economy, all future cash streams d_{-0} can be obtained by trading; but, when the market is incomplete, $\mathbb{M} \subset \mathbb{R}^m$ forms a strict subspace of the Euclidean space.

3 PE and CPE: A Characterization Theorem

An allocation $\mathbf{c} \in \mathbb{A}$ is *constrained feasible* if (1) $c_{-0}^i - e_{-0}^i \in \mathbb{M}$; (2) $\sum_{i \in \mathcal{I}} c^i = e$.[3] A constrained feasible allocation $\mathbf{c}$ is said to be *constrained Pareto efficient* if it is not dominated by other constrained feasible allocations. The set of all constrained feasible allocations is denoted *CF*, and the set of all constraint Pareto efficient allocations is denoted *CPE*.

For complete market, all Pareto efficient allocations can be achieved by trading. So, in this case, the set of *CPE allocations* coincides with that of *PE*, and the set of *CPE* allocations can be fully characterized by Theorem 1. For incomplete market, an allocation $\mathbf{c}^*$ satisfying condition (2) might not belong to *CPE* if it is not constrained feasible. Analoguing to Theorem 1, the set of constrained Pareto efficient allocations can be obtained by solving the following social welfare maximization problem.

Theorem 3 *Under condition* U*, the following conditions are equivalent:*

(a) $\mathbf{c}^* \in$ CPE;

(b) $\exists \{\lambda_i\}_{i \in \mathcal{I}} \gg 0$ such that

$$\mathbf{c}^* = \arg \max_{\substack{c_{-0}^i - e_{-0}^i \in \mathbb{M} \\ \forall i \in \mathcal{I}}} \left\{ \sum_{i \in \mathcal{I}} \lambda_i u^i \left(c^i \right) : \sum_{i \in \mathcal{I}} c^i = e \right\}; \tag{3}$$

(c) $\mathbf{c}^* \in$ CF and $\exists \{\lambda_i\}_{i \in \mathcal{I}} \gg 0$ and $\{\xi^i\}_{i \in \mathcal{I}} \subseteq \mathbb{M}^{\perp}$ such that

$$\mathbf{c}^* = \arg \max \left\{ \sum_{i \in \mathcal{I}} \lambda_i u^i \left(c^i \right) - \xi^i \circ c_{-0}^i : \sum_{i \in \mathcal{I}} c^i = e \right\}; \tag{4}$$

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[3]It is noted that feasibility condition 2 for the financial market, together with the market spanning condition 1, implies feasibility condition $\sum_{i \in \mathcal{I}} c^i = e$ to hold. So, generally speaking, all constrained feasible allocations are feasible; but, not all feasible allocations are constrained feasible. The latter is particularly true for incomplete market economies.

(d) $\mathbf{c}^* \in$ CF *and* $\exists \psi \in \mathbb{R}^m_{++}$ *and* $\{\xi^i\}_{i \in \mathcal{I}} \subseteq \mathbb{M}^\perp$ *such that*

$$MRS^i(c^{i*}) = \psi + \xi^i \text{ for all } i \in \mathcal{I}. \tag{5}$$

Proof. The equivalence between (a) and (b) is well documented in literature (see, Ohlson, 1987). To prove "(b) $\Leftrightarrow$ (c)", we need to proceed with the optimization problem (3). Since $\mathbb{M}^\perp \subseteq \mathbb{R}^m$ is a finite $(k < m)$ dimensional vector space, it can be thus spanned by k independent m-dimensional vectors $\{v_1, \cdots, v_k\}$.[4] The constraints $c^i_{-0} - e^i_{-0} \in \mathbb{M}$ is thus equivalent to

$$v_n \circ (c^i_{-0} - e^i_{-0}) = 0 \text{ for } n = 1, \cdots, k.$$

Accordingly, we may apply the Kuhn-Tucker theorem (Luenberger 1969) to transform the problem (3) with $I \times k$ linear constraints (plus the resource feasibility constraints) into an optimization problem with resource feasibility constraints only. The Lagrangian function for (3) takes the form

$$\sum_{i \in \mathcal{I}} \left[\lambda_i u^i (c^i) - \xi^i \circ (c^i_{-0} - e^i_{-0}) \right]$$

in which $\xi^i = \sum_{n=1}^k \mu_{in} v_n \in \mathbb{M}^\perp$, and $\{\mu_{ni}\}$ are the Lagrangian multipliers for each of the equality constraints. The equivalence between (b) and (c) follows also from the Kuhn-Tucker theorem.

We may apply again the Kuhn-Tucker theorem to establish the equivalence between (c) and (d). This time we proceed with the optimization problem (4). From the first order conditions for (4), we obtain, at the optimal solution $\mathbf{c}^*$, the necessary and sufficient condition

$$\text{MRS}^i(c^{i*}) = \frac{1}{\psi_0} (\xi^i + \psi) \text{ for all } i \in \mathcal{I}$$

in which $(\psi_0, \psi) \in \mathbb{R}^{m+1}_{++}$ corresponds to the positive Lagrangian multipliers for the resource constraints. ∎

From condition (d) of Theorem 3 we see that, even though, at a CPE allocation $\mathbf{c}^*$, agents' marginal rates of substitution may not equal to each other, but their marginal rates of substitutions must share the same projection on the market span! So, as a corollary to Theorem 3, we have:

Corollary 4 *At any constrained Pareto optimal allocation* $\mathbf{c}^*$, *agents' marginal rates of substitution admit the following decomposition on*

$$MRS^i(c^{i*}) = \psi + \xi^i \text{ for all } i \in \mathcal{I} \tag{6}$$

in which $\psi \in \mathbb{M}$ *and* $\{\xi^i\}_{i \in \mathcal{I}} \subseteq \mathbb{M}^\perp$.

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[4]If $\mathbb{M}$ is r-dimensional subspace of $\mathbb{R}^m$, $r \leq m$, then $\mathbb{M}^\perp$ would form a $k = (m-r)$-dimensional vector subspace of $\mathbb{R}^m$.

Proof. Let $\psi \in \mathbb{R}^m_{++}$ and $\left\{\xi^i\right\}_{i \in \mathcal{I}} \subseteq \mathbb{M}^\perp$ be for the validity of condition (d) of Theorem 3. By Lemma 2, with $\psi = \psi_\mathbb{M} + \psi_{\mathbb{M}^\perp}$, we obtain

$$\text{MRS}^i(c^{i*}) = \psi_\mathbb{M} + \eta^i$$

in which $\psi_\mathbb{M} \in \mathbb{M}$ and $\eta^i = \psi_{\mathbb{M}^\perp} + \frac{1}{\lambda^i}\xi^i \in \mathbb{M}^\perp$. ∎

As a final remark, the above characterization theorem for CPE captures clearly the welfare loss resulting from the market incompleteness. The term $\xi^i \circ \left(c^i_{-0} - e^i_{-0}\right)$ in the Lagrangian is known as "penalties" to agent i whenever $c^i_{-0} - e^i_{-0}$ falling outside the market span. Such penalties (whenever not equal to zero) can be thus interpreted as welfare loss to the agent resulting from the market incompleteness — keeping in mind, when market is complete, $\mathbb{M}^\perp = \{\emptyset\}$, agents suffer no welfare losses.

4   Aggregation in Incomplete Markets

This section formulates and studies the aggregation problem in incomplete market. The problem is precisely formulated in section 4.1. An aggregation theorem, the main result of this paper, is proved in section 4.2.

4.1   The Problem

Given a multi-agent stochastic market economy that is summarized by

$$\mathcal{E}\left(p, \left\{e^i\right\}_{i \in \mathcal{I}}\right) = \left(\mathbb{D}(p, \delta); \left\{u^i, e^i\right\}_{i \in \mathcal{I}}\right).$$

In this economy, all agents act as price-takers, and each agent i's problem is to solve

$$\max\left\{u^i\left(c^i\right) : c^i - e^i \in \mathbb{D}(p, \delta)\right\}. \tag{7}$$

By the fundamental theorem, solution to (7) exists if and only if the market satisfies the "no-arbitrage" condition:

Condition NA: $\mathbb{D}(p, \delta) \cap \mathbb{R}^{m+1}_+ = \{\emptyset\}$.

Condition NA implies, by Lemma 2, for all $x \in \mathbb{R}^{m+1}_+$, if $x \neq \emptyset$, then $x \circ x_{\mathbb{D}^\perp} = \|x_{\mathbb{D}^\perp}\|^2 > 0$. In fact, by the Hahn-Banach theorem, condition NA is equivalent to the existence of a positive state price vector $\psi \in \mathbb{R}^m_{++}$ such that $(1, \psi) \in \mathbb{D}^\perp$. This, in turn, is equivalent to existence of a positive linear pricing rule defined over all future cash flows $d_{-0} \in \mathbb{M}$. We may put this formally as follow

Lemma 5 *Condition NA is equivalent to the existence of $\psi \in \mathbb{R}^m_{++}$ such that*

$$p^\phi_0 = \psi \circ d^\phi_{-0}, \; \phi \in \Phi; \tag{8}$$

particularly to those tradable securities $j = 1, \cdots, J$, it yields[5]

$$p^j_0 = \psi \circ \delta^j. \tag{9}$$

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[5]In fact, the two conditions (8) and (9) are equivalent.

A price-allocation bundle $\{p, \overline{\mathbf{c}}\} \in \mathbb{R}^J \times \mathbb{A}$ is a *competitive equilibrium* for $\mathcal{E}\left(p, \{e^i\}_{i \in \mathcal{I}}\right)$ if

- for each i, $\overline{c}^i = \arg\max\left\{u^i\left(c^i\right) : c^i - e^i \in \mathbb{D}(p, \delta)\right\}$;

- for each i, there exists $\phi^i \in \Phi$ financing $\overline{c}^i$, such that $\sum_{i \in \mathcal{I}} \phi^i = \emptyset$.[6]

So, what implicitly assumed for the existence of a competitive equilibrium is that condition NA is satisfied. The set of all competitive equilibrium allocations is denoted CE.

We assume that the multi-agent economy $\mathcal{E}\left(p, \{e^i\}_{i \in \mathcal{I}}\right)$ prescribed in the previous section is at an equilibrium status. Let $\overline{\mathbf{c}} \in CE$ be the corresponding equilibrium allocation so that[7]

$$\overline{c}^i = \arg\max\left\{u^i\left(c^i\right) : c^i - e^i \in \mathbb{D}(p, \delta)\right\}, \forall i \in \mathcal{I}.$$

We consider a hypothetical single agent economy

$$\mathcal{E}\left(p, e\right) = \left(\mathbb{D}\left(p, \delta\right); \{u, e\}\right)$$

with utility function u and initial endowment to be given by the aggregate endowment $e = \sum_{i \in \mathcal{I}} e^i$.

Definition 6 *Let $(p, \overline{\mathbf{c}}) \in \mathbb{R}^J \times \mathbb{A}$ be an equilibrium for $\mathcal{E}\left(p, \{e^i\}_{i \in \mathcal{I}}\right)$. Economy $\mathcal{E}\left(p, e\right)$ is said to be a* representative agent economy *of $\mathcal{E}\left(p, \{e^i\}_{i \in \mathcal{I}}\right)$ if (p, e) (with $\phi = \emptyset$) constitutes an competitive equilibrium for $\mathcal{E}\left(p, e\right)$. In this case, the utility function u is referred to the* representative agent's utility function *of $\mathcal{E}\left(p, \{e^i\}_{i \in \mathcal{I}}\right)$.*

The problem on the existence of representative agent for an arbitrary multi-agent economy is known as "aggregation problem". When the equilibrium allocation $\overline{\mathbf{c}}$ is Pareto efficient, the representative agent exists, and its utility function can be easily constructed. The theorem below is well known, which is due to Nigishi (1960):

Theorem 7 *Suppose condition U is satisfied. If the equilibrium allocation $\overline{\mathbf{c}}$ for $\mathcal{E}\left(p, \{e^i\}_{i \in \mathcal{I}}\right)$ is Pareto efficient, then representative agent exists with utility function $u : \mathbb{R}^{m+1} \mapsto \mathbb{R}$ to be given by*

$$u\left(c\right) = \max\left\{\sum_{i \in \mathcal{I}} \lambda_i u^i\left(c^i\right) : \sum_{i \in \mathcal{I}} c^i = c\right\} \tag{10}$$

where $\lambda_i = \frac{1}{\partial u^i(c^i)/\partial c_0^i} > 0$ for all $i \in \mathcal{I}$. Moreover, the utility u so-defined satisfies condition U.

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[6]This, in turn, implies the validity of the feasibility condition $\sum_{i \in I} \overline{c}^i = \sum_{i \in I} e^i$.

[7]Precisely, we shall put it as $\left\{\overline{\phi}^i, \overline{c}^i\right\}_{i \in \mathcal{I}}$ to constitute an equilibrium allocation respectively for the assets market and cash market, in which, for all i, $\overline{c}^i$ is financed by $\overline{\phi}^i$, and $\sum_i \overline{\phi}^i = \emptyset$.

It is also noted that, for complete market, we have CE to coincide with PE, and the aggregation theorem holds. For incomplete markets, the correspondence between CE and PE, in general, breaks down.[8] Nevertheless, as it is well known, the set of CE allocations admits a correspondence with the set of CPE allocations (Grossman 1977). This forms the foundation for our construction of representative agent's utility function in incomplete market.

4.2   An Aggregation Theorem

We start with an review of the "welfare theorem" in incomplete market. The theorem establishes the correspondence between the set of competitive equilibrium allocations and the set of constrained Pareto efficient allocations. It is originated from Grossman (1977) and is documented in literature (see, Ohlson, 1987). We put it here for reader's convenience.

Theorem 8 (Grossman) *Any CPE allocation* $\mathbf{c}$ *in* $\mathcal{E}\left(p, \{e^i\}_{i \in \mathcal{I}}\right)$ *can be supported as a CE allocation for the economy* $\mathcal{E}\left(p, \{c^i\}_{i \in \mathcal{I}}\right)$ *with* $\mathbf{c}$ *being the initial endowment; and each CE allocation for* $\mathcal{E}\left(p, \{e^i\}_{i \in \mathcal{I}}\right)$ *must constitute a CPE allocation for the same economy.*

With Theorem 8 and Theorem 3 we are ready to introduce the main theorem of this paper, concerning the existence of representative agent and its utility function when the market is not necessarily complete. Here is the aggregation theorem:

Theorem 9 *Suppose condition U is satisfied. Let* $(p, \overline{\mathbf{c}})$ *constitute an equilibrium for the economy* $\mathcal{E}\left(p, \{e^i\}_{i \in \mathcal{I}}\right)$. *Then, there exists a representative agent for* $\mathcal{E}\left(p, \{e^i\}_{i \in \mathcal{I}}\right)$. *Moreover, let* ψ *be any arbitrary positive state price vector for* $\mathbb{D}$, *and let* $\pi^i = \psi + \xi^i \gg 0, \xi^i \in \mathbb{M}^\perp$, *be agent* i's *marginal rate of substitution at* $\overline{c}^i$. *Then, the representative agent's utility function* $u : \mathbb{R}^{m+1} \mapsto \mathbb{R}$ *is given by*

$$u(c) = \max \left\{ \sum_{i \in \mathcal{I}} \lambda_i u^i(c^i) - \xi^i \circ c^i_{-0} : \sum_{i \in \mathcal{I}} c^i = c \right\} \tag{11}$$

for all $c \in \mathbb{R}^{m+1}$, *where* $\lambda_i = \frac{1}{\partial u^i(\overline{c}^i)/\partial c^i_0} > 0, i \in \mathcal{I}$.

Proof. The representative agent's utility function defined by (11) is concave and continuously differentiable. We can further show that the utility function

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[8] The existing aggregation results documented in leterature are either based on utility functions falling into a certain paramatreic class, namely, the HARA class with some homogeneity utility structure (Gorman (1953), Rubinstein (1974), Milne (1979), and Hara 2008), or based on the so-called "effective complete" condition illustrated in Detemple and Gottardi (1998).

u so-defined display strict monotonicity for deviations to be located in $\mathbb{R} \times \mathbb{M}$.[9] Let c be an arbitrary aggregate cash stream with $\{c^i\}_{i \in \mathcal{I}}$ being a solution to (11), and let $\emptyset \neq d \in \mathbb{R} \times \mathbb{M}$ be non-negative. Let $\overline{d}$ be obtained by dividing d fairly among the agents. So, we have $\emptyset \neq \overline{d} \in \mathbb{R}_+ \times \mathbb{M}_+$. Consider the utility incremental when c moves along direction d to $c + d \in \mathbb{R}^{m+1}$. We have:

$$u\left(c + d\right) \geq \sum_{i \in \mathcal{I}} \lambda_i u^i \left(c^i + \overline{d}\right) - \xi^i \circ \left(c^i_{-0} + \overline{d}_{-0}\right)$$

$$\geq \sum_{i \in \mathcal{I}} \lambda_i u^i \left(c^i\right) - \xi^i \circ c^i_{-0} = u\left(c\right)$$

in which, the second inequality holds true because, by assumption $\xi^i \in \mathbb{M}^\perp$, so $\xi^i \circ \overline{d}_{-0} = 0$ for all $i \in \mathcal{I}$, and also because of the strict monotonicity of u^i. The second inequality in the above assessment holds strictly for $d \neq \emptyset$.

Recall that, at equilibrium (p, δ), condition NA is satisfied with $p = \psi \circ \delta$ for $\psi \in \mathbb{R}^m_{++}$. By the fundamental theorem of finance, the representative agent's portfolio choice problem has a solution. In fact, the solution is given by e. First, recall that the representative agent's utility function is strictly increasing and concave within the budget constraint since $c - e$ belongs to the market span $\mathbb{D}$ and $c_{-0} - e_{-0} \in \mathbb{M}$. Second, following the definition of the representative agent's utility function, its marginal rate of substitution at e is given by $\mathrm{MRS}_u(e) = \psi \gg 0$; that is, the necessary condition for optimality is satisfied at $c = e$. This, together with the strict concavity of the utility function u, implies the solution to the optimal portfolio choice problem (7) at p (for the representative agent) be given by the aggregate endowment e, which is financed by $\phi = \emptyset$ — no trading. In consequence, (p, e), along with state price ψ, constitutes an equilibrium for the representative agent economy $\mathcal{E}(p, e)$. ∎

Implicitly determined as a solution to the optimization problem (11), allocation at the aggregate endowment e is given by the equilibrium allocation $\overline{\mathbf{c}}$ in the original multi-agent economy. This is because, at $\{\overline{c}^i\}_{i \in \mathcal{I}}$, it holds true that

$$\mathrm{MRS}^i\left(\overline{c}^i\right) = \psi + \xi^i$$

for all $i \in \mathcal{I}$, which constitutes the first order condition for optimization problem (11). The strict concavity of the utility functions suggests the equilibrium allocation $\{\overline{c}^i\}_{i \in \mathcal{I}}$ to constitute a unique solution to the problem (11).

Moreover, when equilibrium allocation is Pareto efficient, as is the case when the market is complete, we may set $\xi = 0$ for all agents in conforming to Theorem 7. For the case of incomplete market, the term $\sum_{i \in \mathcal{I}} \xi^i \circ c^i_{-0}$ on the right hand side of equation (11), which is referred to as "social welfare function", is interpreted as the "social cost" resulting from market incompleteness.

As a final remark, the marginal rate of substitution for the representative agent at the aggregate consumption is given by the equilibrium state price ψ for

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[9]Here, we prove the utility function to be monotonic on $\mathbb{R} \times \mathbb{M}$, though it is defined for all cash streams in $\mathbb{R}^{m+1}$. Such "constrained monotonicity" is sufficient for our purpose here because all deviations in future cash flows are restricted to fall into the future market span.

the multi-agent economy. So, different state price in supporting the equilibrium would induce to different utility function for the representative agent. This suggests the representative agent for a given economy, in general, not to be unique (if exists); particularly when the economy has more than one equilibria bundles $(p, \overline{\mathbf{c}})$ each being supported by different state price vectors ψ.

5 Concluding Remarks

In this paper, we have accomplished two things. First, we prove a characterization theorem for the set of CPE allocations by extending the existing results documented in literature; and second, we construct the representative agent's utility function when the economy is at an equilibrium status. These are accomplished without imposing any restrictions on agents utility functions and initial resource allocations. Moreover, we do not restrict the equilibrium allocation to be Pareto efficient, nor assume the market to be "effective complete" in the sense of Detemple and Gottardi (1998).

The consumption domain for the set of admissible cash streams is not restricted in this paper. The results, nevertheless, readily extend to economies with externality constraints on the set of cash streams, say, for instance, the case of all cash streams are restricted to be non-negative or to fall into some pre-specified convex set $\mathcal{L}$ of the Euclidean space. In this case, the results hold true by assuming the aggregate endowments to fall into the "interior" of the domain $\mathcal{L}$. It also remains to see to what extent the aggregation theorem holds for infinite economies. This leaves for future research.

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