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BOCHNER TECHNIQUE ON STRONG KÄHLER-FINSLER MANIFOLDS*

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Abstract By using the Chern-Finsler connection and complex Finsler metric, the Bochner technique on strong Kähler-Finsler manifolds is studied. For a strong Kähler-Finsler manifold M, the authors first prove that there exists a system of local coordinate which is normalized at a point $v \in \tilde{M} = T^{1,0}M \backslash o(M)$, and then the horizontal Laplace operator $\square_H$ for differential forms on PTM is defined by the horizontal part of the Chern-Finsler connection and its curvature tensor, and the horizontal Laplace operator $\square_H$ on holomorphic vector bundle over PTM is also defined. Finally, we get a Bochner vanishing theorem for differential forms on PTM. Moreover, the Bochner vanishing theorem on a holomorphic line bundle over PTM is also obtained

Key words Bochner technique; strong Kähler-Finsler manifold; horizontal Hodge-Laplace operator; Weitzenböck formula

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Bochner [1, 2] initiated a method, the well-known "Bochner technique", which used the Laplace operator and the general maximum principle of E.Hopf to deal with the relation between vector or tensor fields and the curvature of manifolds, and got the global properties of manifolds. From then on, the Bochner technique became a very useful method in geometrical study. Such as, both in Riemannian and Kählerian manifolds, the Bochner technique were discussed in details in [3–5]. Recently, under the initiation of S.S. Chern, the global differential geometry of real and complex Finsler manifolds gained a great development [6–8], Abate and Pateizio [8] set up a Cartan-Finsler connection in a real Finsler manifold and a Chern-Finsler connection in a complex Finsler manifold. By using the non linear connection associated to the Cartan-Finsler connection, Zhong Tongde and Zhong Chunping [9] discussed the Bochner technique in a real Finsler manifold. In this article, based on [10], we further discuss the Bochner technique for a strong Kähler-Finsler manifold, and obtain some Weitzenböck formulas on strong Kähler-Finlser manifolds. Using the Weitzenböck formulas, we get the Bochner vanishing theorems on strong Kähler-Finlser manifolds.

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1 The Normalizations of Coordinates and Frames

Definition 1 [8] A strongly pseudoconvex complex Finsler metric on a complex manifold M is a continuous function $F : T^{1,0}M \longrightarrow R^+$ satisfying

(i) $G = F^2$ is smooth on $\tilde{M} = T^{1,0}M\backslash o(M)$;

(ii) $F(v) > 0$ for all $v \in \tilde{M}$;

(iii) $F(\lambda v) = |\lambda|F(v)$ for all $v \in T^{1,0}M$ and $\lambda \in \mathbb{C}$;

(iv) The Hermitian matrix $(G_{\alpha\bar{\beta}})$ is positive definite on $\tilde{M}$, where

$$G_{\alpha\bar{\beta}} = \frac{\partial^2 G}{\partial v^\alpha \partial \overline{v}^\beta}.$$

The derivatives with respect to the z-coordinates will be denoted by indices after a semicolon, for instance,

$$G_{;\mu\nu} = \frac{\partial^2 G}{\partial z^\mu \partial z^\nu} \quad \text{or} \quad G_{\alpha;\bar{\nu}} = \frac{\partial^2 G}{\partial \bar{z}^\nu \partial v^\alpha}.$$

A manifold M endowed with a strongly pseudoconvex complex Finsler metric will be called a strongly pseudoconvex complex Finsler manifold.

Condition (iv) allows us to introduce a Hermitian structure on the vertical bundle $\mathcal{V}$. Indeed, if $v \in \tilde{M}$, and $W_1, W_2 \in \mathcal{V}_v$ with $W_j = W_j^\alpha \dot{\partial}_\alpha$ $(j = 1, 2)$, we set

$$\langle W_1, W_2\rangle_v = G_{\alpha\bar{\beta}}(v) W_1^\alpha \overline{W}_2^\beta, \tag{1}$$

then, there is a unique Chern-Finsler connection D associated to the Hermitian structure induced by F. Being Chern-Finsler connection D a good complex vertical connection, it extends to a complex linear connection on $\tilde{M}$(still called the Chern-Finsler connection in this article). Using the complex horizontal map $\Theta : \mathcal{V} \longrightarrow \mathcal{H}$, we can transfer the Hermitian structure $\langle, \rangle$ on $\mathcal{H}$ just by setting

$$\forall H, K \in \mathcal{H}_v, \quad \langle H, K\rangle_v = \langle \Theta^{-1}(H), \Theta^{-1}(K)\rangle_v, \tag{2}$$

and then, we can define a Hermitian structure on $T^{1,0}\tilde{M}$ by requiring $\mathcal{H}$ to be orthogonal to $\mathcal{V}$. It is easy to check that these definitions are compatible enough so to get

$$X\langle Y, Z\rangle = \langle D_X Y, Z\rangle + \langle Y, D_{\bar{X}} Z\rangle, \tag{3}$$

for any $X \in T^{1,0}\tilde{M}$, and $Y, Z \in \chi(T^{1,0}\tilde{M})$.

The complex nonlinear connection $\tilde{D} : \chi(T^{1,0}M) \longrightarrow \chi(T^*_{\mathbb{C}}M \otimes T^{1,0}M)$ associated to the Chern-Finsler connection is defined as follows.

Take $\xi \in \chi(T^{1,0}M), p \in M$, and set $v = \xi(p)$, then

$$\tilde{D}\xi = \left(\frac{\partial \xi^\alpha}{\partial z^\mu}(p) + \Gamma^\alpha_\mu(\xi(p))\right) \mathrm{d}z^\mu \otimes \frac{\partial}{\partial z^\alpha}\Big|_p, \tag{4}$$

where Γ^α_μ's are the Christoffel symbols of the complex nonlinear connection $\tilde{D}$. In local coordinates, they can be expressed as

$$\Gamma^\alpha_\mu = \tilde{\Gamma}^\alpha_{\beta;\mu} v^\beta = G^{\bar{\tau}\alpha} G_{\bar{\tau};\mu}. \tag{5}$$

Set

$$\Gamma^\alpha_{\beta;\mu} = \dot{\partial}_\beta(\Gamma^\alpha_\mu) = G^{\bar{\tau}\alpha} \delta_\mu(G_{\beta\bar{\tau}}), \tag{6}$$

which is the horizontal part of the Chern-Finsler connection.

Definition 2 [8] Assume that (M,F) is a strongly pseudoconvex complex Finsler manifold, if the horizontal part of (2,0)-torsion θ for the Chern-Finsler connection is zero, that is, for arbitrary $H,K\in\mathcal{H},\theta(H,K)=0$, then F is called a strong Kähler-Finsler metric of M. In local coordinates, the necessary and sufficient conditions of F to be a strong Kähler-Finsler metric of M are

$$\Gamma^{\alpha}_{\mu;\nu}=\Gamma^{\alpha}_{\nu;\mu}. \tag{7}$$

Let (M,F) be a strongly pseudoconvex complex Finsler manifold, then, we may associate the fundamental form

$$\Phi=\mathrm{i}G_{\alpha\bar\beta}\mathrm{d}z^{\alpha}\wedge\mathrm{d}\bar z^{\beta}, \tag{8}$$

which is a well-defined real valued $(1,1)$-form on $\tilde M$.

Proposition 1 [8] The differential form Φ is d_H-closed, that is, $d_H\Phi=0$, if and only if the metric F is a strong Kähler-Finsler metric.

It is easily seen that the condition $d_H\Phi=0$ is equivalent to

$$\frac{\delta G_{\alpha\bar\beta}}{\delta z^{\gamma}}=\frac{\delta G_{\gamma\bar\beta}}{\delta z^{\alpha}},\qquad \frac{\delta G_{\alpha\bar\beta}}{\delta \bar z^{\gamma}}=\frac{\delta G_{\alpha\bar\gamma}}{\delta \bar z^{\beta}}, \tag{9}$$

i.e.,

$$\Gamma^{\alpha}_{\beta;\gamma}=\Gamma^{\alpha}_{\gamma;\beta}.$$

Definition 3 Assume that (M,F) is a strongly pseudoconvex complex Finsler manifold, let $v_0\in\tilde M=T^{1,0}M\backslash o(M)$, then, a complex coordinate system $\{z,v\}$ around v_0 is said to be normal at point v_0 iff

$$\begin{cases} G_{\alpha\bar\beta}(v_0)=\delta_{\alpha\bar\beta},\\ d_HG_{\alpha\bar\beta}(v_0)=0,\end{cases}$$

for all α,β. If a normal complex coordinate system exists at v_0, then obviously, $d_H\Phi=0$ is the additional condition of (M,F) being a strong Kähler-Finsler manifold. Conversely, we have

Lemma 1 Let (M,F) be a strong Kähler-Finsler manifold. Given a point $v_0\in\tilde M=T^{1,0}M\backslash o(M)$, then there exists a complex coordinate system normal at v_0 (cf. [10, 11]).

Proof Let (z^i,v^{α}) be an arbitrary coordinate function around v_0, such that $(z^i,v^{\alpha})(v_0)=(0,v^{\alpha}(v_0))$ for all i,α. Let $\Phi=\mathrm{i}G_{\alpha\bar\beta}\mathrm{d}z^{\alpha}\mathrm{d}\bar z^{\beta}$, where

$$G_{\alpha\bar\beta}=\overline{G_{\beta\bar\alpha}}, \tag{10}$$

for all α,β. It is no matter we may assume

$$G_{\alpha\bar\beta}(v_0)=\delta_{\alpha\bar\beta}. \tag{11}$$

Since $d_H\Phi=0$, by the assumption,

$$\Gamma^{\alpha}_{\beta;\gamma}(v_0)=\Gamma^{\alpha}_{\gamma;\beta}(v_0). \tag{12}$$

We introduce a new coordinate function $({z'}^{\alpha},{v'}^{\alpha})$ with

$${z'}^{\alpha}=z^{\alpha}+\frac{1}{2}\Gamma^{\alpha}_{\beta;\gamma}(v_0)z^{\beta}z^{\gamma},$$

for all α. Then (12) yields immediately for all α

$$\mathrm{d}z'^{\alpha} = \mathrm{d}z^{\alpha} + \Gamma^{\alpha}_{\beta;\gamma}(v_0)z^{\gamma}\mathrm{d}z^{\beta}. \tag{13}$$

Writing $\Phi = \mathrm{i}\tilde{G}_{\alpha\bar{\beta}}\mathrm{d}z'^{\alpha}\mathrm{d}\bar{z}'^{\beta}$, it follows from (9)–(12) and (6) that $\frac{\delta\tilde{G}_{\alpha\bar{\beta}}}{\delta z^{\gamma}} = 0$ for α,β,γ. Thus, $d_H\tilde{G}_{\alpha\bar{\beta}}(v_0) = 0$ for all α,β. It is clear from (10) and (12) that $\tilde{G}_{\alpha\bar{\beta}}(v_0) = \delta_{\alpha\bar{\beta}}$. This completes the proof.

Definition 4 Assume that $\{V_1,\cdots,V_n\}$ is a local horizontal frame field of $(1,0)$-form around $v_0 \in \tilde{M}$, that is, $V_1,\cdots,V_n$ are the horizontal vector fields of $(1,0)$-form defined near v_0, satisfying $\langle V_\alpha, V_\beta\rangle = \delta_{\alpha\bar{\beta}}$ for all α,β (where $\langle,\rangle$ is the Hermitian inner product on $T^{1,0}\tilde{M}$ restricted on $\mathcal{H}$). Then, the local horizontal frame $\{V_\alpha\}$ of $(1,0)$-form is called normalized at v_0, if and only if for all α,β

$$D_{V_\alpha}V_\beta(v_0) = 0, \tag{14}$$

where D is the Chern-Finsler connection. (14) implies that

$$D_{\bar{V}_\alpha}\bar{V}_\beta(v_0) = D_{V_\alpha}\bar{V}_\beta(v_0) = D_{\bar{V}_\alpha}V_\beta(v_0) = 0, \tag{15}$$

for all α,β. Thus, we may write

$$D_{V_\alpha}V_\beta = \Gamma^{\delta}_{\alpha;\beta}V_\delta \tag{16}$$

for some complex-valued functions $\{\Gamma^{\delta}_{\alpha;\beta}\}$, then,

$$D_{\bar{V}_\alpha}\bar{V}_\beta = \overline{\Gamma^{\delta}_{\alpha;\beta}}\bar{V}_\delta. \tag{17}$$

Because $0 = V_\alpha\langle V_\beta, V_\delta\rangle = \langle D_{V_\alpha}V_\beta, V_\delta\rangle + \langle V_\beta, D_{\bar{V}_\alpha}V_\delta\rangle$, we obtain

$$D_{\bar{V}_\alpha}V_\delta = -\sum_\beta \overline{\Gamma^{\delta}_{\alpha;\beta}}V_\beta. \tag{18}$$

So, (15) follows from (17) and (18).

Lemma 2 Let (M,F) be a strong Kähler-Finsler manifold, $v_0 \in \tilde{M}$, then, there exists a frame field of $(1,0)$-form of $\mathcal{H}$ normalized at point v_0 (cf. [4, 11]).

Remark 1 Assume that the local horizontal frame $\{V_\alpha\}$ of (1,0)-form is normal at $v_0 \in \tilde{M}$, and its dual coframe is $\{\omega^\alpha\}$. We have

$$D_{V_\alpha}\omega^\beta(v_0) = D_{\bar{V}_\alpha}\omega^\beta(v_0) = 0.$$

Remark 2 Assume that $\{V_\alpha\}$ is normal at v_0, then for all α,β,

$$[V_\alpha, V_\beta](v_0) = 0, \quad [V_\alpha, V_{\bar{\beta}}](v_0) \neq 0.$$

2 Hodge-Laplace Operator and Weitzenböck Formula on PTM

In the following, we introduce the definition of Hodge-Laplace operator on the projectivized tangent bundle (PTM) in a complex Finsler manifold (cf. [10, 12]).

Let (M,F) be an n-dimensional strongly pseudoconvex compact complex Finsler manifold with a Finsler metric F, then, F induces a natural Hermitian metric on $T^{1,0}\tilde{M}$:

$$\tilde{G} = G_{\alpha\bar{\beta}}\mathrm{d}z^\alpha \otimes \mathrm{d}\bar{z}^\beta + G_{\alpha\bar{\beta}}\delta v^\alpha \otimes \delta\bar{v}^\beta.$$

It descends to a non-degenerated metric (still denote it by $\tilde{G}$)

$$\tilde{G} = G_{\alpha\bar{\beta}}\mathrm{d}z^{\alpha} \otimes \mathrm{d}\bar{z}^{\beta} + (\ln G)_{\alpha\bar{\beta}}\delta v^{\alpha} \otimes \delta\bar{v}^{\beta},$$

on the total space PTM [cf. 12].

Denote by

$$\omega_{\mathcal{V}} = \sqrt{-1}(\ln G)_{\alpha\bar{\beta}}\delta v^{\alpha} \wedge \delta\bar{v}^{\beta}, \quad \omega_H = \sqrt{-1}G_{\alpha\bar{\beta}}\mathrm{d}z^{\alpha} \wedge \mathrm{d}\bar{z}^{\beta}.$$

Then, the invariant volume form of PTM is given by

$$\mathrm{d}v = \frac{\omega_{\mathcal{V}}^{n-1}}{(n-1)!} \wedge \frac{\omega_H^n}{n!}. \tag{19}$$

Since ω_H^n is a horizontal (n,n)-form, the above expression is invariant by replacing δv^{α} and $\delta\bar{v}^{\beta}$ by $\mathrm{d}v^{\alpha}$ and $\mathrm{d}\bar{v}^{\beta}$, respectively.

If we denote by $\mathrm{d}\sigma$ the pure vertical form of the volume form of PTM, then,

$$\mathrm{d}\sigma = \frac{\omega_{\mathcal{V}}^{n-1}}{(n-1)!}.$$

So, we have

$$\mathrm{d}v = \mathrm{d}\sigma \wedge \frac{\omega_H^n}{n!} = G\mathrm{d}\sigma \wedge \mathrm{d}\chi,$$

where

$$G = \det(G_{\alpha\bar{\beta}}), \quad \mathrm{d}\chi = \frac{\tau^n}{n!}, \quad \tau = \sqrt{-1}\sum_{i=1}^{n}\mathrm{d}z^i \wedge \mathrm{d}\bar{z}^i.$$

Let $\mathcal{A}^{p,q}$ be the space of horizontal (p,q)-forms on PTM, i.e., those coefficients of every $\varphi \in \mathcal{A}^{p,q}$ are zero homogeneous with respect to fibre coordinates, the elements of $\mathcal{A}^{p,q}$ in local coordinates are

$$\varphi = \frac{1}{p!q!}\Sigma\varphi_{\alpha_1\cdots\alpha_p\bar{\beta}_1\cdots\bar{\beta}_q}\mathrm{d}z^{\alpha_1} \wedge \cdots \wedge \mathrm{d}z^{\alpha_p} \wedge \mathrm{d}\bar{z}^{\beta_1} \wedge \cdots \wedge \mathrm{d}\bar{z}^{\beta_q},$$

$$\psi = \frac{1}{p!q!}\Sigma\psi_{\lambda_1\cdots\lambda_p\bar{\mu}_1\cdots\bar{\mu}_q}\mathrm{d}z^{\lambda_1} \wedge \cdots \wedge \mathrm{d}z^{\lambda_p} \wedge \mathrm{d}\bar{z}^{\mu_1} \wedge \cdots \wedge \mathrm{d}\bar{z}^{\mu_q}.$$

Then, at each point $(z,v) \in$ PTM, we define

$$\begin{aligned}\langle\varphi,\psi\rangle &= \frac{1}{p!q!}\Sigma\ \varphi_{\alpha_1\cdots\alpha_p\bar{\beta}_1\cdots\bar{\beta}_q}\overline{\psi_{\lambda_1\cdots\lambda_p\bar{\mu}_1\cdots\bar{\mu}_q}}G^{\bar{\lambda}_1\alpha_1}\cdots G^{\bar{\lambda}_p\alpha_p}G^{\bar{\beta}_1\mu_1}\cdots G^{\bar{\beta}_q\mu_q}\\ &= \frac{1}{p!q!}\Sigma\ \varphi_{\alpha_1\cdots\alpha_p\bar{\beta}_1\cdots\bar{\beta}_q}\overline{\psi^{\bar{\alpha}_1\cdots\bar{\alpha}_p\beta_1\cdots\beta_q}},\end{aligned} \tag{20}$$

where

$$\overline{\psi^{\bar{\alpha}_1\cdots\bar{\alpha}_p\beta_1\cdots\beta_q}} = \overline{\psi_{\lambda_1\cdots\lambda_p\bar{\mu}_1\cdots\bar{\mu}_q}}G^{\bar{\lambda}_1\alpha_1}\cdots G^{\bar{\lambda}_p\alpha_p}G^{\bar{\beta}_1\mu_1}\cdots G^{\bar{\beta}_q\mu_q}.$$

Notice that there is a natural Hermitian inner product in $\mathcal{A}^{p,q}$ which is induced by the complex Finsler metric F, i.e.,

$$(\varphi,\psi)_{\mathrm{PTM}} = \int_{\mathrm{PTM}}\langle\varphi,\psi\rangle\mathrm{d}v. \tag{21}$$

In the following, we define the Hodge-Laplace operator for (p,q)-forms in $\mathcal{A}^{p,q}$. First, we define the operator $*:\ \mathcal{A}^{p,q}\longrightarrow\mathcal{A}^{n-q,n-p}$, by the relation

$$\int_{\rm PTM}\varphi\wedge*\bar\psi\wedge{\rm d}\sigma=(\varphi,\psi)_{\rm PTM}. \tag{22}$$

Proposition 2 Assume that (M,F) is a strongly pseudoconvex compact complex Finsler manifold, then, there is a linear map $*:\mathcal{A}^{p,q}\longrightarrow\mathcal{A}^{n-q,n-p}$ satisfying

(i) $\int_{\rm PTM}\varphi\wedge*\bar\psi\wedge{\rm d}\sigma=(\varphi,\psi)_{\rm PTM}$;

(ii) $\overline{*\psi}=*\bar\psi$ (that is, $*$ is a real operator);

(iii) $**\psi=(-1)^{p+q}\psi$.

Proof Before giving the proof, let us fix some notations. Let $n=\dim M$. We denote

$$A_p=(\alpha_1,\cdots,\alpha_p),\quad \alpha_1<\alpha_2<\cdots<\alpha_p,\quad 1\le\alpha_i\le n,$$

$$A_{n-p}=(\alpha_{p+1},\cdots,\alpha_n),\quad \alpha_{p+1}<\cdots<\alpha_n,\quad 1\le\alpha_i\le n,$$

and let $(\alpha_1,\cdots,\alpha_p,\alpha_{p+1},\cdots,\alpha_n)$ be a permutation of $(1,2,\cdots,n)$. Similarly, denote $B_q=(\beta_1,\cdots,\beta_q)$, $B_{n-q}=(\beta_{q+1},\cdots,\beta_n)$. Then, with these notations, we write horizontal (p,q)-form on PTM

$$\psi=\sum_{A_p,B_q}\psi_{A_p\overline{B}_q}{\rm d}z^{A_p}\wedge{\rm d}z^{\overline{B}_q},$$

where ${\rm d}z^{A_p}={\rm d}z^{\alpha_1}\wedge\cdots\wedge{\rm d}z^{\alpha_p}$, ${\rm d}z^{\overline{B}_q}={\rm d}\bar z^{\beta_1}\wedge\cdots\wedge{\rm d}\bar z^{\beta_q}$. Thus,

$$\overline{\psi}=\sum_{A_p,B_q}(\overline{\psi})_{B_q\overline{A}_p}{\rm d}z^{B_q}\wedge{\rm d}z^{\overline{A}_p},$$

where

$$(\overline{\psi})_{B_q\overline{A}_p}=(-1)^{pq}\overline{\psi_{A_p\overline{B}_q}}.$$

Denote

$$\psi^{\overline{A}_pB_q}=\sum_{\lambda,\mu}G^{\bar\alpha_1\lambda_1}\cdots G^{\bar\alpha_p\lambda_p}G^{\bar\mu_1\beta_1}\cdots G^{\bar\mu_q\beta_q}\psi_{\lambda_1\cdots\lambda_p\bar\mu_1\cdots\bar\mu_q},$$

then, we have

$$\overline{\psi^{\overline{A}_pB_q}}=(-1)^{pq}\overline{\psi}^{\overline{B}_qA_p}.$$

Thus, we can rewrite the Hermitian inner product (21) as

$$(\varphi,\psi)_{\rm PTM}=(-1)^{pq}\int_{\rm PTM}\varphi_{A_p\overline{B}_q}\overline{\psi}^{\overline{B}_qA_p}{\rm d}v.$$

Define

$$*\psi=({\rm i})^n(-1)^{\frac{n(n-1)}{2}+pn}\sum_{A_p,B_q}{\rm sgn}(A_pA_{n-p}){\rm sgn}(B_qB_{n-q})\psi^{\overline{A}_pB_q}G{\rm d}z^{B_{n-q}}\wedge{\rm d}z^{\overline{A_{n-p}}}, \tag{23}$$

where

$${\rm sgn}(A_pA_{n-p})={\rm sgn}\begin{pmatrix}1\cdots n\\ \alpha_1\cdots\alpha_n\end{pmatrix},\quad {\rm sgn}(B_qB_{n-q})={\rm sgn}\begin{pmatrix}1\cdots n\\ \beta_1\cdots\beta_n\end{pmatrix}.$$

Then,

$$*\overline{\psi}=(\mathrm{i})^n(-1)^{\frac{n(n-1)}{2}+qn}\sum_{A_p,B_q}\mathrm{sgn}(B_qB_{n-q})\mathrm{sgn}(A_pA_{n-p})\overline{\psi}^{\overline{B}_qA_p}G\mathrm{d}z^{A_{n-p}}\wedge\mathrm{d}z^{\overline{B_{n-q}}}.$$

By (23), we have

$$\begin{aligned}\overline{*\psi}&=(-\mathrm{i})^n(-1)^{\frac{n(n-1)}{2}+pn}\sum_{A_p,B_q}\mathrm{sgn}(A_pA_{n-p})\mathrm{sgn}(B_qB_{n-q})\overline{\psi^{\overline{A}_pB_q}}G\mathrm{d}z^{\overline{B_{n-q}}}\wedge\mathrm{d}z^{A_{n-p}}\\&=(\mathrm{i})^n(-1)^{\frac{n(n-1)}{2}+nq}\sum_{A_p,B_q}\mathrm{sgn}(A_pA_{n-p})\mathrm{sgn}(B_qB_{n-q})\overline{\psi}^{\overline{B}_qA_p}G\mathrm{d}z^{A_{n-p}}\wedge\mathrm{d}z^{\overline{B_{n-q}}}\\&=*\overline{\psi}.\end{aligned}$$

Thus, (ii) is proved.

For (i), let $\varphi=\sum_{A_pB_q}\varphi_{A_p\overline{B}_q}\mathrm{d}z^{A_p}\wedge\mathrm{d}z^{\overline{B}_q}$, by a direct caculation, we have

$$\begin{aligned}\int_{\mathrm{PTM}}\varphi\wedge*\overline{\psi}\wedge\mathrm{d}\sigma&=\int_{\mathrm{PTM}}(\mathrm{i})^n(-1)^{\frac{n(n-1)}{2}+nq}\sum_{A_p,B_q}\mathrm{sgn}(A_pA_{n-p})\mathrm{sgn}(B_qB_{n-q})\\&\qquad\varphi_{A_p\overline{B}_q}\overline{\psi}^{\overline{B}_qA_p}G\mathrm{d}z^{A_p}\wedge\mathrm{d}z^{\overline{B}_q}\wedge\mathrm{d}z^{A_{n-p}}\wedge\mathrm{d}z^{\overline{B_{n-q}}}\wedge\mathrm{d}\sigma\\&=\int_{\mathrm{PTM}}(\mathrm{i})^n(-1)^{\frac{n(n-1)}{2}+pq}\sum_{A_p,B_q}\mathrm{sgn}(A_pA_{n-p})\mathrm{sgn}(B_qB_{n-q})\\&\qquad\varphi_{A_p\overline{B}_q}\overline{\psi}^{\overline{B}_qA_p}G\mathrm{d}z^{A_p}\wedge\mathrm{d}z^{\overline{A_{n-p}}}\wedge\mathrm{d}z^{B_q}\wedge\mathrm{d}z^{\overline{B_{n-q}}}\wedge\mathrm{d}\sigma.\end{aligned}$$

Since

$$\mathrm{d}z^{A_p}\wedge\mathrm{d}z^{\overline{A_{n-p}}}\wedge\mathrm{d}z^{B_q}\wedge\mathrm{d}z^{\overline{B_{n-q}}}=\mathrm{sgn}'(A_pA_{n-p})\mathrm{sgn}'(B_qB_{n-q})(-1)^{\frac{n(n-1)}{2}}\frac{\mathrm{d}\chi}{(\mathrm{i})^n},$$

where

$$\mathrm{sgn}'(A_pA_{n-p})=\mathrm{sgn}\begin{pmatrix}\alpha_1\cdots\alpha_n\\1\cdots n\end{pmatrix},\quad\mathrm{sgn}'(B_qB_{n-q})=\mathrm{sgn}\begin{pmatrix}\beta_1\cdots\beta_n\\1\cdots n\end{pmatrix},$$

and

$$\mathrm{sgn}(A_pA_{n-p})\mathrm{sgn}'(A_pA_{n-p})=1,\quad\mathrm{sgn}(B_qB_{n-q})\mathrm{sgn}'(B_qB_{n-q})=1,$$

then, we have

$$\int_{\mathrm{PTM}}\varphi\wedge*\overline{\psi}\wedge\mathrm{d}\sigma=\int_{\mathrm{PTM}}(-1)^{pq}\sum_{A_p,B_q}\varphi_{A_p\overline{B}_q}\overline{\psi}^{\overline{B}_qA_p}G\mathrm{d}\chi\wedge\mathrm{d}\sigma=(\varphi,\psi)_{\mathrm{PTM}}.$$

Finally, we check (iii) for any point $(z_0;v_0)\in\mathrm{PTM}$. According to Lemma 1, we may assume by a change of coordinates that $G_{\alpha\bar{\beta}}(z_0;v_0)=\delta_{\alpha\bar{\beta}}$, then,

$$G=\det(G_{\alpha\bar{\beta}})=1,$$

and

$$\psi^{\overline{A}_pB_q}=\sum_{\lambda,\mu}G^{\bar{\alpha}_1\lambda_1}\cdots G^{\bar{\alpha}_p\lambda_p}G^{\bar{\mu}_1\beta_1}\cdots G^{\bar{\mu}_q\beta_q}\psi_{\lambda_1\cdots\lambda_p\bar{\mu}_1\cdots\bar{\mu}_q}=\psi_{A_p\overline{B}_q}.\tag{24}$$

Denote

$$*\psi = \sum_{A_{n-p},B_{n-q}} (*\psi)_{B_{n-q}\overline{A_{n-p}}} \mathrm{d}z^{B_{n-q}} \wedge \mathrm{d}z^{\overline{A_{n-p}}},$$

then by (23) and (24), we have

$$(*\psi)_{B_{n-q}\overline{A_{n-p}}} = (\mathrm{i})^n (-1)^{\frac{n(n-1)}{2}+pn} \mathrm{sgn}(A_p A_{n-p}) \mathrm{sgn}(B_q B_{n-q}) \psi_{A_p \overline{B}_q}.$$

Thus, we get

$$(**\psi)_{A_p\overline{B}_q} = (\mathrm{i})^n (-1)^{\frac{n(n-1)}{2}+n(n-q)} \mathrm{sgn}(B_{n-q}B_q) \mathrm{sgn}(A_{n-p}A_p)(*\psi)_{B_{n-q}\overline{A_{n-p}}}. \tag{25}$$

Because

$$\mathrm{sgn}(A_{n-p}A_p) = (-1)^{p(n-p)} \mathrm{sgn}(A_p A_{n-p}), \quad \mathrm{sgn}(B_{n-q}B_q) = (-1)^{q(n-q)} \mathrm{sgn}(B_q B_{n-q}),$$

by (24), we have

$$(**\psi)_{A_p\overline{B}_q} = (-1)^{n+(n-1)n+pn+(n-q)n+p(n-p)+q(n-q)} \psi_{A_p\overline{B}_q} = (-1)^{p+q} \psi_{A_p\overline{B}_q}.$$

And Proposition 2 is proved.

Now, we define the adjoint of $\bar\partial_H, \partial_H$, and d_H

Definition 5 [13–15] $\bar\partial_H^* = -*\partial_H *, \partial_H^* = -*\bar\partial_H *, \delta_H = -*d_H *$.

It is easy to see that

Proposition 3 Let (M,F) be a strongly pseudoconvex compact complex Finsler manifold. Then,

$$\begin{aligned} \bar\partial_H^* &: \ \mathcal{A}^{p,q} \longrightarrow \mathcal{A}^{p,q-1}, \\ \partial_H^* &: \ \mathcal{A}^{p,q} \longrightarrow \mathcal{A}^{p-1,q}, \\ \delta_H &: \ \mathcal{A}^{p} \longrightarrow \mathcal{A}^{p-1}, \end{aligned}$$

$$(\bar\partial_H\varphi,\psi) = (\varphi, \bar\partial_H^*\psi), \quad (\partial_H\varphi,\psi) = (\varphi,\partial_H^*\psi), \quad (d_H\varphi,\psi) = (\varphi,\delta_H\psi).$$

Definition 6 [13–15] $\Box_H = \bar\partial_H\bar\partial_H^* + \bar\partial_H^*\bar\partial_H$, $\Box_H$ maps $\mathcal{A}^{p,q}$ into $\mathcal{A}^{p,q}$, and is called the complex horizontal Laplacian.

Definition 7 [13–15] Let (M,F) be a strongly pseudoconvex compact complex Finsler manifold and $\varphi \in \mathcal{A}^{p,q}$. If $\Box_H\varphi = 0$, then, φ is called a horizontal harmonic (p,q)-form.

If we denote by $\alpha_1\cdots\hat\alpha_i\cdots\alpha_{p+1}$ the sequence which we get from $\alpha_1\cdots\alpha_{p+1}$ by suppressing α_i, then, we have

$$(\partial_H\psi)_{\alpha_1\cdots\alpha_{p+1}\bar\beta_1\cdots\bar\beta_q} = \sum_{i=1}^{p+1}(-1)^{i-1}\delta_{\alpha_i}(\psi_{\alpha_1\cdots\hat\alpha_i\cdots\alpha_{p+1}\bar\beta_1\cdots\bar\beta_q}),$$

$$(\bar\partial_H\psi)_{\alpha_1\cdots\alpha_p\bar\beta_1\cdots\bar\beta_{q+1}} = \sum_{j=1}^{q+1}(-1)^{p+j-1}\delta_{\bar\beta_j}(\psi_{\alpha_1\cdots\alpha_p\bar\beta_1\cdots\hat{\bar\beta}_j\cdots\bar\beta_{q+1}}).$$

If (M,F) is a strong Kähler-Finsler manifold, then by the symmetry of the horizontal part coefficients of the Chern-Finsler connection, we have $\Gamma^\alpha_{\mu;\nu} = \Gamma^\alpha_{\nu;\mu}$. Therefore, we can replace the horizontal derivatives by the covariant derivation in the direction of horizontal vector δ_α, and obtain

$$(\partial_H\psi)_{\alpha_1\cdots\alpha_{p+1}\bar\beta_1\cdots\bar\beta_q}=\sum_{i=1}^{p+1}(-1)^{i-1}D_{\delta_{\alpha_i}}\psi_{\alpha_1\cdots\hat\alpha_i\cdots\alpha_{p+1}\bar\beta_1\cdots\bar\beta_q},$$

$$(\bar\partial_H\psi)_{\alpha_1\cdots\alpha_p\bar\beta_1\cdots\bar\beta_{q+1}}=\sum_{j=1}^{q+1}(-1)^{p+j-1}D_{\delta_{\bar\beta_j}}\psi_{\alpha_1\cdots\alpha_p\bar\beta_1\cdots\hat{\bar\beta}_j\cdots\bar\beta_{q+1}}.$$

Assume that $\{V_\alpha\}$ is a horizontal frame field of (1,0)-form on a strong Kähler-Finsler manifold (M,F), and $\{\omega^\beta\}$ is its dual coframe field, i.e., $\omega^\beta(V_\alpha)=\delta^\beta_\alpha$. Then, we can simply express ∂_H or $\bar\partial_H$ by the Chern-Finsler connection as follows:

$$\partial_H=\sum_\alpha\omega^\alpha\wedge D_{V_\alpha},\quad \bar\partial_H=\sum_\alpha\bar\omega^\alpha\wedge D_{\bar V_\alpha}. \tag{26}$$

It is easy to check that the right-hand sides of these formulas are independent of the choice of $\{V_\alpha\}$. Similar to the case of complex Kähler manifold in [4], using $*$ operator and formula (26), we can also express $\bar\partial_H^*$ or ∂_H^* as follows by the Chern-Finsler connection.

Proposition 4 $\bar\partial_H^*=-\sum\limits_\beta i(\bar V_\beta)D_{V_\beta}$, $\partial_H^*=-\sum\limits_\beta i(V_\beta)D_{\bar V_\beta}$.

Proof It is suffice to prove $\bar\partial_H^*=-\sum_\beta i(\bar V_\beta)D_{V_\beta}$. One can see that the right-hand side of this formula is independent of the choice of the horizontal frame field $\{V_\alpha\}$ for $\mathcal{H}^{1,0}$. To simplify the computations, we can check it at a point $v_0\in$ PTM, and choose $\{V_\alpha\}$ to be a horizontal frame field normal at v_0. Let $\{V_1\cdots V_n\}$ be consistent with the orientation of M, and let $\{\omega^\beta\}$ be the dual coframe field of $\{V_\alpha\}$, then, it suffices to show that, at v_0, $\bar\partial_H^*=-\sum_\beta i(\bar V_\beta)D_{V_\beta}$ is achieved on the (p,q)-form $w=f\omega^1\wedge\cdots\wedge\omega^p\wedge\bar\omega^1\wedge\cdots\wedge\bar\omega^q$, where f is a complex differentiable function defined near v_0 with zero homogeneous with repect to v. Thus at the point v_0, the following calculations are valid:

$$\begin{aligned}*w&=*(f\omega^1\wedge\cdots\wedge\omega^p\wedge\bar\omega^1\wedge\cdots\wedge\bar\omega^q)\\&=(\mathrm{i})^n(-1)^{\frac12 n(n-1)+np}f\omega^{q+1}\wedge\cdots\wedge\omega^n\wedge\bar\omega^{p+1}\wedge\cdots\wedge\bar\omega^n,\\ \partial_H*w&=(\mathrm{i})^n(-1)^{\frac12 n(n-1)+np}(D_{V_\alpha}f)\omega^\alpha\wedge\omega^{q+1}\wedge\cdots\wedge\omega^n\wedge\bar\omega^{p+1}\wedge\cdots\wedge\bar\omega^n,\end{aligned}$$

and

$$\begin{aligned}&*((D_{V_\alpha}f)\omega^\alpha\wedge\omega^{q+1}\wedge\cdots\wedge\omega^n\wedge\bar\omega^{p+1}\wedge\cdots\wedge\bar\omega^n)\\=&(-\mathrm{i})^n(-1)^{\frac12 n(n-1)+\alpha-1+p+np}(D_{V_\alpha}f)\omega^1\wedge\cdots\wedge\omega^p\wedge\bar\omega^1\wedge\cdots\wedge\hat{\bar\omega}^\alpha\wedge\cdots\wedge\bar\omega^q,\end{aligned}$$

then

$$\begin{aligned}\bar\partial_H^*w&=-*\partial_H*w\\&=-(-1)^{p+\alpha-1}(D_{V_\alpha}f)\omega^1\wedge\cdots\wedge\omega^p\wedge\bar\omega^1\wedge\cdots\wedge\hat{\bar\omega}^\alpha\wedge\cdots\wedge\bar\omega^q\\&=-\sum_\alpha i(\bar V_\alpha)D_{V_\alpha}w.\end{aligned}$$

This ends the proof.

Lemma 3 [8] Let $D:\chi(T_C\tilde M)\longrightarrow\chi(T_C^*\tilde M\otimes T_C\tilde M)$ be the Chern-Finsler connectin on $\tilde M$. Then, for any $V_\alpha,V_\beta\in\chi(T^{1,0}\tilde M)$, we have

$$D_{V_\alpha}V_\beta-D_{V_\beta}V_\alpha=[V_\alpha,V_\beta]+\theta(V_\alpha,V_\beta), \tag{27}$$

$$D_{V_\alpha}\bar V_\beta - D_{\bar V_\beta}V_\alpha = [V_\alpha, \bar V_\beta] + \tau(V_\alpha, \bar V_\beta) + \bar\tau(V_\alpha, \bar V_\beta), \tag{28}$$

where θ is the $(2,0)$-torsion of the connection, and τ is the (1,1)-torsion of the connection.

Lemma 4 [8] Let $D : \chi(T^{1,0}\tilde M) \longrightarrow \chi(T_C^*\tilde M \otimes T_C^{1,0}\tilde M)$ be the Chern-Finsler connection on $\tilde M$. Then, for any $V_\alpha, V_\beta \in \chi(T^{1,0}\tilde M)$, we have

$$D_{V_\alpha}D_{V_\beta} - D_{V_\beta}D_{V_\alpha} = D_{[V_\alpha, V_\beta]} + \Omega(V_\alpha, V_\beta), \tag{29}$$

$$D_{V_\alpha}D_{\bar V_\beta} - D_{\bar V_\beta}D_{V_\alpha} = D_{[V_\alpha, \bar V_\beta]} + \Omega(V_\alpha, \bar V_\beta), \tag{30}$$

$$D_{\bar V_\alpha}D_{\bar V_\beta} - D_{\bar V_\beta}D_{\bar V_\alpha} = D_{[\bar V_\alpha, \bar V_\beta]}, \tag{31}$$

where $\Omega(V_\alpha, V_\beta) = 0$ is the (2,0)-form curvature operator, $\Omega(V_\alpha, \bar V_\beta)$ is the (1,1)-form curvature operator.

Theorem 1 (Formula WFI) On an n-dimensional compact strong Kähler-Finsler manifold, if $\{V_\alpha\}$ is a local horizontal frame field, and $\{\omega^\beta\}$ is its dual coframe, then,

$$\Box_H = -\sum_\alpha D^2_{V_\alpha \bar V_\alpha} + \sum_\alpha \bar\omega^\alpha \wedge \sum_\beta i(\bar V_\beta)\big(D_{[V_\beta, \bar V_\alpha]} + \Omega(V_\beta, \bar V_\alpha)\big), \tag{32}$$

$$\overline{\Box}_H = -\sum_\alpha D^2_{\bar V_\alpha V_\alpha} - \sum_\alpha \omega^\alpha \wedge \sum_\beta i(V_\beta)\big(D_{[V_\alpha, \bar V_\beta]} + \Omega(V_\alpha, \bar V_\beta)\big). \tag{33}$$

If the $(1,1)$-torsion τ of the Chern-Finsler connection is zero, $\{V_\alpha\}$ is a normal horizontal frame, that is, $D_{V_\alpha}\bar V_\beta = 0, D_{V_\beta}\bar V_\alpha = 0$, then, by (28), we have $[V_\alpha, \bar V_\beta] = 0$ and $D_{[V_\alpha, \bar V_\beta]} = 0$. Therefore, (32) and (33) reduce respectively to

$$\Box_H = -\sum_\alpha D^2_{V_\alpha \bar V_\alpha} + \sum_\alpha \bar\omega^\alpha \wedge \sum_\beta i(\bar V_\beta)\Omega(V_\beta, \bar V_\alpha), \tag{34}$$

$$\overline{\Box}_H = -\sum_\alpha D^2_{\bar V_\alpha V_\alpha} - \sum_\alpha \omega^\alpha \wedge \sum_\beta i(V_\beta)\Omega(V_\alpha, \bar V_\beta). \tag{35}$$

Remark 3 If the strong Kähler-Finsler manifold is a Kähler manifold, then, (32), (33) and (34), (35) coincide with (2.12), (2.13) of WFIII in [4].

Proof of Theorem 1 Let $p \in M$, and $\{V_\alpha\}$ be normal at $v_0 \in T_p^{1,0}M$, $\pi(v_0) = p$. Because $D^2_{V_\alpha \bar V_\alpha} = D_{V_\alpha}D_{\bar V_\alpha} - D_{D_{V_\alpha}\bar V_\alpha}$, we have $D^2_{V_\alpha \bar V_\alpha} = D_{V_\alpha}D_{\bar V_\alpha}$. By conjugation, (33) can be obtained by (32), so, it suffices to prove (32).

Obviously, the right-hand side of (32) is independent of the choice of the horizontal frame field $\{V_\alpha\}$ of (1,0)-form, and we shall verify the formula at v_0 relative to this $\{V_\alpha\}$. With all computations below understood to hold only at v_0, we have

$$\begin{aligned}
\bar\partial_H \bar\partial_H^* &= \sum_\alpha \bar\omega^\alpha \wedge D_{\bar V_\alpha}\Big(-\sum_\beta i(\bar V_\beta)D_{V_\beta}\Big) = -\sum_\alpha \bar\omega^\alpha \wedge \Big(\sum_\beta i(\bar V_\beta)D_{\bar V_\alpha}D_{V_\beta}\Big),\\
\bar\partial_H^* \bar\partial_H &= -\sum_\beta i(\bar V_\beta)D_{V_\beta}\Big(\sum_\alpha \bar\omega^\alpha \wedge D_{\bar V_\alpha}\Big) = -\sum_\beta i(\bar V_\beta)\Big(\sum_\alpha \bar\omega^\alpha \wedge D_{V_\beta}D_{\bar V_\alpha}\Big)\\
&= -\sum_{\alpha,\beta} i(\bar V_\beta)\bar\omega^\alpha \wedge D_{V_\beta}D_{\bar V_\alpha} + \sum_\alpha \bar\omega^\alpha \wedge \sum_\beta i(\bar V_\beta)D_{V_\beta}D_{\bar V_\alpha}\\
&= -\sum_\alpha D_{V_\alpha}D_{\bar V_\alpha} + \sum_\alpha \bar\omega^\alpha \wedge \sum_\beta i(\bar V_\beta)D_{V_\beta}D_{\bar V_\alpha}.
\end{aligned}$$

Hence, by (30)

$$\begin{aligned}\Box_H &= \bar\partial_H\bar\partial_H^* + \bar\partial_H^*\bar\partial_H \\ &= -\sum_\alpha D_{V_\alpha}D_{\bar V_\alpha} + \sum_\alpha \bar\omega^\alpha \wedge \sum_\beta i(\bar V_\beta)(D_{V_\beta}D_{\bar V_\alpha} - D_{\bar V_\alpha}D_{V_\beta}) \\ &= -\sum_\alpha D^2_{V_\alpha\bar V_\alpha} + \sum_\alpha \bar\omega^\alpha \wedge \sum_\beta i(\bar V_\beta)(D_{[V_\beta,\bar V_\alpha]} + \Omega(V_\beta,\bar V_\alpha)).\end{aligned}$$

This ends the proof.

Corollary 1 Assume that the $(1,1)$-tortion τ of the Chern-Finsler connection is zero. If ϕ is of type $(p,0), 0\le p\le n$, then

$$\Box_H\phi = -\sum_\alpha D^2_{V_\alpha\bar V_\alpha}\phi, \tag{36}$$

and if ϕ is of type $(p,n), 0\le p\le n$, then

$$\Box_H\phi = -\sum_\alpha D^2_{V_\alpha\bar V_\alpha}\phi + \sum_\alpha \Omega(V_\alpha,\bar V_\alpha)\phi. \tag{37}$$

If ϕ is of type $(0,q), 0\le q\le n$, then

$$\overline{\Box}_H\phi = -\sum_\alpha D^2_{V_\alpha\bar V_\alpha}\phi + \sum_\alpha \Omega(V_\alpha,\bar V_\alpha)\phi, \tag{38}$$

and if ϕ is of type $(n,q), 0\le q\le n$, then

$$\overline{\Box}_H\phi = -\sum_\alpha D^2_{V_\alpha\bar V_\alpha}\phi. \tag{39}$$

For applications, it is important to have explicit formula also for $\Box_H|\phi|^2$ for (p,q)-forms ϕ, where $|\phi|^2 = \langle\phi,\phi\rangle$ (where $\langle,\rangle$ is defined by (20)). If $\{V_\alpha\}$ is a horizontal frame field, using (36), we can get

$$-\Box_H|\phi|^2 = \sum_\alpha |D_{\bar V_\alpha}\phi|^2 + \sum_\alpha |D_{V_\alpha}\phi|^2 + \sum_\alpha\langle D^2_{V_\alpha\bar V_\alpha}\phi,\phi\rangle + \sum_\alpha\langle\phi, D^2_{\bar V_\alpha V_\alpha}\phi\rangle. \tag{40}$$

To simplify (40), let ϕ be a harmonic form of type $(p,0)$, then, it follows from Corollary 1, for such ϕ,

$$-\Box_H|\phi|^2 = \sum_\alpha |D_{\bar V_\alpha}\phi|^2 + \sum_\alpha |D_{V_\alpha}\phi|^2 - \sum_\alpha\langle\phi,\Omega(V_\alpha,\bar V_\alpha)\phi\rangle. \tag{41}$$

The last term can be further simplified. Since the consideration will be localized at a point, let $p\in M$, fix $v_0\in T^{1,0}_pM$. On horizontal co-vector of type (1,0) at v_0, the pairing $\xi,\eta\longrightarrow\langle\xi,\eta\rangle$ is hermitian metric defined by (20). Now, under the condition of the (1,1)-tortion τ of the Chern-Finsler connection being zero, $\sum_\alpha\Omega(V_\alpha,\bar V_\alpha)$ is the anti-hermitian transformation relative to the inner product $\langle\xi,\eta\rangle$ at v_0. In fact, from (28) and (30), we get $0=[V_\alpha,\bar V_\alpha] = V_\alpha\bar V_\alpha - \bar V_\alpha V_\alpha$ and $\Omega(V_\alpha,\bar V_\alpha) = D_{V_\alpha}D_{\bar V_\alpha} - D_{\bar V_\alpha}D_{V_\alpha}$, then, we have

$$0=[V_\alpha,\bar V_\alpha]\langle\xi,\eta\rangle = \langle(D_{V_\alpha}D_{\bar V_\alpha} - D_{\bar V_\alpha}D_{V_\alpha})\xi,\eta\rangle + \langle\xi,(D_{\bar V_\alpha}D_{V_\alpha} - D_{V_\alpha}D_{\bar V_\alpha})\eta\rangle,$$

that is,

$$\langle\Omega(V_\alpha,\bar V_\alpha)\xi,\eta\rangle = \langle\xi,\Omega(V_\alpha,\bar V_\alpha)\eta\rangle = -\langle\xi,\overline{\Omega(V_\alpha,\bar V_\alpha)}\eta\rangle.$$

Thus, $(\sum_\alpha \Omega(V_\alpha, \bar{V}_\alpha))$ is diagonalized by some basis $\{\omega^1, \cdots, \omega^n\}$ of horizontal covectors of type $(1,0)$ normalized at v_0, which satisfies $\langle \omega^\alpha, \omega^\beta \rangle = G^{\alpha\bar{\beta}} = \delta^{\alpha\bar{\beta}}$. Let $(\sum_\alpha \Omega(V_\alpha, \bar{V}_\alpha))\omega^\alpha = \lambda_\alpha \omega^\alpha$ for $\alpha = 1, \cdots, n$, i.e., the λ_α's are the (real) eigenvalues of $(\sum_\alpha \Omega(V_\alpha, \bar{V}_\alpha))$. If the horizontal frame $\{V_\alpha\}$ is normalized at v_0 and $\{\omega^\alpha\}$ is its dual frame, we denote $K_F(V_\alpha)$ by the horizontal holomorphic flag curvature of the strongly pseudoconvex Finsler metric F along the horizontal vector $V_\alpha \in \mathcal{H}^{1,0}_{v_0}$, then $\lambda_\alpha = \frac{1}{2} K_F(V_\alpha)$.

Now, return to the last term of (41) and write

$$\phi = \sum_A \phi_A \omega^A,$$

at v_0 on PTM, where A runs through the ordered multi-index $(i_1, \cdots, i_p)$ with $1 < i_1 < \cdots < i_p \leq n$, ϕ_A is zero homogeneous with respect to fibre coordinates, and $\omega^A = \omega^{i_1} \wedge \cdots \wedge \omega^{i_p}$. For simplicity of notation, suppose $A = (1, 2, \cdots, p)$, then

$$\begin{aligned}(\sum_\alpha \Omega(V_\alpha, \bar{V}_\alpha))\omega^A &= \sum_{\alpha=1}^{p} \omega^1 \wedge \cdots \wedge (\sum_\alpha \Omega(V_\alpha, \bar{V}_\alpha)\omega^\alpha) \wedge \cdots \wedge \omega^p \\ &= \sum_{\alpha=1}^{p} \lambda_\alpha \omega^1 \wedge \cdots \wedge \omega^p = \frac{1}{2} \sum_{\alpha \in A} K_F(V_\alpha)\omega^A,\end{aligned}$$

where by abuse of notation, we also write A for the unordered set $(i_1, \cdots, i_p)$, then, we have

$$\langle \phi, \Omega(V_\alpha, \bar{V}_\alpha)\phi \rangle = \langle \Omega(V_\alpha, \bar{V}_\alpha)\phi, \phi \rangle = \frac{1}{2} \sum_{\alpha \in A} K_F(V_\alpha)|\phi_A|^2. \tag{42}$$

It remains to observe that $\langle \sum_\alpha \Omega(V_\alpha, \bar{V}_\alpha)\omega^\alpha, \omega^\alpha \rangle = \langle \sum_\alpha \Omega(V_\alpha, \bar{V}_\alpha)V_\alpha, V_\alpha \rangle$ under the horizontal fame $\{V_\alpha\}$ of type (1,0) normalized at v_0 on PTM, the λ_α's can also be characterized by $\sum_\alpha \Omega(V_\alpha, \bar{V}_\alpha)V_\alpha = \lambda_\alpha V_\alpha$ for $\alpha = 1, \cdots, n$. Thus, (42) together with (41) proves the following.

Theorem 2 (Formula WFII) On an n-dimensional compact strong Kähler-Finsler manifold M, let $p \in M$, assume that $\{V_\alpha\}$ is a local horizontal frame field of $\mathcal{H}^{1,0}_{v_0}$ normalized at v_0, $\pi(v_0) = p$, and $\{\omega^\beta\}$ is its dual coframe field. Furthermore, let ϕ be a horizontal harmonic form of type $(p, 0)$, and write $\phi = \sum_A \phi_A \omega^A$, if the (1,1)-torsion τ of the Chern-Finsler connection is zero, then, we have

$$-\Box_H |\phi|^2 = \sum_\alpha |D_{\bar{V}_\alpha}\phi|^2 + \sum_\alpha |D_{V_\alpha}\phi|^2 - \frac{1}{2} \sum_{\alpha \in A} K_F(V_\alpha)|\phi_A|^2. \tag{43}$$

3 Hodge-Laplace Operator and Weitzenböck Formula on a Holomorphic Vector Bundle Over PTM

Let E be a holomorphic vector bundle of rank r over a complex Finsler manifold M of dimension n with projection π. We identity M with the zero section of E. Let $E' = E \backslash o(E)$. Then C' acts on E' by scalar multiplication. The projective bundle $P(E)$ is defined by $P(E) = E'/C'$ with projection $p : P(E) \longrightarrow M$. The pull-back $\widetilde{E} = p^{-1}(E)$ is a holomorphic vector bundle of rank r over $P(E)$ with projection $\widetilde{p} : \widetilde{E} \longrightarrow E$. Let $L(E)$ be the tautological line subbundle of $\widetilde{E}$.

We summarize the construction in the following diagram:

$$\begin{tikzcd} L(E) \subset \widetilde{E} \arrow[r, "\widetilde{p}"] \arrow[d, "\widetilde{\pi}"] & E \arrow[d, "\pi"] \\ P(E) \arrow[r, "p"] & M \end{tikzcd} \tag{44}$$

Let $L'(E)$ be $L(E)$ minus its zero section $P(E)$. There is a natural map $L(E) \longrightarrow E$, which maps $L'(E)$ biholomorphically to E' and collapses the zero section $P(E)$ of $L(E)$ to the zero section M of E by p; Thus, $L(E)$ is a blow-up of E along the zero section M of E (cf. [16]).

In preparation, we explain local coordinate systems associated to the bundles mentioned in (44). Let $z = (z^1, \cdots, z^n)$ be a local coordinate system in M, and $\xi = (\xi^1, \cdots, \xi^r)$ the local fibre coordinate system defined by a local holomorphic frame field $e = (e^1, \cdots, e^r)$ of E. Then, $(z, \xi) = (z^1, \cdots, z^n, \xi^1, \cdots, \xi^r)$ is local coordinate system for E. This can be considered also as a local coordinate system for $P(E)$ as long as $(\xi^1, \cdots, \xi^r)$ is considered as a homogeneous coordinate system for fibre. Setting

$$Z^i = \xi^i \circ \widetilde{p}. \tag{45}$$

We take $(z, \xi, Z) = (z^1, \cdots, z^n, \xi^1, \cdots, \xi^r, Z^1, \cdots, Z^r)$ as a local coordinate system for $\widetilde{E} = p^{-1}(E)$ with the understanding that $(\xi^1, \cdots, \xi^r)$ is a homogeneous coordinate system. Then the line bundle $L(E)$ is defined by

$$(Z^1 : \cdots : Z^r) = (\xi^1 : \cdots : \xi^r). \tag{46}$$

Given a strongly pseudoconvex complex Finsler metric $F(z, \xi)$ in E, we set $G(z, \xi) = F^2(z, \xi) = G_{\alpha\bar{\beta}}(z, \xi)\xi^\alpha \overline{\xi^\beta}$, then F induces a Hermitian metric $\widehat{G}(z, \xi, Z)$ in the vector bundle $\widetilde{E}$ over $P(E)$, so that $\widehat{G}(z, \xi, Z) = G_{\alpha\bar{\beta}}(z, \xi) Z^\alpha \overline{Z^\beta}$ on the line bundle $L(E)$. By definition, $G(z, \xi) = \widehat{G}(z, \xi, \xi)$ is restricted to $L'(E)$, $\widehat{G}$ coincides with G. Denote the metric of $\widetilde{E}$ by $\langle\rangle$ and its connection by $\bigtriangledown$, and the connection forms $\omega = (\omega^\alpha_\beta)$ defined by

$$\omega^\alpha_\beta = G^{\bar{\tau}\alpha}\partial G_{\beta\bar{\tau}} = \widetilde{\Gamma}^\alpha_{\beta;\mu} \mathrm{d}z^\mu + \widetilde{\Gamma}^\alpha_{\beta\gamma} \mathrm{d}\xi^\gamma, \tag{47}$$

where $\widetilde{\Gamma}^\alpha_{\beta;\mu} = G^{\bar{\tau}\alpha} G_{\beta\bar{\tau};\mu}$, $\widetilde{\Gamma}^\alpha_{\beta\gamma} = G^{\bar{\tau}\alpha} G_{\beta\bar{\tau}\gamma}$, and we have $\widetilde{\Gamma}^\alpha_{\beta\gamma}\xi^\beta = \widetilde{\Gamma}^\alpha_{\beta\gamma}\xi^\gamma = 0$.

The curvature form $\Omega = (\Omega^\alpha_\beta)$ of the connection $\omega = (\omega^\alpha_\beta)$ is given by

$$\Omega^\alpha_\beta = \bar{\partial}\omega^\alpha_\beta. \tag{48}$$

In this article, we shall use only the horizontal part $\bigtriangledown^H$ of the connection $\bigtriangledown$, its horizontal parts of the connection 1-forms are given by

$$(\omega^\alpha_\beta)^H = \widetilde{\Gamma}^\alpha_{\beta;\mu} \mathrm{d}z^\mu. \tag{49}$$

Let $\mathcal{A}^i\ (\widetilde{E})$ (resp. $\mathcal{A}^{p,q}\ (\widetilde{E})$) be the space of complex horizontal i-forms (resp. (p, q)-forms) on PTM with value in $\widetilde{E}$, precisely, if $\mathcal{A}^i$(resp. $\mathcal{A}^{p,q}$) denotes the space of complex horizontal i-forms (resp. (p, q)-forms) on PTM, then by definition $\mathcal{A}^i\ (\widetilde{E}) = \Gamma_\infty(\widetilde{E}) \otimes \mathcal{A}^i$ (resp. $\mathcal{A}^{p,q}\ (\widetilde{E}) = \Gamma_\infty(\widetilde{E}) \otimes \mathcal{A}^{p,q}$). Recall that $\bigtriangledown^H$ is a C-linear map: $\mathcal{A}^0(\widetilde{E}) \longrightarrow \mathcal{A}^1(\widetilde{E})$, and $\bigtriangledown^H$ decomposes into a sum of two operators:$\bigtriangledown'^H : \mathcal{A}^0(\widetilde{E}) \longrightarrow \mathcal{A}^{1,0}(\widetilde{E})$, $\bar{\partial}_H : \mathcal{A}^0(\widetilde{E}) \longrightarrow \mathcal{A}^{0,1}(\widetilde{E})$, that is,

$$\bigtriangledown^H = \bigtriangledown'^H + \bar{\partial}_H. \tag{50}$$

If $\phi = e_\alpha \otimes \phi^\alpha$ is $\widetilde{E}$-valued horizontal (p,q)-forms on PTM, where ϕ^α is a horizontal (p,q)-form on PTM for each α and $e_\alpha \in \Gamma_\infty(\widetilde{E})$, then by definition

$$\bigtriangledown^H \phi = \bigtriangledown^H e_\alpha \otimes \phi^\alpha + e_\alpha \otimes d_H \phi^\alpha. \tag{51}$$

This is well-defined and furthermore, $\bigtriangledown^H$ becomes an antiderivation

$$\bigtriangledown^H (\phi \wedge \mu) = (\bigtriangledown^H \phi) \wedge \mu + (-1)^i \phi \wedge d_H \mu, \tag{52}$$

where $\phi \in \mathcal{A}^i(\widetilde{E})$ and μ is a horizontal form on PTM. Correspondingly, we obtain antiderivations

$$\bigtriangledown'^H : \mathcal{A}^{p,q}(\widetilde{E}) \longrightarrow \mathcal{A}^{p+1,q}(\widetilde{E}), \qquad \bar{\partial}_H : \mathcal{A}^{p,q}(\widetilde{E}) \longrightarrow \mathcal{A}^{p,q+1}(\widetilde{E}).$$

Now, assume that M is a strongly pseudoconvex compact Finsler manifold with a complex Finsler metric F, and the natural Hermitian metric on $\widetilde{E}$ induced by F will also be denoted by $\langle\,\rangle$. Then a pointwise inner product denoted by $(,)$ is defined on $\mathcal{A}^* = \otimes_i \mathcal{A}^i(\widetilde{E})$, namely, if $\sum_\sigma s_\sigma \otimes \phi^\sigma$ and $\sum_\alpha e_\alpha \otimes \psi^\alpha$ are given with $s_\sigma, e_\alpha \in \Gamma_\infty(\widetilde{E})$, and $\phi^\sigma, \psi^\alpha \in \mathcal{A}^*(\equiv \otimes_i \mathcal{A}^i)$ for all σ, α, then by definition

$$(\sum_\sigma s_\sigma \otimes \phi^\sigma, \sum_\alpha e_\alpha \otimes \psi^\alpha) \equiv \sum_{\sigma,\alpha} \langle s_\sigma, e_\alpha\rangle \langle \phi^\sigma, \psi^\alpha\rangle. \tag{53}$$

If $\phi \in \mathcal{A}^*(\widetilde{E})$, define $|\phi|^2 = (\phi,\phi)$. Extending the operator $*$ to $\mathcal{A}^{p,q}(\widetilde{E}) \longrightarrow \mathcal{A}^{n-q,n-p}(\widetilde{E})$ by letting it act on the second variable, then, $\bar{\partial}_H^*$ can now be extended to $\mathcal{A}^{p,q}(\widetilde{E}) \longrightarrow \mathcal{A}^{p,q-1}(\widetilde{E})$ by defining

$$\bar{\partial}_H^* = - * \bigtriangledown'^H * . \tag{54}$$

The horizontal Laplacian $\Box_H$ can consequently be extended to $\mathcal{A}^{p,q}(\widetilde{E}) \longrightarrow \mathcal{A}^{p,q}(\widetilde{E})$ by

$$\Box_H = \bar{\partial}_H \bar{\partial}_H^* + \bar{\partial}_H^* \bar{\partial}_H. \tag{55}$$

The kernel of $\Box_H : \mathcal{A}^{p,q}(\widetilde{E}) \longrightarrow \mathcal{A}^{p,q}(\widetilde{E})$ is the space of harmonic $\widetilde{E}$-valued horizontal (p,q)-forms. To study this space, the following Weitzenböck formula would be useful. For notation, let $\{e_\alpha\}$ be a local holomorphic frame of $\widetilde{E}$ and $\{V_\alpha\}$ be a horizontal frame field of (1,0)-form.

Theorem 3 (Formula WFIII) Let E be a holomorphic vector bunlde of rank r over a complex compact strong Kähler-Finsler manifold M, and the pull-back $\widetilde{E} = p^{-1}(E)$ is a holomorphic vector bundle over $P(E)$. Given $\phi \in \mathcal{A}^{p,q}(\widetilde{E})$, relative to a local holomorphic frame $\{e_\alpha\}$ of $\widetilde{E}$, write $\phi = \sum_\alpha e_\alpha \phi^\alpha$. If $\{V_\alpha\}$ is a horizontal frame field of $(1,0)$-form on PTM, and D is the Chern-Finsler connection, then,

$$\Box_H \phi = \sum_\alpha e_\alpha (\Box_H \phi^\alpha) - \sum_{\alpha,\mu} (\bigtriangledown'^H_{V_\mu} e_\alpha)(D_{\overline{V}_\mu} \phi^\alpha) + \sum_{\alpha,\xi,\mu} e_\xi \{ i(V_\mu)(\Omega^\xi_\alpha)^H \wedge i(\overline{V}_\mu)\phi^\alpha\}. \tag{56}$$

Proof We need to compute the left-hand side directly:

$$\begin{aligned}
\bar{\partial}_H \bar{\partial}_H^* \phi &= \bar{\partial}_H (- * \bigtriangledown'^H *)\phi = \bar{\partial}_H [- * (\sum_\alpha \bigtriangledown'^H e_\alpha \wedge * \phi^\alpha + \sum_\alpha e_\alpha \bar{\partial}_H * \phi^\alpha)] \\
&= - \sum_{\alpha,\xi} e_\xi \bar{\partial}_H * [(\omega^\xi_\alpha)^H \wedge * \phi^\alpha] + \sum_\alpha e_\alpha (\bar{\partial}_H \bar{\partial}_H^* \phi^\alpha), \\
\bar{\partial}_H^* \bar{\partial}_H \phi &= - * \bigtriangledown'^H * (\sum_\alpha e_\alpha \bar{\partial}_H \phi^\alpha) = \sum_\alpha e_\alpha \bar{\partial}_H^* \bar{\partial}_H \phi^\alpha - *(\sum_\alpha \bigtriangledown'^H e_\alpha \wedge * \bar{\partial}_H \phi^\alpha).
\end{aligned}$$

Then,

$$\Box_H\phi = \sum_{\alpha} e_\alpha(\Box_H\phi^\alpha) - \sum_{\alpha,\xi} e_\xi\bar{\partial}_H * [(\omega^\xi_\alpha)^H \wedge *\phi^\alpha] - *(\sum_{\alpha} \bigtriangledown'^H e_\alpha \wedge *\bar{\partial}_H\phi^\alpha). \tag{57}$$

Now, if $\{\omega^\alpha\}$ is the horizontal coframe field of type (1,0) dual to $\{V_\alpha\}$ and $\psi \in \mathcal{A}^{p,q}$, then,

$$*(\omega^\alpha \wedge *\psi) = i(\bar{V}_\alpha)\psi. \tag{58}$$

Hence, writing $(\omega^\xi_\alpha)^H = \widetilde{\Gamma}^\xi_{\alpha;\mu}\omega^\mu$, by (26) and (58), we have

$$\begin{aligned}\bar{\partial}_H * [(\omega^\xi_\alpha)^H \wedge *\phi^\alpha] &= \bar{\partial}_H * (\sum_{\mu,\alpha}\widetilde{\Gamma}^\xi_{\alpha;\mu}\omega^\mu \wedge *\phi^\alpha) = \sum_{\mu,\alpha}\bar{\partial}_H i(V_\mu)(\omega^\xi_\alpha)^H \wedge i(\bar{V}_\mu)\phi^\alpha \\ &= -\sum_{\mu,\alpha} i(V_\mu)(\Omega^\xi_\alpha)^H \wedge i(\bar{V}_\mu)\phi^\alpha, \\ \sum_{\alpha}(\bigtriangledown'^H e_\alpha \wedge *\bar{\partial}_H\phi^\alpha) &= *(\sum_{\alpha,\xi,\mu} e_\xi\widetilde{\Gamma}^\xi_{\alpha;\mu}\omega^\mu \wedge *\bar{\partial}_H\phi^\alpha) = \sum_{\alpha,\xi,\mu} e_\xi\widetilde{\Gamma}^\xi_{\alpha;\mu} i(\bar{V}_\mu)\bar{\partial}_H\phi^\alpha \\ &= \sum_{\alpha,\mu}(\bigtriangledown'^H_{V_\mu} e_\alpha)(D_{\overline{V}_\mu}\phi^\alpha).\end{aligned}$$

This combined with (57) yields (56).

Remark 4 From formula (43) of WFII, one can see that formula (56) of WFIII involves not only the curvature of the strongly pseudoconvex Finsler metric of M, but also the curvature of the holomorphic vector bundle $\widetilde{E}$ on $P(E)$. In many situations, one would like to draw conclusions solely from the given data on $\tilde{E}$ over $P(E)$ without being handicapped by the Finsler manifold M itself. In such case (36) comes in hand because the curvature term of M disappears from that formula. Thus, if $\phi \in \mathcal{A}^{p,0}(\tilde{E})$, then (36) implies

$$\Box_H\phi = -\sum_{\alpha,\beta} e_\alpha(D^2_{V_\alpha\overline{V}_\beta}\phi^\alpha) - \sum_{\alpha,\mu}(\bigtriangledown'^H_{V_\mu} e_\alpha)(D_{\overline{V}_\mu}\phi^\alpha) + \sum_{\alpha,\xi,\mu} e_\xi\{i(V_\mu)(\Omega^\xi_\alpha)^H \wedge i(\overline{V}_\mu)\phi^\alpha\}. \tag{59}$$

In the following, we introduce a Weitzenböck formula for the line bundle $L(E)$, which is useful for proving Bochner-Kodaira type vanishing theorem. Let $\mathcal{A}^{p,q}_o(\tilde{E})$ denote the elements in $\mathcal{A}^{p,q}(\tilde{E})$ with compact support and let $\{V_\alpha\}$ be a horizontal frame field, $\{\omega^\alpha\}$ be the dual coframe field of $\{V_\alpha\}$, and dv be the volume element of PTM. Given $\phi \in \mathcal{A}^{p,q}_o(\tilde{E})$, write locally $\phi = \sum_\alpha e_\alpha\phi^\alpha$, where $\{e_\alpha\}$ is a holomorphic frame field of $\tilde{E}$. Define a global (0,1)-form φ by

$$\varphi \equiv \sum_{\beta}(\sum_{\alpha} e_\alpha(D_{\overline{V}_\beta}\phi^\alpha), \phi)\overline{\omega}^\beta,$$

where $(,)$ is understood in the sense of (53). It is easy to check that ϕ is indeed globally defined, i.e., independent of the choices of $\{e_\alpha\}$, $\{V_\alpha\}$. $\delta_H\varphi$ is a function which, by Proposition 3, is given by

$$\delta_H\varphi = \overline{\partial}^*_H\varphi = -(\sum_{\alpha,\beta} e_\alpha D^2_{V_\beta\overline{V}_\beta}\phi^\alpha, \phi) - (\sum_{\alpha,\beta}(\bigtriangledown_{V_\beta} e_\alpha)(D_{\overline{V}_\beta}\phi^\alpha), \phi) - \sum_{\alpha,\beta}|e_\alpha D_{\overline{V}_\beta}\phi^\alpha|^2.$$

It is easy to prove $\int_{\mathrm{PTM}}\delta_H\varphi \mathrm{d}v = 0$. Hence,

$$\int_{\mathrm{PTM}}(\sum_{\alpha,\beta} e_\alpha D^2_{V_\beta\overline{V}_\beta}\phi^\alpha + \sum_{\alpha,\beta}(\bigtriangledown_{V_\beta} e_\alpha)(D_{\overline{V}_\beta}\phi^\alpha), \phi)\mathrm{d}v = -\int_{\mathrm{PTM}}\sum_{\alpha,\beta}|e_\alpha D_{\overline{V}_\beta}\phi^\alpha|^2\mathrm{d}v. \tag{60}$$

Now, suppose $\phi \in \mathcal{A}_o^{p,0}(\tilde{E})$, then, (59) and (60) yield:

$$\int_{\mathrm{PTM}} (\sum_{\alpha,\xi,\mu} e_\xi \{i(V_\mu)(\Omega_\alpha^\xi)^H \wedge i(\overline{V}_\mu)\phi^\alpha\}, \phi)\mathrm{d}v = \int_{\mathrm{PTM}} (\Box_H \phi, \phi)\mathrm{d}v - \int_{\mathrm{PTM}} \sum_{\alpha,\beta} |e_\alpha D_{\overline{V}_\beta} \phi^\alpha|^2 \mathrm{d}v. \tag{61}$$

Now, we will consider a special case where $\widetilde{E}$ is a line bundle $L(E)$ and ϕ is of type $(p,n)(0 < p \leq n)$. In this case, a local holomorphic frame of $L(E)$ is just a locally nowhere zero holomorphic section e of E. Then, we may simply write $\phi = e\phi'$, where ϕ' is an ordinary horizontal (p,n)-form. The horizontal part of the curvature of $L(E)$ relative to e is just a horizontal $(1,1)$-form Ω. Therefore, it follows from (37), (42), (56), and (60) that

$$\begin{aligned} &\int_{\mathrm{PTM}} |e|^2 (\sum_\beta i(V_\beta)\Omega \wedge i(\overline{V}_\beta)\phi', \phi')\mathrm{d}v \\ =& \int_{\mathrm{PTM}} (\Box_H \phi, \phi)\mathrm{d}v - \int_{\mathrm{PTM}} |e|^2 \sum_\beta |D_{\overline{V}_\beta}\phi'|^2 \mathrm{d}v - \frac{1}{2}\int_{\mathrm{PTM}} |e|^2 \sum_{\alpha\in A} K_F(V_\alpha)|\phi_A|^2 \mathrm{d}v. \end{aligned} \tag{62}$$

Now, $\frac{\sqrt{-1}}{2\pi}\Omega$ is a real $(1,1)$-form on PTM, so, there exist locally n continuous $(1,0)$-forms $\omega^1, \cdots, \omega^n$, such that $\langle \omega^\alpha, \omega^\beta \rangle = \delta^{\alpha\beta}$ and

$$\Omega = \sum_\alpha \mu_\alpha \omega^\alpha \wedge \overline{\omega}^\alpha, \tag{63}$$

where the μ_α's are real-valued continuous functions which, at each point, give the eigenvalues of Ω. The operator $\sum_\beta i(V_\beta)\Omega \wedge i(\overline{V}_\beta)$ in (62) is independent of the choice of $\{V_\alpha\}$. Thus, by computing at some fixed point $v_0 \in \mathrm{PTM}$, we may assume that $\{V_\alpha(v_0)\}$ is dual to $\{\omega^\alpha(v_0)\}$, i.e., $\omega^\alpha(V_\beta)(v_0) = \delta_\beta^\alpha$ for all α, β at v_0, then,

$$\sum_\beta i(V_\beta)\Omega \wedge i(\overline{V}_\beta) = \sum_\beta \mu_\beta \overline{\omega}^\beta \wedge i(\overline{V}_\beta). \tag{64}$$

Furthermore, it is straightforward to check that, if φ, ψ are ordinary horizontal forms and $\omega^\alpha(V_\beta) = \delta_\beta^\alpha$, then,

$$\langle \overline{\omega}^\beta \wedge \varphi, \psi \rangle = \langle \varphi, i(\overline{V}_\beta)\psi \rangle. \tag{65}$$

Using the usual multi-index notation $\omega^A = \omega^{\alpha_1} \wedge \cdots \wedge \omega^{\alpha_p}$ for $A = (\alpha_1, \cdots, \alpha_p)$, $\alpha_1 < \cdots < \alpha_p$ and writing $\phi' = (\sum_A \phi'_A \omega^A) \wedge \bar{\omega}^1 \wedge \cdots \wedge \bar{\omega}^n$, we obtain from (64) and (65)

$$\langle \sum_\beta i(V_\beta)\Omega \wedge i(\overline{V}_\beta)\phi', \phi' \rangle = \sum_\beta \mu_\beta \langle\, i(\overline{V}_\beta)\phi', i(\overline{V}_\beta)\phi' \rangle = \sum_A (\sum_{\beta=1}^n \mu_\beta)|\phi'_A|^2.$$

Note that the last expression is independent of v_0. Thus, by by combining with (64), we have

Theorem 4 (Formula WFIV) Under the assumptions in Formula WFIII, in addition, we assume that the $(1,1)$-tortion τ of the Chern-Finsler connection is zero. Suppose $e\phi'$ is a horizontal (p,n)-form with values in a line bundle $L(E)$ (e being a local holomorphic frame of $L(E)$), Ω is the horizontal part of curvature form of $L(E)$ relative to e which is locally diagonalized as in (63), μ_β's are the eigenvalues of Ω, $K_F(V_\alpha)$ is the horizontal holomorphic flag curvature along its horizontal frame $\{V_\alpha\} \in \mathcal{H}^{1,0}$ and $\phi' = (\sum_A \phi'_A \omega^A) \wedge \bar{\omega}^1 \wedge \cdots \wedge \bar{\omega}^n$. If

$e\phi'$ has a compact support, then

$$\int_{\mathrm{PTM}}|e|^2\sum_A(\sum_{\beta=1}^n\mu_\beta)|\phi'_A|^2\mathrm{d}v=\int_{\mathrm{PTM}}(\Box_H(e\phi'),e\phi')\mathrm{d}v-\int_{\mathrm{PTM}}|e|^2\sum_\beta|D_{\overline{V}_\beta}\phi'|^2\mathrm{d}v$$
$$-\frac{1}{2}\int_{\mathrm{PTM}}|e|^2\sum_{\alpha\in A}K_F(V_\alpha)|\phi_A|^2\mathrm{d}v. \tag{66}$$

4 Bochner Vanishing Theorems

Theorem 5 Assume that M is an n-dimensional compact strong Kähler-Finsler manifold, the $(1,1)$-tortion of the Chern-Finsler connection vanishes. Given a horizontal harmonic form ϕ of type $(p,0)$ on PTM, if the (horizontal) holomorphic flag curvature along its horizontal frame $\{V_\alpha\}\in\mathcal{H}^{1,0}$ is non-positive, then, ϕ has a constant absolute norm.

Proof Let ϕ be a horizontal harmonic $(p,0)$-form on PTM. If $\{V_\alpha\}$ is a local horizontal frame field, then from WFII and (43),we have

$$-\Box_H|\phi|^2=\sum_\alpha|D_{\bar{V}_\alpha}\phi|^2+\sum_\alpha|D_{V_\alpha}\phi|^2-\frac{1}{2}\sum_{\alpha\in A}K_F(V_\alpha)|\phi_A|^2.$$

Since $K_F(V_\alpha)\le 0$, $|\phi|^2$ is subharmonic. By the compactness of M and the general maximum principle, $|\phi|^2$ is constant.

Theorem 6 (Kodaira vanishing theorem) Let M be an n-dimensional compact strong Kähler-Finsler manifold, E be a holomorphic vector bundle of rank r over M, the pull-back $\widetilde{E}=p^{-1}(E)$ be a holomorphic vector bundle over $P(E)$, and $L(E)$ be the line bunlde of $\tilde{E}$. Assumptions are as in Formula WFIV. If the horizontal part of the curvature of $L(E)$ is quasi-positive definite and the horizontal holomorphic flag curvature of M along its horizontal frame $\{V_\alpha\}\in\mathcal{H}^{1,0}$ is quasi-positive definite, then there is no nonzero harmonic horizontal (p,n)-form over PTM with values in $L(E)$ for all $0<p\le n$.

Proof Let ϕ be a harmonic horizontal (p,n)-form over PTM with values in $L(E)$, $0<p\le n$. By (66)

$$\int_{\mathrm{PTM}}|e|^2\sum_A(\sum_{\beta=1}^n\mu_\beta)|\phi'_A|^2\mathrm{d}v+\frac{1}{2}\int_{\mathrm{PTM}}|e|^2\sum_{\alpha\in A}K_F(V_\alpha)|\phi_A|^2\mathrm{d}v\le 0,$$

where ϕ' is a horizontal (p,n)-form over PTM and μ_β's are the eigenvalues of the horizontal part of the curvature form Ω of $L(E)$, $K_F(V_\alpha)$ is the horizontal holomorphic flag curvature along its horizontal frame $\{V_\alpha\}\in\mathcal{H}^{1,0}$. If the horizontal part of the curvature form Ω is positive definite at $v_0\in$ PTM, i.e., $\mu_\beta(v_0)>0$ for all β, and the horizontal holomorphic flag curvature is quasi-positive definite, i.e., $\sum_\alpha K_F(V_\alpha)(v_0)>0$, all the ϕ'_A's must vanish in a neighborhood U of v_0. In other words, $\phi\equiv 0$ on U. If $L(E)$ is positive, i.e., Ω is positive definite everywhere, and the horizontal curvature is positive definite everywhere, then this shows $\phi\equiv 0$ on PTM and the theorem is already proved. To conclude the proof in the general case of the quasi-positive $L(E)$ and the quasi-positive horizontal flag curverture, it suffices to invoke the Aronsajn-Carlemann unique continuation theorem ([4], [17] p.248).

Remark 5 In compact Kähler manifolds, Kodaira vanishing theorem plays an important role in proving the imbedding theorem, that is, every compact Kähler manifold with positive line bundle can be imbeded into some $\mathbb{P}^N$ (cf. [5]).

Remark 6 A strongly pseudoconvex complex Finsler metric F is called Kähler if and only if

$$(\Gamma^{\alpha}_{\mu,\nu} - \Gamma^{\alpha}_{\nu,\mu})v^{\mu} = 0.$$

Recently, Chen and Shen [18] pointed that Kähler Finsler metrics are actually strong Kähler. So, the results of this article are valid in Kähler Finsler manifolds.

References

[1] Bochner S. Vector fields and Ricci curvature. Bull Amer Math Soc, 1946, **52**: 776–797

[2] Bochner S. Curvature in Hermitian metric. Bull Amer Math Soc, 1947, **53**: 179–195

[3] Yano K, Bochner S. Curvature and Betti Numbers. Princeton, New Jersey: Princeton University Press, 1953

[4] Wu H. The Bochner Technique in Differential Geometry. London, Paris: Harwood Academic Publishers, 1988

[5] Morrow J, Kodaira K. Complex Manifold. New York: Holt, Rinehart and Winston, Inc, 1971

[6] Bao D, Chern S S, Shen Z. An Introducetion to Riemann-Finsler Geometry. New York: Spring-Verlag, 2000

[7] Chern S S. Finsler geometry is just Riemannian geometry without the quadratic restriction. AMS Notices, 1996, **43**(9): 959–963

[8] Abate M, Patrizio G. Finsler Metric - A Global Approach. Lecture Notes in Math, Vol 1591. Berlin: Spring-Verlag, 1994

[9] Zhong T D, Zhong C P. Bochner technique on real Finsler manifolds. Acta Math Sci, 2003, **23B**(2): 165–177

[10] Zhong C P. Hodge-Laplace operators on complex Finsler manifolds [D]. Xiamen University, 2003

[11] Aikou T. On complex Finsler manifolds. Rep Fac Sci, Kagoshima Univ (Math, Phys & Chem), 1991, **24**: 9–25

[12] Bland J, Kalka M. Variations of holomorphic curvature for Kähler Finsler metrics. Cont Math, 1996, **196**: 121–132

[13] Zhong C P, Zhong T D. Hodge-Laplace operator on complex Finsler manifolds. Advances in Mathematics (in Chinese), 2006, **35**(4): 415–426

[14] Zhong C P, Zhong T D. Horizontal $\bar{\partial}$-Laplacian on complex Finsler manifolds. Science in China, Ser A, 2005, **48**(Supp): 377–391

[15] Zhong C P, Zhong T D. Hodge decomposition theorem on strongly Kähler Finsler manifolds. Science in China, Ser A, 2006, **49**(11): 1696–1714

[16] Kobayashi S. Complex Finsler vector bundles. Cont Math, Amer Math Soc, 1996, **196**: 145–153

[17] Aronszajn N. A unique continuation theorem for solutions of elliptic partial differential equations or inequalities of second order. J Math Pures Appl, 1957, **36**: 235–249

[18] Chen B, Shen Y. Kähler Finsler metrics are actually strongly Kähler. Chin Ann Math, Ser B, 2009, **30**(2): 173–178