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ROUGH MULTIPLE SINGULAR INTEGRALS ALONG HYPERSURFACES*

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Abstract In this paper, the authors study the mapping properties of singular integrals on product domains with kernels in $L(\log^+ L)^\epsilon(S^{m-1}\times S^{n-1})$ ($\epsilon = 1$ or 2) supported by hyper-surfaces. The L^p bounds for such singular integral operators as well as the related Marcinkiewicz integral operators are established, provided that the lower dimensional maximal function is bounded on $L^q(\mathbb{R}^3)$ for all $q > 1$. The condition on the integral kernels is known to be optimal.

Key words multiple singular integral; Marcinkiewicz integral; maximal function; hyper-surface; rough kernel

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1 Introduction

Let $\mathbb{R}^N$ ($N = m$ or n), $N \geq 2$, be the N-dimensional Euclidean space and S^{N-1} the unit sphere in $\mathbb{R}^N$. For nonzero points $x \in \mathbb{R}^m$ and $y \in \mathbb{R}^n$, we denote $x' = x/|x|$ and $y' = y/|y|$. For $m \geq 2$ and $n \geq 2$, let $\Omega(x', y') \in L^1(S^{m-1}\times S^{n-1})$ be a homogeneous function of degree zero satisfying

$$\int_{S^{m-1}} \Omega(x', y')\mathrm{d}x' = \int_{S^{n-1}} \Omega(x', y')\mathrm{d}y' = 0. \tag{1.1}$$

Let $h(\cdot,\cdot)$ be an appropriate real-valued measurable function defined on $\mathbb{R}^+\times\mathbb{R}^+$. For a suitable continuous function $\gamma(\cdot,\cdot)$ on $\mathbb{R}^+\times\mathbb{R}^+$, let Γ be the hyper-surface given by $\Gamma =$

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$\{(x,y,\gamma(|x|,|y|);\ x\in\mathbb{R}^m, y\in\mathbb{R}^n\}$. For $(x,y,z)\in\mathbb{R}^m\times\mathbb{R}^n\times\mathbb{R}=\mathbb{R}^{m+n+1}$, we define the multiple singular integral operator $T_{\gamma,h}$ in $\mathbb{R}^{m+n+1}$ along Γ by

$$T_{\gamma,h}(f)(x,y,z)=\text{p.v.}\iint_{\mathbb{R}^m\times\mathbb{R}^n}\frac{h(|\xi|,|\eta|)\Omega(\xi',\eta')}{|\xi|^m|\eta|^n}f(x-\xi,y-\eta,z-\gamma(|\xi|,|\eta|))\mathrm{d}\xi\mathrm{d}\eta, \tag{1.2}$$

initially for C_0^∞ function $f(x,y,z)$ on $\mathbb{R}^{m+n+1}$. If $\gamma(s,t)\equiv 0$, we shall let $T_h=T_{0,h}$.

In the one parameter case, the L^p $(1<p<\infty)$ boundedness of such kind of operators $T_{\gamma,h}$ was studied quite extensively. For the relevant results one may consult [14, 4–7, 16, 18], among others. We refer the reader to see Stein-Wainger's report [21] for more background information. In the multiple parameters cases, the study of the L^p boundedness of T_h under various conditions on Ω and h was begun in [12, 13] and continued by many authors (see [1, 3, 5, 9, 10, 19, 26]). In particular, it was shown in [3] (resp., [1]) that T_h is bounded on L^p for $|1/p-1/2|<\min\{1/2,1/\nu'\}$ (resp., $1<p<\infty$) provided that $\Omega\in L(\log^+L)^2(S^{m-1}\times S^{n-1})$, $h\in\Delta_\nu$ for $\nu>1$ (resp., $\Omega\in L\log^+L(S^{m-1}\times S^{n-1})$, $h\in L^2(\mathbb{R}^+\times\mathbb{R}^+,s^{-1}t^{-1}\mathrm{d}s\mathrm{d}t)$). Here we denote $h\in\Delta_\nu$, $\nu>1$, if

$$\|h\|_{\Delta_\nu}:=\sup_{R_1,R_2>0}\left(R_1^{-1}R_2^{-1}\int_0^{R_1}\int_0^{R_2}|h(s,t)|^\nu\mathrm{d}s\mathrm{d}t\right)^{1/\nu}<\infty,$$

and we let $L^2(\mathbb{R}^+\times\mathbb{R}^+,s^{-1}t^{-1}\mathrm{d}s\mathrm{d}t)$ to be the space of all measurable functions $h:\ \mathbb{R}^+\times\mathbb{R}^+\longrightarrow\mathbb{R}$ satisfying

$$\|h\|_{L^2(\mathbb{R}^+\times\mathbb{R}^+,s^{-1}t^{-1}\mathrm{d}s\mathrm{d}t)}:=\left(\int_0^\infty\int_0^\infty|h(s,t)|^2s^{-1}t^{-1}\mathrm{d}s\mathrm{d}t\right)^{1/2}<\infty.$$

In the same paper [3] or [1], the authors also showed that the condition $\Omega\in L(\log^+L)^\epsilon(S^{m-1}\times S^{n-1})$ for $\epsilon=2$ or 1 is nearly optimal in the sense that the exponent ϵ in $L(\log^+L)^\epsilon$ can not be replaced by any smaller number.

In this paper, we will focus our attention on the general operator $T_{\gamma,h}$. Clearly, the operator $T_{\gamma,h}$ is a natural extension of the multiple Hilbert transform along surfaces defined by

$$H_\gamma(f)(x_1,x_2,z)=\text{p.v.}\iint_{\mathbb{R}\times\mathbb{R}}f(s_1-s,x_2-t,z-\gamma(s,t))\frac{\mathrm{d}s\mathrm{d}t}{st}.$$

It is well-known that H_γ is bounded on $L^p(\mathbb{R}^3)$ under various conditions on γ (see [9, 17, 23, 24] and references therein). Our main purpose in this paper is to generalize the results of H_γ to the operator $T_{\gamma,h}$. Precisely, we will establish the L^p-boundedness of $T_{\gamma,h}$ under the optimal size condition $\Omega\in L(\log^+L)^\epsilon(S^{m-1}\times S^{n-1})$ ($\epsilon=1$ or 2) and under certain conditions on γ and h. Before stating our main results, we need to introduce the following maximal function

$$M_\gamma(g)(\iota,\tau,z)=\sup_{R_1>0,R_2>0}R_1^{-1}R_2^{-1}\int_0^{R_1}\int_0^{R_2}|g(\iota-u,\tau-v,z-\gamma(u,v))|\mathrm{d}u\mathrm{d}v, \tag{1.3}$$

where $\iota,\tau,z\in\mathbb{R}$.

Now we can formulate our main results as follows.

Theorem 1 Suppose that $\Omega\in L(\log^+L)^2(S^{m-1}\times S^{n-1})$ is a homogeneous function of degree zero satisfying (1.1), and $h\in\Delta_\nu$ for some $\nu>1$. Then

(i) $\|T_{\gamma,h}(f)\|_{L^2(\mathbb{R}^{m+n+1})} \leq C\|f\|_{L^2(\mathbb{R}^{m+n+1})}$;

(ii) $\|T_{\gamma,h}(f)\|_{L^p(\mathbb{R}^{m+n+1})} \leq C\|f\|_{L^p(\mathbb{R}^{m+n+1})}$ for $|1/p - 1/2| < \min\{1/2, 1/\nu'\}$, provided that for any $q \in (1,\infty)$,

$$\|M_\gamma(g)\|_{L^q(\mathbb{R}^3)} \leq C\|g\|_{L^q(\mathbb{R}^3)}. \tag{1.4}$$

Remark 1.1 By the same arguments as in the proof of Theorem 1.2 (b) in [3], we remark that the condition $\Omega \in L(\log^+ L)^2(S^{m-1}\times S^{n-1})$ is optimal, that is, there exists an Ω that lies in $L(\log^+ L)^{2-\theta}(S^{m-1}\times S^{n-1})$ for all $\theta > 0$ and satisfies (1.1) such that $T_{\gamma,h}$ is not bounded on $L^p(\mathbb{R}^{m+n+1})$ for any $p \in (1,\infty)$. In addition, the condition on h in Theorem 1 is very mild, since by Hölder's inequality it is easy to see that $L^\infty(\mathbb{R}^+\times\mathbb{R}^+) \subset \Delta_{\nu_1} \subset \Delta_{\nu_2}$ if $\nu_1 > \nu_2 > 1$.

Remark 1.2 It is clear that the maximal function in (1.3) is a natural extension of the following maximal function

$$M_\phi(g)(\iota,\tau) = \sup_{R>0} R^{-1}\int_0^R |g(\iota-u, \tau-\phi(u))|\mathrm{d}u,$$

which plays an important role in harmonic analysis and was extensively studied by many authors (see [20]). And the surface γ satisfying (1.4) is easily available. A simple example is $\gamma(s,t) = s^\alpha t^\beta$ with $\alpha > 0$ and $\beta > 0$ (see Corollary 3 in [9]). It will be more interesting to investigate curvature conditions on γ to assert the L^p boundedness of M_γ, similar to those for M_ϕ.

On the other hand, if $h \in L^2(\mathbb{R}^+\times\mathbb{R}^+, s^{-1}t^{-1}\mathrm{d}s\mathrm{d}t)$ then we have the following result.

Theorem 2 Suppose that $\Omega \in L\log^+ L(S^{m-1}\times S^{n-1})$ and satisfies (1.1). Suppose also that $h \in L^2(\mathbb{R}^+\times\mathbb{R}^+, s^{-1}t^{-1}\mathrm{d}s\mathrm{d}t)$. Then

(i) $\|T_{\gamma,h}(f)\|_{L^2(\mathbb{R}^{m+n+1})} \leq C\|f\|_{L^2(\mathbb{R}^{m+n+1})}$;

(ii) $\|T_{\gamma,h}(f)\|_{L^p(\mathbb{R}^{m+n+1})} \leq C\|f\|_{L^p(\mathbb{R}^{m+n+1})}$ for any $p \in (1,\infty)$, provided that the lower dimensional maximal operator M_γ satisfies (1.4) for all $q > 1$.

In order to prove Theorem 2, let $S_{\Omega,\gamma}$ be the operator defined by

$$S_{\Omega,\gamma}(f)(x,y,z)$$
$$:= \left(\int_0^\infty\int_0^\infty \left|\iint_{S^{m-1}\times S^{n-1}} \Omega(u',v')f(x-su', y-tv', z-\gamma(s,t))\mathrm{d}u'\mathrm{d}v'\right|^2 \frac{\mathrm{d}s\mathrm{d}t}{st}\right)^{1/2}.$$

Obviously, if $h \in L^2(\mathbb{R}^+\times\mathbb{R}^+, s^{-1}t^{-1}\mathrm{d}s\mathrm{d}t)$, then

$$|T_{\gamma,h}(f)(x,y,z)| \leq \|h\|_{L^2(\mathbb{R}^+\times\mathbb{R}^+, s^{-1}t^{-1}\mathrm{d}s\mathrm{d}t)} S_{\Omega,\gamma}(f)(x,y,z).$$

Therefore, Theorem 2 can be deduced immediately from the next theorem.

Theorem 3 Let γ, Ω be as in Theorem 2. Then

(i) $\|S_{\Omega,\gamma}(f)\|_{L^2(\mathbb{R}^{m+n+1})} \leq C\|f\|_{L^2(\mathbb{R}^{m+n+1})}$;

(ii) $\|S_{\Omega,\gamma}(f)\|_{L^p(\mathbb{R}^{m+n+1})} \leq C\|f\|_{L^p(\mathbb{R}^{m+n+1})}$, $2 < p < \infty$, provided that the maximal operator M_γ satisfies (1.4).

Remark 1.3 If $\gamma \equiv 0$, then Theorems D and B in [1] immediately follow from Theorems 2 and 3. It should also be pointed out by the same arguments as in [1] that the condition $\Omega \in L\log^+ L(S^{m-1}\times S^{n-1})$ in Theorems 2 and 3 is optimal.

As a simple application of Theorems 1 and 2, we can obtain immediately the following result (also see [1, 3]).

Theorem 4 Let Ω, h, p be as in Theorem 1 or Theorem 2. Then the multiple singular integral operator T_h defined by

$$T_h(f)(x,y) = \text{p.v.} \int_{\mathbb{R}^m \times \mathbb{R}^n} \frac{h(|u|,|v|)\Omega(u',v')}{|x-u|^m |y-v|^n} f(x-u, y-v) \mathrm{d}u\mathrm{d}v$$

is bounded on $L^p(\mathbb{R}^m \times \mathbb{R}^n)$.

Indeed, let $\gamma(u,v) \equiv 0$. Then M_γ satisfies (1.4) in Theorem 1. For any function $f \in \mathcal{S}(\mathbb{R}^m \times \mathbb{R}^n)$, let g be a function on $\mathcal{S}(\mathbb{R})$ such that $\|g\|_p \neq 0$. By the definition and Theorems 1 and 2, it is easy to see that

$$\|g\|_{L^p(\mathbb{R})} \|T_h(f)\|_{L^p(\mathbb{R}^m \times \mathbb{R}^n)} = \|T_{h,\gamma}(f \otimes g)\|_{L^p(\mathbb{R}^m \times \mathbb{R}^n \times \mathbb{R})} \leq C \|f\|_{L^p(\mathbb{R}^m \times \mathbb{R}^n)} \|g\|_{L^p(\mathbb{R})},$$

where $(f \otimes g)(x,y,z) = f(x,y)g(z)$. This implies Theorem 4.

In addition, we also consider the related Marcinkiewicz integral operators $\mu_{\gamma,h}$ along γ defined by

$$\mu_{\gamma,h}(f)(x,y,z) = \left(\int_{\mathbb{R}} \int_{\mathbb{R}} |F^\gamma_{s,t}(x,y,z)|^2 2^{-2s} 2^{-2t} \mathrm{d}s\mathrm{d}t \right)^{1/2},$$

where

$$F^\gamma_{s,t}(x,y,z) = \int\int_{|u| \leq 2^s, |v| \leq 2^t} \frac{h(|u|,|v|)\Omega(u',v')}{|u|^{m-1}|v|^{n-1}} f(x-u, y-v, z-\gamma(|u|,|v|)) \mathrm{d}u\mathrm{d}v.$$

For $h \equiv 1$, Ding, Fan and Pan [8] established the $L^p(\mathbb{R}^{m+n+1})$ boundedness of $\mu_{\gamma,1}$ under the condition: Ω belonging to certain block spaces, $1 < p < \infty$. Recently, the last author [27] gave an improvement of the result in [8] (also see [28] for another related result). On the other hand, from Al-Salman, Al-Qassem, Cheng and Pan's work [2] and Wang, Chen and Fan's work [25], we know that for $\gamma \equiv 0$ and $h \in \Delta_2$, $\mu_{0,h}$ is bounded on $L^p(\mathbb{R}^{m+n+1})$, provided $\Omega \in L\log^+ L(S^{m-1} \times S^{n-1})$, $1 < p < \infty$. Here, we will establish the following result.

Theorem 5 Suppose that $\Omega \in L\log^+ L(S^{m-1} \times S^{n-1})$ and satisfies (1.1), $h \in \Delta_\nu$ for some $\nu > 1$. Then

(i) $\|\mu_{\gamma,h}(f)\|_{L^2(\mathbb{R}^{m+n+1})} \leq C\|f\|_{L^2(\mathbb{R}^{m+n+1})}$;

(ii) $\|\mu_{\gamma,h}(f)\|_{L^p(\mathbb{R}^{m+n+1})} \leq C\|f\|_{L^p(\mathbb{R}^{m+n+1})}$ for $|1/p - 1/2| < \min(1/2, 1/\nu')$, provided that for any $q > 1$, the lower maximal operator M_γ satisfies (1.4).

The remainder of the paper is organized as follows. In Section 2 we shall introduce some notations and establish some estimates which will play key roles in our proofs. After proving Theorem 1 in Section 3 we shall give the proofs of Theorems 2 and 3 in Section 4. Finally, we shall prove Theorem 5 in Section 5. We would like to remark that we are very much motivated by the works [1, 3, 8, 9].

Throughout this paper, C always denotes a positive constant independent of the essential variables, but whose value may vary at each occurrence.

2 Some Notations and Lemmas

Let $\epsilon = 1$ or 2. Assume that $\Omega \in L(\log^+ L)^\epsilon(S^{m-1} \times S^{n-1})$ and satisfies (1.1). Following the notation in [2], for $l \in \mathbb{N}$, let

$$E_l := \{(x',y') \in S^{m-1} \times S^{n-1} : 2^l \leq |\Omega(x',y')| < 2^{l+1}\}.$$

Also, let $E_0 := \{(x', y') \in S^{m-1} \times S^{n-1} :\ |\Omega(x', y')| < 2\}$. Set $D := \{l \in \mathbb{N} :\ |E_l| > 2^{-4l}\}$, and for $l \geq 1$

$$\Omega_l(x', y') := \Omega(x', y')\chi_{E_l}(x', y') + \frac{1}{|S^{m-1}||S^{n-1}|}\iint_{S^{m-1}\times S^{n-1}} \Omega(x', y')\chi_{E_l}(x', y')\mathrm{d}x'\mathrm{d}y'$$
$$-\frac{1}{|S^{m-1}|}\int_{S^{m-1}} \Omega(x', y')\chi_{E_l}(x', y')\mathrm{d}x' - \frac{1}{|S^{n-1}|}\int_{S^{n-1}} \Omega(x', y')\chi_{E_l}(x', y')\mathrm{d}y',$$

and $\Omega_0(x', y') := \Omega(x', y') - \sum\limits_{l\in D} \Omega_l(x', y')$, where $|E_l|$, $|S^{m-1}|$ and $|S^{n-1}|$ denote the Lebesgue measures of E_l, S^{m-1} and S^{n-1}, respectively. It is easy to verify that

$$\int_{S^{m-1}} \Omega_l(x', y')\mathrm{d}x' = \int_{S^{n-1}} \Omega_l(x', y')\mathrm{d}y' = 0, \quad l \geq 0, \tag{2.1}$$

$$\|\Omega_l\|_{L^1(S^{m-1}\times S^{n-1})} \leq 2\|\Omega\chi_{E_l}\|_{L^1(S^{m-1}\times S^{n-1})} = 2A_l, \quad l \in D, \tag{2.2}$$

$$\|\Omega_0\|_{L^1(S^{m-1}\times S^{n-1})} \leq C\|\Omega_0\|_{L^2(S^{m-1}\times S^{n-1})} \leq C < \infty, \tag{2.3}$$

$$\Omega(x', y') = \sum_{l\in D\cup\{0\}} \Omega_l(x', y'), \tag{2.4}$$

$$\sum_{l\in D\cup\{0\}} (l+1)^{\epsilon} A_l \leq C\|\Omega\|_{L(\log^+ L)^{\epsilon}(S^{m-1}\times S^{n-1})}, \tag{2.5}$$

where $A_l := \|\Omega\chi_{E_l}\|_{L^1(S^{m-1}\times S^{n-1})}$ for $l \in D$ and $A_0 = 1$.

For $j, k \in \mathbb{Z}$, $l \in D \cup \{0\}$, we write

$$B^l_{j,k} := \{(x, y) \in \mathbb{R}^m \times \mathbb{R}^n :\ 2^{j(l+1)} \leq |x| < 2^{(j+1)(l+1)}, 2^{k(l+1)} \leq |y| < 2^{(k+1)(l+1)}\}.$$

For each Ω_l, $h \in \Delta_\nu$ with $\nu > 1$, we define the measure $\sigma^l_{j,k}$ by letting its Fourier transform to be

$$\widehat{\sigma^l_{j,k}}(\xi, \eta, \zeta) = \int_{B^l_{j,k}} \frac{h(|u|, |v|)\Omega_l(u', v')}{|u|^m|v|^n}\mathrm{e}^{-\mathrm{i}\{u\cdot\xi+v\cdot\eta+\gamma(|u|,|v|)\zeta\}}\mathrm{d}u\mathrm{d}v. \tag{2.6}$$

It is easy to see that

$$\sigma^l_{j,k} * f(x, y, z) = \int_{B^l_{j,k}} \frac{\Omega_l(u', v')}{|u|^m|v|^n}h(|u|, |v|)f(x-u, y-v, z-\gamma(|u|, |v|))\mathrm{d}u\mathrm{d}v. \tag{2.7}$$

Similarly, we define the measure $|\sigma^l_{j,k}|$ by setting

$$\left|\widehat{\sigma^l_{j,k}}\right|(\xi, \eta, \zeta) = \int_{B^l_{j,k}} \frac{|h(|u|, |v|)||\Omega_l(u', v')|}{|u|^m|v|^n}\mathrm{e}^{-\mathrm{i}\{u\cdot\xi+v\cdot\eta+\gamma(|u|,|v|)\zeta\}}\mathrm{d}u\mathrm{d}v. \tag{2.8}$$

Then

$$\left|\sigma^l_{j,k}\right| * f(x, y, z) = \int_{B^l_{j,k}} \frac{|h(|u|, |v|)||\Omega_l(u', v')|}{|u|^m|v|^n}f(x-u, y-v, z-\gamma(|u|, |v|))\mathrm{d}u\mathrm{d}v.$$

By (2.4), we have

$$T_{\gamma,h}(f)(x, y, z) = \sum_{l\in D\cup\{0\}} \sum_{j=-\infty}^{\infty} \sum_{k=-\infty}^{\infty} \sigma^l_{j,k} * f(x, y, z) := \sum_{l\in D\cup\{0\}} T^l_{\gamma,h}(f)(x, y, z). \tag{2.9}$$

Also, we define the maximal function σ_l^* by

$$\sigma_l^*(f)(x,y,z)=\sup_{j,k\in\mathbb{Z}}\left||\sigma_{j,k}^l|*f(x,y,z)\right|.$$

Then we have the following lemma.

Lemma 2.1 Let $h\in\Delta_\nu$ for $\nu>1$. If $q>\nu'$ and M_γ is bounded on $L^{q/\nu'}(\mathbb{R}^3)$, then for $l\in D\cup\{0\}$,

$$\|\sigma_l^*(f)\|_{L^q(\mathbb{R}^{m+n+1})}\le C(l+1)^2A_l\|f\|_{L^q(\mathbb{R}^{m+n+1})}.$$

Proof For the sake of simplicity, let

$$I_{j,k}^l:=\{(s,t)\in\mathbb{R}^+\times\mathbb{R}^+:\ 2^{j(l+1)}\le s<2^{(j+1)(l+1)},2^{k(l+1)}\le t<2^{(k+1)(l+1)}\}.$$

By the definition, using the spherical coordinate and Hölder's inequality, we have

$$\begin{aligned}\sigma_l^*(f)(x,y,z)\le&\sup_{j,k\in\mathbb{Z}}\bigg\{\iint_{I_{j,k}^l}|h(s,t)|\iint_{S^{m-1}\times S^{n-1}}|\Omega_l(u',v')|\\&\times|f(x-su',y-tv',z-\gamma(s,t))|\mathrm{d}u'\mathrm{d}v'\frac{\mathrm{d}s\mathrm{d}t}{st}\bigg\}\\ \le&\sup_{j,k\in\mathbb{Z}}\bigg\{\Big(\iint_{I_{j,k}^l}|h(s,t)|^\nu\frac{\mathrm{d}s\mathrm{d}t}{st}\Big)^{1/\nu}\Big(\iint_{I_{j,k}^l}\Big|\iint_{S^{m-1}\times S^{n-1}}|\Omega_l(u',v')|\\&\times|f(x-su',y-tv',z-\gamma(s,t))|\mathrm{d}u'\mathrm{d}v'\Big|^{\nu'}\frac{\mathrm{d}s\mathrm{d}t}{st}\Big)^{1/\nu'}\bigg\}\\ \le&C(l+1)^{2/\nu}\|h\|_{\Delta_\nu}\|\Omega_l\|_1^{1/\nu}\sup_{j,k\in\mathbb{Z}}\Big(\iint_{I_{j,k}^l}\iint_{S^{m-1}\times S^{n-1}}|\Omega_l(u',v')|\\&\times|f(x-su',y-tv',z-\gamma(s,t))|^{\nu'}\mathrm{d}u'\mathrm{d}v'\frac{\mathrm{d}s\mathrm{d}t}{st}\Big)^{1/\nu'}\\ \le&C(l+1)^2\|\Omega_l\|_1^{1/\nu}\Big(\iint_{S^{m-1}\times S^{n-1}}|\Omega_l(u',v')|M_{u',v',\gamma}(g)(x,y,z)\mathrm{d}u'\mathrm{d}v'\Big)^{1/\nu'},\end{aligned}$$

where $g=|f|^{\nu'}$, and

$$M_{u',v',\gamma}(g)(x,y,z)=\sup_{R_1>0,R_2>0}\frac{1}{R_1R_2}\int_{R_1}^{2R_1}\int_{R_2}^{2R_2}g(x-su',y-tv',z-\gamma(s,t))\mathrm{d}s\mathrm{d}t.$$

Thus for any $q>\nu'$, $\|\sigma_l^*(f)\|_{L^q(\mathbb{R}^{m+n+1})}^{\nu'}$ is dominated by

$$\Big(C(l+1)^2\|\Omega_l\|_1^{1/\nu}\Big)^{\nu'}\iint_{S^{m-1}\times S^{n-1}}|\Omega_l(u',v')|\|M_{u',v',\gamma}(g)\|_{L^{q/\nu'}(\mathbb{R}^{m+n+1})}\mathrm{d}u'\mathrm{d}v'.$$

Note that $\|g\|_{L^{q/\nu'}(\mathbb{R}^{m+n+1})}=\|f\|_{L^q(\mathbb{R}^{m+n+1})}^{\nu'}$ and $\|\Omega_l\|_1\le 2A_l$. To prove Lemma 2.1, it remains to prove that $M_{u',v',\gamma}$ is bounded on $L^{q/\nu'}(\mathbb{R}^{m+n+1})$ with bound independent of u' and v'. Let $\mathbf{1}=(1,0,\cdots,0)\in S^{m-1}$, $\widetilde{\mathbf{1}}=(1,0,\cdots,0)\in S^{n-1}$. For each fixed (u',v'), choose a rotation $\rho=\rho_1\otimes\rho_2$ such that $\rho_1u'=\mathbf{1}$ and $\rho_2v'=\widetilde{\mathbf{1}}$. Let $\rho^{-1}=\rho_1^{-1}\otimes\rho_2^{-1}$ be the inverse of ρ. We define the function g_ρ by $g_\rho(x,y,z)=f(\rho_1x,\rho_2y,z)$. So

$$g(x-su',y-tv',z-\gamma(s,t))=g_{\rho^{-1}}(\rho_1x-s\mathbf{1},\rho_2y-t\widetilde{\mathbf{1}},z-\gamma(s,t)).$$

This together with the $L^{q/\nu'}(\mathbb{R}^3)$-boundedness of M_γ, and change of variables, shows that

$$\|M_{u',v',\gamma}(g)\|_{L^{q/\nu'}(\mathbb{R}^{m+n+1})} \le C\|g\|_{L^{q/\nu'}(\mathbb{R}^{m+n+1})},$$

where C is independent of (u', v'). Lemma 2.1 is proved.

Lemma 2.2 Let $h \in \Delta_\nu$ for some $\nu \in (1, 2]$. If M_γ is bounded on $L^q(\mathbb{R}^3)$ for all $q > 1$, then for arbitrary functions $\{g_{j,k}\}_{j,k\in\mathbb{Z}}$ on $\mathbb{R}^{m+n+1}$, $l \in D\cup\{0\}$, there exists a positive constant C independent of l, j, k such that

$$\left\|\left(\sum_{j,k\in\mathbb{Z}} \left|\sigma^l_{j,k} * g_{j,k}\right|^2\right)^{1/2}\right\|_p \le C(l+1)^2 A_l \left\|\left(\sum_{j,k\in\mathbb{Z}} |g_{j,k}|^2\right)^{1/2}\right\|_p$$

holds for any p satisfying $|1/p - 1/2| < 1/\nu'$.

By Lemma 2.1 and the similar arguments to the proof of Theorem 7.5 in [11], we easily establish the above lemma. Here the details are omitted.

Lemma 2.3 Let $\Omega = \sum\limits_{l\in D\cup\{0\}} \Omega_l$ be as in (2.4), $h \in \Delta_\nu$ for $1 < \nu \le 2$. Then for each $l \in D \cup \{0\}$, and $j, k \in \mathbb{Z}$, we have

(i) $|\widehat{\sigma^l_{j,k}}(\xi, \eta, \zeta)| \le C(l + 1)^2 A_l$;

(ii) $|\widehat{\sigma^l_{j,k}}(\xi, \eta, \zeta)| \le C(l + 1)^2 A_l \left|2^{j(l+1)}\xi\right|^{1/(l+1)} \left|2^{k(l+1)}\eta\right|^{1/(l+1)}$;

(iii) $|\widehat{\sigma^l_{j,k}}(\xi, \eta, \zeta)| \le C(l + 1)^2 A_l \left|2^{j(l+1)}\xi\right|^{1/4\nu'(l+1)} \left|2^{k(l+1)}\eta\right|^{-1/4\nu'(l+1)}$;

(iv) $|\widehat{\sigma^l_{j,k}}(\xi, \eta, \zeta)| \le C(l + 1)^2 A_l \left|2^{j(l+1)}\xi\right|^{-1/4\nu'(l+1)} \left|2^{k(l+1)}\eta\right|^{1/4\nu'(l+1)}$;

(v) $|\widehat{\sigma^l_{j,k}}(\xi, \eta, \zeta)| \le C(l + 1)^2 A_l \left|2^{j(l+1)}\xi\right|^{-1/4\nu'(l+1)} \left|2^{k(l+1)}\eta\right|^{-1/4\nu'(l+1)}$.

Proof By the definition, Hölder's inequality and (2.2), the proof of (i) is trivial. In what follows, we shall prove (ii)–(v). Set

$$H^l_{j,k;s,t}(\xi, \eta) := \iint_{S^{m-1}\times S^{n-1}} \Omega_l(u', v')\mathrm{e}^{-\mathrm{i}[2^{j(l+1)}su'\cdot\xi + 2^{k(l+1)}tv'\cdot\eta]}\mathrm{d}u'\mathrm{d}v'. \tag{2.10}$$

Then by the spherical coordinate and Hölder's inequality, we have

$$\begin{aligned}|\widehat{\sigma^l_{j,k}}(\xi, \eta, \zeta)| &\le \int_1^{2^{l+1}}\int_1^{2^{l+1}} |h(2^{j(l+1)}s, 2^{k(l+1)}t)||H^l_{j,k;s,t}(\xi, \eta)|s^{-1}t^{-1}\mathrm{d}s\mathrm{d}t \\ &\le \left(\int_1^{2^{l+1}}\int_1^{2^{l+1}} |h(2^{j(l+1)}s, 2^{k(l+1)}t)|^\nu \frac{\mathrm{d}s\mathrm{d}t}{st}\right)^{1/\nu} \\ &\quad\times\left(\int_1^{2^{l+1}}\int_1^{2^{l+1}} |H^l_{j,k;s,t}(\xi, \eta)|^{\nu'} \frac{\mathrm{d}s\mathrm{d}t}{st}\right)^{1/\nu'} \\ &\le C(l + 1)^{2/\nu}\|h\|_{\Delta_\nu}\left(\int_1^{2^{l+1}}\int_1^{2^{l+1}} |H^l_{j,k;s,t}(\xi, \eta)|^{\nu'} \frac{\mathrm{d}s\mathrm{d}t}{st}\right)^{1/\nu'}.\end{aligned} \tag{2.11}$$

Now, we estimate $H^l_{j,k;s,t}(\xi, \eta)$. By (2.1) and (2.2), it is easy to see that

$$|H^l_{j,k;s,t}(\xi, \eta)| \le CA_l|2^{j(l+1)}s\xi||2^{k(l+1)}t\eta|. \tag{2.12}$$

Hence,

$$\begin{aligned}|\widehat{\sigma^l_{j,k}}(\xi, \eta, \zeta)| &\le C(l + 1)^{2/\nu}A_l\left|2^{j(l+1)}\xi\right|\left|2^{k(l+1)}\eta\right|\left(\int_1^{2^{l+1}}\int_1^{2^{l+1}} (st)^{\nu'-1}\mathrm{d}s\mathrm{d}t\right)^{1/\nu'} \\ &\le C(l + 1)^2A_l4^{l+1}\left|2^{j(l+1)}\xi\right|\left|2^{k(l+1)}\eta\right|.\end{aligned} \tag{2.13}$$

Interpolating between (i) and (2.13), we get

$$|\widehat{\sigma^l_{j,k}}(\xi,\eta,\zeta)| \le C(l+1)^2 A_l \left|2^{j(l+1)}\xi\right|^{1/(l+1)} \left|2^{k(l+1)}\eta\right|^{1/(l+1)}.$$

(ii) is proved.

On the other hand, by the fact $|H^l_{j,k;s,t}(\xi,\eta)| \le \|\Omega_l\|_1 \le C\|\Omega_l\|_2$, and the Hölder inequality, it is easy to see that for $1<\nu\le 2$,

$$\begin{aligned}|\widehat{\sigma^l_{j,k}}(\xi,\eta,\zeta)| &\le \left(\int_1^{2^{l+1}}\int_1^{2^{l+1}} |h(2^{j(l+1)}s, 2^{k(l+1)}t)|^\nu \frac{\mathrm{d}s\mathrm{d}t}{st}\right)^{1/\nu}\\ &\quad\times\left(\int_1^{2^{l+1}}\int_1^{2^{l+1}} |H^l_{j,k;s,t}(\xi,\eta)|^{\nu'} \frac{\mathrm{d}s\mathrm{d}t}{st}\right)^{1/\nu'}\\ &\le C(l+1)^{2/\nu}\|\Omega_l\|_2^{1-2/\nu'}\left(\int_1^{2^{l+1}}\int_1^{2^{l+1}} |H^l_{j,k;s,t}(\xi,\eta)|^2 \frac{\mathrm{d}s\mathrm{d}t}{st}\right)^{1/\nu'}. \qquad (2.14)\end{aligned}$$

Now

$$\begin{aligned}|H^l_{j,k;s,t}(\xi,\eta)|^2 &= \iint_{(S^{m-1}\times S^{n-1})^2} \Omega_l(u',v')\overline{\Omega_l(w',z')}\\ &\quad\times \mathrm{e}^{-\mathrm{i}[2^{j(l+1)}s(u'-w')\cdot\xi + 2^{k(l+1)}t(v'-z')\cdot\eta]}\mathrm{d}u'\mathrm{d}v'\mathrm{d}w'\mathrm{d}z', \qquad (2.15)\end{aligned}$$

and by van der Corput's lemma (see [20]) or integration by parts,

$$\begin{aligned}&\left|\int_1^{2^{l+1}}\int_1^{2^{l+1}} \mathrm{e}^{-\mathrm{i}[2^{j(l+1)}s(u'-w')\cdot\xi + 2^{k(l+1)}t(v'-z')\cdot\eta]}\frac{\mathrm{d}s\mathrm{d}t}{st}\right|\\ &\le C\left|2^{j(l+1)}(u'-w')\cdot\xi\right|^{-1}\left|2^{k(l+1)}(v'-z')\cdot\eta\right|^{-1},\end{aligned}$$

which together with the trivial estimate

$$\left|\int_1^{2^{l+1}}\int_1^{2^{l+1}} \mathrm{e}^{-\mathrm{i}[2^{j(l+1)}s(u'-w')\cdot\xi + 2^{k(l+1)}t(v'-z')\cdot\eta]}\frac{\mathrm{d}s\mathrm{d}t}{st}\right| \le C(l+1)^2$$

implies that for any $\theta\in(0,1)$,

$$\begin{aligned}&\left|\int_1^{2^{l+1}}\int_1^{2^{l+1}} \mathrm{e}^{-\mathrm{i}[2^{j(l+1)}s(u'-w')\cdot\xi + 2^{k(l+1)}t(v'-z')\cdot\eta]}\frac{\mathrm{d}s\mathrm{d}t}{st}\right|\\ &\le C(l+1)^{2(1-\theta)}\left|2^{j(l+1)}(u'-w')\cdot\xi\right|^{-\theta}\left|2^{k(l+1)}(v'-z')\cdot\eta\right|^{-\theta}. \qquad (2.16)\end{aligned}$$

Thus, taking $\theta=1/4$, we get

$$\begin{aligned}&\int_1^{2^{l+1}}\int_1^{2^{l+1}} |H^l_{,k;s,t}(\xi,\eta)|^2\frac{\mathrm{d}s\mathrm{d}t}{st}\\ &\le C(l+1)^{3/2}\left|2^{j(l+1)}\xi\right|^{-1/4}\left|2^{k(l+1)}\eta\right|^{-1/4}\iint_{(S^{m-1}\times S^{n-1})^2}\Omega_l(u',v')\overline{\Omega_l(w',z')}\\ &\quad\times|(u'-w')\cdot\xi'|^{-1/4}|(v'-z')\cdot\eta'|^{-1/4}\mathrm{d}u'\mathrm{d}v'\mathrm{d}w'\mathrm{d}z'\\ &\le C(l+1)^{3/2}\left|2^{j(l+1)}\xi\right|^{-1/4}\left|2^{k(l+1)}\eta\right|^{-1/4}\|\Omega_l\|_2^2\\ &\quad\times\iint_{(S^{m-1}\times S^{n-1})^2}|(u'-w')\cdot\xi'|^{-1/2}|(v'-z')\cdot\eta'|^{-1/2}\mathrm{d}u'\mathrm{d}v'\mathrm{d}w'\mathrm{d}z'\\ &\le C(l+1)^{3/2}\left|2^{j(l+1)}\xi\right|^{-1/4}\left|2^{k(l+1)}\eta\right|^{-1/4}\|\Omega_l\|_2^2. \qquad (2.17)\end{aligned}$$

Consequently,

$$\widehat{\sigma^l_{j,k}}(\xi,\eta,\zeta)| \le C(l+1)^2\|\Omega_l\|_2 \left|2^{j(l+1)}\xi\right|^{-1/4\nu'} \left|2^{k(l+1)}\eta\right|^{-1/4\nu'}.$$

Note that $\|\Omega_l\|_0 \le C = CA_0$, and for $l \in D$, $A_l \ge C2^l|E_l| \ge C2^{-3l}$, we have

$$\|\Omega_l\|_2 \le C2^{l+1}|E_l|^{1/2} \le C2^{2(l+1)}A_l.$$

So,

$$|\widehat{\sigma^l_{j,k}}(\xi,\eta,\zeta)| \le C(l+1)^2 A_l 2^{2(l+1)} \left|2^{j(l+1)}\xi\right|^{-1/4\nu'} \left|2^{k(l+1)}\eta\right|^{-1/4\nu'}. \tag{2.18}$$

Employing the interpolation theorem, it follows from (i) and (2.18) that

$$|\widehat{\sigma^l_{j,k}}(\xi,\eta,\zeta)| \le C(l+1)^2 A_l \left|2^{j(l+1)}\xi\right|^{-1/4(l+1)\nu'} \left|2^{k(l+1)}\eta\right|^{-1/4(l+1)\nu'}.$$

This proves (v).

It remains to prove (iii) and (iv). Notice that

$$\begin{aligned}|H^l_{j,k;s,t}(\xi,\eta)|^2 = &\iint_{(S^{m-1}\times S^{n-1})^2} \Omega_l(u',v')\overline{\Omega_l(w',z')}\left[\mathrm{e}^{-\mathrm{i}2^{j(l+1)}s(u'-w')\cdot\xi}-1\right]\\ &\times \mathrm{e}^{-\mathrm{i}2^{k(l+1)}t(v'-z')\cdot\eta}\mathrm{d}u'\mathrm{d}v'\mathrm{d}w'\mathrm{d}z',\end{aligned} \tag{2.19}$$

and

$$\begin{aligned}|H^l_{j,k;s,t}(\xi,\eta)|^2 = &\iint_{(S^{m-1}\times S^{n-1})^2} \Omega_l(u',v')\overline{\Omega_l(w',z')}\mathrm{e}^{-\mathrm{i}2^{j(l+1)}s(u'-w')\cdot\xi}\\ &\times \left[\mathrm{e}^{-\mathrm{i}2^{k(l+1)}t(v'-z')\cdot\eta}-1\right]\mathrm{d}u'\mathrm{d}v'\mathrm{d}w'\mathrm{d}z'.\end{aligned} \tag{2.20}$$

Similarly to (2.16), it is easy to verify that for any $\theta \in (0,1)$,

$$\begin{aligned}&\left|\int_1^{2^{l+1}}\int_1^{2^{l+1}}\left[\mathrm{e}^{-\mathrm{i}2^{j(l+1)}s(u'-w')\cdot\xi}-1\right]\mathrm{e}^{-\mathrm{i}2^{k(l+1)}t(v'-z')\cdot\eta}\frac{\mathrm{d}s\mathrm{d}t}{st}\right|\\ &\le C(l+1)^{2(1-\theta)}2^{(l+1)\theta}\left|2^{j(l+1)}(u'-w')\cdot\xi\right|^{\theta}\left|2^{k(l+1)}(v'-z')\cdot\eta\right|^{-\theta},\end{aligned} \tag{2.21}$$

and

$$\begin{aligned}&\left|\int_1^{2^{l+1}}\int_1^{2^{l+1}}\mathrm{e}^{-\mathrm{i}2^{j(l+1)}s(u'-w')\cdot\xi}\left[\mathrm{e}^{-\mathrm{i}2^{k(l+1)}t(v'-z')\cdot\eta}-1\right]\frac{\mathrm{d}s\mathrm{d}t}{st}\right|\\ &\le C(l+1)^{2(1-\theta)}2^{(l+1)\theta}\left|2^{j(l+1)}(u'-w')\cdot\xi\right|^{-\theta}\left|2^{k(l+1)}(v'-z')\cdot\eta\right|^{\theta}.\end{aligned} \tag{2.22}$$

Therefore, by the same arguments as those used in proving (2.18), we obtain

$$|\widehat{\sigma^l_{j,k}}(\xi,\eta,\zeta)| \le C(l+1)^2 A_l 2^{(l+1)/4\nu'} \left|2^{j(l+1)}\xi\right|^{1/4\nu'} \left|2^{k(l+1)}\eta\right|^{-1/4\nu'}, \tag{2.23}$$

and

$$|\widehat{\sigma^l_{j,k}}(\xi,\eta,\zeta)| \le C(l+1)^2 A_l 2^{(l+1)/4\nu'} \left|2^{j(l+1)}\xi\right|^{-1/4\nu'} \left|2^{k(l+1)}\eta\right|^{1/4\nu'}. \tag{2.24}$$

Invoking interpolation theorem again, (iii)–(iv) follow from (i) and (2.21)–(2.22). This completes the proof of Lemma 2.3.

3 Proof of Theorem 1

Take two radial Schwartz functions $\phi \in \mathcal{S}(\mathbb{R}^m)$ and $\psi \in \mathcal{S}(\mathbb{R}^n)$ such that

(a) $0 \le \phi, \psi \le 1$;

(b) $\operatorname{supp}(\phi) \subseteq \{x \in \mathbb{R}^m;\ \ 1/2 \le |x| \le 2\}$ and $\operatorname{supp}(\psi) \subseteq \{y \in \mathbb{R}^n;\ \ 1/2 \le |y| \le 2\}$;

(c) $\sum\limits_{d\in\mathbb{Z}} \big(\phi(2^{\mathrm{d}}x)\big)^2 \equiv 1$ for all $x \in \mathbb{R}^m \setminus \{0\}$ and $\sum\limits_{d\in\mathbb{Z}} \big(\psi(2^d y)\big)^2 \equiv 1$ for all $y \in \mathbb{R}^n \setminus \{0\}$.

For any $l \in \mathbb{N}$, set

$$\phi^{(l)}(x) = \bigg(\sum_{d=-l}^{0} \phi(2^{\mathrm{d}}x)^2\bigg)^{1/2}, \qquad \psi^{(l)}(y) = \bigg(\sum_{d=-l}^{0} \psi(2^d y)^2\bigg)^{1/2},$$

and let $\phi_j^{(l)}(\xi) = \phi^{(l)}(2^{j(l+1)}\xi)$, $\psi_k^{(l)} = \psi^{(l)}(2^{k(l+1)}\eta)$. Then

(a′) both $\phi^{(l)}$ and $\psi^{(l)}$ are radial Schwartz functions, and $0 \le \phi^{(l)}$, $\psi^{(l)} \le 1$;

(b′) $\operatorname{supp}(\phi^{(l)}) \subset \{\xi \in \mathbb{R}^m :\ 1/2 \le |\xi| \le 2^{l+1}\}$, $\operatorname{supp}(\psi^{(l)}) \subset \{\eta \in \mathbb{R}^n :\ 1/2 \le \eta \le 2^{l+1}\}$;

(c′) $\sum\limits_{j\in\mathbb{Z}} \phi_j^{(l)}(\xi)^2 = \sum\limits_{j\in\mathbb{Z}} \sum\limits_{d=-l}^{0} \phi(2^{j(l+1)+d}\xi)^2 = \sum\limits_{j\in\mathbb{Z}} \phi(2^j\xi)^2 = 1,$

$\sum\limits_{k\in\mathbb{Z}} \psi_k^{(l)}(\eta)^2 = \sum\limits_{k\in\mathbb{Z}} \sum\limits_{d=-l}^{0} \psi(2^{k(l+1)+d}\eta)^2 = \sum\limits_{k\in\mathbb{Z}} \psi(2^k\eta)^2 = 1.$

Define the multiplier operator $S_{j,k}^{(l)}$ in $\mathbb{R}^{m+n+1}$ by

$$\widehat{S_{j,k}^{(l)}(f)}(\xi,\eta,\zeta) = \phi_j^{(l)}(\xi)\psi_k^{(l)}(\eta)\widehat{f}(\xi,\eta,\zeta). \tag{3.1}$$

Then, in the sense of $L^2(\mathbb{R}^m \times \mathbb{R}^n \times \mathbb{R})$

$$\sum_{j\in\mathbb{Z}}\sum_{k\in\mathbb{Z}} S_{j,k}^{(l)}(S_{j,k}^{(l)}(f))(x,y,z) = f(x,y,z). \tag{3.2}$$

And by the Littlewood-Paley theory (also see [15]) we have

$$\bigg\|\bigg(\sum_{j,k\in\mathbb{Z}} \Big|S_{j,k}^{(l)}(f)\Big|^2\bigg)^{1/2}\bigg\|_p \le C\|f\|_p \tag{3.3}$$

and

$$\bigg\|\bigg(\sum_{j,k\in\mathbb{Z}} |S_{j,k}^{(l)}(f_{j,k})|^2\bigg)^{1/2}\bigg\|_p \le C\bigg\|\bigg(\sum_{j,k\in\mathbb{Z}} |f_{j,k}|^2\bigg)^{1/2}\bigg\|_p, \qquad 1<p<\infty \tag{3.4}$$

with C independent of l.

Thus, we can write

$$\begin{aligned} T_{\gamma,h}^l(f)(x,y,z) &= \sum_{j,k\in\mathbb{Z}} \sigma_{j,k}^l * f(x,y,z) = \sum_{j,k\in\mathbb{Z}}\sum_{j',k'\in\mathbb{Z}} S_{j+j',k+k'}^{(l)}\Big(\sigma_{j,k}^l * S_{j+j',k+k'}^{(l)} f\Big)(x,y,z) \\ &:= \sum_{j',k'\in\mathbb{Z}} T_{j',k'}^l f(x,y,z). \end{aligned} \tag{3.5}$$

Note that for $\nu > 2$, $\triangle_\nu \subset \triangle_2$, we may assume that $1 < \nu \le 2$ and $|1/p - 1/2| < 1/\nu'$. Applying Plancherel's theorem, we know that

$$\|T_{j',k'}^l(f)\|_2^2 = \int_{\mathbb{R}^{m+n+1}} \bigg|\sum_{j,k\in\mathbb{Z}} S_{j+j',k+k'}^{(l)}\Big(\sigma_{j,k}^l * S_{j+j',k+k'}^{(l)} f\Big)(x,y,z)\bigg|^2 \mathrm{d}x\mathrm{d}y\mathrm{d}z$$

$$
\begin{aligned}
&= \int_{\mathbb{R}^{m+n+1}} \Big| \sum_{j,k\in\mathbb{Z}} \widehat{\sigma^l_{j,k}}(\xi,\eta,\zeta)\phi^{(l)}_{j+j'}(\xi)^2\psi^{(l)}_{k+k'}(\eta)^2\widehat{f}(\xi,\eta,\zeta)\Big|^2 \mathrm{d}\xi\mathrm{d}\eta\mathrm{d}\zeta \\
&\le C\sum_{j,k\in\mathbb{Z}} \int_{E^l_{j,k;j',k'}} |\widehat{f}(\xi,\eta,\zeta)|^2 \Big|\widehat{\sigma^l_{j,k}}(\xi,\eta,\zeta)\Big|^2 \mathrm{d}\xi\mathrm{d}\eta\mathrm{d}\zeta,
\end{aligned}
$$

where

$$
\begin{aligned}
E^l_{j,k;j',k'} = \Big\{&(\xi,\eta)\in\mathbb{R}^m\times\mathbb{R}^n;\ 2^{-(j+j')(l+1)-1}\le|\xi|\le 2^{-(j+j'-1)(l+1)}, \\
&2^{-(k+k')(l+1)-1}\le|\eta|\le 2^{-(k+k'-1)(l+1)}\Big\}\times\mathbb{R}. \qquad (3.6)
\end{aligned}
$$

Also, by invoking Lemma 2.3, it is easy to see that for $(\xi,\eta,\zeta)\in E^l_{j,k;j',k'}$ and some $\theta\in(0,1)$,

$$
\begin{aligned}
\Big|\widehat{\sigma^l_{j,k}}(\xi,\eta,\zeta)\Big| \le C(l+1)^2A_l\min\Big\{&1, \Big|2^{j(l+1)}\xi\Big|^{1/(l+1)}\Big|2^{k(l+1)}\eta\Big|^{1/(l+1)}, \\
&\Big|2^{j(l+1)}\xi\Big|^{\theta/(l+1)}\Big|2^{j(l+1)}\eta\Big|^{-\theta/(l+1)}, \Big|2^{j(l+1)}\xi\Big|^{-\theta/(l+1)}\Big|2^{k(l+1)}\eta\Big|^{\theta/(l+1)}, \\
&\Big|2^{j(l+1)}\xi\Big|^{-\theta/(l+1)}\Big|2^{k(l+1)}\eta\Big|^{-\theta/(l+1)}\Big\} \\
\le C(l+1)^2A_l\min\Big\{&1, 2^{-j'-k'}, 2^{-\theta j'+\theta k'}, 2^{\theta j'-\theta k'}, 2^{\theta(j'+k')}\Big\}.
\end{aligned}
$$

So,

$$
\begin{aligned}
\big\|T^l_{j',k'}(f)\big\|_2^2 &\le C\sum_{j,k\in\mathbb{Z}}\int_{E^l_{j,k;j',k'}} |\widehat{f}(\xi,\eta,\zeta)|^2(l+1)^4A_l^2 \\
&\quad\times\min\Big\{1, 2^{-j'-k'}, 2^{-\theta j'+\theta k'}, 2^{\theta j'-\theta k'}, 2^{\theta(j'+k')}\Big\}^2 \mathrm{d}\xi\mathrm{d}\eta\mathrm{d}\zeta \\
&\le C(l+1)^4A_l^2\min\Big\{1, 2^{-j'-k'}, 2^{-\theta j'+\theta k'}, 2^{\theta j'-\theta k}, 2^{\theta(j'+k')}\Big\}^2\|f\|_2^2. \qquad (3.7)
\end{aligned}
$$

Consequently,

$$
\begin{aligned}
\big\|T^l_{\gamma,h}f\big\|_2 &\le \sum_{j',k'}\big\|T^l_{j',k'}f\big\|_2 \\
&\le C(l+1)^2A_l\sum_{j',k'\in\mathbb{Z}}\min\Big\{1, 2^{-j'-k'}, 2^{-\theta j'+\theta k'}, 2^{\theta j'-\theta k}, 2^{\theta(j'+k')}\Big\}\|f\|_2 \\
&\le C(l+1)^2A_l\|f\|_2\Big\{\sum_{j',k'\ge0}2^{-j'-k'} + \sum_{j'\ge0,k'<0}2^{-\theta j'+\theta k'} \\
&\quad+\sum_{j'<0,k'\ge0}2^{\theta j'-\theta k'} + \sum_{j'<0,k'<0}2^{\theta(j'+k')}\Big\} \\
&\le C(l+1)^2A_l\|f\|_2. \qquad (3.8)
\end{aligned}
$$

Therefore, by (2.9) and (2.5) we get

$$
\begin{aligned}
\|T_{\gamma,h}(f)\|_2 &\le \sum_{l\in D\cup\{0\}}\|T^l_{\gamma,h}(f)\|_2 \le C\sum_{l\in D\cup\{0\}}(l+1)^2A_l\|f\|_2 \\
&\le C\|\Omega\|_{L(\log^+L)(S^{m-1}\times S^{n-1})}\|f\|_2.
\end{aligned}
$$

This proves (i).

On the other hand, by (3.3) and Lemma 2.2, we have

$$\left\|T^l_{j',k'}(f)\right\|_p \le C(l+1)^2 A_l \|f\|_p, \qquad |1/p-1/2|<1/\nu'. \tag{3.9}$$

This together with (3.7) and the interpolation theorem implies that

$$\left\|T^l_{j',k'}(f)\right\|_p \le C(l+1)^2 A_l \min\{2^{-\delta(j'+k')}, 2^{-\delta\theta(j'-k')}, 2^{\delta\theta(j'-k')}, 2^{\delta\theta(j'+k')}\}\|f\|_p \tag{3.10}$$

with $\delta\in(0,1]$ and $|1/p-1/2|<1/\nu'$. Thus

$$\|T^l_{\gamma,h}(f)\|_p \le \sum_{j',k'\in\mathbb{Z}} \|T^l_{j',k'}(f)\|_p \le C(l+1)^2 A_l\|f\|_p, \qquad |1/p-1/2|<1/\nu',$$

which together with (2.9) and (2.5) completes the proof of (ii). Theorem 1 is proved.

4 Proofs of Theorems 2 and 3

Assume that $\Omega\in L\log^+L(S^{m-1}\times S^{n-1})$ and satisfies (1.1). Decompose $\Omega(u',v')=\sum_{l\in D\cup\{0\}}\Omega_l(u',v')$ as in (2.4), where D,Ω_l are as before. Ω_l satisfies (2.1)–(2.3) and (2.5) holds for $\epsilon=1$. In what follows, we prove Theorems 2 and 3, respectively.

Proof of Theorem 3 For each $l\in D\cup\{0\}$, let

$$\begin{aligned}&S_{\Omega_l,\gamma}(f)(x,y,z)\\ :=&\left(\int_0^\infty\int_0^\infty\left|\iint_{S^{m-1}\times S^{n-1}}\Omega_l(u',v')f(x-su',y-tv',z-\gamma(s,t))\mathrm{d}u'\mathrm{d}v'\right|^2\frac{\mathrm{d}s\mathrm{d}t}{st}\right)^{1/2}.\end{aligned}$$

By (2.4) and Minkowski's inequality, we have

$$S_{\Omega,\gamma}(f)(x,y,z)\le\sum_{l\in D\cup\{0\}}S_{\Omega_l,\gamma}(f)(x,y,z). \tag{4.1}$$

For each $l\in D\cup\{0\}$, $j,k\in\mathbb{Z}$, let $S^{(l)}_{j,k}$ be as in (3.1). Then by (3.2) and the Minkowski inequality

$$\begin{aligned}S_{\Omega_l,\gamma}(f)(x,y,z)\le&\sum_{j',k'\in\mathbb{Z}}\Bigg(\sum_{j,k\in\mathbb{Z}}\int_1^{2^{l+1}}\int_1^{2^{l+1}}\Bigg|\iint_{S^{m-1}\times S^{n-1}}\Omega_l(u;,v')S^{(l)}_{j+j',k+k'}S^{(l)}_{j+j',k+k'}\\ &\times f(x-2^{j(l+1)}su',y-2^{k(l+1)}tv',z-\gamma(2^{j(l+1)}s,2^{k(l+1)}t))\mathrm{d}u'\mathrm{d}v'\Bigg|^2\frac{\mathrm{d}s\mathrm{d}t}{st}\Bigg)^{1/2}\\ :=&\sum_{j',k'\in\mathbb{Z}}I_{l;j',k'}(f)(x,y,z),\end{aligned}$$

from which and (4.1), we know that

$$\|S_{\Omega,\gamma}(f)\|_{L^p(\mathbb{R}^{m+n+1})}\le\sum_{l\in D\cup\{0\}}\sum_{j',k'\in\mathbb{Z}}\|I_{l;j',k'}(f)\|_{L^p(\mathbb{R}^{m+n+1})}. \tag{4.2}$$

Now we estimate $\|I_{l;j',k'}(f)\|_{L^p(\mathbb{R}^{m+n+1})}$ in the following cases:

Case 1 For $p=2$, we claim that there exists $\delta>0$ such that for $l\in D\cup\{0\}$,

$$\|I_{l;j',k'}(f)\|_{L^2(\mathbb{R}^{m+n+1})}\le C2^{-\delta(|j'|+|k'|)}aA_l\|f\|_{L^2(\mathbb{R}^{m+n+1})}, \tag{4.3}$$

where C is independent of l and j',k'.

Indeed, by Plancherel's theorem and Fubini's theorem,

$$\|I_{l;j',k'}(f)\|_2^2\le\sum_{j,k\in\mathbb{Z}}\int_{E^l_{j,k;j',k'}}|\widehat{f}(\xi,\eta,\zeta)|^2J_{l;j,k}(\xi,\eta)\mathrm{d}\xi\mathrm{d}\eta\mathrm{d}\zeta, \tag{4.4}$$

where $E^l_{j,k;j',k'}$ is as in (3.6) and

$$\begin{aligned}J_{l;j,k}(\xi,\eta)&=\int_1^{2^{l+1}}\int_1^{2^{l+1}}\left|\iint_{S^{m-1}\times S^{n-1}}\Omega_l(u',v')\mathrm{e}^{-\mathrm{i}[2^{j(l+1)}su'\cdot\xi+2^{k(l+1)}tv'\cdot\eta]}\mathrm{d}u'\mathrm{d}v'\right|^2\frac{\mathrm{d}s\mathrm{d}t}{st}\\&=\int_1^{2^{l+1}}\int_1^{2^{l+1}}|H^l_{j,k;s,t}(\xi,\eta)|^2\frac{\mathrm{d}s\mathrm{d}t}{st},\end{aligned} \tag{4.5}$$

where $H^l_{j,k;st}(\xi,\eta)$ is just as in (2.10).

By the arguments similar to those in the proof of Lemma 2.3, we easily deduce that

$$\begin{aligned}J_{l;j,k}(\xi,\eta)\le C(l+1)^2A_l^2\min\Big\{&1,|2^{j(l+1)}\xi|^{2/(l+1)}|2^{k(l+1)}\eta|^{2/(l+1)},\\&|2^{j(l+1)}\xi|^{1/4(l+1)}|2^{k(l+1)}\eta|^{-1/4(l+1)},|2^{j(l+1)}\xi|^{-1/4(l+1)}|2^{k(l+1)}\eta|^{1/4(l+1)},\\&|2^{j(l+1)}\xi|^{-1/4(l+1)}|2^{k(l+1)}\eta|^{-1/4(l+1)}\Big\}.\end{aligned} \tag{4.6}$$

Then by the fact that $E^l_{j,k;j',k'}\cap E^l_{j'',k'';j',k'}=\emptyset$ whenever $(j'',k'')\notin\{j-1,j,j+1\}\times\{k-1,k,k+1\}$, (4.3) follows from (4.4) and (4.6).

Case 2 For $p>2$, we claim that there exists $\theta>0$ such that for $l\in D\cup\{0\}$,

$$\|I_{l;j',k'}(f)\|_{L^p(\mathbb{R}^{m+n+1})}\le C2^{-\theta(|j'|+|k'|)}(l+1)A_l\|f\|_{L^p(\mathbb{R}^{m+n+1})} \tag{4.7}$$

with C independent of l, j' and k'.

Indeed, choose $g\in L^{(p/2)'}(\mathbb{R}^{m+n+1})$ such that $\|g\|_{L^{(p/2)'}(\mathbb{R}^{m+n+1})}=1$ and

$$\begin{aligned}\|I_{l;j',k'}(f)\|_p^2=\int_{\mathbb{R}^{m+n+1}}\sum_{j,k\in\mathbb{Z}}\int_1^{2^{l+1}}\int_1^{2^{l+1}}\Big|\iint_{S^{m-1}\times S^{n-1}}&\Omega_l(u',v')S^{(l)}_{j+j',k+k'}S^{(l)}_{j+j',k+k'}\\&\times f(x-2^{j(l+1)}su',y-2^{k(l+1)}tv',z-\gamma(s,t))\mathrm{d}u'\mathrm{d}v'\Big|^2\frac{\mathrm{d}s\mathrm{d}t}{st}g(x,y,z)\mathrm{d}x\mathrm{d}y\mathrm{d}z.\end{aligned}$$

By Hölder's inequality and change of variables, we get

$$\begin{aligned}&\|I_{l;j',k'}(f)\|_p^2\\&\le\int_{\mathbb{R}^{m+n+1}}\sup_{j,k\in\mathbb{Z}}\int_1^{2^{l+1}}\int_1^{2^{l+1}}\iint_{S^{m-1}\times S^{n-1}}|g(x+2^{j(l+1)}su',y+2^{k(l+1)}tv',z+\gamma(s,t))|\\&\quad\times|\Omega_l(u',v')|\mathrm{d}u'\mathrm{d}v'\frac{\mathrm{d}s\mathrm{d}t}{st}\sum_{j,k\in\mathbb{Z}}|S^{(l)}_{j+j',k+k'}S^{(l)}_{j+j',k+k'}f(x,y,z)|^2\mathrm{d}x\mathrm{d}y\mathrm{d}z\|\Omega_l\|_1.\end{aligned}$$

Notice that

$$\begin{aligned}&\sup_{j,k\in\mathbb{Z}}\int_1^{2^{l+1}}\int_1^{2^{l+1}}\iint_{S^{m-1}\times S^{n-1}}|\Omega_l(u',v')||g(x+2^{j(l+1)}su',y+2^{k(l+1)}tv',z+\gamma(s,t))|\\&\times \mathrm{d}u'\mathrm{d}v'\frac{\mathrm{d}s\mathrm{d}t}{st}\\ \le\ &(l+1)^2M_{\Omega_l,\gamma}(g)(x,y,z),\end{aligned}$$

where

$$\begin{aligned}&M_{\Omega_l,\gamma}(g)(x,y,z)\\:=&\sup_{R_1>0,R_2>0}\iint_{R_1<|u|<2R_1,R_2<|v|<2R_2}\frac{\Omega_l(u',v')}{|u|^m|v|^n}|g(x+u,y+v,z+\gamma(|u|,|v|))|\mathrm{d}u\mathrm{d}v.\end{aligned}$$

Employing the arguments similar to those in proving Lemma 2.1 with the $L^q(\mathbb{R}^3)$-boundedness of M_γ for all $q>1$, it is not difficult to see that

$$\|M_{\Omega_l,\gamma}(g)\|_{p_0}\le C\|\Omega_l\|_{L^1(S^{m-1}\times S^{n-1})}\|f\|_{p_0},\quad \text{for any}\quad 1<p_0<\infty. \tag{4.8}$$

Applying Hölder's inequality, (4.8), (3.3) and (3.4), we obtain

$$\begin{aligned}\|I_{l;j',k'}(f)\|_p^2&\le C(l+1)^2\|\Omega_l\|^2_{L^1(S^{m-1}\times S^{n-1})}\|g\|_{(p/2)'}\Big\|\Big(\sum_{j,k\in\mathbb{Z}}|S^{(l)}_{j+j',k+k'}S^{(l)}_{j+j',k+k'}f|^2\Big)^{1/2}\Big\|_p^2\\&\le C(l+1)^2A_l^2\Big\|\Big(\sum_{j,k\in\mathbb{Z}}|S^{(l)}_{j+j',k+k'}f|^2\Big)^{1/2}\Big\|_p^2\le C(l+1)^2A_l^2\|f\|_p^2,\end{aligned}$$

which together with (4.3) and an interpolation implies (4.7).

Therefore, by (4.2), (4.3) and (2.5), we have

$$\begin{aligned}\|S_{\Omega,\gamma}(f)\|_{L^2(\mathbb{R}^{m+n+1})}&\le C\sum_{l\in D\cup\{0\}}\sum_{j',k'\in\mathbb{Z}}(l+1)A_l2^{-\delta(|j'|+|k'|)}\|f\|_{L^2(\mathbb{R}^{m+n+1})}\\&\le C\sum_{l\in D\cup\{0\}}(l+1)A_l\|f\|_{L^2(\mathbb{R}^{m+n+1})}\\&\le C\|\Omega\|_{L\log^+L(S^{m-1}\times S^{n-1})}\|f\|_{L^2(\mathbb{R}^{m+n+1})}.\end{aligned}$$

This prove (i) of Theorem 3.

If M_γ is bounded on $L^q(\mathbb{R}^3)$ for $1<q<\infty$, then by (4.2), (4.7) and (2.5), we get

$$\|S_{\Omega,\gamma}(f)\|_{L^p(\mathbb{R}^{m+n+1})}\le C\|\Omega\|_{L\log^+L(S^{m-1}\times S^{n-1})}\|f\|_{L^p(\mathbb{R}^{m+n+1})},\qquad p>2,$$

which completes the proof of Theorem 3.

Proof of Theorem 2 By the fact

$$|T_{\gamma,h}(f)(x,y,z)|\le\|h\|_{L^2(\mathbb{R}^+\times\mathbb{R}^+,s^{-1}t^{-1}\mathrm{d}s\mathrm{d}t)}S_{\Omega,\gamma}(f)(x,y,z),$$

it follows from Theorem 3 that $T_{\gamma,h}$ is bounded on $L^p(\mathbb{R}^{m+n+1})$ for $2\le p<\infty$. On the other hand, by duality we can establish the L^p-boundedness of $T_{\gamma,h}$ for $1<p<2$. Theorem 2 is proved.

5 Proof of Theorem 5

Let Ω, h, γ be as in Theorem 5. We decompose $\Omega(u',v') = \sum_{l\in D\cup\{0\}} \Omega_l(u',v')$ as in (2.4). Then by the definition and Minkowski's inequality,

$$\begin{aligned}\mu_{\gamma,h}(f)(x,y,z) &\le \sum_{l\in D\cup\{0\}} \left(\int_{\mathbb{R}}\int_{\mathbb{R}} |F^{\gamma}_{l;s,t}(x,y,z)|^2 2^{-2s}2^{-2t}\mathrm{d}s\mathrm{d}t\right)^{1/2} \\ &:= \sum_{l\in D\cup\{0\}} \mu^{l}_{\gamma,h}(f)(x,y,z), \qquad (5.1)\end{aligned}$$

where

$$F^{\gamma}_{l;j,k}(x,y,z) = \iint_{|u|\le 2^s, |v|\le 2^t} \frac{h(|u|,|v|)\Omega_l(u',v')}{|u|^{m-1}|v|^{n-1}} f(x-u, y-v, z-\gamma(|u|,|v|))\mathrm{d}u\mathrm{d}v.$$

For $l \in D\cup\{0\}$, $s,t\in\mathbb{R}$, we denote

$$B^{l}_{s,t} = \{(u,v)\in \mathbb{R}^m\times\mathbb{R}^n:\ |u|<2^{s(l+1)}, |v|<2^{t(l+1)}\}.$$

For each Ω_l, we define the measures $\tau^{l}_{s,t}$ by setting

$$\widehat{\tau^{l}_{s,t}}(\xi,\eta,\zeta) = \frac{1}{2^{(s+t)(l+1)}}\int_{B^{l}_{s,t}} \frac{h(|u|,|v|)\Omega_l(u',v')}{|u|^{m-1}|v|^{n-1}} \mathrm{e}^{-\mathrm{i}[u\cdot\xi+v\cdot\eta+\gamma(|u|,|v|)\zeta]}\mathrm{d}u\mathrm{d}v.$$

It is easy to see that

$$\tau^{l}_{s,t} * f(x,y,z) = \frac{1}{2^{(s+t)(l+1)}}\int_{B^{l}_{s,t}} \frac{h(|u|,|v|)\Omega_l(u',v')}{|u|^{m-1}|v|^{n-1}} f(x-u,y-v,z-\gamma(|u|,|v|))\mathrm{d}u\mathrm{d}v.$$

Consequently, a simple calculation shows that

$$\mu^{l}_{\gamma,h}(f)(x,y,z) = (l+1)\left(\int_{\mathbb{R}}\int_{\mathbb{R}} |\tau^{l}_{s,t} * f(x,y,z)|^2\mathrm{d}s\mathrm{d}t\right)^{1/2} := (l+1)\mathcal{I}_l(f)(x,y,z). \qquad (5.2)$$

Therefore, by (5.1)–(5.2) and Minkowski's inequality, we have

$$\|\mu_{\gamma,h}(f)\|_p \le \sum_{l\in D\cup\{0\}} \|\mu^{l}_{\gamma,h}(f)\|_p = \sum_{l\in D\cup\{0\}} (l+1)\|\mathcal{I}_l(f)\|_p \qquad (5.3)$$

from which and (2.5), to prove Theorem 5, it suffices to show that

$$\|\mathcal{I}_l(f)\|_p \le CA_l\|f\|_p.$$

By (3.2) and Minkowski's inequality, we have

$$\begin{aligned}\mathcal{I}_l(f)(x,y,z) &= \left(\int_{\mathbb{R}}\int_{\mathbb{R}} |\tau^{l}_{s,t} * f(x,y,z)|^2\mathrm{d}s\mathrm{d}t\right)^{1/2} \\ &= \left(\sum_{j,k\in\mathbb{Z}}\int_0^1\int_0^1 |\tau^{l}_{s+j,t+k} * \sum_{j',k'\in\mathbb{Z}} S^{(l)}_{j+j',k+k'}(S^{(l)}_{j+j',k+k'}f)(x,y,z)|^2\mathrm{d}s\mathrm{d}t\right)^{1/2} \\ &\le \sum_{j',k'\in\mathbb{Z}}\left(\sum_{j,k\in\mathbb{Z}}\int_0^1\int_0^1 |\tau^{l}_{s+j,t+k} * S^{(l)}_{j+j',k+k'}(S^{(l)}_{j+j',k+k'}f)(x,y,z)|^2\mathrm{d}s\mathrm{d}t\right)^{1/2} \\ &:= \sum_{j',k'\in\mathbb{Z}} \mathcal{J}_{l;,j',k'}(f)(x,y,z). \qquad (5.4)\end{aligned}$$

Also, by the arguments similar to those in proving Lemma 2.2 and Lemma 2.3, we can deduce the following results.

Lemma 5.1 Let $h \in \Delta_\nu$ for some $\nu \in (1,2]$. If M_γ is bounded on $L^q(\mathbb{R}^3)$ for all $q > 1$, then for arbitrary functions $\{g_{j,k}\}_{j,k\in\mathbb{Z}}$ on $\mathbb{R}^{m+n+1}$, $l \in D \cup \{0\}$, there exists a positive constant C independent of l, j, k such that

$$\bigg\|\bigg(\sum_{j,k\in\mathbb{Z}}\int_0^1\int_0^1 \left|\tau^l_{s+j,t+k} * g_{j,k}\right|^2 \mathrm{d}s\mathrm{d}t\bigg)^{1/2}\bigg\|_p \le CA_l \bigg\|\bigg(\sum_{j,k\in\mathbb{Z}}|g_{j,k}|^2\bigg)^{1/2}\bigg\|_p \tag{5.5}$$

holds for any p satisfying $|1/p - 1/2| < 1/\nu'$.

Lemma 5.2 Let $\Omega = \sum\limits_{l\in D\cup\{0\}} \Omega_l$ be as in (2.4), $h \in \Delta_\nu$ for $1 < \nu \le 2$. Then for each $l \in D \cup \{0\}$, and $s, t \in \mathbb{R}$, we have

(i) $|\widehat{\tau^l_{s,t}}(\xi,\eta,\zeta)| \le CA_l$;

(ii) $|\widehat{\tau^l_{s,t}}(\xi,\eta,\zeta)| \le CA_l \left|2^{s(l+1)}\xi\right|^{1/(l+1)} \left|2^{t(l+1)}\eta\right|^{1/(l+1)}$;

(iii) $|\widehat{\tau^l_{s,t}}(\xi,\eta,\zeta)| \le CA_l \left|2^{s(l+1)}\xi\right|^{1/4\nu'(l+1)} \left|2^{t(l+1)}\eta\right|^{-1/4\nu'(l+1)}$;

(iv) $|\widehat{\tau^l_{s,t}}(\xi,\eta,\zeta)| \le CA_l \left|2^{s(l+1)}\xi\right|^{-1/4\nu'(l+1)} \left|2^{t(l+1)}\eta\right|^{1/4\nu'(l+1)}$;

(v) $|\widehat{\tau^l_{s,t}}(\xi,\eta,\zeta)| \le CA_l \left|2^{s(l+1)}\xi\right|^{-1/4\nu'(l+1)} \left|2^{t(l+1)}\eta\right|^{-1/4\nu'(l+1)}$.

Now we estimate $\|\mathcal{I}_l(f)\|_p$. Note that for $\nu > 2$, $\triangle_\nu \subset \triangle_2$, we may assume that $1 < \nu \le 2$ and $|1/p - 1/2| < 1/\nu'$.

Applying Plancherel's theorem, we know from (5.4) that

$$\begin{aligned}
\|\mathcal{J}_{l;j',k'}(f)\|_2^2 &= \int_{\mathbb{R}^{m+n+1}} \sum_{j,k\in\mathbb{Z}} \int_0^1\int_0^1 \left|\tau^l_{s+j,t+k} * S^{(l)}_{j+j',k+k'}(S^{(l)}_{j+j',k+k'}f)(x,y,z)\right|^2 \mathrm{d}s\mathrm{d}t\mathrm{d}x\mathrm{d}y\mathrm{d}z \\
&= \sum_{j,k\in\mathbb{Z}} \int_0^1\int_0^1\int_{\mathbb{R}^{m+n+1}} \left|\widehat{\tau^l_{s+j,t+k}}(\xi,\eta,\zeta)\phi^{(l)}_{j+j'}(\xi)^2\psi^{(l)}_{k+k'}(\eta)^2\widehat{f}(\xi,\eta,\zeta)\right|^2 \mathrm{d}\xi\mathrm{d}\eta\mathrm{d}\zeta\mathrm{d}s\mathrm{d}t \\
&\le C\sum_{j,k\in\mathbb{Z}} \int_0^1\int_0^1\int_{E^l_{j,k;j',k'}} |\widehat{f}(\xi,\eta,\zeta)|^2 \left|\widehat{\tau^l_{j,k}}(\xi,\eta,\zeta)\right|^2 \mathrm{d}\xi\mathrm{d}\eta\mathrm{d}\zeta\mathrm{d}s\mathrm{d}t,
\end{aligned}$$

where $E^l_{j,k;j',k'}$ is as in (3.6). Also, by invoking Lemma 5.2, it is easy to see that for $(\xi,\eta,\zeta) \in E^l_{j,k;j',k'}$ and some $\theta \in (0,1)$,

$$\begin{aligned}
\left|\widehat{\tau^l_{s+j,t+k}}(\xi,\eta,\zeta)\right| &\le CA_l \min\Big\{1, \left|2^{(s+j)(l+1)}\xi\right|^{1/(l+1)} \left|2^{(t+k)(l+1)}\eta\right|^{1/(l+1)}, \\
&\qquad \left|2^{(s+j)(l+1)}\xi\right|^{\theta/(l+1)} \left|2^{(t+k)(l+1)}\eta\right|^{-\theta/(l+1)}, \\
&\qquad \left|2^{(s+j)(l+1)}\xi\right|^{-\theta/(l+1)} \left|2^{(t+k)(l+1)}\eta\right|^{\theta/(l+1)}, \\
&\qquad \left|2^{(s+j)(l+1)}\xi\right|^{-\theta/(l+1)} \left|2^{(t+k)(l+1)}\eta\right|^{-\theta/(l+1)}\Big\} \\
&\le CA_l \min\Big\{1, 2^{s+t-j'-k'}, 2^{\theta(s-j'-t+k')}, 2^{-\theta(s-j'-t+k')}, 2^{-\theta(s-j'+t-k')}\Big\}.
\end{aligned}$$

So,

$$\|\mathcal{J}_{l;j',k'}(f)\|_2^2 \le C\sum_{j,k\in\mathbb{Z}} \int_{E^l_{j,k;j',k'}} \int_0^1\int_0^1 |\widehat{f}(\xi,\eta,\zeta)|^2 A_l^2 \min\Big\{1, 2^{s+t-j'-k'},$$

$$2^{\theta(s-j'-t+k')}, 2^{-\theta(s-j'-t+k')}, 2^{-\theta(s-j'+t-k')}\Big\}^2 \mathrm{d}s\mathrm{d}t\mathrm{d}\xi\mathrm{d}\eta\mathrm{d}\zeta$$

$$\leq CA_l^2 \min\Big\{1, 2^{-j'-k'}, 2^{-\theta j'+\theta k'}, 2^{\theta j'-\theta k'}, 2^{\theta(j'+k')}\Big\}^2 \|f\|_2^2. \tag{5.6}$$

Consequently,

$$\begin{aligned}\|\mathcal{I}_l(f)\|_2 &\leq \sum_{j',k'} \|\mathcal{J}_{l;j',k'}(f)\|_2 \qquad (5.7)\\ &\leq CA_l \sum_{j',k'\in\mathbb{Z}} \min\Big\{1, 2^{-j'-k'}, 2^{-\theta j'+\theta k'}, 2^{\theta j'-\theta k}, 2^{\theta(j'+k')}\Big\} \|f\|_2\\ &\leq CA_l\|f\|_2\Big\{\sum_{j',k'\geq 0} 2^{-j'-k'} + \sum_{j'\geq 0,k'<0} 2^{-\theta j'+\theta k'}\\ &\quad + \sum_{j'<0,k'\geq 0} 2^{\theta j'-\theta k'} + \sum_{j'<0,k'<0} 2^{\theta(j'+k')}\Big\}\\ &\leq CA_l\|f\|_2, \qquad (5.8)\end{aligned}$$

which together with (5.3) and (2.5) implies (i) of Theorem 5.

On the other hand, by (3.3) and Lemma 5.1, we have

$$\|\mathcal{J}_{l;j',k'}(f)\|_p \leq CA_l\|f\|_p, \qquad |1/p-1/2| < 1/\nu'. \tag{5.8}$$

This together with (5.6) and the interpolation theorem implies that

$$\|\mathcal{J}_{l;j',k'}(f)\|_p \leq CA_l \min\{2^{-\delta(j'+k')}, 2^{-\delta\theta j'+\delta\theta k'}, 2^{\delta\theta j'-\delta\theta k'}, 2^{\delta\theta(j'+k')}\}\|f\|_p \tag{5.9}$$

with $\delta \in (0,1]$ and $|1/p-1/2| < 1/\nu'$. Thus

$$\|\mathcal{I}_l(f)\|_p \leq \sum_{j',k'\in\mathbb{Z}} \|\mathcal{J}_{l;j',k'}(f)\|_p \leq CA_l\|f\|_p, \qquad |1/p-1/2| < 1/\nu',$$

which together with (5.3) and (2.5) completes the proof of (ii). Theorem 5 is proved.

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