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Beurling-Ahlfors 的一个积分之估值的改进及其应用[1]

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摘要　得到 Beurling-Ahlfors 的一个积分不等式, 以及 ρ-拟对称函数的 Beurling-Ahlfors 的 k-拟共形扩张的估值. 这些结果改进了已有的一些相关定理.

关键词　拟共形映照, 拟共形扩张, ρ-拟对称函数

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设 $\mu(x)$ 是一个 ρ-拟对称函数, 也即 $\mu(x)$ 是实轴到自身的, 保持定向的一个同胚, 满足条件 $\frac{1}{\rho} \leq \frac{\mu(x+t)-\mu(x)}{\mu(x)-\mu(x-t)} \leq \rho$ 对于一切 x 及 $t \neq 0$. Beruling 与 Ahlfors[1] 证明对于这样的函数 $\mu(x)$, 有估计式

$$\frac{1}{\rho+1} \leq \int_0^1 \mu(x)\,dx \leq \frac{\rho}{\rho+1} \tag{1}$$

刘勇和李伟[2] 把估计式(1) 改进为

$$\frac{1}{\rho+1} + \frac{1}{15}\frac{\rho^2-\rho}{(\rho+1)^4} \leq \int_0^1 \mu(x)\,dx \leq \frac{\rho}{\rho+1} - \frac{1}{15}\frac{\rho^2-\rho}{(\rho+1)^4} \tag{2}$$

并利用这一结果于 ρ-拟对称函数的 Beruling-Ahlfors 的 k 拟共形扩张, 得到

$$k \leq 2\rho - 2 + \frac{58}{15\rho}, \ \rho \text{ 充分大} \tag{3}$$

本文把式(2) 中的 $\frac{1}{15}$ 代以 $\frac{5}{48}$, 把式(3) 中的 $\frac{58}{15}$ 代以 $\frac{91}{24}$, 确切地, 我们的结果是

定理 1　设 $\mu(x)$ 是一个满足规范化条件的 ρ-拟对称函数, 也即 $\mu(x)$ 满足式(1) 和 $\mu(0) = 0$, $\mu(1) = 1$, 则有 $\frac{1}{\rho+1} + \frac{5}{48}\frac{\rho^2-\rho}{(\rho+1)^4} \leq \int_0^1 \mu(x)\,dx \leq \frac{\rho}{\rho+1} - \frac{5}{48}\frac{\rho^2-\rho}{(\rho+1)^4}$.

定理 2　任意一个 ρ-拟对称函数 $\mu(x)$, 都存在着上半平面到自身的 k 拟共形映照, 以 $\mu(x)$ 为其边界值, 且有估计式 $k \leq 2\rho - 2 + \frac{91}{24\rho}$, ρ 充分大.

1　两个引理

引理 1　设 C_ρ 是规范化的 ρ-拟对称函数全体所成的类, 则对 $\mu \in C_\rho$ 有

$$\int_0^1 \mu(x)\,dx \leq \frac{1}{2^k}\left[\sum_{i=1}^{2^k-1} \mu\left(\frac{i}{2^k}\right) + \frac{\rho}{\rho+1}\right]$$

[1] 本文 1998-03-27 收到;　国家自然科学基金和福建省自然基金资助项目

证　令 $A_\rho = \sup_{\mu\in C_\rho}\{\int_0^1 \mu(x)\,dx\}$, 由式(1) 有 $A_\rho \le \frac{\rho}{1+\rho}$. 又 $\frac{\mu[(b-a)t+a]-\mu(a)}{\mu(b)-\mu(a)}$ 也是规范化的 ρ-拟对称函数, 因而有 $\int_0^1 \frac{\mu[(b-a)t+a]-\mu(a)}{\mu(b)-\mu(a)}\,dt \le A_\rho$, 由此易得 $\int_a^b \mu(x)\,dx \le [A_\rho\mu(b)+(1-A_\rho)\mu(a)](b-a)$. 从而 $\int_0^1 \mu(x)\,dx = \sum_{i=0}^{2^k-1}\int_{\frac{i}{2^k}}^{\frac{i+1}{2^k}}\mu(x)\,dx \le \frac{1}{2^k}\sum_{i=0}^{2^k-1}[A_\rho\mu(\frac{i+1}{2^k}) + (1-A_\rho)\mu(\frac{i}{2^k})] = \frac{1}{2^k}[\sum_{i=1}^{2^k-1}\mu(\frac{i}{2^k}) + A_\rho] \le \frac{1}{2^k}[\sum_{i=1}^{2^k-1}\mu(\frac{i}{2^k}) + \frac{\rho}{\rho+1}]$.

引理 2　$\sum_{i=0}^{2^k-1}\mu(\frac{i}{2^k}) \le \sum_{i=0}^{2^k-1}G(\frac{i}{2^k}) = (2^k-1)\frac{\rho}{\rho+1}$.

为了引理 2 的证明, 我们把区间[0, 1] 分为 2^k 等分, 并置 $G(0) = 0, G(1) = 1$, 由 ρ-拟对称条件, 对于任意的 $a, b, a < b$, 都有

$$\mu(\frac{a+b}{2}) \le \frac{1}{1+\rho}\mu(a) + \frac{\rho}{1+\rho}\mu(b) \tag{4}$$

容易得到　$\mu(\frac{1}{2}) \le \frac{\rho}{1+\rho} = G(\frac{1}{2})$, $\mu(\frac{1}{4}) \le \frac{1}{1+\rho}\mu(0) + \frac{\rho}{1+\rho}\mu(\frac{2}{4}) \le (\frac{\rho}{1+\rho})^2 = G(\frac{1}{4})$, $\mu(\frac{3}{4}) \le \frac{\rho^2+2\rho}{(1+\rho)^2} = G(\frac{3}{4})$, $\mu(\frac{1}{8}) \le \frac{\rho^3}{(1+\rho)^3} = G(\frac{1}{8})$, $\mu(\frac{7}{8}) \le \frac{\rho^3+3\rho^2+3\rho}{(1+\rho)^3} = G(\frac{7}{8})$. 但 $\mu(\frac{3}{8})$ 有下面两种估计法:　$\mu(\frac{3}{8}) \le \frac{1}{1+\rho}\mu(0) + \frac{\rho}{1+\rho}\mu(\frac{6}{8}) \le \frac{1}{1+\rho}G(0) + \frac{\rho}{1+\rho}G(\frac{6}{8})\ \frac{\rho^3+2\rho^2}{(1+\rho)^3}$,　和　$\mu(\frac{3}{8}) \le \frac{1}{1+\rho}\mu(\frac{2}{8}) + \frac{\rho}{1+\rho}\mu(\frac{4}{8}) \le \frac{1}{1+\rho}G(\frac{2}{8}) + \frac{\rho}{1+\rho}G(\frac{4}{8}) = \frac{\rho^3+2\rho^2}{(1+\rho)^3}$, 这两种估计法得到的上界均为 $\frac{\rho^3+2\rho^2}{(1+\rho)^3}$, 记作 $G(\frac{3}{8})$. 同样, $\mu(\frac{5}{8})$ 也有两种估计法, 两种估计法得到的上界均为 $\frac{\rho^3+3\rho^2+\rho}{(1+\rho)^3}$, 记作 $G(\frac{5}{8})$.

一般地, 假定 $\mu(\frac{i}{2^{l-1}})$ ($l \le k, i = \frac{a+b}{2}$ 为小于 2^{l-1} 的数, a, b 为整数, $a < b$) 的按式(4) 表出的各种不同的估计法得到的上界均为 $G(\frac{i}{2^{l-1}})$, 下面我们证明 $\mu(\frac{i}{2^k})$ ($i = \frac{a+b}{2}$ 为小于 2^k 的数, a, b 为整数, $a < b$) 也有一个统一的上界, 记作 $G(\frac{i}{2^k})$. 事实上, 设 $i = \frac{a+b}{2}$ 为小于 2^{k-1} 的数, a, b 为偶数 $a < b$,　$\mu(\frac{i}{2^{k-1}}) \le \frac{1}{1+\rho}\mu(\frac{a}{2^{k-1}}) + \frac{\rho}{1+\rho}\mu(\frac{b}{2^{k-1}}) \le \frac{1}{1+\rho}G(\frac{a}{2^{k-1}}) + \frac{\rho}{1+\rho}G(\frac{b}{2^{k-1}}) = G(\frac{i}{2^{k-1}})$. 那么 $\mu(\frac{i}{2^k}) \le \frac{1}{1+\rho}\mu(\frac{a}{2^k}) + \frac{\rho}{1+\rho}\mu(\frac{b}{2^k}) \le \frac{\rho}{1+\rho}[\frac{1}{1+\rho}\mu(\frac{a}{2^{k-1}}) + \frac{\rho}{1+\rho}\mu(\frac{b}{2^{k-1}})] \le \frac{\rho}{1+\rho}[\frac{1}{1+\rho}G(\frac{a}{2^{k-1}}) + \frac{\rho}{1+\rho}G(\frac{b}{2^{k-1}})] = \frac{\rho}{1+\rho}G(\frac{i}{2^{k-1}}) = G(\frac{i}{2^{k-1}})$. 至于 $\mu(\frac{2^{k-1}+i}{2^k})$, 则利用式(4) 有 $\mu(\frac{2^{k-1}+i}{2^k}) \le \frac{\rho}{1+\rho} + \frac{1}{1+\rho}\mu(\frac{i}{2^{k-1}})$, $i = 1, 2, \cdots, 2^{k-1}$ 从而 $\mu(\frac{2^{k-1}+i}{2^k}) \le \frac{1}{1+\rho}\mu(\frac{2^{k-1}+a}{2^k}) + \frac{\rho}{1+\rho}\mu(\frac{2^{k-1}+b}{2^k}) \le \frac{1}{1+\rho}[\frac{\rho}{1+\rho} + \frac{1}{1+\rho}\mu(\frac{a}{2^{k-1}})] + \frac{\rho}{1+\rho}[\frac{\rho}{1+\rho} + \frac{1}{1+\rho}\mu(\frac{b}{2^{k-1}})] \le \frac{\rho}{1+\rho} + \frac{1}{1+\rho}G(\frac{i}{2^{k-1}}) = G(\frac{2^{k-1}+i}{2^k})$.

引理 2 的证明　前一个不等式显然, 只证明后一个不等式, 由于 $G(\frac{1}{2^k})=\frac{\rho}{1+\rho}G(\frac{1}{2^{k-1}})$, $G(\frac{3}{2^k})=\frac{\rho}{1+\rho}G(\frac{3}{2^{k-1}})$, $\cdots$, $G(\frac{2^{k-1}-1}{2^k})=\frac{\rho}{1+\rho}G(\frac{2^{k-1}-1}{2^k})$, $G(\frac{2^{k-1}+1}{2^k})=\frac{\rho}{1+\rho}+\frac{1}{1+\rho}G(\frac{1}{2^{k-1}})$, $G(\frac{2^{k-1}+3}{2^k})=\frac{\rho}{1+\rho}+\frac{1}{1+\rho}G(\frac{3}{2^{k-1}})$, $\cdots$, $G(\frac{2^{k-1}+2^{k-1}-1}{2^k})=\frac{\rho}{1+\rho}+\frac{1}{1+\rho}G(\frac{2^{k-1}-1}{2^{k-1}})$ 记 $S_1^{(k)}=G(\frac{1}{2^k})+G(\frac{3}{2^k})+\cdots+G(\frac{2^k-1}{2^k})$ 和 $S_2^{(k)}=G(\frac{2}{2^k})+G(\frac{4}{2^k})+\cdots+G(\frac{2^k-2}{2^k})$, 显然有 $S_2^{(k)}=G(\frac{1}{2^{k-1}})+G(\frac{2}{2^{k-1}})+\cdots+G(\frac{2^{k-1}-2}{2^{k-1}})+G(\frac{2^{k-1}-1}{2^{k-1}})=\{G(\frac{1}{2^{k-1}})+G(\frac{3}{2^{k-1}})+\cdots+G(\frac{2^{k-1}-1}{2^{k-1}})\}+\{G(\frac{2}{2^{k-1}})+G(\frac{4}{2^{k-1}})+\cdots+G(\frac{2^{k-1}-2}{2^{k-1}})\}=S_1^{(k-1)}+S_2^{(k-1)}$. $S_1^{(k)}=2^{k-2}\frac{\rho}{1+\rho}+S_1^{(k-1)}$. 反复利用这个递推关系式, 有 $S_1^{(k)}=(2^{k-2}+2^{k-3})\frac{\rho}{1+\rho}+S_1^{(k-2)}=\cdots=(2^{k-2}+2^{k-3}+\cdots+1)\frac{\rho}{1+\rho}+S_1^{(1)}=(2^{k-1}-1)\frac{\rho}{\rho+1}+G(\frac{1}{2})=2^{k-1}\frac{\rho}{1+\rho}$. 把上式代入 $S_1^{(k)}=2^{k-2}\frac{\rho}{1+\rho}+S_1^{(k-1)}$, 即得 $S_2^{(k)}=2^{k-2}\frac{\rho}{\rho+1}+S_2^{(k-1)}=\cdots=(2^{k-2}+2^{k-3}+\cdots+1)\frac{\rho}{\rho+1}+S_2^{(1)}=(2^{k-1}-1)\frac{\rho}{\rho+1}+0=(2^{k-1}-1)\frac{\rho}{\rho+1}$. 从而 $\sum_{i=1}^{2^k-1}G(\frac{i}{2^k})=S_1^{(k)}+S_2^{(k)}=(2^k-1)\frac{\rho}{\rho+1}$.

2　定理 1 和定理 2 的证明

$x=\frac{7}{16},\frac{8}{16},\frac{9}{16}$ 是比较特殊的点. 在三点, $\mu(\frac{7}{16})+\mu(\frac{8}{16})+\mu(\frac{9}{16})$ 的最小上界的估计可以不用 $G(\frac{7}{16})+G(\frac{8}{16})+G(\frac{9}{16})$, 而有稍优的估计式 $\frac{(1+\rho)G(\frac{8}{16})-G(\frac{9}{16})}{\rho}+G(\frac{8}{16})+G(\frac{9}{16})$. 实际上[2], $G(\frac{7}{16})-\frac{(1+\rho)G(\frac{8}{16})-G(\frac{9}{16})}{\rho}=\frac{\rho^2-\rho}{(1+\rho)^4}$, 亦即 $[G(\frac{7}{16})+G(\frac{8}{16})+G(\frac{9}{16})]-[\mu(\frac{7}{16})+\mu(\frac{8}{16})+\mu(\frac{9}{16})]\geq[G(\frac{7}{16})+G(\frac{8}{16})+G(\frac{9}{16})]-\left[\frac{(1+\rho)G(\frac{8}{16})-G(\frac{9}{16})}{\rho}+G(\frac{8}{16})+G(\frac{9}{16})\right]=(\rho^2-\rho)/(1+\rho)^4$.

为方便计, 我们把不等式的右端简记为 $S(\frac{7,8,9}{16})$. 我们称使得 $\sum_i\mu(x_i)$ 的最小上界小于 $\sum_i G(x_i)$ 的点组 $\{x_i\}$ 为特殊点组. 例如, 上面已经看到, 在 $\{\frac{i}{2^4}, i=1,2,\cdots,15\}$ 中, $\{x_1=\frac{7}{16}, x_2=\frac{8}{16}, x_3=\frac{9}{16}\}$ 就是一特殊点组.

下面来看 $\{\frac{i}{2^5}, i=1,2,\cdots,31\}$ 中的特殊点组. 首先, $\{\frac{7}{32},\frac{8}{32},\frac{9}{32}\}$ 是一特殊点组, 这是因为 $G(\frac{7}{32})=\frac{\rho}{1+\rho}G(\frac{7}{16})$, $G(\frac{8}{32})=\frac{\rho}{1+\rho}G(\frac{8}{16})$, $G(\frac{9}{32})=\frac{\rho}{1+\rho}G(\frac{9}{16})$, $\mu(\frac{7}{32})\leq\frac{\rho}{1+\rho}\mu(\frac{7}{16})$,

$\mu(\frac{8}{32}) \le \frac{\rho}{1+\rho}\mu(\frac{8}{16}), \mu(\frac{9}{32}) \le \frac{\rho}{1+\rho}\mu(\frac{9}{16})$. 故 $S(\frac{7,8,9}{16}) = G(\frac{7}{32}) + G(\frac{8}{32}) + G(\frac{9}{32}) - [\mu(\frac{7}{32}) + \mu(\frac{8}{32}) + \mu(\frac{9}{32})] \ge \frac{\rho}{1+\rho}[G(\frac{7}{16}) + G(\frac{8}{16}) + G(\frac{9}{16})] - \frac{\rho}{1+\rho}[\mu(\frac{7}{16}) + \mu(\frac{8}{16}) + \mu(\frac{9}{16})] = \frac{\rho}{1+\rho}S(\frac{7,8,9}{16})$, 其次还可以看出$\{\frac{23}{32}, \frac{24}{32}, \frac{25}{32}\}$ 也是一特殊点组. 事实上, $G(\frac{23}{32}) = \frac{\rho}{1+\rho} + \frac{1}{1+\rho}G(\frac{7}{16}), G(\frac{24}{32}) = \frac{\rho}{1+\rho} + \frac{1}{1+\rho}G(\frac{8}{16}), G(\frac{25}{32}) = \frac{\rho}{1+\rho} + \frac{1}{1+\rho}G(\frac{9}{16})$ 而 $\mu(\frac{23}{32}) \le \frac{\rho}{1+\rho} + \frac{1}{1+\rho}\mu(\frac{7}{16}), \mu(\frac{24}{32}) \le \frac{\rho}{1+\rho} + \frac{1}{1+\rho}\mu(\frac{8}{16}), \mu(\frac{25}{32}) \le \frac{\rho}{1+\rho} + \frac{1}{1+\rho}\mu(\frac{9}{16})$. 所以 $S(\frac{23,24,25}{32}) = G(\frac{23}{32}) + G(\frac{24}{32}) + G(\frac{25}{32}) - [\mu(\frac{23}{32}) + \mu(\frac{24}{32}) + \mu(\frac{25}{32})] \ge \frac{1}{1+\rho}[G(\frac{7}{16}) + G(\frac{8}{16}) + G(\frac{9}{16})] - \frac{1}{1+\rho}[\mu(\frac{7}{16}) + \mu(\frac{8}{16}) + \mu(\frac{9}{16})] = \frac{1}{16}S(\frac{7,8,9}{16})$. 当然$\{\frac{14}{32}, \frac{16}{32}, \frac{18}{32}\}$ 是一特殊点组, 由此可得另一特殊点组$\{\frac{13}{32}, \frac{15}{32}, \frac{17}{32}, \frac{19}{32}\}$. 事实上 $\mu(\frac{13}{32}) \le \frac{1}{1+\rho}G(\frac{12}{32}) + \frac{\rho}{1+\rho}\mu(\frac{14}{32}) \le \frac{1}{1+\rho}G(\frac{12}{32}) + \frac{\rho}{1+\rho}G(\frac{14}{32}) = G(\frac{13}{32}), \mu(\frac{15}{32}) \le \frac{1}{1+\rho}\mu(\frac{14}{32}) + \frac{\rho}{1+\rho}\mu(\frac{16}{32}) \le \frac{1}{1+\rho}G(\frac{14}{32}) + \frac{\rho}{1+\rho}G(\frac{16}{32}) = G(\frac{15}{32})$. $\mu(\frac{17}{32}) \le \frac{1}{1+\rho}\mu(\frac{16}{32}) + \frac{\rho}{1+\rho}\mu(\frac{18}{32}) \le \frac{1}{1+\rho}G(\frac{16}{32}) + \frac{\rho}{1+\rho}G(\frac{18}{32}) = G(\frac{17}{32}), \mu(\frac{19}{32}) \le \frac{1}{1+\rho}\mu(\frac{18}{32}) + \frac{\rho}{1+\rho}G(\frac{20}{32}) \le \frac{1}{1+\rho}G(\frac{18}{32}) + \frac{\rho}{1+\rho}\mu(\frac{20}{32}) = G(\frac{19}{32})$. 上面四式相加, 有 $\mu(\frac{13}{32}) + \mu(\frac{15}{32}) + \mu(\frac{17}{32}) + \mu(\frac{19}{32}) \le \frac{1}{1+\rho}G(\frac{12}{32}) + \mu(\frac{14}{32}) + \mu(\frac{16}{32}) + \mu(\frac{18}{32}) + \frac{\rho}{1+\rho}G(\frac{20}{32}) = \frac{1}{1+\rho}G(\frac{12}{32}) + G(\frac{14}{32}) + G(\frac{16}{32}) + G(\frac{18}{32}) + \frac{\rho}{1+\rho}G(\frac{20}{32}) - [G(\frac{14}{32}) + G(\frac{16}{32}) + G(\frac{18}{32}) - \mu(\frac{14}{32}) - \mu(\frac{16}{32}) - \mu(\frac{18}{32})] = G(\frac{13}{32}) + G(\frac{15}{32}) + G(\frac{17}{32}) + G(\frac{19}{32}) - [G(\frac{14}{32}) + G(\frac{16}{32}) + G(\frac{18}{32}) - \mu(\frac{14}{32}) - \mu(\frac{16}{32}) - \mu(\frac{18}{32})]$. 故 $S(\frac{13,15,17,19}{32}) = G(\frac{13}{32}) + G(\frac{15}{32}) + G(\frac{17}{32}) + G(\frac{19}{32}) - [\mu(\frac{13}{32}) + \mu(\frac{15}{32}) + \mu(\frac{17}{32}) + \mu(\frac{19}{32})] \ge [G(\frac{14}{32}) + G(\frac{16}{32}) + G(\frac{18}{32})] - \mu(\frac{14}{32}) - \mu(\frac{16}{32}) - \mu(\frac{18}{32}) = S(\frac{14,16,18}{32}) = S(\frac{7,8,9}{16})$. 如上推出特殊点组的过程可以简明地用图格表出("=" 号表示"等价", "⇒" 号表示"推出"):

$$\text{I}\quad \left\langle\frac{7,8,9}{16}\right\rangle - \frac{14,16,18}{32} = \left\langle\frac{13,15,17,19}{32}\right\rangle;\quad \text{II}\begin{cases}\left\langle\frac{7,8,9}{32}\right\rangle \\ \left\langle\frac{23,24,25}{32}\right\rangle \text{即} \frac{16+7, 16+8, 16+9}{32}\end{cases}$$

对于$\{\frac{i}{2^6}, i = 1, 2, \cdots, 63\}$, 我们仍用上图格找出特殊点组:

$$\text{I}\quad \left\langle\frac{7,8,9}{16}\right\rangle \left\{\begin{array}{l}\frac{28,32,36}{64} \\ \frac{14,16,18}{32} \Rightarrow \left\langle\frac{13,15,17,19}{32}\right\rangle = \frac{26,30,34,38}{64}\end{array}\right\} \Rightarrow \left\langle\frac{25,27,29,31,33,35,37,39}{64}\right\rangle$$

$$\text{II}\begin{cases}\left\langle\frac{7,8,9}{32}\right\rangle=\frac{14,16,18}{64}\Rightarrow\left\langle\frac{13,15,17,19}{64}\right\rangle\\ \left\langle\frac{23,24,25}{32}\right\rangle=\frac{46,48,50}{64}\Rightarrow\left\langle\frac{45,47,49,51}{64}\right\rangle,\end{cases}$$

$$\text{III}\begin{cases}\left\langle\frac{7,8,9}{32}\right\rangle\\ \left\langle\frac{55,56,57}{64}\right\rangle(\text{即}\frac{32+23,32+24,32+25}{64})\end{cases}$$

容易知道: $S(\frac{25,27,29,31,33,35,37,39}{64})\geq S(\frac{26,28,30,32,34,36,38}{64})=S(\frac{26,30,34,38}{64})+S(\frac{28,32,36}{64})=S(\frac{13,15,17,19}{32})+S(\frac{7,8,9}{16})\geq S(\frac{14,16,18}{32})+S(\frac{7,8,9}{16})=2S(\frac{7,8,9}{16})$.

又 $S(\frac{55,56,57}{64})\geq\frac{1}{1+\rho}S(\frac{23,24,25}{32})\geq(\frac{1}{1+\rho})^2S(\frac{7,8,9}{16})$. 故由全部特殊点组可以得到以下改进值. $4S(\frac{7,8,9}{16})+2S(\frac{7,8,9}{32})+S(\frac{7,8,9}{64})+2S(\frac{23,24,25}{32})+S(\frac{55,56,57}{64})\geq 4S(\frac{7,8,9}{16})+2\{\frac{\rho}{1+\rho}S(\frac{7,8,9}{16})+\frac{1}{1+\rho}S(\frac{7,8,9}{16})\}+\{(\frac{\rho}{1+\rho})^2S(\frac{7,8,9}{16})+(\frac{1}{1+\rho})^2S(\frac{7,8,9}{16})\}=\{2^2+2(\frac{\rho}{1+\rho}+\frac{1}{1+\rho})+[(\frac{\rho}{1+\rho})^2+(\frac{1}{1+\rho})^2]\}S(\frac{7,8,9}{16})$.

事实上, 一般地在$\{\frac{i}{2^k},i=1,2,\cdots,2^k-1\}$情形下, 由$\frac{7,8,9}{16}$推出的特殊点组可以导出$2^{k-4}S(\frac{7,8,9}{16})$的改进值. 同样道理, 在$\{\frac{i}{2^k},i=1,2,\cdots,2^k-1\}$情形下, 由$\frac{7,8,9}{32}$推出的特殊点组可以导出$2^{k-5}S(\frac{7,8,9}{32})$的改进值, 由$\frac{23,24,25}{32}$推出的特殊点组可以导出$2^{k-5}S(\frac{23,24,25}{32})$的改进值, 以此类推. 故对于一般的$\{\frac{i}{2^k},i=1,2,\cdots,2^k-1\}$, 我们从所有这些由图格找出的特殊点组中得到的改进值为$S^{k-4}S(\frac{7,8,9}{16})+S^{k-5}S(\frac{7,8,9}{32})+2^{k-6}S(\frac{7,8,9}{64})+\cdots+2S(\frac{7,8,9}{2^{k-1}})+S(\frac{7,8,9}{2^k})+2^{k-5}S(\frac{23,24,25}{32})+2^{k-6}S(\frac{55,56,57}{64})+\cdots+2S(\frac{2^{k-1}-9,2^{k-1}-8,2^{k-1}-7}{2^{k-1}})+S(\frac{2^k-9,2^k-8,2^k-7}{2^k})\geq\{2^{k-4}+2^{k-5}(\frac{\rho}{1+\rho}+\frac{1}{1+\rho})+2^{k-6}[(\frac{\rho}{1+\rho})^2+(\frac{1}{1+\rho})^2+\cdots+2[(\frac{\rho}{1+\rho})^{k-5}+(\frac{1}{1+\rho})^{k-5}]+[(\frac{\rho}{1+\rho})^{k-4}+(\frac{1}{1+\rho})^{k-4}]\}S(\frac{7,8,9}{16})=2^{k-4}\{1+\frac{\rho}{2+\rho}[1-(\frac{\rho}{2(1+\rho)})^{k-4}]+\frac{1}{1+2\rho}[1-(\frac{\rho}{2(1+\rho)})^{k-4}]\}S(\frac{7,8,9}{16})$.

从而由引理 1, 可得 $\int_0^1\mu(x)\,dx\leq\frac{1}{2^k}[\sum_{i=1}^{2^k-1}\mu(\frac{i}{2^k})+\frac{\rho}{1+\rho}]\leq\frac{1}{2^k}\{\sum_{i=1}^{2^k-1}G(\frac{i}{2^k})-2^{k-4}[1+\frac{\rho}{2+\rho}[1-(\frac{\rho}{2(1+\rho)})^{k-4}]-\frac{1}{1+2\rho}[1-(\frac{1}{2(1+\rho)})^{k-4}]]S(\frac{7,8,9}{16})+\frac{\rho}{1+\rho}\}=\frac{1}{2^k}\{\frac{(2^k-1)\rho}{1+\rho}-2^{k-4}[1+\frac{\rho}{2+\rho}[1-(\frac{\rho}{2(1+\rho)})^{k-4}]+\frac{1}{1+2\rho}[1-(\frac{1}{2(1+\rho)})^{k-4}]]$

$S(\frac{7,8,9}{16})+\frac{\rho}{1+\rho}\}$. 记 $M_0=\frac{(2^k-1)\rho}{1+\rho}-2^{k-4}\{1+\frac{\rho}{2+\rho}[1-(\frac{\rho}{2(1+\rho)})^{k-4}]+\frac{1}{1+2\rho}[1-(\frac{1}{2(1+\rho)})^{k-4}]\}S(\frac{7,8,9}{16})$, 并令 $\frac{1}{S^k}(M_0+\frac{\rho}{1+\rho})=B_1$. 易见 $B_1<\frac{\rho}{1+\rho}$. 构造序列 $\frac{1}{2^k}(M_0+B_1)=B_2$, $\frac{1}{2^k}(M_0+B_2)=B_3$ ………… $\frac{1}{2^k}(M_0+B_n)=B_{n+1}$ ………… 显然 $B_1,B_2,\cdots,B_n,\cdots$ 都是 $\int\mu(x)\mathrm{d}x$ 的上界, 并且 B_n 是单调下降序列, 因而 B_n 的极限存在. 记 $\lim_{n\to\infty}B_n=B$. 我们得到 $\frac{1}{2^k}(M_0+B)=B$. 解出 $B=\frac{M_0}{2^k-1}$. 由于 $\int\mu(x)\mathrm{d}x\le B_n$, 令 $n\to\infty$ 就有 $\int\mu(x)\mathrm{d}x\le B=\frac{M_0}{2^k-1}=\frac{\rho}{1+\rho}-\frac{2^{k-4}}{2^k-1}\{1+\frac{\rho}{2+\rho}[1-(\frac{\rho}{2(1+\rho)})^{k-4}]+\frac{1}{1+2\rho}[1-(\frac{1}{2(1+\rho)})^{k-4}]\}S(\frac{7,8,9}{16})$. 在上式中令 $k\to\infty$ 得到 $\int\mu(x)\mathrm{d}x\le\frac{\rho}{1+\rho}-\frac{1}{16}(1+\frac{\rho}{2+\rho}+\frac{1}{1+2\rho})S(\frac{7,8,9}{16})$. 容易知道 $1+\frac{\rho}{2+\rho}+\frac{1}{1+2\rho}\ge1+\frac{1}{3}+\frac{1}{3}=\frac{5}{3}$. 故 $\int\mu(x)\mathrm{d}x\le\frac{\rho}{1+\rho}-\frac{5}{48}\frac{\rho^2-\rho}{(\rho+1)^4}$. 由于 $h(t)=1-\mu(1-t)$ 也是规范化的 ρ 拟对函数, 因此由上式有 $\int[1-\mu(1-t)\mathrm{d}t\le\frac{\rho}{1+\rho}-\frac{5}{48}\frac{\rho^2-\rho}{(\rho+1)^4}$. 而 $\int[1-(1-t)\mathrm{d}t=\int\mu(t)\mathrm{d}t$, 所以 $\int\mu(x)\mathrm{d}x\ge\frac{1}{\rho+1}+\frac{1}{16}(1+\frac{\rho}{2+\rho}+\frac{1}{1+2\rho})\frac{\rho^2-\rho}{(\rho+1)^4}\ge\frac{1}{\rho+1}+\frac{5}{48}\frac{\rho^2-\rho}{(\rho+1)^4}$

有了定理 1 的证明, 定理 2 可仿文[2] 中 §3 定理证明.

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The Improvement of the Estimate of Beurling-Ahlfors's Integral and its Application

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Abstract　The estimate of Beurling-Ahlfors´s integral inequality and the Beurling-Ahlfors´s k-quasiconformal dilatation of ρ-quasisymetry function are obtained. These results improve some related theorems.

key words　K-quasiconformal mappings, Quasiconformal dilatation, ρ-quasisymetry function