Optimization Approaches for Inverse Quadratic Eigenvalue Problems

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Introduction

In many engineering application, we often need to solve a linear second-order differential equation (e.g. Finite Element Model)

$$M\ddot{\mathbf{u}}(t) + C\dot{\mathbf{u}}(t) + K\mathbf{u}(t) = \mathbf{0},$$

where M, C, K are n-by-n matrices and $\mathbf{u}(t)$ is an nth-order vector.

The separation of variables $\underline{\mathbf{u}(t) = \mathbf{u}e^{\lambda t}}$ leads to the quadratic eigenvalue problem (QEP)

$$Q(\lambda)\mathbf{u} \equiv (\lambda^2 M + \lambda C + K)\mathbf{u} = \mathbf{0}.$$

The scalar λ and the corresponding nonzero vector $\mathbf{u}$ are called the eigenvalue and eigenvector of the quadratic pencil $Q(\lambda)$.

(See Tisseur'01 for detail)

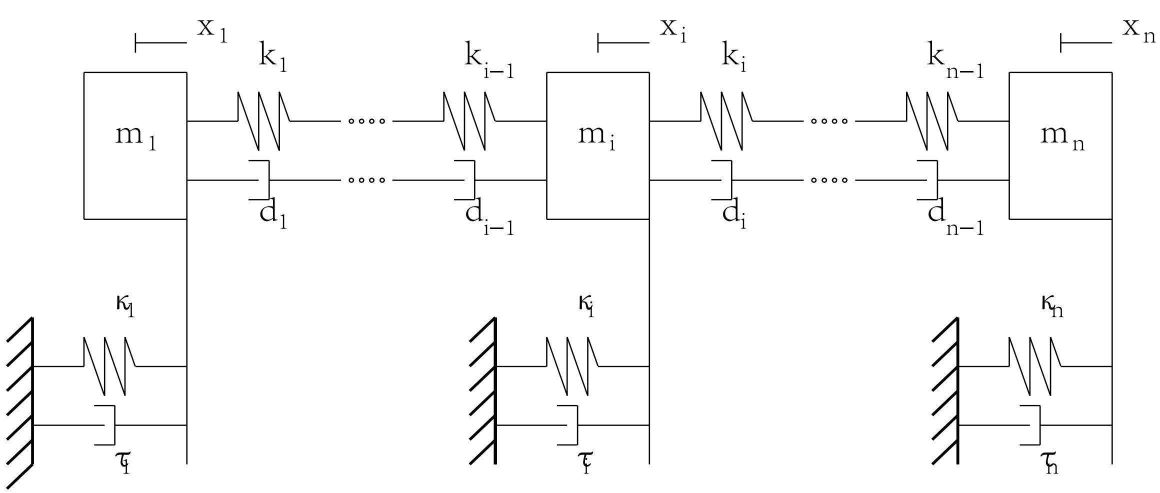
Applications of QEPs:

- Vibrating Analysis of Structural Mechanical and Acoustic Systems
- Electrical Circuit Simulation
- Fluid Mechanics
- Modeling Microelectronic Mechanic Systems
- Linear algebra Problems and Signal Processing

Example: Damped mass-spring system

- The ith mass of weight m_i is connected to the $(i+1)$th mass by a spring and a damper with constants k_i and d_i, respectively.

- The ith mass is also connected to the ground by a spring and a damper with constants κ_i and τ_i, respectively.


x 1
k1
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τ1
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τn

The vibration of this system is governed by the following QEP

$$(\lambda^2 M + \lambda C + K)\mathbf{u} = \mathbf{0},$$

where M, C, K are called the mass, damping, and stiffness matrices which are defined by

$$M = \text{diag}(m_1, \cdots, m_n)$$

$$C = \begin{bmatrix} d_1 + d_2 + \tau_1 & -d_2 & & & \\ -d_2 & d_2 + d_3 + \tau_2 & -d_3 & & \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ & & & -d_{n-1} & \tau_n \end{bmatrix},$$

$$K = \begin{bmatrix} k_1 + k_2 + \kappa_1 & -k_2 & & & \\ -k_2 & k_2 + k_3 + \kappa_2 & -k_3 & & \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ & & & -k_{n-1} & \kappa_n \end{bmatrix}.$$

The dynamics is governed by

- Natural Frequencies $\Longleftrightarrow$ Eigenvalues of the QEP

- Model Shapes $\Longleftrightarrow$ Eigenvectors of the QEP

Disadvantage:

- Only partial frequencies & model shapes accurately predicted
- The predicted frequencies & model shapes often disagree with that of experimentally measured from a realized practical structure

Inverse Quadratic eigenvalue Problems (IQEP):

Reconstructing the quadratic pencil

$$Q(\lambda) \equiv \lambda^2 M + \lambda C + K$$

from the prescribed eigenvalues/eigenvectors.

Applications of IQEP:

- Finite Element Model Updating (Friswell and Mottershead'95)
- Partial Eigenstructure Assignment Problem (Datta'02)

Previous Approaches:

- Optimization procedures (Baruch'78, Berman & Nagy'83, Caesar'86)

- Eigenstructure assignment techniques (Minas & Inman'90, Zimmerman & Widengren'90)

- Eigenvalue embedding methods (Ferng'01, Carvalho'01)

Disadvantage:

- The damping matrix is just proportional or even ignored.

- Exploitable structural properties (e.g., symmetry, definiteness, sparsity and bandedness) of the original model are not preserved.

Main Problems

In our talk, we consider two types of IQEPs:

- Find the real and symmetric mass, damping, and stiffness matrices with the mass and the stiffness matrices positive definite and positive semidefinite, respectively such that they are closest to the given analytical matrices and satisfy the measured eigendata;

- Constructing physical parameters of a damped mass-spring system from measured eigendata.

PART I.

A Dual Optimization Approach
for
Inverse Quadratic Eigenvalue Problems

PART I: Statement of Problem

The general IQEP can be defined as follows:

- Given a measured partial eigenpair $(\Lambda, X) \in \mathbb{R}^{k\times k} \times \mathbb{R}^{n\times k}$ with $k \leq n$ and X full column rank.

$$\Lambda = \mathrm{diag}\{\Lambda_1, \ldots, \Lambda_\mu, \Lambda_{\mu+1}, \ldots, \Lambda_\nu\},$$

$$\Lambda_i = \mathrm{diag}\{\overbrace{\lambda_i^{[2]}, \ldots, \lambda_i^{[2]}}^{s_i}\},\ \lambda_i^{[2]} = \begin{bmatrix} \alpha_i & \beta_i \\ -\beta_i & \alpha_i \end{bmatrix} \in \mathbb{R}^{2\times 2},\ 1 \leq i \leq \mu,$$

$$\Lambda_i = \lambda_i I_{s_i},\ \mu + 1 \leq i \leq \nu,$$

$$\sigma(\Lambda_i) \cap \sigma(\Lambda_j) = \emptyset, \forall\, i \neq j$$

- Find $M, C, K \in \mathcal{S}^n$ with $M \succ 0$ and $K \succeq 0$ such that

$$M X \Lambda^2 + C X \Lambda + K X = 0.$$

Chu, Kuo, and Lin (2004) showed that the general IQEP admits a nontrivial solution, i.e, there exist

$$M \succ 0, C = C^T, K \succeq 0 \text{ satisfying}$$

$$M X \Lambda^2 + C X \Lambda + K X = 0.$$

Optimization Problem: For given $M_a, C_a, K_a \in \mathcal{S}^n$, which are called the estimated analytic mass, damping, and stiffness matrix, the IQEP is

$$\begin{array}{ll} \text{inf} & \frac{c_1}{2}\|M - M_a\|^2 + \frac{c_2}{2}\|C - C_a\|^2 + \frac{1}{2}\|K - K_a\|^2 \\ \text{s.t.} & MX\Lambda^2 + CX\Lambda + KX = 0, \\ & M \succ 0 (M \succeq 0), \quad C = C^T, \quad K \succeq 0. \end{array}$$

where c_1 and c_2 are two positive parameters.

PART I: Our Approach

Let the QR factorization of X be given by

$$X = Q \begin{bmatrix} R \\ 0 \end{bmatrix},$$

where $Q \in \mathbb{R}^{n \times n}$: orthogonal and $R \in \mathbb{R}^{k \times k}$: nonsingular and upper triangular.

By doing variables substitution,

$$M := \sqrt{c_1}Q^T MQ, M_a := \sqrt{c_1}Q^T M_a Q, etc.$$

The IQEP becomes

$$\begin{array}{ll} \min & \frac{1}{2}\|M - M_a\|^2 + \frac{1}{2}\|C - C_a\|^2 + \frac{1}{2}\|K - K_a\|^2 \\ \text{s.t.} & \frac{1}{\sqrt{c_1}} M \begin{bmatrix} R \\ 0 \end{bmatrix} \Lambda^2 + \frac{1}{\sqrt{c_2}} C \begin{bmatrix} R \\ 0 \end{bmatrix} \Lambda + K \begin{bmatrix} R \\ 0 \end{bmatrix} = 0, \\ & (M, C, K) \in \Omega, \end{array}$$

where Ω is a convex cone defined by

$$\Omega_0 := \mathcal{S}^n \times \mathcal{S}^n \times \mathcal{S}^n$$

$$\Omega := \{(M, C, K) \in \Omega_0 \, : \, M \succeq 0, \quad K \succeq 0\},$$

Let

$$S := R\Lambda R^{-1}.$$

Partition

$$M =: \begin{bmatrix} M_1 & M_2 \\ M_2^T & M_4 \end{bmatrix},\ C := \begin{bmatrix} C_1 & C_2 \\ C_2^T & C_4 \end{bmatrix}, K := \begin{bmatrix} K_1 & K_2 \\ K_2^T & K_4 \end{bmatrix},$$

where $M_1, C_1, K_1 \in \mathcal{S}^k$, M_2, C_2, $K_2 \in \mathbb{R}^{k\times(n-k)}$, and $M_4, C_4, K_4 \in \mathcal{S}^{(n-k)}$.

For $(M, C, K) \in \Omega_0$, let $\underline{\mathcal{H}(M, C, K)}$ be given by

$$\boxed{\frac{1}{\sqrt{c_1}}(\Lambda^2)^T \left(R^T M_1 R\right) + \frac{1}{\sqrt{c_2}}\Lambda^T \left(R^T C_1 R\right) + \left(R^T K_1 R\right)}$$

and $\underline{\mathcal{G}(M, C, K)}$ be given by

$$\boxed{\frac{1}{\sqrt{c_1}}(S^2)^T M_2 + \frac{1}{\sqrt{c_2}} S^T C_2 + K_2.}$$

While $\mathcal{G} : \Omega_0 \to \mathbb{R}^{k\times(n-k)}$ is onto, $\mathcal{H} : \Omega_0 \to \mathbb{R}^{k\times k}$ is not.

Let

$$\text{Range}(\mathcal{H}) := \{\mathcal{H}(M, C, K) \,:\, (M, C, K) \in \Omega_0\} \subseteq \mathbb{R}^{k \times k}.$$

Then $\mathcal{H} : \Omega_0 \to \text{Range}(\mathcal{H})$ is surjective. The dimension of Range($\mathcal{H}$) is given by

$$k^2 - \sum_{i=1}^{\mu} s_i(s_i - 1) - \frac{1}{2} \sum_{i=\mu+1}^{\nu} s_i(s_i - 1).$$

In particular, if $s_1 = \cdots = s_\mu = s_{\mu+1} = \cdots = s_\nu = 1$, it is equal to k^2.

Define the linear operator $\mathcal{A} : \Omega_0 \to \text{Range}(\mathcal{H}) \times \mathbb{R}^{k\times(n-k)}$ by

$$\mathcal{A}(M, C, K) := (\mathcal{H}(M, C, K), \mathcal{G}(M, C, K)) .$$

The IQEP takes the following compact form

$$\begin{array}{ll} \text{min} & \frac{1}{2}\|(M, C, K) - (M_a, C_a, K_a)\|^2 \\ \text{s.t.} & \mathcal{A}(M, C, K) = 0, \\ & (M, C, K) \in \Omega. \end{array}$$

Its dual problem is

$$\begin{array}{ll} \min & \theta(Y, Z) \\ \text{s.t.} & (Y, Z) \in \text{Range}(\mathcal{H}) \times \mathbb{R}^{k\times(n-k)}, \end{array}$$

where

$$\theta(Y, Z) := \frac{1}{2}\|\Pi_\Omega((M_a, C_a, K_a) + \mathcal{A}^*(Y, Z))\|^2 - \frac{1}{2}\|(M_a, C_a, K_a)\|^2.$$

Under Slater's condition

$$\begin{cases} \mathcal{A} : \mathcal{X} \rightarrow \mathcal{Y} \text{ is onto,} \\ \exists\ \bar{x} \in \mathcal{X} \text{ such that } \ \mathcal{A}\bar{x} = b,\ \bar{x} \in \text{int}\,(\Omega)\,, \end{cases}$$

where "int" denotes the topological interior, the classical duality theorem [Rockafellar'74] says that

$x^* := \Pi_\Omega(x^0 + \mathcal{A}^* y^*)$ solves the original problem if y^* solves the dual problem.

From our construction, we know that

$$\mathcal{A} : \Omega_0 \to \text{Range}(\mathcal{H}) \times \mathbb{R}^{k\times(n-k)} \text{ is onto.}$$

Moreover, we have shown that

Theorem 1. The IQEP has a strictly feasible solution iff

$$\text{Det}(\Lambda) \neq 0.$$

Remark: If $\text{Det}(\Lambda) = 0$, we do not lose generality as we can reduce the IQEP to another problem with a strictly feasible solution.

Thus the Slater condition is satisfied.

In addition, the gradient of the dual function is given by

$$F(Y, Z) := \nabla\theta(Y, Z) = \mathcal{A}\,\Pi_\Omega\left((M_a, C_a, K_a) + \mathcal{A}^*(Y, Z)\right),$$

where $(Y, Z) \in \text{Range}(\mathcal{H}) \times \mathbb{R}^{k\times(n-k)}$. Therefore,

• Gradient based methods (e.g., BFGS method) can be used to find the optimal solution (Y^*, Z^*) of the dual problem.

• We can't directly use Newton's method to solve the dual problem since $\Pi_\Omega(\cdot)$ is not continuously differential.

Notice that $\Pi_\Omega(\cdot)$ is globally Lipschitz continuous. Then we can apply Clarke's General Jacobian based Newton methods for locally Lipschitz equations. We first recall the definition of Clarke's General Jacobian (Clarke'83).

Let $\mathcal{Y}$ and $\mathcal{Z}$ be arbitrary finite dimensional real vector spaces.

Let $\mathcal{O}$ be an open set in $\mathcal{Y}$ and $\Psi : \mathcal{O} \subseteq \mathcal{Y} \to \mathcal{Z}$ be a locally Lipschitz continuous function on the open set $\mathcal{O}$.

Rademacher's theorem says that Ψ is almost everywhere Fréchet differentiable in $\mathcal{O}$.

We denote by $\mathcal{O}_\Psi$ the set of points in $\mathcal{O}$ where Ψ is Fréchet differentiable.

Let $\Psi'(y)$ denote the Jacobian of Ψ at $y \in \mathcal{O}_\Psi$.

Then Clarke's generalized Jacobian of Ψ at $y \in \mathcal{O}$ is defined by [Clarke'83]

$$\partial\Psi(y) := \text{conv}\{\partial_B\Psi(y)\},$$

where "conv" denotes the convex hull and

$$\partial_B\Psi(y) := \left\{V \,:\, V = \lim_{j\to\infty} \Psi'(y^j)\,,\, y^j \in \mathcal{O}_\Psi\right\}.$$

When $F : \mathcal{O} \subseteq \mathcal{Y} \to \mathcal{Y}$ is continuously differentiable (smooth), the most effective approach for solving

$$F(y) = 0$$

is probably Newton's method. For example, in 1987, S. Smale wrote

> If any algorithm has proved itself for the problem of nonlinear systems, it is Newton's method and its many modifications..."

The extension of Newton's methods to Lipschitz systems:

- Friedland, Nocedal, and Overton [87] for inverse eigenvalue problems.

- Kojima and Shindoh [86] for piecewise smooth equations.

- Kummer [88] proposed a condition

(ii) for any $x \to y$ and $V \in \partial \Psi(x)$,

$$\Psi(x) - \Psi(y) - V(x-y) = o(||x-y||)\,.$$

- Finally, Qi and J. Sun [93] showed what needed is semismoothness.

The function Ψ is (strongly) *semismooth* at a point $y \in \mathcal{O}$ if

(i) Ψ is directionally differentiable at y; and

(ii) for any $x \to y$ and $V \in \partial\Psi(x)$,

$$\Psi(x) - \Psi(y) - V(x-y) = o(||x-y||) \ (O(||x-y||^2)).$$

Let $A \in \mathcal{S}^n$. Then A admits the following spectral decomposition

$$A = P\Sigma P^T,$$

where Σ is the diagonal matrix of eigenvalues of A and P is a corresponding orthogonal matrix of orthonormal eigenvectors.

Define three index sets of positive, zero, and negative eigenvalues of A, respectively, as

$$\alpha := \{i : \sigma_i > 0\},$$

$$\beta := \{i : \sigma_i = 0\},$$

$$\gamma := \{i : \sigma_i < 0\}.$$

Write

$$\Sigma = \text{diag}\left(\Sigma_\alpha, \Sigma_\beta, \Sigma_\gamma\right) \quad \text{and} \quad P = [\ P_\alpha \ \ P_\beta \ \ P_\gamma\ ]$$

with $P_\alpha \in \mathbb{R}^{n\times|\alpha|}$, $P_\beta \in \mathbb{R}^{n\times|\beta|}$, and $P_\gamma \in \mathbb{R}^{n\times|\gamma|}$.

Define the matrix $U \in \mathbb{R}^{|\alpha|\times|\gamma|}$ with entries

$$U_{ij} := \frac{\max\{\sigma_i, 0\} + \max\{\sigma_j, 0\}}{|\sigma_i| + |\sigma_j|}, \quad i \in \alpha, j \in \gamma$$

where 0/0 is defined to be 1.

Sun and Sun [02] showed $\Pi_{\mathcal{S}^n_+}(\cdot)$ is strongly semismooth everywhere and the directional derivative $\Pi'_{\mathcal{S}^n_+}(A;H)$ is given by

$$P\begin{bmatrix} P_\alpha^T H P_\alpha & P_\alpha^T H P_\beta & U \circ P_\alpha^T H P_\gamma \\ P_\beta^T H P_\alpha & \Pi_{\mathcal{S}^{|\beta|}_+}\left(P_\beta^T H P_\beta\right) & 0 \\ P_\gamma^T H P_\alpha \circ U^T & 0 & 0 \end{bmatrix} P^T,$$

where $\circ$ denotes the Hadamard product.

When A is nonsingular, i.e., $|\beta| = 0$, $\Pi_{\mathcal{S}^n_+}(\cdot)$ is continuously differentiable around A and the above formula reduces to the classical result of Löwner [34].

The tangent cone of $\mathcal{S}^n_+$ at $A_+ = \Pi_{\mathcal{S}^n_+}(A)$:

$$\mathcal{T}_{\mathcal{S}^n_+}(A_+) = \{B \in \mathcal{S}^n \,:\, B = \Pi'_{\mathcal{S}^n_+}(A_+; H)\} = \{B \in \mathcal{S}^n \,:\, P_{\bar{\alpha}}^T B P_{\bar{\alpha}} \succeq 0\},$$

where $P_{\bar{\alpha}} := [P_\beta \; P_\gamma]$ and the lineality space of $\mathcal{T}_{\mathcal{S}^n_+}(A_+)$, i,e, the largest linear space in $\mathcal{T}_{\mathcal{S}^n_+}(A_+)$,

$$\text{lin}\left(\mathcal{T}_{\mathcal{S}^n_+}(A_+)\right) = \{B \in \mathcal{S}^n \,:\, P_{\bar{\alpha}}^T B P_{\bar{\alpha}} = 0\},$$

Let $W(H)$ be defined by

$$P \begin{bmatrix} P_\alpha^T H P_\alpha & P_\alpha^T H P_\beta & U \circ P_\alpha^T H P_\gamma \\ P_\beta^T H P_\alpha^T & 0 & 0 \\ P_\gamma^T H P_\alpha \circ U^T & 0 & 0 \end{bmatrix} P^T$$

for all $H \in \mathcal{S}^n$. Then W is an element in $\partial_B \Pi_{\mathcal{S}_+^n}(A)$.

(Newton's Method for solving $F(Y,Z) = \nabla\theta(Y,Z) = 0$)

[**Step 0.**] Given $(Y^0, Z^0) \in \text{Range}(\mathcal{H}) \times \mathbb{R}^{k\times(n-k)}$, $\eta \in (0,1)$, $\rho, \delta \in (0, 1/2)$. $j := 0$.

[**Step 1.**] (Newton's Iteration) Select an element

$$\underline{W_j \in \partial\Pi_\Omega\Big((M_a, C_a, K_a) + \mathcal{A}^*(Y^j, Z^j)\Big)}$$

and let

$$\underline{V_j := \mathcal{A}W_j\mathcal{A}^*}.$$

Apply the conjugate gradient method to find an approximate solution

$$(\Delta Y^j, \Delta Z^j) \in \text{Range}(\mathcal{H}) \times \mathbb{R}^{k\times(n-k)}$$

to the linear system

$$F(Y^j, Z^j) + V_j(\Delta Y, \Delta Z) = 0 \qquad (1)$$

such that

$$\|F(Y^j, Z^j) + V_j(\Delta Y^j, \Delta Z^j)\| \le \eta_j \|F(Y^j, Z^j)\| \qquad (2)$$

and

$$\begin{aligned} &\left\langle F(Y^j, Z^j), (\Delta Y^j, \Delta Z^j)\right\rangle \\ &\le -\eta_j \left\langle (\Delta Y^j, \Delta Z^j), (\Delta Y^j, \Delta Z^j)\right\rangle, \end{aligned} \qquad (3)$$

where $\eta_j := \min\{\eta, \|F(Y^j, Z^j)\|\}$.

If (2) and (3) are not achievable, let

$$
\begin{aligned}
&(\Delta Y^j, \Delta Z^j) := -F(Y^j, Z^j) \\
&= -\mathcal{A}\,\Pi_\Omega\left((M_a, C_a, K_a) + \mathcal{A}^*(Y^j, Z^j)\right).
\end{aligned}
$$

[**Step 2.**] (Line Search) Let m_j be the smallest nonnegative integer m such that

$$
\begin{aligned}
&\theta\left((Y^j, Z^j) + \rho^m(\Delta Y^j, \Delta Z^j)\right) - \theta(Y^j, Z^j) \\
&\leq \delta\rho^m \left\langle F(Y^j, Z^j), (\Delta Y^j, \Delta Z^j)\right\rangle.
\end{aligned}
$$

Set

$$
(Y^{j+1}, Z^{j+1}) := (Y^j, Z^j) + \rho^{m_j}(\Delta Y^j, \Delta Z^j).
$$

[**Step 3.**] Replace j by $j+1$ and go to Step 1.

Main Results

Global convergence:

Theorem 2. The algorithm generates an infinite sequence $\{(Y^j, Z^j)\}$ with the properties that for each $j \geq 0$, $(Y^j, Z^j) \in \text{Range}(\mathcal{H}) \times \mathbb{R}^{k\times(n-k)}$, $\{(Y^j, Z^j)\}$ is bounded, and any accumulation point of $\{(Y^j, Z^j)\}$ is a solution to the dual problem.

For discussions on the rate of convergence, we need the constraint nondegenerate condition ("LICQ")

$$\mathcal{A}\left(\text{lin}\left(\mathcal{T}_{\mathcal{S}^n_+}(\overline{M})\right), \mathcal{S}^n, \text{lin}\left(\mathcal{T}_{\mathcal{S}^n_+}(\overline{K})\right)\right) = \text{Range}(\mathcal{H}) \times \mathbb{R}^{k\times(n-k)},$$

where $(\overline{M}, \overline{C}, \overline{K}) \in \Omega_0$ is a feasible solution to the original problem.

Theorem 3. Let $(\overline{Y}, \overline{Z})$ be an accumulation point of the infinite sequence $\{(Y^j, Z^j)\}$ generated by the algorithm. Let

$$(\overline{M}, \overline{C}, \overline{K}) := \Pi_{\Omega}\Big((M_a, C_a, K_a) + \mathcal{A}^*(\overline{Y}, \overline{Z})\Big).$$

Assume that the constraint nondegenerate condition holds at $(\overline{M}, \overline{C}, \overline{K})$. Then the whole sequence $\{(Y^j, Z^j)\}$ converges to $(\bar{Y}, \bar{Z})$ quadratically.

Numerical Experiments

The stopping criterion is

$$\texttt{Tol}. := \frac{\|\nabla\theta(Y_k, Z_k)\|}{\max\left\{1, \left\|(\frac{1}{\sqrt{c_1}}M_a, \frac{1}{\sqrt{c_2}}C_a, K_a)\right\|\right\}} \leq 10^{-7}\,.$$

We set other parameters used in our algorithm as $\eta = 10^{-6}$, $\rho = 0.5$, and $\delta = 10^{-4}$.

$k = 30, \quad c_1 = c_2 = 1.0$				
n	cputime	It.	Func.	Tol.
100	01 m 26 s	18	24	3.9×10^{-11}
200	04 m 39 s	14	15	3.9×10^{-11}
500	21 m 16 s	11	12	1.3×10^{-10}
1,000	44 m 13 s	9	10	1.1×10^{-9}
1,500	08 h 49 m 11 s	7	8	1.6×10^{-8}
2,000	05 h 24 m 37 s	9	10	3.3×10^{-8}

$k \approx n/3, \quad c_1 = 10.0, c_2 = 0.10$					
n	k	cputime	It.	Func.	Tol.
100	33	46.1 s	9	11	1.4×10^{-9}
200	66	42 m 42 s	13	15	5.8×10^{-8}
300	100	02 h 24 m 23 s	17	20	6.5×10^{-9}
400	133	04 h 38 m 42 s	10	11	4.0×10^{-8}
450	150	12 h 23 m 44 s	13	14	8.8×10^{-9}

The largest numerical examples that we tested in this paper are:

- (i) $n = 2,000$ and $k = 30$ and

- (ii) $n = 450$ and $k = 150$.

For case (i), there are roughly $6,000,000$ unknowns in the primal problem and $60,000$ unknowns in the dual problem while for case (ii), these numbers are roughly $300,000$ and $67,000$, respectively.

- In consideration of the scales of problems solved, our algorithm is very effective.

PART II.

Reconstruction of the Physical Parameters of a Damped Vibrating System From Eigendata

In structural mechanics, a damped vibrating model is governed by the equation

$$(\lambda^2 M + \lambda C + K)\mathbf{u} = \mathbf{0},$$

where

$$M = \begin{bmatrix} 2m_1 + 2m_2 & m_2 & & & \\ m_2 & 2m_2 + 2m_3 & m_3 & & \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ & & & m_n & 2m_n \end{bmatrix},$$

$$C = \begin{bmatrix} c_1 + c_2 & -c_2 & & & \\ -c_2 & c_2 + c_3 & -c_3 & & \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ & & & -c_n & c_n \end{bmatrix},$$

$$K = \begin{bmatrix} k_1 + k_2 & -k_2 & & & \\ -k_2 & k_2 + k_3 & -k_3 & & \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ & & & -k_n & k_n \end{bmatrix}.$$

(See Ram and Gladwell'94 for undamped case, i.e., $C = O$)

PART II: Statement of Problem

Inverse problems can be stated as follows:

Problem A. Construct the parameters $(m_j, c_j, k_j)_1^n$ from $w = \sum_1^n m_j$ and two real eigenvalues $(\lambda_j)_1^2$ and three real eigenvectors $(\mathbf{u}^{(j)})_1^3$.

Problem B. Construct the parameters $(m_j, c_j, k_j)_1^n$ from $w = \sum_1^n m_j$ and one real eigenvector $\mathbf{u}^{(1)}$ and a complex conjugate eigenpair $(\lambda_{2,3} = \alpha \pm \beta i, \mathbf{u}^{(2,3)} = \mathbf{u}_R \pm \mathbf{u}_I i)$, where $i = \sqrt{-1}$.

PART II: Our Approach

Let $(\lambda, \mathbf{u})$ be any eigenpair of the equation

$$(\lambda^2 M + \lambda C + K)\mathbf{u} = \mathbf{0},$$

Rewrite this equation so that $(u_j)_1^n$ appear in matrices and $(m_j, c_j, k_j)_1^n$ in the vectors:

$$\lambda^2 A\mathbf{m} + \lambda B\mathbf{c} + B\mathbf{k} = 0,$$

where

$$A = \begin{bmatrix} 2u_1 & 2u_1 + u_2 & & & \\ & u_1 + 2u_2 & 2u_2 + u_3 & & \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ & & & u_{n-2} + 2u_{n-1} & 2u_{n-1} + u_n \\ & & & & u_{n-1} + 2u_n \end{bmatrix},$$

$$B = \begin{bmatrix} u_1 & u_1 - u_2 & & & \\ & -u_1 + u_2 & u_2 - u_3 & & \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ & & & -u_{n-2} + u_{n-1} & u_{n-1} - u_n \\ & & & & -u_{n-1} + u_n \end{bmatrix}.$$

Let

$$(a_j)_1^n = u_{j-1} + 2u_j,\ (b_j)_1^{n-1} = 2u_j + u_{j+1},\ (d_j)_1^n = u_j - u_{j-1}.$$

Then

$$A = \begin{bmatrix} a_1 & b_1 & & & \\ & a_2 & b_2 & & \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ & & & a_{n-1} & b_{n-1} \\ & & & & a_n \end{bmatrix},\ B = \begin{bmatrix} d_1 & -d_2 & & & \\ & d_2 & -d_3 & & \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ & & & d_{n-1} & -d_n \\ & & & & d_n \end{bmatrix}.$$

Suppose that we have three real eigenpairs: $\{\lambda_j, \mathbf{u}^{(j)}\}_1^3$. Then

$$\lambda_j^2 A^{(j)}\mathbf{m} + \lambda_j B^{(j)}\mathbf{c} + B^{(j)}\mathbf{k} = 0, \quad j = 1, 2, 3. \quad (4)$$

The last rows of above expression give

$$\begin{bmatrix} \lambda_1^2 a_n^{(1)} & \lambda_1 d_n^{(1)} & d_n^{(1)} \\ \lambda_2^2 a_n^{(2)} & \lambda_2 d_n^{(2)} & d_n^{(2)} \\ \lambda_3^2 a_n^{(3)} & \lambda_3 d_n^{(3)} & d_n^{(3)} \end{bmatrix} \begin{pmatrix} m_n \\ c_n \\ k_n \end{pmatrix} = 0.$$

To ensure the existence of a nontrivial solution, we let

$$\det \begin{bmatrix} \lambda_1^2 a_n^{(1)} & \lambda_1 d_n^{(1)} & d_n^{(1)} \\ \lambda_2^2 a_n^{(2)} & \lambda_2 d_n^{(2)} & d_n^{(2)} \\ \lambda_3^2 a_n^{(3)} & \lambda_3 d_n^{(3)} & d_n^{(3)} \end{bmatrix} = 0.$$

If this condition is satisfied, or alternatively if $\{\lambda_j\}_1^2$ and $\{\mathbf{u}^{(j)}\}_1^3$ are given and λ_3 is determined by above equation, then the ratio c_n/m_n and k_n/m_n are determined by

$$\begin{bmatrix} \lambda_1 d_n^{(1)} & d_n^{(1)} \\ \lambda_2 d_n^{(2)} & d_n^{(2)} \end{bmatrix} \begin{pmatrix} c_n/m_n \\ k_n/m_n \end{pmatrix} = \begin{pmatrix} -\lambda_1^2 a_n^{(1)} \\ -\lambda_2^2 a_n^{(2)} \end{pmatrix}.$$

The other $n-1$ rows of expression (4) yield

$$\begin{bmatrix} \lambda_1^2 a_j^{(1)} & \lambda_1 d_j^{(1)} & d_j^{(1)} \\ \lambda_2^2 a_j^{(2)} & \lambda_2 d_j^{(2)} & d_j^{(2)} \\ \lambda_3^2 a_j^{(3)} & \lambda_3 d_j^{(3)} & d_j^{(3)} \end{bmatrix} \begin{pmatrix} m_j \\ c_j \\ k_j \end{pmatrix}$$

$$= \begin{pmatrix} -\lambda_1^2 b_j^{(1)} m_{j+1} + \lambda_1 d_{j+1}^{(1)} c_{j+1} + d_{j+1}^{(1)} k_{j+1} \\ -\lambda_2^2 b_j^{(2)} m_{j+1} + \lambda_2 d_{j+1}^{(2)} c_{j+1} + d_{j+1}^{(2)} k_{j+1} \\ -\lambda_3^2 b_j^{(3)} m_{j+1} + \lambda_3 d_{j+1}^{(3)} c_{j+1} + d_{j+1}^{(3)} k_{j+1} \end{pmatrix}$$

$$j = n-1, n-2, \ldots, 1.$$

Let

$$\widetilde{m_j} = m_j/m_n,\ \widetilde{c_j} = c_j/m_n,\ \widetilde{k_j} = k_j/m_n.$$

Then

$$\begin{bmatrix} \lambda_1^2 a_j^{(1)} & \lambda_1 d_j^{(1)} & d_j^{(1)} \\ \lambda_2^2 a_j^{(2)} & \lambda_2 d_j^{(2)} & d_j^{(2)} \\ \lambda_3^2 a_j^{(3)} & \lambda_3 d_j^{(3)} & d_j^{(3)} \end{bmatrix} \begin{pmatrix} \widetilde{m}_j \\ \widetilde{c}_j \\ \widetilde{k}_j \end{pmatrix}$$

$$= \begin{pmatrix} -\lambda_1^2 b_j^{(1)} \widetilde{m}_{j+1} + \lambda_1 d_{j+1}^{(1)} \widetilde{c}_{j+1} + d_{j+1}^{(1)} \widetilde{k}_{j+1} \\ -\lambda_2^2 b_j^{(2)} \widetilde{m}_{j+1} + \lambda_2 d_{j+1}^{(2)} \widetilde{c}_{j+1} + d_{j+1}^{(2)} \widetilde{k}_{j+1} \\ -\lambda_3^2 b_j^{(3)} \widetilde{m}_{j+1} + \lambda_3 d_{j+1}^{(3)} \widetilde{c}_{j+1} + d_{j+1}^{(3)} \widetilde{k}_{j+1} \end{pmatrix}.$$

Notice that the total mass $w = \sum_1^n m_j$ is known. Therefore, one can obtain the parameters $(m_j, c_j, k_j)_1^n$ by $m_j = \widetilde{m}_j w/\widetilde{w}$, $c_j = \widetilde{c}_j w/\widetilde{w}$, and $k_j = \widetilde{k}_j w/\widetilde{w}$, where $\widetilde{w} = \sum_1^n \widetilde{m}_j$.

Problem A is solved by the constructive proof. We can solve Problem B by the same way. Here, we only note the following fact.

For the complex conjugate eigenpair $(\lambda_{2,3} = \alpha \pm \beta i, \mathbf{u}^{(2,3)} = \mathbf{u}_R \pm \mathbf{u}_I i)$, we have

$$(\lambda_j^2 M + \lambda_j C + K)\,\mathbf{u}^{(j)} = 0, \quad j = 2, 3.$$

The real form:

$$M \begin{bmatrix} \mathbf{u}_R & \mathbf{u}_I \end{bmatrix} \begin{bmatrix} \alpha & \beta \\ -\beta & \alpha \end{bmatrix}^2 + C \begin{bmatrix} \mathbf{u}_R & \mathbf{u}_I \end{bmatrix} \begin{bmatrix} \alpha & \beta \\ -\beta & \alpha \end{bmatrix} + K \begin{bmatrix} \mathbf{u}_R & \mathbf{u}_I \end{bmatrix} = 0$$

i.e.,

$$\begin{cases} M\left[(\alpha^2-\beta^2)\mathbf{u}_R - 2\alpha\beta\mathbf{u}_I\right] + C\left(\alpha\mathbf{u}_R - \beta\mathbf{u}_I\right) + K\mathbf{u}_R = 0 \\ M\left[2\alpha\beta\mathbf{u}_R + (\alpha^2-\beta^2)\mathbf{u}_I\right] + C\left(\beta\mathbf{u}_R + \alpha\mathbf{u}_I\right) + K\mathbf{u}_I = 0 \end{cases}$$

In terms of the parameters $(m_j, c_j, k_j)_1^n$:

$$\begin{cases} \left[(\alpha^2-\beta^2)A_R - 2\alpha\beta A_I\right]\mathbf{m} + \left[\alpha B_R - \beta B_I\right]\mathbf{c} + B_R\mathbf{k} = 0 \\ \left[2\alpha\beta A_R + (\alpha^2-\beta^2)A_I\right]\mathbf{m} + \left[\beta B_R + \alpha B_I\right]\mathbf{c} + B_I\mathbf{k} = 0 \end{cases}$$

where

$$A_R = \begin{bmatrix} a_{1R} & b_{1R} & & & \\ & a_{2R} & b_{2R} & & \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ & & & a_{n-1,R} & b_{n-1,R} \\ & & & & a_{nR} \end{bmatrix}, A_I = \begin{bmatrix} a_{1I} & b_{1I} & & & \\ & a_{2I} & b_{2I} & & \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ & & & a_{n-1,I} & b_{n-1,I} \\ & & & & a_{nI} \end{bmatrix}$$

$$B_R = \begin{bmatrix} d_{1R} & -d_{2R} & & & \\ & d_{2R} & -d_{3R} & & \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ & & & d_{n-1,R} & -d_{n,R} \\ & & & & d_{nR} \end{bmatrix}, B_I = \begin{bmatrix} d_{1I} & -d_{2I} & & & \\ & d_{2I} & -d_{3I} & & \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ & & & d_{n-1,I} & -d_{n,I} \\ & & & & d_{nI} \end{bmatrix}$$

with

$$
\begin{aligned}
(a_{jR})_1^n &= u_{j-1,R} + 2u_{jR}, & (a_{jI})_1^n &= u_{j-1,I} + 2u_{jI}, \\
(b_{jR})_1^{n-1} &= 2u_{jR} + u_{j+1,R}, & (b_{jI})_1^{n-1} &= 2u_{jI} + u_{j+1,I} \\
(d_{jR})_1^n &= u_{jR} - u_{j-1,R}, & (d_{jI})_1^n &= u_{jI} - u_{j-1,I},
\end{aligned}
$$

Disadvantage:

• Physical realistic (i.e., positive) mass, damping, and stiffness not guaranteed

• Sensitive to Perturbations.

Aim:

• To reduce the sensitivity, we find the solution in the least squares sense.

Given $\underline{w = \sum_1^n m_j}$ and k noise corrupted eigenpairs

$$\left\{\lambda_{2j-1,2j} := \alpha_j \pm \beta_j i, \mathbf{u}^{(2j-1,2j)} := \mathbf{u}_{jR}^{(j)} \pm \mathbf{u}_{jI}^{(j)} i\right\}_1^{\ell}$$

$$\left\{\lambda_j, \mathbf{u}^{(j)}\right\}_{2\ell+1}^{k}$$

Solving the least squares problems successively:

$$\min \quad \frac{1}{2}\left\| G_n \begin{pmatrix} \widetilde{c}_n \\ \widetilde{k}_n \end{pmatrix} - \mathbf{f}^{(n)} \right\|^2,$$

$$G_n = \begin{bmatrix} \alpha_1 d_{nR}^{(1)} - \beta_1 d_{nI}^{(1)} & d_{nR}^{(1)} \\ \beta_1 d_{nR}^{(1)} + \alpha_1 d_{nI}^{(1)} & d_{nI}^{(1)} \\ \cdots & \cdots \\ \alpha_\ell d_{nR}^{(\ell)} - \beta_\ell d_{nI}^{(\ell)} & d_{nR}^{(\ell)} \\ \beta_\ell d_{nR}^{(\ell)} + \alpha_\ell d_{nI}^{(\ell)} & d_{nI}^{(\ell)} \\ \lambda_{2\ell+1} d_n^{(2\ell+1)} & d_n^{(2\ell+1)} \\ \cdots & \cdots \\ \lambda_k d_n^{(k)} & d_n^{(k)} \end{bmatrix}, \mathbf{f}^{(n)} = \begin{pmatrix} -[(\alpha_1^2 - \beta_1^2) a_{nR}^{(1)} - 2\alpha_1\beta_1 a_{nI}^{(1)}] \\ -[2\alpha_1\beta_1 a_{nR}^{(1)} + (\alpha_1^2 - \beta_1^2) a_{nI}^{(1)}] \\ \cdots \\ -[(\alpha_\ell^2 - \beta_\ell^2) a_{nR}^{(\ell)} - 2\alpha_\ell\beta_\ell a_{nI}^{(\ell)}] \\ -[2\alpha_\ell\beta_\ell a_{nR}^{(\ell)} + (\alpha_\ell^2 - \beta_\ell^2) a_{nI}^{(\ell)}] \\ -\lambda_{2\ell+1}^2 a_n^{(2\ell+1)} \\ \cdots \\ -\lambda_k^2 a_n^{(k)} \end{pmatrix}$$

$$\min \quad \frac{1}{2}\left\| G_j \begin{pmatrix} \widetilde{m}_j \\ \widetilde{c}_j \\ \widetilde{k}_j \end{pmatrix} - \mathbf{f}^{(j)} \right\|^2, \quad j = n-1, n-2, \ldots, 1$$

$$G_j = \begin{bmatrix} (\alpha_1^2 - \beta_1^2)a_{jR}^{(1)} - 2\alpha_1\beta_1 a_{jI}^{(1)} & \alpha_1 d_{jR}^{(1)} - \beta_1 d_{jI}^{(1)} & d_{jR}^{(1)} \\ 2\alpha_1\beta_1 a_{jR}^{(1)} + (\alpha_1^2 - \beta_1^2)a_{jI}^{(1)} & \beta_1 d_{jR}^{(1)} + \alpha_1 d_{jI}^{(1)} & d_{jI}^{(1)} \\ \cdots & \cdots & \cdots \\ (\alpha_\ell^2 - \beta_\ell^2)a_{jR}^{(\ell)} - 2\alpha_\ell\beta_\ell a_{jI}^{(\ell)} & \alpha_\ell d_{jR}^{(\ell)} - \beta_\ell d_{jI}^{(\ell)} & d_{jR}^{(\ell)} \\ 2\alpha_\ell\beta_\ell a_{jR}^{(\ell)} + (\alpha_\ell^2 - \beta_\ell^2)a_{jI}^{(\ell)} & \beta_\ell d_{jR}^{(\ell)} + \alpha_\ell d_{jI}^{(\ell)} & d_{jI}^{(\ell)} \\ \\ \lambda_{2\ell+1}^2 a_j^{(2\ell+1)} & \lambda_{2\ell+1} d_j^{(2\ell+1)} & d_j^{(2\ell+1)} \\ \cdots & \cdots & \cdots \\ \lambda_k^2 a_j^{(k)} & \lambda_k d_j^{(k)} & d_j^{(k)} \end{bmatrix}$$

$$
\mathbf{f}^{(j)} = \begin{pmatrix} -[(\alpha_1^2-\beta_1^2)b_{jR}^{(1)} - 2\alpha_1\beta_1 b_{jI}^{(1)}]\widetilde{m}_{j+1} + [\alpha_1 d_{j+1,R}^{(1)} - \beta_1 d_{j+1,I}^{(1)}]\widetilde{c}_{j+1} + d_{j+1,R}^{(1)}\widetilde{k}_{j+1} \\ -[2\alpha_1\beta_1 b_{jR}^{(1)} + (\alpha_1^2-\beta_1^2)b_{jI}^{(1)}]\widetilde{m}_{j+1} + [\beta_1 d_{j+1,R}^{(1)} + \alpha_1 d_{j+1,I}^{(1)}]\widetilde{c}_{j+1} + d_{j+1,I}^{(1)}\widetilde{k}_{j+1} \\ \cdots \\ -[(\alpha_\ell^2-\beta_\ell^2)b_{jR}^{(\ell)} - 2\alpha_\ell\beta_\ell b_{jI}^{(\ell)}]\widetilde{m}_{j+1} + [\alpha_\ell d_{j+1,R}^{(\ell)} - \beta_\ell d_{j+1,I}^{(\ell)}]\widetilde{c}_{j+1} + d_{j+1,R}^{(\ell)}\widetilde{k}_{j+1} \\ -[2\alpha_\ell\beta_\ell b_{jR}^{(\ell)} + (\alpha_\ell^2-\beta_\ell^2)b_{jI}^{(\ell)}]\widetilde{m}_{j+1} + [\beta_\ell d_{j+1,R}^{(\ell)} + \alpha_\ell d_{j+1,I}^{(\ell)}]\widetilde{c}_{j+1} + d_{j+1,I}^{(\ell)}\widetilde{k}_{j+1} \\ \\ -\lambda_1^2 b_j^{(1)}\widetilde{m}_{j+1} + \lambda_1 d_{j+1}^{(1)}\widetilde{c}_{j+1} + d_{j+1}^{(1)}\widetilde{k}_{j+1} \\ \cdots \\ -\lambda_k^2 b_j^{(k)}\widetilde{m}_{j+1} + \lambda_k d_{j+1}^{(k)}\widetilde{c}_{j+1} + d_{j+1}^{(k)}\widetilde{k}_{j+1} \end{pmatrix}.
$$

Advantage:

• Yield practically acceptable results for minor changes in eigen-data

Drawback:

• Not theoretically ensure that the mass, damping, and stiffness are positive.

Our Goal:

• Reconstructing Physical Model over experimentally measured data

Given $\underline{w = \sum_1^n m_j}$ and k noise corrupted eigenpairs

$$\left\{\lambda_{2j-1,2j} := \alpha_j \pm \beta_j i, \mathbf{u}^{(2j-1,2j)} := \mathbf{u}_{jR}^{(j)} \pm \mathbf{u}_{jI}^{(j)} i\right\}_1^{\ell}$$

$$\left\{\lambda_j, \mathbf{u}^{(j)}\right\}_{2\ell+1}^{k}$$

Solving the following positivity-constrained least squares optimization problems successively:

$$\min \quad \frac{1}{2}\left\| G_n \begin{pmatrix} \widetilde{c}_n \\ \widetilde{k}_n \end{pmatrix} - \mathbf{f}^{(n)} \right\|^2$$

$$\text{s.t.} \quad \widetilde{c}_n \geq \epsilon, \widetilde{k}_n \geq \epsilon,$$

$$\min \quad \frac{1}{2}\left\| G_j \begin{pmatrix} \widetilde{m}_j \\ \widetilde{c}_j \\ \widetilde{k}_j \end{pmatrix} - \mathbf{f}^{(j)} \right\|^2$$

$$\text{s.t.} \quad \widetilde{m}_j \geq \epsilon, \widetilde{c}_j \geq \epsilon, \widetilde{k}_j \geq \epsilon,$$

$$\text{for} \quad j = n-1, n-2, \ldots, 1$$

where $\epsilon > 0$ is a parameter determined by practical requirements.

Att:

- Can be solved fast by the active/passive set related methods

[Lawson & Hanson'74, Bro & Jong'97, Benthem & Keenan'04]

PART II: Numerical Example

- Randomly generate the quadratic pencil

$$Q(\lambda) := \lambda^2 \overline{M} + \lambda \overline{C} + \overline{K}$$

with

$$\begin{aligned}
(\overline{m}_j)_1^n &= (1.4360, 1.5401, 1.1141, 1.0754, 1.4964, 1.3537, 1.8337, 1.3974, 1.2314, 1.1680), \\
(\overline{c}_j)_1^n &= (4.3780, 4.0110, 3.1299, 5.6259, 5.2197, 5.0297, 5.9495, 3.6815, 3.4181, 5.9454), \\
(\overline{k}_j)_1^n &= (12.7586, 10.6233, 7.8552, 13.6456, 13.4818, 10.0050, 11.5915, 9.4480, 10.1156, \\
&\quad 7.3799)
\end{aligned}$$

Thus $\underline{w = \sum_1^n \overline{m}_j = 13.6462}.$

• Its nine eigenvalues λ_j with smallest absolute value of imaginary parts :

$$-2.0214, -0.0066 \pm 0.1815i, -0.0687 \pm 0.5642i,$$

$$-0.1753 \pm 0.9249i, -0.3435 \pm 1.2335i.$$

Their corresponding eigenvectors $\mathbf{u}^{(j)}$ omitted here.

• Perturb the eigenvectors $\mathbf{u}^{(j)}$ by a uniform distribution between -0.001 and 0.001 (denoted by $\tilde{\mathbf{u}}^{(j)}$: minor error) or between -0.1 and 0.1 (denoted by $\hat{\mathbf{u}}^{(j)}$: large error)

Direct construction with $\lambda_{2,3}$ and $\{\widetilde{u}^{(j)}\}_1^3$

			Exact			$\lambda_1 = -1.2311$
j	m_j	c_j	k_j	$\tilde{m}_j$	$\tilde{c}_j$	$\tilde{k}_j$
1	1.4360	4.3780	12.7586	5.6389	3.1818	8.8605
2	1.5401	4.0110	10.6233	−5.6475	3.5939	8.0443
3	1.1141	3.1299	7.8552	8.0544	2.3894	5.4086
4	1.0754	5.6259	13.6456	−6.7987	4.4939	10.2333
5	1.4964	5.2197	13.4818	6.3876	4.6543	11.0499
6	1.3537	5.0297	10.0050	12.1230	−0.6583	−1.8995
7	1.8337	5.9495	11.5915	2.7431	−11.4309	−18.0664
8	1.3974	3.6815	9.4480	−9.3768	−3.6187	−11.7761
9	1.2314	3.4181	10.1156	−0.0105	0.8531	3.1674
10	1.1680	5.9454	7.3799	0.5327	3.1846	3.1356
			Exact			$\lambda_1 = -4.9175$
1	1.4360	4.3780	12.7586	6.3704	3.0292	8.9758
2	1.5401	4.0110	10.6233	−4.5867	3.3911	7.9768
3	1.1141	3.1299	7.8552	6.5166	2.2947	5.5019
4	1.0754	5.6259	13.6456	−6.9295	4.5947	11.0647
5	1.4964	5.2197	13.4818	11.9582	3.3720	8.5015
6	1.3537	5.0297	10.0050	0.1812	0.1549	0.3327
7	1.8337	5.9495	11.5915	0.0867	0.1380	0.2173
8	1.3974	3.6815	9.4480	0.0412	0.0256	0.0832
9	1.2314	3.4181	10.1156	0.0049	0.0092	0.0340
10	1.1680	5.9454	7.3799	0.0033	0.0195	0.0192

Least Squares Solution with data $\{\lambda_j, \widetilde{\mathrm{u}}^{(j)}\}_1^s$ (minor error)

	Exact	$s = 3$	$s = 5$	$s = 7$	$s = 9$
m_1	1.4360	2.4850	0.8194	1.5355	1.4389
m_2	1.5401	−2.3071	1.7277	1.4646	1.5282
m_3	1.1141	3.2693	1.1624	1.1760	1.1249
m_4	1.0754	−3.0153	1.1483	1.0041	1.0629
m_5	1.4964	3.7243	1.5204	1.5229	1.5057
m_6	1.3537	5.2184	1.5387	1.3474	1.3504
m_7	1.8337	3.1366	1.6805	1.8116	1.8309
m_8	1.3974	1.7486	1.5538	1.3985	1.4018
m_9	1.2314	−0.2668	1.2761	1.2259	1.2332
m_{10}	1.1680	−0.3468	1.2190	1.1597	1.1693
c_1	4.3780	4.4897	4.6555	4.3227	4.3538
c_2	4.0110	5.0003	4.3081	3.9459	3.9552
c_3	3.1299	3.6256	3.4385	3.0901	3.1263
c_4	5.6259	6.8113	5.9581	5.6565	5.6443
c_5	5.2197	6.8906	5.3820	5.2141	5.2535
c_6	5.0297	4.6982	5.3242	4.9680	5.0151
c_7	5.9495	3.9867	6.3193	5.8908	5.9908
c_8	3.6815	0.2915	3.9451	3.6603	3.6896
c_9	3.4181	−0.5826	3.5109	3.4021	3.4102
c_{10}	5.9454	−1.6932	6.2091	5.9120	5.9582

Least Squares Solution with data $\{\lambda_j, \widetilde{\mathrm{u}}^{(j)}\}_1^s$ (minor error)

	Exact	$s=3$	$s=5$	$s=7$	$s=9$
k_1	12.7586	13.6628	13.2621	12.6760	12.7209
k_2	10.6233	11.9998	11.2915	10.5859	10.6227
k_3	7.8552	8.8830	8.2406	7.7914	7.8335
k_4	13.6456	16.7363	14.3518	13.6887	13.6628
k_5	13.4818	17.7558	14.1613	13.3454	13.4253
k_6	10.0050	9.9822	10.5008	9.9262	10.0035
k_7	11.5915	6.1531	12.2609	11.5661	11.6238
k_8	9.4480	0.6472	9.9126	9.3929	9.4562
k_9	10.1156	−2.8647	10.5204	10.0472	10.1181
k_{10}	7.3799	−2.0449	7.7082	7.3428	7.3947

Relative error	$s=3$	$s=5$	$s=7$	$s=9$
$\frac{\|\mathbf{m}_{\text{appr}}-\mathbf{m}_{\text{ex}}\|}{\|\mathbf{m}_{\text{ex}}\|}$	1.8240	0.1635	0.0368	0.0054
$\frac{\|\mathbf{c}_{\text{appr}}-\mathbf{c}_{\text{ex}}\|}{\|\mathbf{c}_{\text{ex}}\|}$	0.6503	0.0584	0.0092	0.0057
$\frac{\|\mathbf{k}_{\text{appr}}-\mathbf{k}_{\text{ex}}\|}{\|\mathbf{k}_{\text{ex}}\|}$	0.5778	0.0502	0.0064	0.0024

Comparison using data $\{\lambda_j, \widehat{\mathrm{u}}^{(j)}\}_1^s$ (large error)

					LS.
	Exact	$s = 3$	$s = 5$	$s = 7$	$s = 9$
m_1	1.4360	12.9057	10.9010	6.8952	1.7398
m_2	1.5401	−8.2321	−10.0628	−4.8752	−0.9282
m_3	1.1141	8.7035	13.3456	5.9059	2.2882
m_4	1.0754	−7.8355	−12.6486	−4.9820	−0.9015
m_5	1.4964	7.4295	14.0211	6.0671	2.5244
m_6	1.3537	0.6405	−12.0545	−2.4680	0.4464
m_7	1.8337	0.0262	6.2632	4.4962	2.0392
m_8	1.3974	−0.0016	−2.1849	−2.2597	1.8613
m_9	1.2314	0.0047	2.8448	2.5797	2.3767
m_{10}	1.1680	0.0052	3.2213	2.2868	2.1998
					LSP. with $\epsilon = 0.5$
m_1	1.4360	0.7274	1.0949	1.0949	1.0949
m_2	1.5401	0.7274	1.0949	1.0949	1.0949
m_3	1.1141	0.7274	1.0949	1.0949	1.0949
m_4	1.0754	0.7274	1.0949	1.0949	1.0949
m_5	1.4964	0.7274	1.5467	1.5467	1.5467
m_6	1.3537	0.7274	1.0949	1.0949	1.0949
m_7	1.8337	5.0765	1.0949	1.0949	1.0949
m_8	1.3974	0.7274	1.0949	1.0949	1.0949
m_9	1.2314	2.0229	2.2457	2.2457	2.2457
m_{10}	1.1680	1.4549	2.1897	2.1897	2.1897

Comparison using data $\{\lambda_j, \widehat{\mathrm{u}}^{(j)}\}_1^s$ (large error)

					LS.
	Ex	$s = 3$	$s = 5$	$s = 7$	$s = 9$
c_1	4.3780	−1.8481	−6.1334	−0.2534	−0.0546
c_2	4.0110	33.9379	−5.0655	0.5326	−1.0893
c_3	3.1299	1.3655	−15.1989	−1.7273	1.2321
c_4	5.6259	−6.6282	27.1637	6.1650	5.2688
c_5	5.2197	−3.2575	−16.1711	2.4353	5.0864
c_6	5.0297	−1.4070	9.0094	3.1657	4.5888
c_7	5.9495	0.0985	15.2712	7.4984	12.8800
c_8	3.6815	−0.0104	13.6657	5.1899	7.1767
c_9	3.4181	−0.0210	−2.1159	6.2126	4.0477
c_{10}	5.9454	0.0123	12.2913	11.3894	11.0405
					LSP. with $\epsilon = 0.5$
c_1	4.3780	6.7868	9.7403	9.7403	9.7403
c_2	4.0110	0.7274	1.0949	1.0949	1.0949
c_3	3.1299	0.7274	1.0949	1.0949	1.0949
c_4	5.6259	14.3470	6.4661	6.4661	6.4661
c_5	5.2197	10.2324	2.9924	2.9924	2.9924
c_6	5.0297	0.7274	6.5217	6.5217	6.5217
c_7	5.9495	24.3331	13.1154	13.1154	13.1154
c_8	3.6815	0.7274	11.6797	11.6797	11.6797
c_9	3.4181	0.7274	1.0949	1.0949	1.0949
c_{10}	5.9454	3.4087	8.3552	8.3552	8.3552

Comparison using data $\{\lambda_j, \widehat{\mathrm{u}}^{(j)}\}_1^s$ (large error)

					LS.
	Exact	$s = 3$	$s = 5$	$s = 7$	$s = 9$
k_1	12.7586	2.4771	2.3744	−0.2132	1.7622
k_2	10.6233	−1.0043	7.5856	4.0691	4.6821
k_3	7.8552	2.2522	−1.9176	−0.1602	1.9632
k_4	13.6456	5.6359	6.1022	13.9451	10.3818
k_5	13.4818	5.1054	9.2500	2.5029	6.0708
k_6	10.0050	0.3162	11.1928	6.4722	9.8678
k_7	11.5915	0.0074	9.3144	11.9589	12.5322
k_8	9.4480	0.0065	7.9727	12.4826	12.2428
k_9	10.1156	−0.0095	13.4619	15.2008	15.0072
k_{10}	7.3799	0.0031	11.5439	13.4617	12.5696
					LSP. with $\epsilon = 0.5$
k_1	12.7586	0.7274	4.5823	4.5823	4.5823
k_2	10.6233	1.6018	5.0356	5.0356	5.0356
k_3	7.8552	7.7955	3.5757	3.5757	3.5757
k_4	13.6456	18.1074	1.0949	1.0949	1.0949
k_5	13.4818	16.3208	7.9030	7.9030	7.9030
k_6	10.0050	15.6725	3.7800	3.7800	3.7800
k_7	11.5915	1.0196	9.9123	9.9123	9.9123
k_8	9.4480	1.6321	7.1326	7.1326	7.1326
k_9	10.1156	0.7274	9.8310	9.8310	9.8310
k_{10}	7.3799	0.8555	7.8472	7.8472	7.8472

Concluding Remarks

In this talk, we considered two types of IQEPs:

For the first IQEP:

- Express the IQEP as a semidefinite constraint nonlinear optimization problem.

- A dual optimization method proposed

- Quadratically convergent Newton's method

- Efficiency observed from our numerical experiments

- Positive Semidefiniteness of mass and stiffness preserved

For the second IQEP:

- Direct Construction

- Data with minor error: Least squares solution, feasible in practice but physical realistic model not guaranteed

- Data with large error: Positivity-constrained least squares solution, the constructed model is physical realizable.

Future Work:

- Sensitivity analysis in the case of a unique solution

- Robustness in the case of multiple solutions

- Existence theory where M, C or K is other specially structured

- The necessary and sufficient conditions for the mass, damping, and stiffness to be positive