非定常边界层问题整体解的存在唯一性*

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摘要 研究粘性不可压缩流体中的边界层问题. 通过引进变换, 将原来边界层问题化为仅含有一个方程的拟线性抛物方程的初边值问题. 对于不同情形, 分别证明了该问题整体解与局部解的存在唯一性.

关键词 边界层 BV空间 整体解 稳定性与唯一性

1 引言

在1904年于Heidelberg召开的国际数学家大会, Prandtl[1] 提出了一个新的理论——边界层理论, 并利用这个新理论描述了粘性流体运动的主要特点. 在固体周围运动的流体可分为两部分: 一部分是在固体表面附近的一个薄层中运动的流体, 另一部分为在层外运动的流体. 在薄层中, 粘性对于流体的运动状态起着重要的影响作用, 该薄层称为Prandtl边界层; 在边界层外, 粘性对于流体的运动状态的影响是可以忽略不计的. 在适当的物理实验假设下, 边界层问题可由下列方程组描述:

$$\begin{cases} u_t + uu_x + vu_y = \nu(|u_y|^{n-1}u_y)_y + U_t + UU_x, \\ u_x + v_y = 0, \end{cases} \tag{1.1}$$

其中 $(t,x,y) \in D = \{0 < t < T, 0 < x < L, 0 < y < +\infty\}$ 且

$$\begin{cases} u(0,x,y) = u_0(x,y),\ u(t,0,y) = u_1(t,y),\ v(t,x,0) = v_0(t,x), \\ u(t,x,0) = 0,\ \lim\limits_{y\to\infty} u(t,x,y) = U(t,x), \end{cases} \tag{1.2}$$

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其中 ν 是粘性系数， $U=U(t,x)$ 是层外速度.

当 $0<n<1$ 时，方程 (1.1) 描述拟可塑流在对称边界层中的运动状态 (见文献 [2, 3]); 当 $n>1$ 时，方程 (1.1) 描述可膨胀流在对称边界层中的运动状态 (见文献 [4]); 当 $n=1$ 时，方程 (1.1) 描述 Newton 流在对称边界层中的运动状态 (见文献 [1]), 且此时一般称方程 (1.1) 为 Prandtl 边界层方程组.

最近几十年以来，许多数学家和流体力学专家围绕着边界层这一方面问题进行了大量的研究，无论理论上还是数值计算及物理实验方面都有许多文献 (见文献 [2, 5–7]). 在文献 [2] 中， Oleinik 与 Samokhin 系统地研究了定常与非定常的 Prandtl 方程组及拟可塑流边界层问题以及定常情形的可膨胀流边界层问题. 对于非定常情形， Oleinik[7] 与 Samokhin[3] 在一定条件下分别证明了 Prandtl 方程组与拟可塑流边界层问题局部解的存在性. Xin 等 [8]1) 证明了 Prandtl 方程组整体解的存在性与唯一性. 对于非定常可膨胀流边界层的所有问题一直没有解决，这些问题作为未解决问题列在文献 [2] 后面的附录中. 本文通过引进变换，证明了变换后的非定常拟可塑流边界层问题 (包括 Prandtl 方程组) 整体弱解以及非定常可膨胀流边界层问题局部弱解的存在唯一性.

2　定义及结果

不失一般性，假设 $\nu\equiv 1$. 引进新的变量 (τ,ξ,η) 及未知函数

$$w=w(\tau,\xi,\eta),\quad \tau=t,\ \xi=x,\ \eta=u/U;\quad w(\tau,\xi,\eta)=|u_y|^{n-1}u_y/U^{\frac{2n}{1+n}},$$

则方程 (1.1) 和 (1.2) 可化为如下初边值问题:

$$w_\tau+\eta Uw_\xi+Aw_\eta+Bw-n|w|^{\frac{1+n}{n}}w_{\eta\eta}=0,\quad (\tau,\xi,\eta)\in\Omega_T,\tag{2.1}$$

$$w(0,\xi,\eta)=w_0(\xi,\eta),\quad w(\tau,0,\eta)=w_1(\tau,\eta),\quad w(\tau,\xi,1)=0,\tag{2.2}$$

$$|w|^{\frac{1-n}{n}}ww_\eta-v_0U^{\frac{1-n}{1+n}}|w|^{\frac{1-n}{n}}w+(U_\xi+U_\tau/U)=0,\quad \eta=0,\tag{2.3}$$

其中 $\Omega_T=\{(\tau,\xi,\eta)|\,0<\tau<T,0<\xi<L,0<\eta<1\}$ 且

$$\begin{cases} A=(1-\eta^2)U_\xi+(1-\eta)\dfrac{U_\tau}{U},\ B=\dfrac{2n}{1+n}\Big(\eta U_\xi+\dfrac{U_\tau}{U}\Big),\\[2ex] w_0(\xi,\eta)=\dfrac{|u_{0y}|^{n-1}u_{0y}}{U^{\frac{2n}{1+n}}(0,x)},\ w_1(\tau,\eta)=\dfrac{|u_{1y}|^{n-1}u_{1y}}{U^{\frac{2n}{1+n}}(t,0)}.\end{cases}\tag{2.4}$$

令

$$\mathrm{sgn}_\beta s=\int_0^s h_\beta(\tau)d\tau,\quad 其中\ \beta>0,\ h_\beta(s)=\frac{2}{\beta}\Big(1-\frac{|s|}{\beta}\Big)_+.$$

1) Xin Z, Zhang L, Zhao J. Global well-posedness for the two dimensional Prandtl's boundary layer equations, preprint

显然 $h_\beta(s)\in C(\mathbb{R})$ 且

$$h_\beta(s)\geqslant 0,\quad |sh_\beta(s)|\leqslant 1,\quad |\mathrm{sgn}_\beta s|\leqslant 1,\quad \lim_{\beta\to 0}\mathrm{sgn}_\beta s=\mathrm{sgn}s,\quad \lim_{\beta\to 0}sh_\beta(s)=0.$$

定义 2.1 函数 $w\in \mathrm{BV}(\Omega_T)\cap L^\infty(\Omega_T)$ 称为问题 (2.1)–(2.3) 的弱解，如果 $w=w(\tau,\xi,\eta)$ 满足以下条件:

1) $w_\eta\in L^2_{\mathrm{loc}}(\Omega_T)$ 且存在常数 $C>0$, 使得

$$C^{-1}(1-\eta)^\alpha\leqslant w\leqslant C(1-\eta)^\alpha,\quad \forall\,(\tau,\xi,\eta)\in\Omega_T,\tag{2.5}$$

其中 $\alpha=\frac{2n}{1+n}$ 当 $0<n\leqslant 1$ 时， $\alpha=n$ 当 $n>1$ 时;

2) 对任意 $\varphi\in C_0^2(\Omega_T)$, $\varphi\geqslant 0$, w 满足

$$\int_{\Omega_T} I_\beta(V-k)[\varphi_\tau+(\eta U\varphi)_\xi+(A\varphi)_\eta]dX+\int_{\Omega_T}\frac{B}{n}\mathrm{sgn}_\beta(V-k)V\varphi dX$$
$$+\int_{\Omega_T}G_\beta(V,k)\varphi_{\eta\eta}dX-\int_{\Omega_T}nV^{-(1+n)}V_\eta^2\mathrm{sgn}'_\beta(V-k)\varphi dX\geqslant 0,\tag{2.6}$$

其中 $X=(\tau,\xi,\eta)$, $dX=d\tau d\xi d\eta$ 及

$$V=w^{-\frac{1}{n}},\ I_\beta(s)=\int_0^s\mathrm{sgn}_\beta\tau d\tau,\ G_\beta(V,k)=\int_k^V ns^{-(1+n)}\mathrm{sgn}_\beta(s-k)ds;$$

3) $w=w(\tau,\xi,\eta)$ 在迹的意义下满足初边值条件.

注 2.1 (2.6) 式蕴涵着 w 在广义意义下满足问题 (2.1), 从而 $w^{\frac{1+n}{n}}w_{\eta\eta}$ 是 Ω_T 上的有限测度. 事实上，由于 (2.6) 式的最后一项积分非负，令 $\beta\to 0$, 则有

$$\int_{\Omega_T}|V-k|[\varphi_\tau+(\eta U\varphi)_\xi+(A\varphi)_\eta]dX+\int_{\Omega_T}\frac{B}{n}\mathrm{sgn}(V-k)V\varphi dX$$
$$-\int_{\Omega_T}nV^{-(1+n)}V_\eta\mathrm{sgn}(V-k)\varphi_\eta dX\geqslant 0.\tag{2.7}$$

在 (2.7) 式中分别选取 $k>\sup_{supp\varphi}V$ 以及 $k<0$, 则有

$$\int_{\Omega_T}V\Big[\varphi_\tau+(\eta U\varphi)_\xi+(A\varphi)_\eta+\frac{B}{n}\varphi\Big]dX-\int_{\Omega_T}nV^{-(1+n)}V_\eta\varphi_\eta dX=0.\tag{2.8}$$

令 Γ_u 表示 $u\in\mathrm{BV}(\Omega_T)$ 的全体间断点构成的集合，ν 表示 Γ_u 在 $X=(\tau,\xi,\eta)$ 的单位法方向，u^+ 与 u^- 表示 u 在 $X\in\Gamma_u$ 上对应于 $(\nu,Y-X)>0$ 与 $(\nu,Y-X)<0$ 的渐进极限. 对于连续函数 $P(u)$ 以及函数 $u\in\mathrm{BV}(\Omega_T)$, 称

$$\widehat{P(u)}=\int_0^1 P(\tau u^++(1-\tau)u^-)d\tau$$

为 P 与 u 的复合均值. 由文献 [9], 对于 BV 函数成立如下复合求导公式：如果 $f(s)\in C^1(R)$, $u\in\mathrm{BV}(\Omega_T)\cap L^\infty(\Omega_T)$, 则 $f(u)\in\mathrm{BV}(\Omega_T)$ 且

$$\frac{\partial f(u)}{\partial x_i}=\widehat{f'(u)}\frac{\partial u}{\partial x_i},\ \ i=1,2,\cdots,N,$$

因此 (2.8) 式可改写为

$$\int_{\Omega_T}(\widehat{w^{\frac{1}{n}}})^{-1}\Big[\varphi_\tau+(\eta U\varphi)_\xi+(A\varphi)_\eta+\frac{B}{n}\varphi\Big]dX+\int_{\Omega_T}w_\eta\varphi_\eta dX=0. \tag{2.9}$$

由于 $w\in \mathrm{BV}(\Omega_T), w_\eta\in L^2_{\mathrm{loc}}(\Omega_T)$, 通过逼近，可以在 (2.9) 式中选取 $\varphi=\varphi_1 w^{\frac{1+n}{n}}$, 其中 $\varphi_1\in C_0^2(\Omega_T), \varphi_1\geqslant 0$, 由分部积分可得

$$\int_{\Omega_T}w\Big[\varphi_{1\tau}+(\eta U\varphi_1)_\xi+(A\varphi_1)_\eta-B\varphi_1\Big]dX-\int_{\Omega_T}n\varphi_{1\eta}w^{\frac{1+n}{n}}w_\eta dX$$

$$-\int_{\Omega_T}(1+n)w^{\frac{1}{n}}(w_\eta)^2\varphi_1 dX=0.$$

上式表明 w 在广义意义下满足问题 (2.1), 从而 $w^{\frac{1+n}{n}}w_{\eta\eta}$ 是 Ω_T 上的有限测度.

注 2.2　定义 2.1 比通常熵解的定义要强，但是根据定义 2.1 可以证明问题 (2.1)–(2.3) 弱解的唯一性，因此定义 2.1 对于问题 (2.1)–(2.3) 来说是合理的.

本文的主要结果如下：

定理 2.1　假设 $0<n\leqslant 1$, $U(\tau,\xi), v_0(\tau,\xi), w_0(\xi,\eta), w_1(\tau,\eta)$ 充分光滑,

$$U(\tau,\xi)>0,\quad U_\tau+UU_\xi\geqslant 0,\quad v_0(\tau,\xi)\leqslant 0,\quad w_0(\tau,\xi)\geqslant 0,\quad w_1(\tau,\eta)\geqslant 0, \tag{2.10}$$

且存在常数 $C_0>0$, 使得

$$C_0^{-1}(1-\eta)^{\frac{2n}{1+n}}\leqslant w_0(\xi,\eta),\quad w_1(\tau,\eta)\leqslant C_0(1-\eta)^{\frac{2n}{1+n}}, \tag{2.11}$$

其中 w_0, w_1 为 (2.4) 式中定义的函数. 则对任意给定的 $T>0$, 问题 (2.1)–(2.3) 存在弱解 $w\in \mathrm{BV}(\Omega_T)\cap L^\infty(\Omega_T)$.

定理 2.2　假设 $n>1$, (2.10) 式成立且存在常数 $C_0>0$, 使得

$$C_0^{-1}(1-\eta)^n\leqslant w_0(\xi,\eta),\ w_1(\tau,\eta)\leqslant C_0(1-\eta)^n, \tag{2.12}$$

则问题 (2.1)–(2.3) 存在着局部 (即时间 T 足够小) 弱解 $w\in \mathrm{BV}(\Omega_T)\cap L^\infty(\Omega_T)$.

定理 2.3　问题 (2.1)–(2.3) 的弱解是唯一的.

注 2.3　假设 (2.10) 式满足边界层问题的物理条件. 假设 (2.11) 及 (2.12) 式对于证明定理 2.1 和 2.2 非常重要. 当 $n=1$ 时，文献 [2] 与文献 [8] 也提出类似的假设. 由 (2.11) 及 (2.12) 式，可知 w 满足 (2.5) 式 (见引理 3.1 和 4.1), 而由 (2.5) 式可以推出 $u(t,x,y)\to U(t,x)$ 当 $y\to\infty$.

3　定理 2.1 的证明

我们利用抛物正则化方法证明问题 (2.1)–(2.3) 解的存在性. 考虑正则化问题

$$w_\tau+\eta U w_\xi+Aw_\eta+Bw-n\Psi_\varepsilon^{\frac{1+n}{n}}(w)w_{\eta\eta}-\varepsilon n\Psi_\varepsilon^{\frac{1+n}{n}}(w)w_{\xi\xi}=0,\ (\tau,\xi,\eta)\in\Omega_T, \tag{3.1}$$

$$w|_{\tau=0}=w_{0\varepsilon}(\xi,\eta),\ w|_{\xi=0}=w_{1\varepsilon}(\tau,\eta),\ w|_{\xi=L}=w_{2\varepsilon}(\tau,\eta),\ w|_{\eta=1}=\varepsilon^{\frac{2n}{1+n}}, \tag{3.2}$$

$$\Big[\Psi_\varepsilon^{\frac{1}{n}}(w)w_\eta-v_0U^{\frac{1-n}{1+n}}\Psi_\varepsilon^{\frac{1}{n}}(w)+(U_\xi+U_\tau/U)\Big]\Big|_{\eta=0}=0, \tag{3.3}$$

其中 $0<n\leqslant 1$, A 和 B 如 (2.4) 式所定义, $w_{0\varepsilon}, w_{1\varepsilon}, w_{2\varepsilon}$ 充分光滑,

$$\begin{cases} w_{0\varepsilon}\to w_0,\ w_{1\varepsilon}\to w_1, & \text{当 } \varepsilon\to 0 \text{ 时};\\ w_{0\varepsilon}|_{\eta=1}=w_{1\varepsilon}|_{\eta=1}=w_{2\varepsilon}|_{\eta=1}=\varepsilon^{\frac{2n}{1+n}};\\ C_0^{-1}(1+\varepsilon-\eta)^{\frac{2n}{1+n}}\leqslant w_{0\varepsilon}, w_{1\varepsilon}, w_{2\varepsilon}\leqslant C_0(1+\varepsilon-\eta)^{\frac{2n}{1+n}}. \end{cases} \tag{3.4}$$

函数 $\Psi_\varepsilon(s)\in C^1(R)$, $0\leqslant \Psi'_\varepsilon(s)\leqslant 1$ 定义如下:

$$\Psi_\varepsilon(w)=\begin{cases} w, & \text{当 } w\geqslant \mu\varepsilon^{\frac{2n}{1+n}} \text{ 时};\\ \dfrac{\mu}{2}\varepsilon^{\frac{2n}{1+n}}, & \text{当 } w\leqslant 0 \text{ 时}, \end{cases} \tag{3.5}$$

其中 $\mu>0$ 是待定常数.

由文献 [10] 可知问题 (3.1)–(3.3) 存在唯一解 $w_\varepsilon\in C^2(\overline{\Omega}_T)\cap C^3(\Omega_T)$, 且由极值原理有 $w_\varepsilon\geqslant C\varepsilon^{\frac{2n}{1+n}}$, 其中 C 为与 ε 无关的常数. 在 (3.5) 式中选取 $\mu=C$, 则问题 (3.1) 与 (3.3) 可改写为

$$w_\tau+\eta U w_\xi+Aw_\eta+Bw-nw^{\frac{1+n}{n}}w_{\eta\eta}-\varepsilon n w^{\frac{1+n}{n}}w_{\eta\eta}=0,\quad (\tau,\xi,\eta)\in\Omega_T, \tag{3.6}$$

$$\left[w^{\frac{1}{n}}w_\eta-v_0U^{\frac{1-n}{1+n}}w^{\frac{1}{n}}+(U_\xi+U_\tau/U)\right]\Big|_{\eta=0}=0. \tag{3.7}$$

引理 3.1 存在与 ε 无关的常数 C_1, 使得

$$C_1^{-1}(1+\varepsilon-\eta)^{\frac{2n}{1+n}}\leqslant w_\varepsilon\leqslant C_1(1+\varepsilon-\eta)^{\frac{2n}{1+n}}.$$

证 定义

$$L_\varepsilon(V)=V_\tau+\eta UV_\xi+AV_\eta+BV-nV^{\frac{1+n}{n}}V_{\eta\eta}-\varepsilon nV^{\frac{1+n}{n}}V_{\xi\xi},$$

$$l_\varepsilon(V)=V^{\frac{1}{n}}V_\eta-v_0U^{\frac{1-n}{1+n}}V^{\frac{1}{n}}+\left(U_\xi+\frac{U_\tau}{U}\right),\quad \text{当 } \eta=0 \text{ 时}.$$

令

$$V_1=C(1+\varepsilon-\eta)^{\frac{2n}{1+n}}e^{\beta\tau},$$

其中 C,β 待定. 注意到 $|A|\leqslant M(1+\varepsilon-\eta)$ 及 $0<n\leqslant 1$, 选取 C, β 充分大, 使得

$$L_\varepsilon(V_1)=Ce^{\beta\tau}(1+\varepsilon-\eta)^{\frac{2n}{1+n}}\left[\beta+\frac{2n^2(1-n)}{(1+n)^2}(Ce^{\beta\tau})^{\frac{1+n}{n}}-\frac{2n}{1+n}\frac{A}{1+\varepsilon-\eta}+B\right]>0,$$

$$l_\varepsilon(V_1)=-\frac{2n}{1+n}(Ce^{\beta\tau})^{\frac{1+n}{n}}(1+\varepsilon)-(Ce^{\beta\tau})^{\frac{1}{n}}v_0U^{\frac{1-n}{1+n}}(1+\varepsilon)^{\frac{2}{1+n}}+(U_\xi+U_\tau/U)<0.$$

令 $Z=V_1-w$, 则有

$$\begin{aligned} &L_\varepsilon(V_1)-L_\varepsilon(w)\\ &=Z_\tau+\eta UZ_\xi+AZ_\eta+BZ-n(V_1^{\frac{1+n}{n}}V_{1\eta\eta}-w^{\frac{1+n}{n}}w_{\eta\eta})-\varepsilon nw^{\frac{1+n}{n}}Z_{\xi\xi} \end{aligned}$$

$$= Z_\tau + \eta U Z_\xi + A Z_\eta + \Big[B - (1+n)\int_0^1 (\theta V_1 + (1-\theta)w)^{\frac{1}{n}} d\theta V_{1\eta\eta}\Big] Z$$

$$-nw^{\frac{1+n}{n}} Z_{\eta\eta} - \varepsilon n w^{\frac{1+n}{n}} Z_{\xi\xi} > 0,$$

$$l_\varepsilon(V_1)/V_1^{\frac{1}{n}} - l_\varepsilon(w)/w^{\frac{1}{n}}$$

$$= Z_\eta + (U_\xi + U_\tau/U)(V_1^{-\frac{1}{n}} - w^{-\frac{1}{n}})$$

$$= Z_\eta - \frac{1}{n}(U_\xi + U_\tau/U)\Big[\int_0^1 (\theta V_1 + (1-\theta)w)^{-\frac{1+n}{n}} d\theta\Big] Z < 0.$$

由 (3.4) 式可知，如果选取 $C>0$ 足够大，则有

$$Z|_{\tau=0} \geqslant 0,\quad Z|_{\xi=0} \geqslant 0,\quad Z|_{\eta=1} \geqslant 0,\quad Z|_{\xi=L} \geqslant 0,$$

因此，由极值原理可得 $Z \geqslant 0$, 即

$$w \leqslant C_1(1+\varepsilon-\eta)^{\frac{2n}{1+n}}.$$

为了得到 w_ε 的下界估计，考虑函数

$$V_2 = C(1+\varepsilon-\eta)^{\frac{2n}{1+n}} e^{\alpha\eta-\beta\tau},\quad \text{其中 } C,\ \alpha,\ \beta \text{ 待定.}$$

选取 $\alpha > \frac{2n}{1+n}(1+\varepsilon)^{-1}$ 且 β 足够大，使得

$$L_\varepsilon(V_2) = C(1+\varepsilon-\eta)^{\frac{2n}{1+n}} e^{\alpha\eta-\beta\tau}\Big[-\beta + \alpha A - \frac{2n}{1+n}\frac{A}{1+\varepsilon-\eta} + B\Big]$$

$$-nV_2^{\frac{1+n}{n}}\Big[\frac{2n(n-1)}{(1+n)^2}(1+\varepsilon-\eta)^{-\frac{2}{1+n}} + \alpha^2(1+\varepsilon-\eta)^{\frac{2n}{1+n}}$$

$$-\frac{4n}{1+n}\alpha(1+\varepsilon-\eta)^{\frac{n-1}{1+n}}\Big] C e^{\alpha\eta-\beta\tau}$$

$$= C(1+\varepsilon-\eta)^{\frac{2n}{1+n}} e^{\alpha\eta-\beta\tau}\Big\{-\beta + \alpha A - \frac{2n}{1+n}\frac{A}{1+\varepsilon-\eta} + B$$

$$-n(Ce^{\alpha\eta-\beta\tau})^{\frac{1+n}{n}}\Big[\frac{2n(n-1)}{(1+n)^2} + \alpha^2(1+\varepsilon-\eta)^2 - \frac{4n\alpha}{1+n}(1+\varepsilon-\eta)\Big]\Big\}$$

$$< 0,$$

$$l_\varepsilon(V_2) = V_2^{\frac{1}{n}} C e^{-\beta\tau}\Big[\alpha(1+\varepsilon)^{\frac{2n}{1+n}} - \frac{2n}{1+n}(1+\varepsilon)^{\frac{n-1}{1+n}}\Big] - v_0 U^{\frac{1-n}{1+n}} V_2^{\frac{1}{n}}$$

$$+\Big(U_\xi + \frac{U_\tau}{U}\Big) > 0.$$

另外，选取 C 充分小，使得

$$V_2 \leqslant w,\quad \text{当 } \tau=0,\ \xi=0, L,\ \eta=1 \text{ 时.}$$

令 $Z = V_2 - w$, 类似地可证明 $Z \leqslant 0$, 即

$$w \geqslant C(1+\varepsilon-\eta)^{\frac{2n}{1+n}} e^{\alpha\eta-\beta\tau} \geqslant C_1(1+\varepsilon-\eta)^{\frac{2n}{1+n}}.$$

引理证毕.

为了估计 $w_{\varepsilon\eta}$, 令

$$V = w_\varepsilon^{-\frac{1}{n}}, \quad \varphi = \frac{\xi}{L} w_{2\varepsilon} + \frac{L-\xi}{L} w_{1\varepsilon}.$$

由引理 3.1 及 (3.4) 式可知 V 与 φ 满足

$$\begin{cases} C^{-1}(1+\varepsilon-\eta)^{-\frac{2}{1+n}} \leqslant V \leqslant C(1+\varepsilon-\eta)^{-\frac{2}{1+n}}, \\ V^{-n} - \varphi = 0, \quad \text{当 } \xi = 0, L, \ \eta = 1 \text{ 时}; \end{cases} \tag{3.8}$$

及

$$\begin{cases} C^{-1}(1+\varepsilon-\eta)^{\frac{2n}{1+n}} \leqslant \varphi \leqslant C(1+\varepsilon-\eta)^{\frac{2n}{1+n}}, \\ |\varphi_\tau|, |\varphi_\xi| \leqslant C(1+\varepsilon-\eta)^{\frac{2n}{1+n}}, \ |\varphi_\eta| \leqslant C(1+\varepsilon-\eta)^{\frac{n-1}{1+n}}. \end{cases} \tag{3.9}$$

且由 (3.6) 式可知 V 满足

$$V_\tau + \eta U V_\xi + A V_\eta - \frac{1}{n} B V + (V^{-n})_{\eta\eta} + \varepsilon (V^{-n})_{\xi\xi} = 0. \tag{3.10}$$

引理 3.2 设 $0 < n < 1$, 则

$$\int_{\Omega_T} (1+\varepsilon-\eta)^\alpha [(V^{-n})_\eta^2 + \varepsilon (V^{-n})_\xi^2] dX \leqslant C,$$

其中 $\frac{1-3n}{1+n} < \alpha < 1$, $C > 0$ 与 ε 无关.

证 将 (3.10) 式两端同乘 $(1+\varepsilon-\eta)^\alpha (V^{-n} - \varphi)$ 并在 Ω_T 上积分, 可得

$$\begin{aligned} &\int_{\Omega_T} (1+\varepsilon-\eta)^\alpha (V^{-n}-\varphi) V_\tau dX + \int_{\Omega_T} \eta U (1+\varepsilon-\eta)^\alpha (V^{-n}-\varphi) V_\xi dX \\ &+ \int_{\Omega_T} A (1+\varepsilon-\eta)^\alpha (V^{-n}-\varphi) V_\eta dX - \int_{\Omega_T} \frac{1}{n} B V (1+\varepsilon-\eta)^\alpha (V^{-n}-\varphi) dX \\ &+ \int_{\Omega_T} (1+\varepsilon-\eta)^\alpha (V^{-n}-\varphi) \big[(V^{-n})_{\eta\eta} + \varepsilon (V^{-n})_{\xi\xi} \big] dX = \sum_{i=1}^{6} I_i = 0. \end{aligned} \tag{3.11}$$

现在估计 (3.11) 式中的每一项. 首先由 (3.8) 与 (3.9) 式及注意到 $\frac{1-3n}{1+n} < \alpha < 1$, 有

$$\begin{aligned} I_1 &= \int_{\Omega_T} (1+\varepsilon-\eta)^\alpha \frac{1}{1-n} (V^{1-n})_\tau dX - \int_{\Omega_T} (1+\varepsilon-\eta)^\alpha V_\tau \varphi dX \\ &= \int_{\Omega} (1+\varepsilon-\eta)^\alpha \frac{1}{1-n} V^{1-n} d\xi d\eta \Big|_{\tau=0}^{\tau=T} - \int_{\Omega} (1+\varepsilon-\eta)^\alpha \varphi V d\xi d\eta \Big|_{\tau=0}^{\tau=T} \\ &\quad + \int_{\Omega_T} (1+\varepsilon-\eta)^\alpha V \varphi_\tau dX \\ &\leqslant C \int_{\Omega} (1+\varepsilon-\eta)^{\alpha + \frac{2(n-1)}{1+n}} d\xi d\eta + C \int_{\Omega_T} (1+\varepsilon-\eta)^{\alpha + \frac{2(n-1)}{1+n}} dX \leqslant C_1, \end{aligned}$$

其中 $\Omega = \{0 < \xi < L, 0 < \eta < 1\}$. 类似地可证明 $I_2 \leqslant C$. 由分部积分, 利用 (3.8)

与 (3.9) 式及 Cauchy-Schwarz 不等式，可得

$$
\begin{aligned}
I_3 = & \int_{\Omega_1} A(1+\varepsilon-\eta)^\alpha (V^{-n}-\varphi) V d\tau d\xi \Big|_{\eta=0}^{\eta=1} \\
& - \int_{\Omega_T} A(1+\varepsilon-\eta)^\alpha V (V^{-n}-\varphi)_\eta dX \\
& - \int_{\Omega_T} [A(1+\varepsilon-\eta)^\alpha]_\eta V (V^{-n}-\varphi) dX \\
\leqslant & - \int_{\Omega_1} A(1+\varepsilon-\eta)^\alpha (V^{-n}-\varphi) V d\tau d\xi \Big|_{\eta=0} \\
& - \int_{\Omega_T} A(1+\varepsilon-\eta)^\alpha V (V^{-n})_\eta dX \\
& + \int_{\Omega_T} A(1+\varepsilon-\eta)^\alpha V \varphi_\eta dX + C \int_{\Omega_T} (1+\varepsilon-\eta)^{\alpha+\frac{2(n-1)}{1+n}} dX \\
\leqslant & - \int_{\Omega_1} A(1+\varepsilon-\eta)^\alpha (V^{-n}-\varphi) V d\tau d\xi \Big|_{\eta=0} \\
& + \delta \int_{\Omega_T} (1+\varepsilon-\eta)^\alpha (V^{-n})_\eta^2 dX + C(\delta),
\end{aligned}
$$

其中 $\Omega_1 = \{0<\tau<T, 0<\xi<L\}$. 由 (3.8) 与 (3.9) 式，有

$$
I_4 \leqslant C \int_{\Omega_T} (1+\varepsilon-\eta)^{\alpha+\frac{2(n-1)}{1+n}} dX \leqslant C.
$$

最后还需估计 I_5 以及 I_6. 由分部积分，有

$$
\begin{aligned}
I_5 = & \int_{\Omega_1} (1+\varepsilon-\eta)^\alpha (V^{-n}-\varphi)(V^{-n})_\eta d\tau d\xi \Big|_{\eta=0}^{\eta=1} \\
& - \int_{\Omega_T} (1+\varepsilon-\eta)^\alpha (V^{-n})_\eta^2 dX + \int_{\Omega_T} (1+\varepsilon-\eta)^\alpha (V^{-n})_\eta \varphi_\eta dX \\
& + \alpha \int_{\Omega_T} (1+\varepsilon-\eta)^{\alpha-1} (V^{-n}-\varphi)(V^{-n})_\eta dX.
\end{aligned}
$$

由于 $\frac{1-3n}{1+n} < \alpha < 1$, 利用 (3.8) 与 (3.9) 式以及 Cauchy-Schwarz 不等式，可得

$$
\begin{aligned}
& \left| \int_{\Omega_T} (1+\varepsilon-\eta)^\alpha (V^{-n})_\eta \varphi_\eta dX \right| \\
\leqslant & \delta \int_{\Omega_T} (1+\varepsilon-\eta)^\alpha (V^{-n})_\eta^2 dX + C(\delta) \int_{\Omega_T} (1+\varepsilon-\eta)^\alpha \varphi_\eta^2 dX \\
\leqslant & C(\delta) + \delta \int_{\Omega_T} (1+\varepsilon-\eta)^\alpha (V^{-n})_\eta^2 dX,
\end{aligned}
$$

$$\left|\int_{\Omega_T}\alpha(1+\varepsilon-\eta)^{\alpha-1}(V^{-n}-\varphi)(V^{-n})_\eta dX\right|$$
$$\leqslant \delta\int_{\Omega_T}(1+\varepsilon-\eta)^{\alpha}(V^{-n})_\eta^2 dX + C(\delta)\int_{\Omega_T}(1+\varepsilon-\eta)^{\alpha-2}(V^{-n}-\varphi)^2 dX$$
$$\leqslant C(\delta)+\delta\int_{\Omega_T}(1+\varepsilon-\eta)^{\alpha}(V^{-n})_\eta^2 dX,$$

因此

$$-I_5 \geqslant \int(1+\varepsilon-\eta)^{\alpha}(V^{-n}-\varphi)(V^{-n})_\eta d\tau d\xi\Big|_{\eta=0}$$
$$+(1-2\delta)\int_{\Omega_T}(1+\varepsilon-\eta)^{\alpha}(V^{-n})_\eta^2 dX - C(\delta).$$

注意到 $V^{-n}-\varphi=0$ 当 $\xi=0, L$ 时，类似地可证明

$$-I_6 \geqslant \varepsilon(1-\delta)\int_{\Omega_T}(1+\varepsilon-\eta)^{\alpha}(V^{-n})_\xi^2 dX - C(\delta).$$

由 (3.7) 及 (3.8) 式可知 I_3 和 I_5 中的边界积分是有界的．选取 $\delta=1/4$, 由 I_1–I_6 的估计以及 (3.11) 式即可证明引理．

引理 3.3 设 $n=1$, 则存在与 ε 无关的常数 C, 使得

$$\int_{\Omega_T}(1+\varepsilon-\eta)^{\beta}[(V^{-1})_\eta^2+\varepsilon(V^{-1})_\xi^2]dX \leqslant C, \quad 其中 \quad \beta\in(0,1).$$

证 (3.10) 式两端同乘 $(1+\varepsilon-\eta)^{\beta}(V^{-1}-\varphi)$, 并注意到

$$|(1+\varepsilon-\eta)^{\beta}\ln V| \leqslant C|(1+\varepsilon-\eta)^{\beta}\ln(1+\varepsilon-\eta)| \leqslant C,$$

类似引理 3.2 的证明即可得引理 3.3.

利用引理 3.1–3.3 以及 Höder 不等式，可得如下推论：

推论 3.1 设 $0<n\leqslant 1$, w_ε 是问题 (3.6)、(3.7)、(3.2) 的解，则存在与 ε 无关的常数 $C>0$, 使得

$$\int_{\Omega_T}|w_{\varepsilon\eta}|d\tau d\xi d\eta \leqslant C.$$

接下来估计 $w_{\varepsilon\tau}$ 和 $w_{\varepsilon\xi}$. 为此令

$$\overline{w}=w-\varphi, \quad \varphi=\frac{\xi}{L}w_{2\varepsilon}+\frac{L-\xi}{L}w_{1\xi},$$

则显然

$$\overline{w}=0, \quad 当\ \xi=0, L,\ \eta=1\ 时.$$

且由 (3.4) 式可知 φ 满足不等式

$$\begin{cases} C_0^{-1}(1+\varepsilon-\eta)^{\frac{2n}{1+n}} \leqslant \varphi \leqslant C_0(1+\varepsilon-\eta)^{\frac{2n}{1+n}}; \\ |\varphi_\tau|, |\varphi_\xi| \leqslant M(1+\varepsilon-\eta)^{\frac{2n}{1+n}},\ |\varphi_\eta| \leqslant M(1+\varepsilon-\eta)^{\frac{n-1}{1+n}}; \\ |\varphi_{\tau\tau}|, |\varphi_{\tau\xi}|, |\varphi_{\xi\xi}| \leqslant M(1+\varepsilon-\eta)^{\frac{2n}{1+n}};\ |\varphi_{\tau\eta}|, |\varphi_{\xi\eta}| \leqslant M(1+\varepsilon-\eta)^{\frac{n-1}{1+n}}; \\ |\varphi_{\xi\xi\tau}|, |\varphi_{\xi\xi\xi}| \leqslant M(1+\varepsilon-\eta)^{\frac{2n}{1+n}};\ |\varphi_{\tau\eta\eta}|, |\varphi_{\xi\eta\eta}| \leqslant M(1+\varepsilon-\eta)^{-\frac{2}{1+n}}. \end{cases} \tag{3.12}$$

另外由 (3.6) 式可知 $\overline{w}$ 满足

$$\frac{1}{n}w^{-\frac{1+n}{n}}\overline{w}_\tau=\overline{w}_{\eta\eta}+\varepsilon\overline{w}_{\xi\xi}+\varphi_{\eta\eta}+\varepsilon\varphi_{\xi\xi}-\frac{1}{n}\eta Uw^{-\frac{1+n}{n}}w_\xi-\frac{A}{n}w^{-\frac{1+n}{n}}w_\eta$$
$$-\frac{B}{n}w^{-\frac{1}{n}}-\frac{1}{n}w^{-\frac{1+n}{n}}\varphi_\tau. \tag{3.13}$$

将 (3.13) 式两端关于 τ 求导一次，然后同乘 $(1+\varepsilon-\eta)^2\mathrm{sgn}_\beta\overline{w}_\tau$, 其中 $\beta>0$, 并在 $\Omega_t=\{0<\tau<t,0<\xi<L,0<\eta<1\}$ 上积分，其中 $0<t<T$, 可得

$$\begin{aligned}L=&\int_{\Omega_t}\frac{1}{n}w^{-\frac{1+n}{n}}\overline{w}_{\tau\tau}(1+\varepsilon-\eta)^2\mathrm{sgn}_\beta\overline{w}_\tau dX\\
&-\int_{\Omega_t}\frac{1+n}{n^2}w^{-\frac{1+2n}{n}}\overline{w}_\tau w_\tau(1+\varepsilon-\eta)^2\mathrm{sgn}_\beta\overline{w}_\tau dX\\
=&\int_{\Omega_t}\overline{w}_{\eta\eta\tau}(1+\varepsilon-\eta)^2\mathrm{sgn}_\beta\overline{w}_\tau dX+\varepsilon\int_{\Omega_t}\overline{w}_{\xi\xi\tau}(1+\varepsilon-\eta)^2\mathrm{sgn}_\beta\overline{w}_\tau dX\\
&-\frac{1}{n}\int_{\Omega_t}(w^{-\frac{1+n}{n}}\varphi_\tau)_\tau(1+\varepsilon-\eta)^2\mathrm{sgn}_\beta\overline{w}_\tau dX\\
&+\int_{\Omega_t}(\varphi_{\eta\eta}+\varepsilon\varphi_{\xi\xi})_\tau(1+\varepsilon-\eta)^2\mathrm{sgn}_\beta\overline{w}_\tau dX\\
&-\frac{1}{n}\int_{\Omega_t}(\eta Uw^{-\frac{1+n}{n}}w_\xi)_\tau(1+\varepsilon-\eta)^2\mathrm{sgn}_\beta\overline{w}_\tau dX\\
&-\frac{1}{n}\int_{\Omega_t}(Aw^{-\frac{1+n}{n}}w_\eta)_\tau(1+\varepsilon-\eta)^2\mathrm{sgn}_\beta\overline{w}_\tau dX\\
&-\frac{1}{n}\int_{\Omega_t}(Bw^{-\frac{1+n}{n}}w)_\tau(1+\varepsilon-\eta)^2\mathrm{sgn}_\beta\overline{w}_\tau dX=\sum_{i=1}^{7}I_i.\end{aligned}\tag{3.14}$$

由分部积分，(3.14) 式的左端可估计如下：

$$\begin{aligned}L=&\int_{\Omega_t}\frac{1}{n}w^{-\frac{1+n}{n}}\overline{w}_{\tau\tau}(1+\varepsilon-\eta)^2\mathrm{sgn}_\beta\overline{w}_\tau dX\\
&-\int_{\Omega_t}\frac{1+n}{n^2}w^{-\frac{1+2n}{n}}\overline{w}_\tau w_\tau(1+\varepsilon-\eta)^2\mathrm{sgn}_\beta\overline{w}_\tau dX\\
=&\int_{\Omega_t}\frac{1}{n}w^{-\frac{1+n}{n}}\Big(\frac{\partial}{\partial\tau}\int_0^{\overline{w}_\tau}\mathrm{sgn}_\beta sds\Big)(1+\varepsilon-\eta)^2dX\\
&-\int_{\Omega_t}\frac{1+n}{n^2}w^{-\frac{1+2n}{n}}\overline{w}_\tau w_\tau(1+\varepsilon-\eta)^2\mathrm{sgn}_\beta\overline{w}_\tau dX\\
=&\int_{\Omega}\frac{1}{n}w^{-\frac{1+n}{n}}\Big(\int_0^{\overline{w}_\tau}\mathrm{sgn}_\beta sds\Big)(1+\varepsilon-\eta)^2d\xi d\eta\Big|_{\tau=0}^{\tau=t}\end{aligned}$$

$$+\int_{\Omega_t}\frac{1+n}{n^2}w^{-\frac{1+2n}{n}}w_\tau\Big(\int_0^{\overline{w}_\tau}\mathrm{sgn}_\beta s ds\Big)(1+\varepsilon-\eta)^2dX$$

$$-\int_{\Omega_t}\frac{1+n}{n^2}w^{-\frac{1+2n}{n}}\overline{w}_\tau w_\tau(1+\varepsilon-\eta)^2\mathrm{sgn}_\beta\overline{w}_\tau dX$$

$$\to\int_{\Omega}\frac{1}{n}w^{-\frac{1+n}{n}}|\overline{w}_\tau|(1+\varepsilon-\eta)^2d\xi d\eta\Big|_{\tau=0}^{\tau=t},\quad 当\ \beta\to 0\ 时. \tag{3.15}$$

下面估计 (3.14) 式的右端各项：首先注意到

$$I_1=\int_{\Omega_t}\overline{w}_{\eta\eta\tau}(1+\varepsilon-\eta)^2\mathrm{sgn}_\beta\overline{w}_\tau dX$$

$$=\int(1+\varepsilon-\eta)^2\overline{w}_{\tau\eta}\mathrm{sgn}_\beta\overline{w}_\tau d\tau d\xi\Big|_{\eta=0}^{\eta=1}$$

$$-\int_{\Omega_t}(1+\varepsilon-\eta)^2\overline{w}_{\tau\eta}^2\mathrm{sgn}'_\beta\overline{w}_\tau dX+2\int_{\Omega_t}(1+\varepsilon-\eta)\overline{w}_{\tau\eta}\mathrm{sgn}_\beta\overline{w}_\tau dX$$

及

$$\int_{\Omega_t}(1+\varepsilon-\eta)\overline{w}_{\tau\eta}\mathrm{sgn}_\beta\overline{w}_\tau dX$$

$$=\int(1+\varepsilon-\eta)\Big(\int_0^{\overline{w}_\tau}\mathrm{sgn}_\beta s ds\Big)d\xi d\tau\Big|_{\eta=0}^{\eta=1}+\int_{\Omega_t}\Big(\int_0^{\overline{w}_\tau}\mathrm{sgn}_\beta s ds\Big)dX$$

$$\leqslant\int_{\Omega_t}\Big(\int_0^{\overline{w}_\tau}\mathrm{sgn}_\beta s ds\Big)dX,$$

因此令 $\beta\to 0$, 则有

$$\lim_{\beta\to 0}I_1\leqslant 2\int_{\Omega_t}|\overline{w}_\tau|dX-\lim_{\beta\to 0}\int(1+\varepsilon-\eta)^2\overline{w}_{\tau\eta}\mathrm{sgn}_\beta\overline{w}_\tau d\tau d\xi\Big|_{\eta=0}. \tag{3.16}$$

由于 $\overline{w}=0$ 当 $\xi=0,L$ 时，因此类似地有

$$I_2=\varepsilon\int\overline{w}_{\xi\tau}(1+\varepsilon-\eta)^2\mathrm{sgn}_\beta\overline{w}_\tau d\tau d\eta\Big|_{\xi=0}^{\xi=L}$$

$$-\varepsilon\int_{\Omega_t}(1+\varepsilon-\eta)^2\overline{w}_{\xi\tau}^2\mathrm{sgn}'_\beta\overline{w}_\tau dX\leqslant 0. \tag{3.17}$$

由引理 3.1 及 (3.12) 式，可得

$$I_3=-\frac{1}{n}\int_{\Omega_t}w^{-\frac{1+n}{n}}\varphi_{\tau\tau}(1+\varepsilon-\eta)^2\mathrm{sgn}_\beta\overline{w}_\tau dX$$

$$+\frac{1+n}{n^2}\int_{\Omega_t}w^{-\frac{1+2n}{n}}\varphi_\tau(\overline{w}_\tau+\varphi_\tau)(1+\varepsilon-\eta)^2\mathrm{sgn}_\beta\overline{w}_\tau dX$$

$$\leqslant C\Big(1+\int_{\Omega_t}|\overline{w}_\tau|dX\Big), \tag{3.18}$$

$$I_4\leqslant C,\quad I_7\leqslant C\Big(1+\int_{\Omega_t}|\overline{w}_\tau|dX\Big). \tag{3.19}$$

下面估计 I_5. 显然

$$I_5 = -\frac{1}{n}\int_{\Omega_t} \eta U_\tau w^{-\frac{1+n}{n}} w_\xi (1+\varepsilon-\eta)^2 sgn_\beta \overline{w}_\tau dX$$
$$-\frac{1}{n}\int_{\Omega_t} \eta U (w^{-\frac{1+n}{n}} w_\xi)_\tau (1+\varepsilon-\eta)^2 sgn_\beta \overline{w}_\tau dX$$
$$= I_5^1 + I_5^2.$$

由 (3.12) 式及引理 3.1, 有

$$I_5^1 = -\frac{1}{n}\int_{\Omega_t} \eta U_\tau w^{-\frac{1+n}{n}} (\overline{w}_\xi + \varphi_\xi)(1+\varepsilon-\eta)^2 \mathrm{sgn}_\beta \overline{w}_\tau dX$$
$$\leqslant C\Big(1 + \int_{\Omega_t} |\overline{w}_\xi| dX\Big).$$

对于 I_5^2 这项，有

$$I_5^2 = -\frac{1}{n}\int_{\Omega_t} \eta U (w^{-\frac{1+n}{n}} \overline{w}_\tau)_\xi (1+\varepsilon-\eta)^2 \mathrm{sgn}_\beta \overline{w}_\tau dX$$
$$-\frac{1}{n}\int_{\Omega_t} \eta U (w^{-\frac{1+n}{n}} \varphi_\tau)_\xi (1+\varepsilon-\eta)^2 \mathrm{sgn}_\beta \overline{w}_\tau dX$$
$$= I_5^{2,1} + I_5^{2,2}.$$

利用 (3.12) 式及引理 3.1, 容易得到

$$I_5^{2,2} \leqslant C\Big(1 + \int_{\Omega_t} |\overline{w}_\xi| dX\Big).$$

且由于 $\overline{w} = 0$ 当 $\xi = 0, L$ 时，分部积分后可得

$$I_5^{2,1} = -\frac{1}{n}\int \eta U (w^{-\frac{1+n}{n}} \overline{w}_\tau)(1+\varepsilon-\eta)^2 \mathrm{sgn}_\beta \overline{w}_\tau d\tau d\eta \Big|_{\xi=0}^{\xi=L}$$
$$+\frac{1}{n}\int_{\Omega_t} \eta U_\xi w^{-\frac{1+n}{n}} \overline{w}_\tau (1+\varepsilon-\eta)^2 \mathrm{sgn}_\beta \overline{w}_\tau dX$$
$$+\frac{1}{n}\int_{\Omega_t} \eta U w^{-\frac{1+n}{n}} \overline{w}_\tau (1+\varepsilon-\eta)^2 \overline{w}_{\tau\xi} \mathrm{sgn}'_\beta \overline{w}_\tau dX,$$

因此

$$\lim_{\beta\to 0} I_5^{2,1} \leqslant C\Big(1 + \int_{\Omega_t} |\overline{w}_\tau| dX\Big).$$

由 I_5^i 的估计式，可得到

$$\lim_{\beta\to 0} I_5 \leqslant C\Big(1 + \int_{\Omega_t} (|\overline{w}_\tau| + |\overline{w}_\xi|) dX\Big). \tag{3.20}$$

最后估计 I_6. 类似地

$$I_6 = -\frac{1}{n}\int_{\Omega_t} A_\tau (w^{-\frac{1+n}{n}} w_\eta)(1+\varepsilon-\eta)^2 \mathrm{sgn}_\beta \overline{w}_\tau dX$$
$$-\frac{1}{n}\int_{\Omega_t} A (w^{-\frac{1+n}{n}} w_\eta)_\tau (1+\varepsilon-\eta)^2 \mathrm{sgn}_\beta \overline{w}_\tau dX$$
$$= I_6^1 + I_6^2.$$

利用引理 3.1–3.3 以及 (3.12) 式，有

$$I_6^1 \leqslant C\Big(1+\int_{\Omega_t}|\overline{w}_\eta|dX\Big)\leqslant C,$$

$$\begin{aligned}I_6^2 &= -\frac{1}{n}\int_{\Omega_t}A(w^{-\frac{1+n}{n}}w_\tau)_\eta(1+\varepsilon-\eta)^2\mathrm{sgn}_\beta\overline{w}_\tau dX\\&= -\frac{1}{n}\int_{\Omega_t}A(w^{-\frac{1+n}{n}}\overline{w}_\tau)_\eta(1+\varepsilon-\eta)^2\mathrm{sgn}_\beta\overline{w}_\tau dX\\&\quad -\frac{1}{n}\int_{\Omega_t}A(w^{-\frac{1+n}{n}}\varphi_\tau)_\eta(1+\varepsilon-\eta)^2\mathrm{sgn}_\beta\overline{w}_\tau dX =: I_6^{2,1}+I_6^{2,2},\end{aligned}$$

$$I_6^{2,2}\leqslant C\Big(1+\int_{\Omega_t}|\overline{w}_\eta|dX\Big)\leqslant C,$$

$$\begin{aligned}I_6^{2,1} &= -\frac{1}{n}\int A(1+\varepsilon-\eta)^2w^{-\frac{1+n}{n}}\overline{w}_\tau sgn_\beta\overline{w}_\tau d\tau d\xi\Big|_{\eta=0}^{\eta=1}\\&\quad+\frac{1}{n}\int_{\Omega_t}[A(1+\varepsilon-\eta)^2]_\eta w^{-\frac{1+n}{n}}\overline{w}_\tau\mathrm{sgn}_\beta\overline{w}_\tau dX\\&\quad+\frac{1}{n}\int_{\Omega_t}A(1+\varepsilon-\eta)^2w^{-\frac{1+n}{n}}\overline{w}_\tau\overline{w}_{\tau\eta}\mathrm{sgn}'_\beta\overline{w}_\tau dX\\&\leqslant C\int_{\Omega_t}|\overline{w}_\tau|dX+\frac{1}{n}\int A(1+\varepsilon-\eta)^2w^{-\frac{1+n}{n}}\overline{w}_\tau\mathrm{sgn}_\beta\overline{w}_\tau d\tau d\xi\Big|_{\eta=0},\end{aligned}$$

因此

$$\begin{aligned}\lim_{\beta\to0}I_6\leqslant\ & C\Big(1+\int_{\Omega_t}|\overline{w}_\tau|dX\Big)\\&+\frac{1}{n}\lim_{\beta\to0}\int A(1+\varepsilon-\eta)^2w^{-\frac{1+n}{n}}\overline{w}_\tau\mathrm{sgn}_\beta\overline{w}_\tau d\tau d\xi\Big|_{\eta=0}.\end{aligned}\tag{3.21}$$

注意到在 $\eta=0$ 上，

$$w_{\tau\eta}=(v_0U^{\frac{1-n}{1+n}})_\tau+\frac{1}{n}\Big(U_\xi+\frac{U_\tau}{U}\Big)w^{-\frac{1+n}{n}}w_\tau-\Big(U_\xi+\frac{U_\tau}{U}\Big)_\tau w^{-\frac{1}{n}},$$

因此 (3.16) 与 (3.21) 两式中的边界积分之和是有界的，即

$$\lim_{\beta\to0}(I_1+I_6)\leqslant C\Big(1+\int_{\Omega_t}|\overline{w}_\tau|dX\Big).$$

由上式与 (3.14)–(3.21) 式，可得

$$\int_\Omega|\overline{w}_\tau|d\xi d\eta\Big|_{\tau=t}\leqslant C\Big(1+\int_{\Omega_t}(|\overline{w}_\tau|+|\overline{w}_\xi|)dX\Big).\tag{3.22}$$

为了得到 w_ε 的估计，还需估计 $\int_\Omega|\overline{w}_\xi|d\xi d\eta|_{\tau=t}$. 为此将 (3.13) 式两端关于 ξ

同时求导一次，然后同乘 $(1+\varepsilon-\eta)^2\mathrm{sgn}_\beta\overline{w}_\xi$，并在 Ω_t 上积分，可得

$$\begin{aligned}
L=&\int_{\Omega_t}\frac{1}{n}w^{-\frac{1+n}{n}}\overline{w}_{\tau\xi}(1+\varepsilon-\eta)^2\mathrm{sgn}_\beta\overline{w}_\xi dX\\
&-\int_{\Omega_t}\frac{1+n}{n^2}w^{-\frac{1+2n}{n}}\overline{w}_\tau w_\xi(1+\varepsilon-\eta)^2\mathrm{sgn}_\beta\overline{w}_\xi dX\\
=&\int_{\Omega_t}\overline{w}_{\eta\eta\xi}(1+\varepsilon-\eta)^2\mathrm{sgn}_\beta\overline{w}_\xi dX+\varepsilon\int_{\Omega_t}\overline{w}_{\xi\xi\xi}(1+\varepsilon-\eta)^2\mathrm{sgn}_\beta\overline{w}_\xi dX\\
&-\frac{1}{n}\int_{\Omega_t}(w^{-\frac{1+n}{n}}\varphi_\tau)_\xi(1+\varepsilon-\eta)^2\mathrm{sgn}_\beta\overline{w}_\xi dX\\
&+\int_{\Omega_t}(\varphi_{\eta\eta}+\varepsilon\varphi_{\xi\xi})_\xi(1+\varepsilon-\eta)^2\mathrm{sgn}_\beta\overline{w}_\xi dX\\
&-\frac{1}{n}\int_{\Omega_t}(\eta Uw^{-\frac{1+n}{n}}w_\xi)_\xi(1+\varepsilon-\eta)^2\mathrm{sgn}_\beta\overline{w}_\xi dX\\
&-\frac{1}{n}\int_{\Omega_t}(Aw^{-\frac{1+n}{n}}w_\eta)_\xi(1+\varepsilon-\eta)^2\mathrm{sgn}_\beta\overline{w}_\xi dX\\
&-\frac{1}{n}\int_{\Omega_t}(Bw^{-\frac{1+n}{n}}w)_\xi(1+\varepsilon-\eta)^2\mathrm{sgn}_\beta\overline{w}_\xi dX=\sum_{i=1}^{7}I_i.
\end{aligned}\tag{3.23}$$

类似 (3.15)、(3.16)、(3.18)、(3.19)、(3.21) 式的证明，可得

$$\begin{aligned}
L=&\int_{\Omega_t}\frac{1}{n}w^{-\frac{1+n}{n}}\overline{w}_{\tau\xi}(1+\varepsilon-\eta)^2\mathrm{sgn}_\beta\overline{w}_\xi dX\\
&-\frac{1+n}{n^2}\int_{\Omega_t}w^{-\frac{1+2n}{n}}\overline{w}_\tau w_\xi(1+\varepsilon-\eta)^2\mathrm{sgn}_\beta\overline{w}_\xi dX\\
\geqslant&\frac{1}{n}\int_{\Omega}w^{-\frac{1+n}{n}}|\overline{w}_\xi|(1+\varepsilon-\eta)^2d\xi d\eta\Big|_{\tau=0}^{\tau=t}\\
&-C\Big(1+\int_{\Omega_t}(|\overline{w}_\tau|+|\overline{w}_\xi|)dX\Big),
\end{aligned}\tag{3.24}$$

$$I_1+I_6\leqslant C\Big(1+\int_{\Omega_t}|\overline{w}_\xi|dX\Big),\tag{3.25}$$

$$I_3\leqslant C\Big(1+\int_{\Omega_t}|\overline{w}_\xi|dX\Big),\tag{3.26}$$

$$I_4\leqslant C,\quad I_7\leqslant C\Big(1+\int_{\Omega_t}|\overline{w}_\xi|dX\Big).\tag{3.27}$$

还需估计 I_2 与 I_5. 利用分部积分，可得

$$I_2=\varepsilon\int\overline{w}_{\xi\xi}(1+\varepsilon-\eta)^2\mathrm{sgn}_\beta\overline{w}_\xi d\tau d\eta|_{\xi=0}^{\xi=L}-\varepsilon\int_{\Omega_t}(1+\varepsilon-\eta)^2\overline{w}_{\xi\xi}^2\mathrm{sgn}'_\beta\overline{w}_\xi dX$$

$$\leqslant \varepsilon \int \overline{w}_{\xi\xi}(1+\varepsilon-\eta)^2 \mathrm{sgn}_\beta \overline{w}_\xi d\tau d\eta\Big|_{\xi=0}^{\xi=L},$$

$$I_5 = -\frac{1}{n}\int_{\Omega_t} \eta U_\xi w^{-\frac{1+n}{n}} w_\xi (1+\varepsilon-\eta)^2 \mathrm{sgn}_\beta \overline{w}_\xi dX$$
$$-\frac{1}{n}\int_{\Omega_t} \eta U (w^{-\frac{1+n}{n}} w_\xi)_\xi (1+\varepsilon-\eta)^2 \mathrm{sgn}_\beta \overline{w}_\xi dX = I_5^1 + I_5^2.$$

由 (3.12) 式及引理 3.1, 有

$$I_5^1 = -\frac{1}{n}\int_{\Omega_t} \eta U_\xi w^{-\frac{1+n}{n}} (\overline{w}_\xi + \varphi_\xi)(1+\varepsilon-\eta)^2 \mathrm{sgn}_\beta \overline{w}_\xi dX$$
$$\leqslant C\Big(1 + \int_{\Omega_t} |\overline{w}_\xi| dX\Big),$$

$$I_5^2 = -\frac{1}{n}\int_{\Omega_t} \eta U (w^{-\frac{1+n}{n}} \overline{w}_\xi)_\xi (1+\varepsilon-\eta)^2 \mathrm{sgn}_\beta \overline{w}_\xi dX$$
$$-\frac{1}{n}\int_{\Omega_t} \eta U (w^{-\frac{1+n}{n}} \varphi_\xi)_\xi (1+\varepsilon-\eta)^2 \mathrm{sgn}_\beta \overline{w}_\xi dX = I_5^{2,1} + I_5^{2,2}.$$

显然

$$I_5^{2,2} \leqslant C\Big(1 + \int_{\Omega_t} |\overline{w}_\xi| dX\Big)$$

且

$$I_5^{2,1} = -\frac{1}{n}\int \eta U (w^{-\frac{1+n}{n}} \overline{w}_\xi)(1+\varepsilon-\eta)^2 \mathrm{sgn}_\beta \overline{w}_\xi d\eta d\tau\Big|_{\xi=0}^{\xi=L}$$
$$+\frac{1}{n}\int_{\Omega_t} \eta U_\xi w^{-\frac{1+n}{n}} \overline{w}_\xi (1+\varepsilon-\eta)^2 \mathrm{sgn}_\beta \overline{w}_\xi dX$$
$$+\frac{1}{n}\int_{\Omega_t} \eta U w^{-\frac{1+n}{n}} \overline{w}_\xi (1+\varepsilon-\eta)^2 \overline{w}_{\xi\xi} \mathrm{sgn}'_\beta \overline{w}_\xi dX.$$

因此由 (3.13) 式，有

$$\lim_{\beta\to 0}(I_2 + I_5) \leqslant C\Big(1 + \int_{\Omega_t} |\overline{w}_\xi| dX\Big). \tag{3.28}$$

由上式及 (3.23)–(3.27) 式，可得

$$\int_\Omega |\overline{w}_\xi| d\xi d\eta\Big|_{\tau=t} \leqslant C\Big(1 + \int_{\Omega_t} (|\overline{w}_\tau| + |\overline{w}_\xi|) dX\Big). \tag{3.29}$$

利用 (3.22) 和 (3.29) 式及 Gronwall 不等式可得

引理 3.4 设 $0 < n \leqslant 1$, w_ε 是问题 (3.6)、(3.2)、(3.7) 的解，则对任意的 $t \in (0,T)$, 存在着与 ε 无关的常数 $C > 0$, 使得

$$\int_\Omega (|w_{\varepsilon\tau}| + |w_{\varepsilon\xi}|) d\xi d\eta\Big|_{\tau=t} \leqslant C.$$

为了证明定理 2.1, 还需要下面引理 (见文献 [11]):

引理 3.5　假设 $U \subset \mathbb{R}^N$ 是有界开集，$f_k, f \in L^q(U)(1 \leqslant q < \infty)$ 且当 $k \to \infty$ 时，f_k 在 $L^q(U)$ 中弱收敛于 f, 则

$$\liminf_{k\to\infty} \|f_k\|^q_{L^q(U)} \geqslant \|f\|^q_{L^q(U)}.$$

定理 2.1 的证明　由引理 3.1–3.4, 存在 $\{w_\varepsilon\}$ 的一个子列 $\{w_{\varepsilon j}\}$ 及 $w \in BV(\Omega_T) \cap L^\infty(\Omega_T)$, 使得当 $\varepsilon_j \to 0$ 时，

$$w_{\varepsilon_j} \to w, \quad \text{几乎处处于 } \Omega_T,$$
$$\frac{\partial w_{\varepsilon_j}}{\partial \eta} \rightharpoonup \frac{\partial w}{\partial \eta}, \quad \text{在 } L^2_{\text{loc}}(\Omega_T) \text{ 中},$$

且

$$C^{-1}(1-\eta)^{\frac{2n}{1+n}} \leqslant w \leqslant C(1-\eta)^{\frac{2n}{1+n}}, \text{ 其中 } C>0 \text{ 为常数}.$$

下面证明 w 是问题 (2.1)–(2.3) 的解．由引理 3.2 和 3.3, 对任意的 $\varphi \in C^1(\overline{\Omega}_T)$ 且 $\varphi = 0$ 在 $\eta = 1$ 附近，有

$$\begin{aligned}
&\int_{\Omega_T} (w^\alpha)_\eta \varphi d\xi d\eta d\tau \\
&= \lim_{\varepsilon_j \to 0} \int_{\Omega_T} (w^\alpha_{\varepsilon_j})_\eta \varphi d\xi d\eta d\tau \\
&= -\int_{\Omega_T} (w^\alpha)\varphi_\eta d\xi d\eta d\tau - \lim_{\varepsilon_j\to 0}\int w^\alpha_{\varepsilon_j}\varphi\Big|_{\eta=0} d\xi d\tau \\
&= \int_{\Omega_T} (w^\alpha)_\eta \varphi d\xi d\eta d\tau + \int \mathrm{r}(w)^\alpha \varphi\Big|_{\eta=0} d\xi d\tau - \lim_{\varepsilon_j\to 0}\int w^\alpha_{\varepsilon_j}\varphi\Big|_{\eta=0} d\xi d\tau,
\end{aligned}$$

其中 $\alpha = 1, 2$, $\mathrm{r}(w)$ 表示 w 在边界上的迹，因此有

$$\lim_{\varepsilon_j\to 0}\int_0^T\int_0^L w^\alpha_{\varepsilon_j}\varphi\Big|_{\eta=0} d\xi d\tau = \int_0^T\int_0^L \mathrm{r}(w)^\alpha\varphi\Big|_{\eta=0} d\xi d\tau, \ \alpha = 1,2,$$

及

$$\lim_{\varepsilon_j\to 0}\int_0^T\int_0^L |w_{\varepsilon_j} - \mathrm{r}(w)|^2 d\xi d\tau = 0, \text{ 当 } \eta = 0 \text{ 时}. \tag{3.30}$$

由引理 3.4, 对几乎所有的 $t \in (0,T)$, 有

$$\begin{aligned}
&\int_0^L\int_0^1 |w(t,\xi,\eta) - w_0(\xi,\eta)| d\xi d\eta \\
&= \lim_{\varepsilon_j\to 0}\int_0^L\int_0^1 |w_{\varepsilon_j}(t,\xi,\eta) - w_{0\varepsilon_j}(\xi,\eta)| d\xi d\eta \\
&\leqslant \lim_{\varepsilon_j\to 0}\int_0^t\int_0^L\int_0^1 \left|\frac{\partial w_{\varepsilon_j}}{\partial \tau}\right| d\xi d\eta d\tau \leqslant Ct \to 0, \qquad t \to 0.
\end{aligned}$$

上式表示 w 在迹的意义下满足边值条件．接下来证明 w 满足 (2.6) 式．令

$$V_\varepsilon = w_\varepsilon^{-\frac{1}{n}}, \ V = w^{-\frac{1}{n}},$$

则 (3.6) 式可改写为

$$\frac{\partial V_\varepsilon}{\partial \tau}+\eta U\frac{\partial V_\varepsilon}{\partial \xi}+A\frac{\partial V_\varepsilon}{\partial \eta}-\frac{1}{n}BV_\varepsilon+(V_\varepsilon^{-n})_{\eta\eta}+\varepsilon(V_\varepsilon^{-n})_{\xi\xi}=0. \tag{3.31}$$

令 $\varphi\in C^2(\overline{\Omega}_T)$, $\varphi\geqslant 0$, $\varphi|_{\xi=L}=0$, $\varphi(0,\xi,\eta)=\varphi(T,\xi,\eta)=0$, $\varphi=0$ 在 $\eta=1$ 附近. 将 (3.31) 式两端同乘 $\varphi\mathrm{sgn}_\beta(V_\varepsilon-k)$ $(k\in R)$ 并在 Ω_T 上积分, 则有

$$\begin{aligned}&\int_{\Omega_T}I_\beta(V_\varepsilon-k)\big[\varphi_\tau+(\eta U\varphi)_\xi+(A\varphi)_\eta\big]dX+\frac{1}{n}\int_{\Omega_T}BV_\varepsilon\mathrm{sgn}_\beta(V_\varepsilon-k)\varphi dX\\&+\int_{\Omega_T}\mathrm{sgn}'_\beta(V_\varepsilon-k)(V_\varepsilon^{-n})_\eta V_{\varepsilon\eta}\varphi dX+\int_{\Omega_T}(V_\varepsilon^{-n})_\eta\mathrm{sgn}_\beta(V_\varepsilon-k)\varphi_\eta dX\\&+\varepsilon\int_{\Omega_T}\mathrm{sgn}'_\beta(V_\varepsilon-k)(V_\varepsilon^{-n})_\xi V_{\varepsilon\xi}\varphi dX+\varepsilon\int_{\Omega_T}(V_\varepsilon^{-n})_\xi\mathrm{sgn}_\beta(V_\varepsilon-k)\varphi_\xi dX\\=&-\int_0^T\int_0^1\eta U\varphi I_\beta(V_\varepsilon-k)d\eta d\tau\Big|_{\xi=0}-\int_0^T\int_0^L(V_\varepsilon^{-n})_\eta\mathrm{sgn}_\beta(V_\varepsilon-k)\varphi d\xi d\tau\Big|_{\eta=0}\\&-\int_0^T\int_0^L A\varphi I_\beta(V_\varepsilon-k)d\xi d\tau\Big|_{\eta=0}\\&-\varepsilon\int_0^T\int_0^1(V_\varepsilon^{-n})_\xi\mathrm{sgn}_\beta(V_\varepsilon-k)\varphi d\tau d\eta\Big|_{\xi=0}.\end{aligned}\tag{3.32}$$

下面估计 (3.22) 式中的每一项. 首先, 当 $\varepsilon\to 0$ 时, 有

$$\int_0^T\int_0^1\eta U\varphi I_\beta(V_\varepsilon-k)d\eta d\tau\Big|_{\xi=0}\to\int_0^T\int_0^1\eta U\varphi I_\beta(w_1^{-\frac{1}{n}}-k)d\eta d\tau, \tag{3.33}$$

利用 (3.30) 与 (3.7) 式, 当 $\varepsilon\to 0$ 时, 有

$$\int_0^T\int_0^L A\varphi I_\beta(V_\varepsilon-k)d\xi d\tau\Big|_{\eta=0}\to\int_0^T\int_0^L A\varphi I_\beta(V-k)d\xi d\tau\Big|_{\eta=0}. \tag{3.34}$$

$$\begin{aligned}&\int_0^T\int_0^L(V_\varepsilon^{-n})_\eta\mathrm{sgn}_\beta(V_\varepsilon-k)\varphi d\xi d\tau\Big|_{\eta=0}\\&=\int_0^T\int_0^L\Big[v_0U^{\frac{1-n}{1+n}}-\Big(U_\xi+\frac{U_\tau}{U}\Big)V_\varepsilon\Big]\mathrm{sgn}_\beta(V_\varepsilon-k)\varphi d\xi d\tau\Big|_{\eta=0}\\&\to\int_0^T\int_0^L\Big[v_0U^{\frac{1-n}{1+n}}-\Big(U_\xi+\frac{U_\tau}{U}\Big)V\Big]\mathrm{sgn}_\beta(V-k)\varphi d\xi d\tau\Big|_{\eta=0}.\end{aligned}\tag{3.35}$$

下面估计 $\lim_{\varepsilon\to 0}\varepsilon\int_0^T\int_0^1(V_\varepsilon^{-n})_\xi\mathrm{sgn}_\beta(V_\varepsilon-k)\varphi d\eta d\tau|_{\xi=0}$. 设 $\delta(s)\in C^\infty(\mathbb{R})$, $\delta(s)\geqslant 0$, $\delta(s)=0$ 当 $|s|\geqslant 1$, $\int_{-\infty}^{\infty}\delta(s)ds=1$. 令

$$\delta_h(s)=\frac{1}{h}\delta\Big(\frac{s}{h}\Big),\quad \lambda_h(s)=1-\rho_h(s-2h),\quad \rho_h(s)=\int_{-\infty}^s\delta_h(\tau)d\tau,$$

则 $-\varepsilon\int_0^T\int_0^1(V_\varepsilon^{-n})_\xi \mathrm{sgn}_\beta(V_\varepsilon-k)\varphi d\eta d\tau|_{\xi=0}$ 可写为

$$\begin{aligned}&-\varepsilon\int_0^T\int_0^1(V_\varepsilon^{-n})_\xi \mathrm{sgn}_\beta(V_\varepsilon-k)\varphi d\eta d\tau\Big|_{\xi=0}\\&=-\int_0^T\int_0^1\varepsilon(V_\varepsilon^{-n})_\xi \mathrm{sgn}_\beta(w_{1\varepsilon}^{-\frac{1}{n}}-k)\varphi\lambda_h(\xi)d\eta d\tau\Big|_{\xi=0}\\&=\varepsilon\int_{\Omega_T}\mathrm{sgn}_\beta(w_{1\varepsilon}^{-\frac{1}{n}}-k)((V_\varepsilon^{-n})_\xi\varphi\lambda_h)_\xi dX\\&=\varepsilon\int_{\Omega_T}\mathrm{sgn}_\beta(w_{1\varepsilon}^{-\frac{1}{n}}-k)(V_\varepsilon^{-n})_{\xi\xi}\varphi\lambda_h dX+\varepsilon\int_{\Omega_T}\mathrm{sgn}_\beta(w_{1\varepsilon}^{-\frac{1}{n}}-k)(V_\varepsilon^{-n})_\xi\varphi_\xi\lambda_h dX\\&\quad+\varepsilon\int_{\Omega_T}\mathrm{sgn}_\beta(w_{1\varepsilon}^{-\frac{1}{n}}-k)(V_\varepsilon^{-n})_\xi\varphi\lambda_h' dX\\&=I_1+I_2+I_3.\end{aligned}$$

由引理3.2和3.3可得

$$\lim_{\varepsilon\to0}I_2=0,\quad \lim_{\varepsilon\to0}I_3=0.$$

由(3.31)式，有

$$I_1=\int_{\Omega_T}\mathrm{sgn}_\beta(w_{1\varepsilon}^{-\frac{1}{n}}-k)\Big[\frac{1}{n}BV_\varepsilon-(V_\varepsilon^{-n})_{\eta\eta}-A\frac{\partial V_\varepsilon}{\partial\eta}-\eta U\frac{\partial V_\varepsilon}{\partial\xi}-\frac{\partial V_\varepsilon}{\partial\tau}\Big]\varphi\lambda_h dX.$$

现在需要估计 I_1 中的每一项．首先当 $\varepsilon\to0$, $h\to0$ 时，有

$$\frac{1}{n}\int_{\Omega_T}\mathrm{sgn}_\beta(w_{1\varepsilon}^{-\frac{1}{n}}-k)BV_\varepsilon\lambda_h dX\to0,$$

由引理3.2和3.3, 有

$$\begin{aligned}&\int_{\Omega_T}\mathrm{sgn}_\beta(w_{1\varepsilon}^{-\frac{1}{n}}-k)(V_\varepsilon^{-n})_{\eta\eta}\varphi\lambda_h(\xi)dX\\&=-\int_0^T\int_0^L\mathrm{sgn}_\beta(w_{1\varepsilon}^{-\frac{1}{n}}-k)(V_\varepsilon^{-n})_\eta\varphi\lambda_h(\xi)d\xi d\tau\Big|_{\eta=0}\\&\quad-\int_{\Omega_T}\mathrm{sgn}'_\beta(w_{1\varepsilon}^{-\frac{1}{n}}-k)(w_{1\varepsilon}^{-\frac{1}{n}})_\eta(V_\varepsilon^{-n})_\eta\varphi\lambda_h(\xi)dX\\&\quad-\int_{\Omega_T}\mathrm{sgn}_\beta(w_{1\varepsilon}^{-\frac{1}{n}}-k)(V_\varepsilon^{-n})_\eta\varphi_\eta\lambda_h(\xi)dX\\&=-\int_0^T\int_0^L\mathrm{sgn}_\beta(w_{1\varepsilon}^{-\frac{1}{n}}-k)\Big[v_0U^{\frac{1-n}{1+n}}-\Big(U_\xi+\frac{U_\tau}{U}\Big)r(w)^{-\frac{1}{n}}\Big]\varphi\lambda_h(\xi)d\xi d\tau\Big|_{\eta=0}\\&\quad-\int_{\Omega_T}\mathrm{sgn}'_\beta(w_{1\varepsilon}^{-\frac{1}{n}}-k)(w_{1\varepsilon}^{-\frac{1}{n}})_\eta(V_\varepsilon^{-n})_\eta\varphi\lambda_h(\xi)dX\\&\quad-\int_{\Omega_T}\mathrm{sgn}_\beta(w_{1\varepsilon}^{-\frac{1}{n}}-k)(V_\varepsilon^{-n})_\eta\varphi_\eta\lambda_h(\xi)dX\\&\to0,\end{aligned}$$

$$\int_{\Omega_T} \mathrm{sgn}_\beta(w_{1\varepsilon}^{-\frac{1}{n}} - k) A \frac{\partial V_\varepsilon}{\partial \eta} \varphi \lambda_h dX \to 0.$$

利用分部积分并令 $\varepsilon \to 0, h \to 0$, 可得

$$\begin{aligned}
&\int_{\Omega_T} \mathrm{sgn}_\beta(w_{1\varepsilon}^{-\frac{1}{n}} - k) \frac{\partial V_\varepsilon}{\partial \tau} \varphi \lambda_h(\xi) dX \\
&= -\int_{\Omega_T} \mathrm{sgn}'_\beta(w_{1\varepsilon}^{-\frac{1}{n}} - k)(w_{1\varepsilon}^{-\frac{1}{n}})_\tau V_\varepsilon \varphi \lambda_h(\xi) dX \\
&\quad - \int_{\Omega_T} \mathrm{sgn}_\beta(w_{1\varepsilon}^{-\frac{1}{n}} - k) V_\varepsilon \varphi_\tau \lambda_h(\xi) dX \\
&\to 0,
\end{aligned}$$

$$\begin{aligned}
&\int_{\Omega_T} \eta U \frac{\partial V_\varepsilon}{\partial \xi} \mathrm{sgn}_\beta(w_{1\varepsilon}^{-\frac{1}{n}} - k) \varphi \lambda_h(\xi) dX \\
&= -\int_0^T \int_0^1 \eta U(w_{1\varepsilon}^{-\frac{1}{n}} - k) \mathrm{sgn}_\beta(w_{1\varepsilon}^{-\frac{1}{n}} - k) \varphi d\eta d\tau \Big|_{\xi=0} \\
&\quad - \int_{\Omega_T} \eta U_\xi (V_\varepsilon - k) \mathrm{sgn}_\beta(w_{1\varepsilon}^{-\frac{1}{n}} - k) \varphi \lambda_h(\xi) dX \\
&\quad - \int_{\Omega_T} \eta U \varphi_\xi (V_\varepsilon - k) \mathrm{sgn}_\beta(w_{1\varepsilon}^{-\frac{1}{n}} - k) \lambda_h(\xi) dX \\
&\quad - \int_{\Omega_T} \eta U (V_\varepsilon - k) \mathrm{sgn}_\beta(w_{1\varepsilon}^{-\frac{1}{n}} - k) \varphi \lambda'_h(\xi) dX \\
&\to -\int_0^T \int_0^1 \eta U(w_1^{-\frac{1}{n}} - k) \mathrm{sgn}_\beta(w_1^{-\frac{1}{n}} - k) \varphi d\eta d\tau \Big|_{\xi=0} \\
&\quad + \int_0^T \int_0^1 \eta U(\mathrm{r}(V) - k) \mathrm{sgn}_\beta(w_1^{-\frac{1}{n}} - k) \varphi d\eta d\tau \Big|_{\xi=0},
\end{aligned}$$

因此 I_1 可估计如下:

$$\begin{aligned}
\lim_{\varepsilon \to 0} I_1 = &\int_0^T \int_0^1 \eta U(w_1^{-\frac{1}{n}} - k) \mathrm{sgn}_\beta(w_1^{-\frac{1}{n}} - k) \varphi d\eta d\tau \Big|_{\xi=0} \\
&- \int_0^T \int_0^1 \eta U(\mathrm{r}(V) - k) \mathrm{sgn}_\beta(w_1^{-\frac{1}{n}} - k) \varphi d\eta d\tau \Big|_{\xi=0}.
\end{aligned}$$

由此及 I_2 和 I_3 的估计，可得

$$\begin{aligned}
&\lim_{\varepsilon \to 0} \varepsilon \int_0^T \int_0^1 (V_\varepsilon^{-n})_\xi \mathrm{sgn}_\beta(V_\varepsilon - k) \varphi d\eta d\tau \Big|_{\xi=0} \\
&= \int_0^T \int_0^1 \eta U(\mathrm{r}(V) - k) \mathrm{sgn}_\beta(w_1^{-\frac{1}{n}} - k) \varphi d\eta d\tau \Big|_{\xi=0} \\
&\quad - \int_0^T \int_0^1 \eta U(w_1^{-\frac{1}{n}} - k) \mathrm{sgn}_\beta(w_1^{-\frac{1}{n}} - k) \varphi d\eta d\tau \Big|_{\xi=0}. \qquad (3.36)
\end{aligned}$$

另外由引理 3.5, 有

$$\liminf_{\varepsilon\to 0}\int_{\Omega_T}\mathrm{sgn}'_\beta(V_\varepsilon-k)V_\varepsilon^{-(1+n)}(V_{\varepsilon\eta})^2\varphi dX$$
$$\geqslant\int_{\Omega_T}\mathrm{sgn}'_\beta(V-k)V^{-(1+n)}(V_\eta)^2\varphi dX, \tag{3.37}$$

由引理 3.2–3.3, 有

$$\lim_{\varepsilon\to 0}\varepsilon\int_{\Omega_T}(V_\varepsilon^{-n})_\xi\mathrm{sgn}_\beta(V_\varepsilon-k)\varphi_\xi dX=0. \tag{3.38}$$

注意到 (3.32) 式中左端的第 5 项积分非正，在 (3.32) 式中令 $\varepsilon\to 0$, 由 (3.32)–(3.38) 式，可得

$$\int_{\Omega_T}I_\beta(V-k)[\varphi_\tau+(\eta U\varphi)_\xi+(A\varphi)_\eta]dX+\frac{1}{n}\int_{\Omega_T}\varphi VB\mathrm{sgn}_\beta(V-k)dX$$
$$-n\int_{\Omega_T}\mathrm{sgn}'_\beta(V-k)V^{-(1+n)}(V_\eta)^2\varphi dX+\int_{\Omega_T}(V^{-n})_\eta\mathrm{sgn}_\beta(V-k)\varphi_\eta dX$$
$$+\int_0^T\int_0^1\eta U\varphi[\mathrm{sgn}_\beta(w_1^{-\frac{1}{n}}-k)(\mathrm{r}(V)-k)-(w_1^{-\frac{1}{n}}-k))]d\eta d\tau\Big|_{\xi=0}$$
$$+\int_0^T\int_0^1\eta U\varphi I_\beta(w_1^{-\frac{1}{n}}-k)d\eta d\tau\Big|_{\xi=0}+\int_0^T\int_0^L A\varphi I_\beta(V-k)d\xi d\tau\Big|_{\eta=0}$$
$$+\int_0^T\int_0^L\Big[v_0U^{\frac{1-n}{1+n}}-\Big(U_\xi+\frac{U_\tau}{U}\Big)V\Big]\mathrm{sgn}_\beta(V-k)\varphi d\xi d\tau\Big|_{\eta=0}\geqslant 0. \tag{3.39}$$

在 (3.39) 式中选取 $\varphi\in C_0^2(\Omega_T),\varphi\geqslant 0$, 则可知 V 满足 (2.6) 式. 在 (3.39) 式中选取 $\varphi=\phi(\tau,\xi)\lambda_h(\eta)$, 其中 $\phi(\tau,\xi)\in C_0^2((0,T)\times(0,L))$, $\phi(\tau,\xi)\geqslant 0$, 作为检验函数并令 $h\to 0$, 则 $\forall\, k\in\mathbb{R}$, 有

$$\int_0^T\int_0^L\Big[\mathrm{r}(V^{-n})_\eta\mathrm{sgn}_\beta(V-k)-\Big(v_0U^{\frac{1-n}{1+n}}-\Big(U_\xi+\frac{U_\tau}{U}\Big)V\Big)\mathrm{sgn}_\beta(V-k)\Big]\phi d\tau d\xi\Big|_{\eta=0}\leqslant 0.$$

分别选取 $k>\|V(\tau,\xi,0)\|_{L^\infty}$ 以及 $k<0$, 由 $\phi(\tau,\xi)$ 的任意性，可得

$$\mathrm{r}(w^{\frac{1}{n}}w_\eta)=U^{\frac{1-n}{1+n}}v_0w^{\frac{1}{n}}-\Big(U_\xi+\frac{U_\tau}{U}\Big),\quad \text{当 }\eta=0\text{ 时}.$$

另外, 在 (3.39) 式中选取 $\varphi=\phi(\tau,\eta)\lambda_h(\xi)$, 其中 $\phi(\tau,\eta)\in C_0^2((0,T)\times(0,1))$, $\phi(\tau,\eta)\geqslant 0$, 作为检验函数并令 $h\to 0$, 则 $\forall\, k\in\mathbb{R}$, 有

$$\int_0^T\int_0^1\eta U\phi[I_\beta(w_1^{-\frac{1}{n}}-k)+\mathrm{sgn}_\beta(w_1^{-\frac{1}{n}}-k)(\mathrm{r}(V)-k)$$
$$-\mathrm{sgn}_\beta(w_1^{-\frac{1}{n}}-k)(w_1^{-\frac{1}{n}}-k)-I_\beta(\mathrm{r}(V)-k)]d\tau d\eta\Big|_{\xi=0}\geqslant 0.$$

令 $\beta\to 0$ 可得

$$\int_0^T\int_0^1\eta U\phi[\mathrm{sgn}(\mathrm{r}(V)-k)-\mathrm{sgn}(w_1^{-\frac{1}{n}}-k)](\mathrm{r}(V)-k)d\eta d\tau\Big|_{\xi=0}\leqslant 0,$$

由于 $U,\ \phi \geqslant 0$, 由上式可得

$$[\mathrm{sgn}(\mathrm{r}(V)-k)-\mathrm{sgn}(w_1^{-\frac{1}{n}}-k)](\mathrm{r}(V)-k)\Big|_{\xi=0}\leqslant 0,\quad \text{几乎处处于 } (0,T)\times(0,1).$$

上式表明

$$\mathrm{r}(V)|_{\xi=0}=w_1^{-\frac{1}{n}},\qquad \mathrm{r}(w)|_{\xi=0}=w_1,\quad \text{几乎处处于 } (0,T)\times(0,1).$$

定理证毕.

4 定理 2.2 的证明

假设 (2.10) 和 (2.12) 式成立，本节证明问题 (2.1)–(2.3) 当 $n>1$ 时局部解的存在性. 由文献 [10]，正则化问题

$$w_\tau+\eta Uw_\xi+Aw_\eta+Bw-nw^{\frac{1+n}{n}}w_{\eta\eta}-\varepsilon nw^{\frac{1+n}{n}}w_{\xi\xi}=0,\quad (\tau,\xi,\eta)\in\Omega_T, \tag{4.1}$$

$$w|_{\tau=0}=w_{0\varepsilon}(\xi,\eta),\ w|_{\xi=0}=w_{1\varepsilon}(\tau,\eta),\ w|_{\xi=L}=w_{2\varepsilon}(\tau,\eta),\ w|_{\eta=1}=\varepsilon^n, \tag{4.2}$$

$$[w^{\frac{1}{n}}w_\eta-v_0U^{\frac{1-n}{1+n}}w^{\frac{1}{n}}+(U_\xi+U_\tau/U)]|_{\eta=0}=0 \tag{4.3}$$

存在唯一解 $w_\varepsilon\geqslant\varepsilon^n$, 其中 $n>1$, A, B 如 (2.4) 式所定义； $w_{0\varepsilon}, w_{1\varepsilon}, w_{2\varepsilon}$ 充分光滑且满足

$$\begin{cases} w_{0\varepsilon}\to w_0, w_{1\varepsilon}\to w_1, w_{0\varepsilon}|_{\eta=1}=w_{1\varepsilon}|_{\eta=1}=w_{2\varepsilon}|_{\eta=1}=\varepsilon^n;\\ w_{0\varepsilon}\geqslant\varepsilon^n, w_{1\varepsilon}\geqslant\varepsilon^n, w_{2\varepsilon}\geqslant\varepsilon^n;\\ C_0^{-1}(1+\varepsilon-\eta)^n\leqslant w_{0\varepsilon}, w_{1\varepsilon}, w_{2\varepsilon}\leqslant C_0(1+\varepsilon-\eta)^n. \end{cases} \tag{4.4}$$

类似定理 2.1 的证明，需对逼近问题 (4.1)–(4.3) 的解 $\{w_\varepsilon\}$ 做一致估计.

引理 4.1 设 w_ε 是 (4.1)–(4.3) 的解，则存在与 ε 无关的常数 C_1 及 T_0 (充分小), 使得

$$C_1^{-1}(1+\varepsilon-\eta)^n\leqslant w_\varepsilon\leqslant C_1(1+\varepsilon-\eta)^n,\quad \forall\,\tau\in(0,T_0).$$

证 定义

$$L_\varepsilon(V)=V_\tau+\eta UV_\xi+AV_\eta+BV-nV^{\frac{1+n}{n}}V_{\eta\eta}-\varepsilon nV^{\frac{1+n}{n}}V_{\xi\xi},$$

$$l_\varepsilon(V)=w^{\frac{1}{n}}V_\eta-v_0U^{\frac{1-n}{1+n}}w^{\frac{1}{n}}+\left(U_\xi+\frac{U_\tau}{U}\right),\quad \text{当 } \eta=0 \text{ 时}.$$

考虑函数

$$V_1=C(1+\varepsilon-\eta)^ne^{\alpha\eta-\beta\tau},\ \alpha>n(1+\varepsilon)^{-1},$$

其中 $C, \beta > 0$ 待定. 选取 β 充分大，使得

$$l_\varepsilon(V_1) = w^{\frac{1}{n}} C e^{\alpha\eta - \beta\tau}[\alpha(1+\varepsilon)^n - n(1+\varepsilon)^{n-1}] - v_0 U^{\frac{1-n}{1+n}} w^{\frac{1}{n}} + (U_\xi + U_\tau/U) > 0,$$

$$\begin{aligned}L_\varepsilon(V_1) &= C(1+\varepsilon-\eta)^n e^{\alpha\eta-\beta\tau}[-\beta + \alpha A - nA(1+\varepsilon-\eta)^{-1} + B] \\ &\quad - nV_1^{\frac{1+n}{n}}[n(n-1)(1+\varepsilon-\eta)^{n-2} + \alpha^2(1+\varepsilon-\eta)^n \\ &\quad - 2n\alpha(1+\varepsilon-\eta)^{n-1}]Ce^{\alpha\eta-\beta\tau} \\ &= C(1+\varepsilon-\eta)^n e^{\alpha\eta-\beta\tau}\{-\beta + \alpha A - nA(1+\varepsilon-\eta)^{-1} + B \\ &\quad - (Ce^{\alpha\eta-\beta\tau})^{\frac{1+n}{n}}[n^2(n-1)(1+\varepsilon-\eta)^{n-1} + \alpha^2(1+\varepsilon-\eta)^{1+n} \\ &\quad - 2n\alpha(1+\varepsilon-\eta)^n]\} < 0,\end{aligned}$$

令 $Z = V_1 - w$, 选取 C 充分小，则有

$$\begin{aligned}&Z_\tau + \eta U Z_\xi + AZ_\eta + BZ - nV_1^{\frac{1+n}{n}} V_{1\eta\eta} + nw^{\frac{1+n}{n}} w_{\eta\eta} - \varepsilon n w^{\frac{1+n}{n}} Z_{\xi\xi} \\ &= Z_\tau + \eta U Z_\xi + AZ_\eta + \left[B - (1+n)\int_0^1 (\theta V_1 + (1-\theta)w)^{\frac{1}{n}} d\theta V_{1\eta\eta}\right] Z \\ &\quad - nw^{\frac{1+n}{n}} Z_{\eta\eta} - \varepsilon n w^{\frac{1+n}{n}} Z_{\xi\xi} < 0,\end{aligned}$$

$$Z_\eta|_{\eta=0} = l_\varepsilon(V_1)/w^{\frac{1}{n}} - l_\varepsilon(w)/w^{\frac{1}{n}} > 0,$$

$$Z \leqslant 0 \quad 当\ \tau = 0, \xi = 0, L, \eta = 1 时.$$

因此由极值原理，有

$$w \geqslant C(1+\varepsilon-\eta)^n e^{\alpha\eta-\beta\tau} = C_1(1+\varepsilon-\eta)^n.$$

为了证明引理 4.1 的第 2 部分，考虑函数

$$V_2 = C(1+\varepsilon-\eta)^n e^{\alpha\tau},$$

其中 C, α 待定. 选取 C 充分大，使得

$$\begin{aligned}\bar{l}_\varepsilon(V_2) &\equiv [V_2^{\frac{1}{n}} V_{2\eta} - v_0 U^{\frac{1-n}{1+n}} V_2^{\frac{1}{n}} + (U_\xi + U_\tau/U)]|_{\eta=0} \\ &= -nCe^{\alpha\tau}(1+\varepsilon)^{n-1}(C(1+\varepsilon)^n e^{\alpha\tau})^{\frac{1}{n}} - v_0 U^{\frac{1-n}{1+n}}(C(1+\varepsilon)^n e^{\alpha\tau})^{\frac{1}{n}} \\ &\quad + \left(U_\xi + \frac{U_\tau}{U}\right) < 0,\end{aligned}$$

$$V_2 \geqslant w \quad 当\ \tau = 0,\ \xi = 0,\ L,\ \eta = 1.$$

由于 $|A(1+\varepsilon-\eta)^{-1}|\leqslant C$, 选取 α 充分大，使得对任意的 $\tau\in(0,T_0)$, 有

$$\begin{aligned}L_\varepsilon(V_2) &= Ce^{\alpha\tau}(1+\varepsilon-\eta)^n[\alpha-nA(1+\varepsilon-\eta)^{-1}+B]\\&\quad -n[Ce^{\alpha\tau}(1+\varepsilon-\eta)^n]^{\frac{1+n}{n}}[n(n-1)Ce^{\alpha\tau}(1+\varepsilon-\eta)^{n-2}]\\&= Ce^{\alpha\tau}(1+\varepsilon-\eta)^n\Big[\alpha-\frac{nA}{1+\varepsilon-\eta}+B\\&\quad -n^2(n-1)(Ce^{\alpha\tau})^{\frac{1+n}{n}}(1+\varepsilon-\eta)^{n-1}\Big]>0,\end{aligned}$$

其中 $T_0>0$ 充分小，使得

$$(Ce^{\alpha\tau})^{\frac{1+n}{n}}\leqslant\frac{\alpha}{2},\quad\forall\tau\in(0,T_0).$$

令 $Z=V_2-w$, 由极值原理即可证明

$$w\leqslant C(1+\varepsilon-\eta)^n e^{\alpha\tau}\leqslant C_1(1+\varepsilon-\eta)^n,\ \forall\tau\in(0,T_0).$$

引理证毕.

令

$$V_\varepsilon=w_\varepsilon^{-\frac{1}{n}},\quad \varphi=\frac{\xi}{L}w_{2\varepsilon}+\frac{L-\xi}{L}w_{1\varepsilon}.$$

由引理 4.1 及 (4.4) 式，有

$$\begin{cases}M_1(1+\varepsilon-\eta)^{-1}\leqslant V_\varepsilon\leqslant M_2(1+\varepsilon-\eta)^{-1};\\ V^{-n}-\varphi,\quad \text{当 } \xi=0,L,\eta=1 \text{ 时};\\ C^{-1}(1+\varepsilon-\eta)^n\leqslant\varphi\leqslant C(1+\varepsilon-\eta)^n;\\ |\varphi_\tau|,|\varphi_\xi|\leqslant C(1+\varepsilon-\eta)^n, |\varphi_\eta|\leqslant C(1+\varepsilon-\eta)^{n-1}.\end{cases}\tag{4.5}$$

由 (4.1) 式可知 V_ε 满足

$$V_\tau+\eta UV_\xi+AV_\eta-\frac{1}{n}BV+(V^{-n})_{\eta\eta}+\varepsilon(V^{-n})_{\xi\xi}=0.\tag{4.6}$$

引理 4.2 设 $n>1$, w_ε 为 (4.1)–(4.3) 式定义在 Ω_{T_0} 上的解，则存在与 ε 无关的常数 C, 使得 $V_\varepsilon=w_\varepsilon^{-\frac{1}{n}}$ 满足

$$\int_{\Omega_T}[(V_\varepsilon^{-n})_\eta^2+\varepsilon(V_\varepsilon^{-n})_\xi^2]dX\leqslant C.$$

证 (4.6) 式两端同乘 $(V_\varepsilon^{-n}-\varphi)$ 并在 Ω_T 上积分，可得

$$\begin{aligned}&\int_{\Omega_{T_0}}(V_\varepsilon^{-n}-\varphi)\Big[V_{\varepsilon\tau}+\eta UV_{\varepsilon\xi}+AV_{\varepsilon\eta}-\frac{1}{n}BV_\varepsilon\Big]dX\\&=-\int_{\Omega_{T_0}}(V_\varepsilon^{-n}-\varphi)[(V_\varepsilon^{-n})_{\eta\eta}+\varepsilon(V_\varepsilon^{-n})_{\xi\xi}]dX.\end{aligned}$$

利用引理 4.1 及 (4.5) 式，类似引理 3.2 的证明即可证明引理 4.2.

由引理 4.2 以及 Hölder 不等式，可得到如下推论:

推论 4.1　设 w_ε 为 (4.1)–(4.3) 式在 Ω_{T_0} 上的解，则存在与 ε 的常数 C, 使得

$$\int_{\Omega_{T_0}} |w_{\varepsilon\eta}| d\tau d\xi d\eta \leqslant C.$$

引理 4.3　设 w_ε 为 (4.1)–(4.3) 式在 Ω_{T_0} 上的解，则存在与 ε 的常数 M, 使得

$$\int_0^L \int_0^1 (|w_{\varepsilon\tau}| + |w_{\varepsilon\xi}|) d\xi d\eta|_{\tau=t} \leqslant M, \quad \forall\, t \in (0, T_0).$$

证　令

$$\overline{w}_\varepsilon = w_\varepsilon - \varphi,$$

则显然 $\overline{w}_\varepsilon = 0$ 当 $\xi = 0, L, \eta = 1$ 时，且

$$\begin{aligned}\frac{1}{n} w^{-\frac{1+n}{n}} \overline{w}_\tau = {} & \overline{w}_{\eta\eta} + \varepsilon \overline{w}_{\xi\xi} + \varphi_{\eta\eta} + \varepsilon \varphi_{\xi\xi} - \frac{1}{n} \eta U w^{-\frac{1+n}{n}} w_\xi - \frac{1}{n} A w^{-\frac{1+n}{n}} w_\eta \\ & - \frac{B}{n} w^{-\frac{1}{n}} - \frac{1}{n} w^{-\frac{1+n}{n}} \varphi_\tau. \end{aligned} \tag{4.7}$$

将 (4.7) 式两端同时关于 τ 求导 1 次，然后同乘 $(1+\varepsilon-\eta)^{1+n} \mathrm{sgn}_\beta \overline{w}_\tau$ $(\beta > 0)$, 并在 $\Omega_t = \{0 < \tau < t, 0 < \xi < L, 0 < \eta < 1\}$ 上积分，其中 $0 < t < T_0$, 可得

$$\begin{aligned} & \int_{\Omega_t} \frac{1}{n} w^{-\frac{1+n}{n}} \overline{w}_{\tau\tau} (1+\varepsilon-\eta)^{1+n} \mathrm{sgn}_\beta \overline{w}_\tau dX \\ & - \int_{\Omega_t} \frac{1+n}{n^2} w^{-\frac{1+2n}{n}} \overline{w}_\tau w_\tau (1+\varepsilon-\eta)^{1+n} \mathrm{sgn}_\beta \overline{w}_\tau dX \\ = & \int_{\Omega_t} \overline{w}_{\eta\eta\tau} (1+\varepsilon-\eta)^{1+n} \mathrm{sgn}_\beta \overline{w}_\tau dX \\ & + \varepsilon \int_{\Omega_t} \overline{w}_{\xi\xi\tau} (1+\varepsilon-\eta)^{1+n} \mathrm{sgn}_\beta \overline{w}_\tau dX \\ & - \frac{1}{n} \int_{\Omega_t} \left(w^{-\frac{1+n}{n}} \varphi_\tau \right)_\tau (1+\varepsilon-\eta)^{1+n} \mathrm{sgn}_\beta \overline{w}_\tau dX \\ & + \int_{\Omega_t} \left(\varphi_{\eta\eta} + \varepsilon \varphi_{\xi\xi} \right)_\tau (1+\varepsilon-\eta)^{1+n} \mathrm{sgn}_\beta \overline{w}_\tau dX \\ & - \frac{1}{n} \int_{\Omega_t} \left(\eta U w^{-\frac{1+n}{n}} w_\xi \right)_\tau (1+\varepsilon-\eta)^{1+n} \mathrm{sgn}_\beta \overline{w}_\tau dX \\ & - \frac{1}{n} \int_{\Omega_t} \left(B w^{-\frac{1}{n}} \right)_\tau (1+\varepsilon-\eta)^{1+n} \mathrm{sgn}_\beta \overline{w}_\tau dX \\ & - \frac{1}{n} \int_{\Omega_t} \left(A w^{-\frac{1+n}{n}} w_\eta \right)_\tau (1+\varepsilon-\eta)^{1+n} \mathrm{sgn}_\beta \overline{w}_\tau dX. \end{aligned} \tag{4.8}$$

类似 (3.22) 式的证明，有

$$\int_\Omega |\overline{w}_{\varepsilon\tau}| d\xi d\eta \Big|_{\tau=t} \leqslant C\Big(1 + \int_{\Omega_t} (|\overline{w}_{\varepsilon\tau}| + |\overline{w}_{\varepsilon\xi}|) dX\Big). \tag{4.9}$$

类似地，将 (4.7) 式两端同时关于 ξ 求导 1 次，然后同乘 $(1+\varepsilon-\eta)^{1+n}\mathrm{sgn}_\beta \overline{w}_\xi$ 并在 $\Omega_t=\{0<\tau<t,0<\xi<L,0<\eta<1\}$ 上积分，类似 (3.29) 式的证明，有

$$\int_\Omega |\overline{w}_{\varepsilon\xi}|d\xi d\eta\Big|_{\tau=t} \leqslant C\Big(1+\int_{\Omega_t}(|\overline{w}_{\varepsilon\tau}|+|\overline{w}_{\varepsilon\xi}|)dX\Big). \tag{4.10}$$

利用 (4.9) 和 (4.10) 式以及 Gronwall 不等式即可得到引理．证毕．

利用引理 4.1–4.3, 类似定理 2.1 的证明，我们可以证明定理 2.2.

5 定理 2.3 的证明

本节证明问题 (2.1)–(2.3) 弱解的唯一性．首先需要下面一个技术性引理．

引理 5.1 设 w_1 和 w_2 是问题 (2.1)–(2.3) 的两个解，$V_1=w_1^{-\frac{1}{n}}$ 和 $V_2=w_2^{-\frac{1}{n}}$，则对任意的 $\varphi\in C_0^\infty(\Omega_T)$，有

$$\int_{\Omega_T}|V_1-V_2|\Big\{\varphi_\tau+(\eta U\varphi)_\xi+(A\varphi)_\eta+\frac{B}{n}\varphi\Big\}dX+\int_{\Omega_T}|V_1^{-n}-V_2^{-n}|\varphi_{\eta\eta}dX\geqslant 0.$$

证 令

$$X=(\tau,\xi,\eta),\quad \overline{X}=(\overline{\tau},\overline{\xi},\overline{\eta}),\quad V_1=w_1^{-\frac{1}{n}}(\tau,\xi,\eta),\quad \overline{V}_2=w_2^{-\frac{1}{n}}(\overline{\tau},\overline{\xi},\overline{\eta}).$$

由定义 2.1, 对任意的 $\varphi,\phi\in C_0^\infty(\Omega_T)$, $k,l\in\mathbb{R}$, 有

$$\int_{\Omega_T}I_\beta(V_1-k)[\varphi_\tau+(\eta U\varphi)_\xi+(A\varphi)_\eta]dX+\int_{\Omega_T}\frac{1}{n}BV_1\mathrm{sgn}_\beta(V_1-k)\varphi dX$$
$$-\int_{\Omega_T}nV_1^{-(1+n)}V_{1\eta}^2\mathrm{sgn}'_\beta(V_1-k)\varphi dX+\int_{\Omega_T}G_\beta(V_1,k)\varphi_{\eta\eta}dX\geqslant 0, \tag{5.1}$$

$$\int_{\Omega_T}I_\beta(\overline{V}_2-l)[\phi_{\overline{\tau}}+(\overline{\eta}\overline{U}\phi)_{\overline{\xi}}+(\overline{A}\phi)_{\overline{\eta}}]d\overline{X}+\int_{\Omega_T}\frac{1}{n}\overline{B}\overline{V}_2\mathrm{sgn}_\beta(\overline{V}_2-l)\phi d\overline{X}$$
$$-\int_{\Omega_T}n\overline{V}_2^{-(1+n)}\overline{V}_{2\overline{\eta}}^2\mathrm{sgn}'_\beta(\overline{V}_2-l)\phi d\overline{X}+\int_{\Omega_T}G_\beta(\overline{V}_2,l)\phi_{\overline{\eta}\overline{\eta}}d\overline{X}\geqslant 0, \tag{5.2}$$

其中

$$\overline{U}=U(\overline{\tau},\overline{\xi}),\ \overline{A}=(1-\overline{\eta}^2)\overline{U}_{\overline{\xi}}+(1-\overline{\eta})\overline{U}_\tau/\overline{U},\ \overline{B}=\frac{2n}{1+n}(\overline{\eta}\overline{U}_{\overline{\xi}}+\overline{U}_{\overline{\tau}}/\overline{U}),$$

$$I_\beta(V-k)=\int_0^{V-k}\mathrm{sgn}_\beta s ds,\ G_\beta(V,k)=\int_k^V ns^{-(1+n)}\mathrm{sgn}_\beta(s-k)ds.$$

令 $\Psi(X,\overline{X})\geqslant 0,\Psi\in C^2(\Omega_T\times\Omega_T)$ 且

$$\mathrm{supp}\Psi(X,\cdot)\subset\Omega_T \text{ 若 } X\in\Omega_T;\quad \mathrm{supp}\Psi(\cdot,\overline{X})\subset\Omega_T \text{ 若 } \overline{X}\in\Omega_T.$$

在 (5.1) 和 (5.2) 式中分别选取 $k=V_2(\overline{X})$, $l=V_1(X)$, $\varphi=\Psi$, $\phi=\Psi$, 并分别在

Ω_T 中关于 $\overline{X}$ 和 X 积分，相加后可得

$$\begin{aligned}&\int_{\Omega_T}\int_{\Omega_T} I_\beta(V_1-\overline{V}_2)\{(\Psi_\tau+\Psi_{\overline{\tau}})+[(\eta U\Psi)_\xi+(\overline{\eta}\overline{U}\Psi)_{\overline{\xi}}]\\&+[(A\Psi)_\eta+(\overline{A}\Psi)_{\overline{\eta}}]\}dXd\overline{X}\\&+\int_{\Omega_T}\int_{\Omega_T}\Big(\frac{B}{n}V_1-\frac{\overline{B}}{n}\overline{V}_2\Big)\mathrm{sgn}_\beta(V_1-\overline{V}_2)\Psi dXd\overline{X}\\&+\int_{\Omega_T}\int_{\Omega_T}[G_\beta(V_1,\overline{V}_2)\Psi_{\eta\eta}+G_\beta(\overline{V}_2,V_1)\Psi_{\overline{\eta}\overline{\eta}}]dXd\overline{X}\\&-\int_{\Omega_T}\int_{\Omega_T}\Big(nV_1^{-(1+n)}V_{1\eta}^2+n\overline{V}_2^{-(1+n)}\overline{V}_{2\overline{\eta}}^2\Big)\mathrm{sgn}'_\beta(V_1-\overline{V}_2)\Psi dXd\overline{X}\geqslant 0.\end{aligned}\tag{5.3}$$

特别地选取

$$\Psi(X,\overline{X})=\varphi_1(\tau,\xi,\eta)j_h(\tau-\overline{\tau})j_h(\xi-\overline{\xi})j_h(\eta-\overline{\eta}),$$

其中 $\varphi_1\geqslant 0$, $\varphi_1\in C_0^\infty(\Omega_T)$ 且

$$j_h(s)=\frac{1}{h}j\Big(\frac{s}{h}\Big),\ j(s)\in C_0^\infty(R), j(s)\geqslant 0;\ j(s)=0\ 若\ |s|\geqslant 1;$$

$$\int_{-\infty}^{\infty}j(s)ds=1.$$

显然 Ψ 满足

$$\begin{cases}\Psi_\tau+\Psi_{\overline{\tau}}=\dfrac{\partial\varphi_1}{\partial\tau}j_h(\tau-\overline{\tau})j_h(\xi-\overline{\xi})j_h(\eta-\overline{\eta}),\\ \Psi_\xi+\Psi_{\overline{\xi}}=\dfrac{\partial\varphi_1}{\partial\xi}j_h(\tau-\overline{\tau})j_h(\xi-\overline{\xi})j_h(\eta-\overline{\eta}),\\ \Psi_\eta+\Psi_{\overline{\eta}}=\dfrac{\partial\varphi_1}{\partial\eta}j_h(\tau-\overline{\tau})j_h(\xi-\overline{\xi})j_h(\eta-\overline{\eta}).\end{cases}\tag{5.4}$$

现在估计 (5.3) 式中的各项. 首先由 (5.4) 式，当 $h\to 0$, $\beta\to 0$ 时，有

$$\int_{\Omega_T}\int_{\Omega_T}I_\beta(V_1-\overline{V}_2)(\Psi_\tau+\Psi_{\overline{\tau}})dXd\overline{X}\to\int_{\Omega_T}|V_1-V_2|\varphi_{1\tau}dX.\tag{5.5}$$

注意到

$$\begin{aligned}&\int_{\Omega_T}\int_{\Omega_T}I_\beta(V_1-\overline{V}_2)(\overline{\eta}\overline{U}-\eta U)\Psi_{\overline{\xi}}dXd\overline{X}\\&=\int_{\Omega_T}\int_{\Omega_T}I'_\beta(\widehat{V_1-\overline{V}_2})(\overline{\eta}\overline{U}-\eta U)\frac{\partial\overline{V}_2}{\partial\overline{\xi}}\Psi dXd\overline{X}\\&\quad-\int_{\Omega_T}\int_{\Omega_T}I_\beta(V_1-\overline{V}_2)\overline{\eta}\overline{U}_{\overline{\xi}}\Psi dXd\overline{X}\end{aligned}$$

及

$$\lim_{h\to 0}\int_{\Omega_T}\Big[\int_{\Omega_T}I'_\beta(\widehat{V_1-\overline{V}_2})(\overline{\eta}\overline{U}-\eta U)\Psi dX\Big]\frac{\partial\overline{V}_2}{\partial\overline{\xi}}d\overline{X}=0,$$

从而当 $h\to 0,\beta\to 0$ 时，有

$$\begin{aligned}&\int_{\Omega_T}\int_{\Omega_T} I_\beta(V_1-\overline{V}_2)\Big[(\eta U\Psi)_\xi+(\overline{\eta}\,\overline{U}\Psi)_{\overline{\xi}}\Big]dXd\overline{X}\\
=&\int_{\Omega_T}\int_{\Omega_T} I_\beta(V_1-\overline{V}_2)\eta U(\Psi_\xi+\Psi_{\overline{\xi}})dXd\overline{X}\\
&+\int_{\Omega_T}\int_{\Omega_T} I_\beta(V_1-\overline{V}_2)(\overline{\eta}\,\overline{U}-\eta U)\Psi_{\overline{\xi}}dXd\overline{X}\\
&+\int_{\Omega_T}\int_{\Omega_T} I_\beta(V_1-\overline{V}_2)(\eta U_\xi+\overline{\eta}\,\overline{U}_\xi)\Psi dXd\overline{X}\\
\to&\int_{\Omega_T}|V_1-V_2|(\eta U\varphi_{1\xi}+\eta U_\xi\varphi_1)dX,\end{aligned}\tag{5.6}$$

其中

$$I'_\beta(\widehat{V_1-\overline{V}_2})=\int_0^1 \mathrm{sgn}_\beta(V_1-s\overline{V}_2^+-(1-s)\overline{V}_2^-)ds$$

是 I'_β 与 $\overline{V}_2$ 的复合均值. 类似地，当 $h\to 0,\beta\to 0$ 时，

$$\int_{\Omega_T}\int_{\Omega_T} I_\beta(V_1-\overline{V}_2)[(A\Psi)_\eta+(\overline{A}\Psi)_{\overline{\eta}}]dXd\overline{X}\to\int_{\Omega_T}|V_1-V_2|(A\varphi_{1\eta}+A_\eta\varphi_1)dX,\tag{5.7}$$

$$\int_{\Omega_T}\int_{\Omega_T}\Big(\frac{B}{n}V_1-\frac{\overline{B}}{n}\overline{V}_2\Big)\mathrm{sgn}_\beta(V_1-\overline{V}_2)\Psi dXd\overline{X}\to\int_{\Omega_T}\frac{B}{n}|V_1-V_2|\varphi_1 dX.\tag{5.8}$$

由于 $w_{1\eta},\overline{w}_{2\overline{\eta}}\in L^2_{\mathrm{loc}}(\Omega_T)$ 及 $V_{1\eta},\overline{V}_{2\overline{\eta}}\in L^2_{\mathrm{loc}}(\Omega_T)$, 因此

$$\begin{aligned}&-\int_{\Omega_T}\int_{\Omega_T}(nV_1^{-(1+n)}V_{1\eta}^2+n\overline{V}_2^{-(1+n)}\overline{V}_{2\overline{\eta}}^2)\mathrm{sgn}'_\beta(V_1-\overline{V}_2)\Psi dXd\overline{X}\\
=&-\int_{\Omega_T}\int_{\Omega_T} n(V_1^{-\frac{1+n}{2}}V_{1\eta}-\overline{V}_2^{-\frac{1+n}{2}}\overline{V}_{2\overline{\eta}})^2\mathrm{sgn}'_\beta(V_1-\overline{V}_2)\Psi dXd\overline{X}\\
&-2\int_{\Omega_T}\int_{\Omega_T} nV_1^{-\frac{1+n}{2}}V_{1\eta}\overline{V}_2^{-\frac{1+n}{2}}\overline{V}_{2\overline{\eta}}\mathrm{sgn}'_\beta(V_1-\overline{V}_2)\Psi dXd\overline{X}\\
\leqslant&-2n\int_{\Omega_T}\int_{\Omega_T} V_1^{-\frac{1+n}{2}}V_{1\eta}\overline{V}_2^{-\frac{1+n}{2}}\overline{V}_{2\overline{\eta}}\mathrm{sgn}'_\beta(V_1-\overline{V}_2)\Psi dXd\overline{X}\\
=&-2n\int_{\Omega_T}\int_{\Omega_T}\frac{\partial}{\partial\eta}\frac{\partial}{\partial\overline{\eta}}\int_{\overline{V}_2}^{V_1}\Big(s^{-\frac{1+n}{2}}\int_s^{\overline{V}_2}\sigma^{-\frac{1+n}{2}}\mathrm{sgn}'_\beta(\sigma-s)d\sigma\Big)ds\Psi dXd\overline{X}\\
=&\,2n\int_{\Omega_T}\int_{\Omega_T} V_1^{-\frac{1+n}{2}}V_{1\eta}\int_{V_1}^{\overline{V}_2}\sigma^{-\frac{1+n}{2}}\mathrm{sgn}'_\beta(\sigma-V_1)d\sigma\Psi_{\overline{\eta}}dXd\overline{X}\\
=&-2n\int_{\Omega_T}\int_{\Omega_T} V_1^{-\frac{1+n}{2}}V_{1\eta}\Big(\int_{V_1}^{\overline{V}_2}\sigma^{-\frac{1+n}{2}}\mathrm{sgn}'_\beta(\sigma-V_1)d\sigma\Big)\varphi_1\\
&\times j_h(\tau-\overline{\tau})j_h(\xi-\overline{\xi})\partial_\eta j_h(\eta-\overline{\eta})dXd\overline{X}.\end{aligned}\tag{5.9}$$

由分部积分可得

$$
\begin{aligned}
&\int_{\Omega_T}\int_{\Omega_T}[G_\beta(V_1,\overline{V}_2)\Psi_{\eta\eta}+G_\beta(\overline{V}_2,V_1)\Psi_{\overline{\eta}\overline{\eta}}]dXd\overline{X}\\
=&\int_{\Omega_T}\int_{\Omega_T}G_\beta(V_1,\overline{V}_2)\varphi_{1\eta\eta}j_h(\tau-\overline{\tau})j_h(\xi-\overline{\xi})j_h(\eta-\overline{\eta})dXd\overline{X}\\
&+2\int_{\Omega_T}\int_{\Omega_T}G_\beta(V_1,\overline{V}_2)\varphi_{1\eta}j_h(\tau-\overline{\tau})j_h(\xi-\overline{\xi})\frac{\partial}{\partial\eta}j_h(\eta-\overline{\eta})dXd\overline{X}\\
&+\int_{\Omega_T}\int_{\Omega_T}G_\beta(V_1,\overline{V}_2)\varphi_1 j_h(\tau-\overline{\tau})j_h(\xi-\overline{\xi})\frac{\partial^2}{\partial\eta^2}j_h(\eta-\overline{\eta})dXd\overline{X}\\
&+\int_{\Omega_T}\int_{\Omega_T}G_\beta(\overline{V}_2,V_1)\varphi_1 j_h(\tau-\overline{\tau})j_h(\xi-\overline{\xi})\frac{\partial^2}{\partial\eta^2}j_h(\eta-\overline{\eta})dXd\overline{X}\\
=&\int_{\Omega_T}\int_{\Omega_T}G_\beta(V_1,\overline{V}_2)\varphi_{1\eta\eta}j_h(\tau-\overline{\tau})j_h(\xi-\overline{\xi})j_h(\eta-\overline{\eta})dXd\overline{X}\\
&-\int_{\Omega_T}\int_{\Omega_T}\Big(\frac{\partial}{\partial\eta}G_\beta(V_1,\overline{V}_2)+\frac{\partial}{\partial\eta}G_\beta(\overline{V}_2,V_1)\Big)\varphi_1\\
&\times j_h(\tau-\overline{\tau})j_h(\xi-\overline{\xi})\frac{\partial}{\partial\eta}j_h(\eta-\overline{\eta})dXd\overline{X}\\
&+\int_{\Omega_T}\int_{\Omega_T}(G_\beta(V_1,\overline{V}_2)-G_\beta(\overline{V}_2,V_1))\varphi_{1\eta}\\
&\times j_h(\tau-\overline{\tau})j_h(\xi-\overline{\xi})\frac{\partial}{\partial\eta}j_h(\eta-\overline{\eta})dXd\overline{X},
\end{aligned}\tag{5.10}
$$

$$
\begin{aligned}
&\frac{\partial}{\partial\eta}G_\beta(V_1,\overline{V}_2)+\frac{\partial}{\partial\eta}G_\beta(\overline{V}_2,V_1)\\
=&\,nV_1^{-(1+n)}V_{1\eta}\mathrm{sgn}_\beta(V_1-\overline{V}_2)-n\int_{V_1}^{\overline{V}_2}\mathrm{sgn}'_\beta(\sigma-V_1)\sigma^{-(1+n)}d\sigma V_{1\eta}\\
=&-n\int_{V_1}^{\overline{V}_2}(V_1^{-\frac{1+n}{2}}-\sigma^{-\frac{1+n}{2}})^2\mathrm{sgn}'_\beta(\sigma-V_1)d\sigma V_{1\eta}\\
&-2n\int_{V_1}^{\overline{V}_2}\sigma^{-\frac{1+n}{2}}\mathrm{sgn}'_\beta(\sigma-V_1)d\sigma V_1^{-\frac{1+n}{2}}V_{1\eta}.
\end{aligned}\tag{5.11}
$$

注意到

$$\lim_{\beta\to0}G_\beta(V_1,\overline{V}_2)=\lim_{\beta\to0}G_\beta(\overline{V}_2,V_1)=|V_1^{-n}-V_2^{-n}|$$

以及

$$\int_{V_1}^{\overline{V}_2}(V_1^{-\frac{1+n}{2}}-\sigma^{-\frac{1+n}{2}})^2\mathrm{sgn}'_\beta(\sigma-V_1)d\sigma$$

关于 β 一致有界且趋向于零当 $\beta\to 0$, 在 (5.3) 式中令 $h\to 0,\beta\to 0$, 由 (5.5)–(5.11) 式可得

$$\int_{\Omega_T}|V_1-V_2|\Big[\varphi_{1\tau}+(\eta U\varphi_1)_\xi+(A\varphi_1)_\eta+\frac{B}{n}\varphi_1\Big]dX+\int_{\Omega_T}|V_1^{-n}-V_2^{-n}|\varphi_{1\eta\eta}dX\geqslant 0.$$

引理证毕.

定理 2.3 的证明 设 $V_1=w_1^{-\frac{1}{n}},V_2=w_2^{-\frac{1}{n}}$ 是问题 (2.1)–(2.3) 的两个解. 由引理 5.1, 有

$$\int_{\Omega_T}|V_1-V_2|\Big[\varphi_\tau+(\eta U\varphi)_\xi+(A\varphi)_\eta+\frac{B}{n}\varphi\Big]dX+\int_{\Omega_T}|V_1^{-n}-V_2^{-n}|\varphi_{\eta\eta}dX\geqslant 0. \quad (5.12)$$

定义

$$\begin{cases}\mu_h^1(\tau)=\displaystyle\int_{-\infty}^{\tau-s-2h}j_h(\sigma)d\sigma-\int_{-\infty}^{\tau-t+2h}j_h(\sigma)d\sigma,\\ \mu_h^2(\xi)=\displaystyle\int_{-\infty}^{\xi-2h}j_h(\sigma)d\sigma-\int_{-\infty}^{\xi-L+2h}j_h(\sigma)d\sigma,\\ \mu_h^3(\eta)=\displaystyle\int_{-\infty}^{\eta-2h}j_h(\sigma)d\sigma-\int_{-\infty}^{\eta-1+2h}j_h(\sigma)d\sigma,\end{cases}$$

其中 $j_h(s)$ 是引理 5.1 中所定义的函数, $0<s<t<T$. 在 (5.12) 式中选取 $\varphi=(1-\eta)\mu_h^1(\tau)\mu_h^2(\xi)\mu_h^3(\eta)$, 则可得

$$\begin{aligned}&\int_{\Omega_T}(1-\eta)|V_1-V_2|\Big[\frac{\partial\mu_h^1}{\partial\tau}\mu_h^2\mu_h^3+\eta U\frac{\partial\mu_h^2}{\partial\xi}\mu_h^1\mu_h^3+A\frac{\partial\mu_h^3}{\partial\eta}\mu_h^1\mu_h^2\Big]dX\\&+\int_{\Omega_T}(1-\eta)|V_1-V_2|\Big\{\eta U_\xi-\Big[(1+\eta)U_\xi+\frac{U_\tau}{U}\Big]+A_\eta+\frac{B}{n}\Big\}\mu_h^1\mu_h^2\mu_h^3dX\\&+\int_{\Omega_T}|V_1^{-n}-V_2^{-n}|\Big\{(1-\eta)\mu_h^1\mu_h^2\frac{\partial^2}{\partial\eta^2}\mu_h^3-2\mu_h^1\mu_h^2\frac{\partial}{\partial\eta}\mu_h^3\Big\}dX\geqslant 0.\end{aligned}\quad (5.13)$$

注意到当 $h\to 0$ 时,

$$\int_{\Omega_T}(1-\eta)|V_1-V_2|\frac{\partial}{\partial\tau}\mu_h^1\mu_h^2\mu_h^3dX\to\int_\Omega(1-\eta)|V_1-V_2|d\xi d\eta\Big|_{\tau=t}^{\tau=s}, \quad (5.14)$$

$$\int_{\Omega_T}\eta U(1-\eta)|V_1-V_2|\frac{\partial}{\partial\xi}\mu_h^2\mu_h^1\mu_h^3dX\to\int\eta U(1-\eta)|V_1-V_2|d\tau d\eta\Big|_{\xi=L}^{\xi=0}\leqslant 0, \quad (5.15)$$

$$\begin{aligned}&\int_{\Omega_T}A(1-\eta)|V_1-V_2|\frac{\partial}{\partial\eta}\mu_h^3\mu_h^1\mu_h^2dX\\&\to\int A(1-\eta)|V_1-V_2|d\tau d\xi\Big|_{\eta=1}^{\eta=0}=\int\Big(U_\xi+\frac{U_\tau}{U}\Big)|V_1-V_2|d\tau d\xi\Big|_{\eta=0},\end{aligned}\quad (5.16)$$

$$\begin{aligned}&\int_{\Omega_T}(1-\eta)|V_1-V_2|\Big\{\eta U_\xi-\Big[(1+\eta)U_\xi+\frac{U_\tau}{U}\Big]+A_\eta+\frac{B}{n}\Big\}\mu_h^1\mu_h^2\mu_h^3dX\\&\leqslant C\int_s^t\int_\Omega(1-\eta)|V_1-V_2|dX.\end{aligned}\quad (5.17)$$

利用

$$\frac{\partial}{\partial\eta}|V_1^{-n}-V_2^{-n}|=\mathrm{sgn}(V_1^{-n}-V_2^{-n})((V_1^{-n})_\eta-(V_2^{-n})_\eta),$$

可得到

$$\begin{aligned}
&\int_{\Omega_T}(1-\eta)|V_1^{-n}-V_2^{-n}|\mu_h^1\mu_h^2\frac{\partial^2}{\partial\eta^2}\mu_h^3dX-2\int_{\Omega_T}|V_1^{-n}-V_2^{-n}|\mu_h^1\mu_h^2\frac{\partial}{\partial\eta}\mu_h^3dX\\
=&-\int_{\Omega_T}(1-\eta)\mathrm{sgn}(V_1^{-n}-V_2^{-n})((V_1^{-n})_\eta-(V_2^{-n})_\eta)\mu_h^1\mu_h^2\frac{\partial}{\partial\eta}\mu_h^3dX\\
&-\int_{\Omega_T}|V_1^{-n}-V_2^{-n}|\mu_h^1\mu_h^2\frac{\partial}{\partial\eta}\mu_h^3dX\\
\to&-\int\mathrm{sgn}(V_1^{-n}-V_2^{-n})((V_1^{-n})_\eta-(V_2^{-n})_\eta)d\tau d\xi\Big|_{\eta=0}\\
&-\int|V_1^{-n}-V_2^{-n}|d\tau d\xi\Big|_{\eta=0}\\
\leqslant&-\int\mathrm{sgn}(V_1^{-n}-V_2^{-n})((V_1^{-n})_\eta-(V_2^{-n})_\eta)d\tau d\xi\Big|_{\eta=0}\\
=&-\int\mathrm{sgn}(V_1-V_2)\Big(U_\xi+\frac{U_\tau}{U}\Big)(V_1-V_2)d\tau d\xi\Big|_{\eta=0}\\
=&-\int\Big(U_\xi+\frac{U_\tau}{U}\Big)|V_1-V_2|d\tau d\xi\Big|_{\eta=0}.
\end{aligned}\tag{5.18}$$

由 (5.13)–(5.18) 式，有

$$\int_\Omega(1-\eta)|V_1-V_2|d\xi d\eta\Big|_{\tau=t}\leqslant\int_\Omega(1-\eta)|V_1-V_2|d\xi d\eta\Big|_{\tau=s}+C\int_s^t\int_\Omega(1-\eta)|V_1-V_2|dX.$$

由上式及 Gronwall 不等式，可得

$$\int(1-\eta)|V_1-V_2|d\xi d\eta\Big|_{\tau=t}\leqslant e^{c(t-s)}\int_\Omega(1-\eta)|V_1-V_2|d\xi d\eta\Big|_{\tau=s},$$

令 $s\to 0$, 即可证明定理 2.3.

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