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Approximation of Integerated Szász-Bézier Operators for Local Bounded Functions

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Abstract In this paper the approximation properties of integerated Szász-Bézier operators $\hat{S}_{n,\alpha}$ are studied. The rate of convergence of pointwise approximation for local bounded functions are obtained.

Key words Rate of approximation; Szász-Bézier operators; local bounded functions; Lebesgue-Stieltjes integration

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1 Introduction

The article [1] introduced Szász-Bézier operators $S_{n,\alpha}(f, x)$:

$$S_{n,\alpha}(f, x) = \sum_{k=0}^{+\infty} f\left(\frac{k}{n}\right) Q_{nk}^{(\alpha)}(x), \tag{1}$$

where $\alpha \geq 1$, $Q_{nk}^{(\alpha)}(x) = J_{nk}^{\alpha}(x) - J_{n,k+1}^{\alpha}(x)$, $J_{nk}(x) = \sum_{j=k}^{+\infty} \frac{(nx)^j}{j!} e^{-nx}$, $q_{nk}(x) = \frac{(nx)^k}{k!} e^{-nx}$ is Szăsz basis function. When $\alpha = 1$, we get Szász operators $S_{n,1}(f, x) = \sum_{k=0}^{+\infty} f\left(\frac{k}{n}\right) \frac{(nx)^k}{k!} e^{-nx}$.

In this paper we will consider a kind of integerated Szász-Bézier operator $S_{n,\alpha}$ and study its approximation behaviors. For a function $f(x)$ defined on the infinite interval $[0, +\infty)$, the integerated Szász-Bézier type operators applied to $f(x)$ are

$$\hat{S}_{n,\alpha}(f, x) = \sum_{k=0}^{+\infty} \frac{Q_{nk}^{(\alpha)}(x)}{\int_{I_k} \mathrm{d}t} \int_{I_k} f(t)\,\mathrm{d}t, \tag{2}$$

where $Q_{nk}^{(\alpha)}(x)$ is defined as in (1), and $I_k = \left[\frac{k}{n}, \frac{k+1}{n}\right]$, $k \in N$. In this paper we shall establish general estimate formulas on the rates of convergence of $\hat{S}_{n,\alpha}$ for a kind of fundtions I_{locB} defined as follows:

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$$I_{locB} = \{f \mid f \text{ is bounded in every finite subinterval of } [0, +\infty)\}.$$

For $f \in I_{locB}$ and arbitrily fixed $x \in (0, +\infty)$, let $\Omega_x(f, \lambda) = \sup_{t\in[x-\lambda, x+\lambda]} |f(t) - f(x)|$.

It is clear that

(1) $\Omega_x(f, \lambda)$ is monotone non-decreasing with respect to λ.

(2) $\lim_{\lambda\to 0}\Omega_x(f, \lambda) = 0$ if $f(x)$ is continuous at point x.

Our main results can be stated as follows.

Theorem 1 Let $f \in I_{locB}$, $f(t) = O(t^{\beta})$ for some $\beta > 0$, as $t \to +\infty$ and $f(x+)$, $f(x-)$ exist at $x \in (0, +\infty)$, then for n sufficiently large, we have

$$\left|\hat{S}_{n,\alpha}(f, x) - \frac{f(x+) + (2^{\alpha} - 1)f(x-)}{2^{\alpha}}\right| \le \left(\frac{4\alpha + x}{nx} + \frac{4\alpha}{n^2x^2}\right)\sum_{k=1}^{n}\Omega_x\left(g_x, \frac{x}{\sqrt{k}}\right)$$
$$+ O(1)\frac{\alpha(2x+1)^{(2x+1)\beta}}{1+\sqrt{nx}}\left(\frac{e}{4}\right)^{nx} + \alpha\left[Z(n, x) + \frac{1}{\sqrt{2enx}}\right]|f(x+) - f(x-)|, \quad (3)$$

where

$$g_x(t) = \begin{cases} f(t) - f(x+), & x < t < +\infty \\ 0, & t = x \\ f(t) - f(x-), & 0 \le t < x \end{cases}, \qquad Z(n, x) = \frac{0.8\sqrt{3x+1}}{\sqrt{nx}}. \quad (4)$$

2 Some Lammas

We need some lemmas for Proving Theorem 1.

Lemma 1 Let $K_{n,\alpha,2}(x, t) = \sum_{k=0}^{+\infty}\frac{Q_{nk}^{(\alpha)}(x)\chi_k(t)}{\int_0^{\infty}\chi_k(u)\,du}$, then for $x \in (0, +\infty)$ and n sufficiently large, we have

(i) for $0 \le y < x$, there holds

$$\int_0^y K_{n,\alpha,2}(x, t)\,dt \le \left(\frac{\alpha x}{n} + \frac{\alpha}{n^2}\right)\frac{1}{(x-y)^2}, \quad (5)$$

(ii) for $x < z$, there holds

$$\int_z^{\infty} K_{n,\alpha,2}(x, t)\,dt \le \left(\frac{\alpha x}{n} + \frac{\alpha}{n^2}\right)\frac{1}{(x-y)^2}, \quad (6)$$

where χ_k is the characteristic function of the interval $I_k = \left[\frac{k}{n}, \frac{k}{n+1}\right]$ with respect to $I = [0, +\infty)$.

Proof First we prove (i). By direct calculation

$$\hat{S}_{n,\alpha}((x-t)^2, x) \le \sum_{k=0}^{+\infty}\frac{\alpha q_{nk}(x)}{\int_{I_k} dt}\int_{I_k}(x-t)^2\,dt \le \frac{\alpha x}{n} + \frac{\alpha}{n^2}.$$

For $0 \le t \le y < x$, we have $(x-t)^2 > (x-y)^2$ and $\hat{S}_{n,\alpha}((x-t)^2, x) \le \frac{\alpha x}{n} + \frac{\alpha}{n^2}$. Hence

$$\int_0^y K_{n,\alpha,2}(x, t)\,dt \le \int_0^y \frac{(x-t)^2}{(x-y)^2}K_{n,\alpha,2}(x, t)\,dt = \frac{1}{(x-y)^2}\int_0^y (x-t)^2 K_{n,\alpha,2}(x, t)\,dt$$

$$\le \frac{1}{(x-y)^2}\hat{S}_{n,\alpha}((t-x)^2, x) \le \left(\frac{\alpha x}{n}+\frac{\alpha}{n^2}\right)\frac{1}{(x-y)^2}.$$

Similarly, for $x\in z$, we have

$$\int_z^{\infty} K_{n,\alpha,2}(x,t)\,\mathrm{d}t \le \left(\frac{\alpha x}{n}+\frac{\alpha}{n^2}\right)\frac{1}{(x-z)^2}.$$

Lemma 2　For $\alpha\ge 1$, $\beta>0$, we have the following.

(i) we get $Q_{nk}^{(\alpha)}(x)\le \alpha q_{nk}(x)$, $Q_{nk}^{(\alpha)}(x)$ defined as (2).

(ii) $\int_{\frac{k}{n}}^{\frac{k+1}{n}} t^{\beta t}\,\mathrm{d}t \le \max\left\{\left(\frac{k+1}{n}\right)^{\beta\left(\frac{k+1}{n}\right)}, \left(\frac{k}{n}\right)^{\beta\left(\frac{k}{n}\right)}\right\}$.

(iii) $\sum_{k=[2nx]+1}^{+\infty} q_{nk}(x)\max\left\{\left(\frac{k+1}{n}\right)^{\beta\left(\frac{k+1}{n}\right)}, \left(\frac{k}{n}\right)^{\beta\left(\frac{k}{n}\right)}\right\}\le 3\frac{(2x+1)^{(2x+1)\beta}}{1+\sqrt{nx}}\left(\frac{e}{4}\right)^{nx}$.

Proof　Using the differential mean value theorem, we get (i).

(ii) can be proved by differential method.

(iii) refer to lemma 5 of [1], we get

$$\sum_{k=[2nx]+1}^{+\infty} q_{nk}(x)\left(\frac{k}{n}\right)^{\beta\left(\frac{k}{n}\right)} \le 3\frac{(2x+1)^{(2x+1)\beta}}{1+\sqrt{nx}}\left(\frac{e}{4}\right)^{nx},$$

by a similar method, we have

$$\sum_{k=[2nx]+1}^{+\infty} q_{nk}(x)\left(\frac{k+1}{n}\right)^{\beta\left(\frac{k+1}{n}\right)} \le 3\frac{(2x+1)^{(2x+1)\beta}}{1+\sqrt{nx}}\left(\frac{e}{4}\right)^{nx}.$$

Therefore,

$$\sum_{k=[2nx]+1}^{+\infty} q_{nk}(x)\max\left\{\left(\frac{k+1}{n}\right)^{\beta\left(\frac{k+1}{n}\right)}, \left(\frac{k}{n}\right)^{\beta\left(\frac{k}{n}\right)}\right\}\le 3\frac{(2x+1)^{(2x+1)\beta}}{1+\sqrt{nx}}\left(\frac{e}{4}\right)^{nx}.$$

Lemma 3　Let f is satisfied the conditions of Theorem 1 and g_x is defined by (4). Then for n suffciently large, we have

$$\left|\hat{S}_{n,\alpha}(g_x, x)\right| \le \left(\frac{4\alpha+x}{nx}+\frac{4\alpha}{n^2x^2}\right)\sum_{k=1}^{n}\Omega_x\left(g_x, \frac{x}{\sqrt{k}}\right). \tag{7}$$

Proof　We recall the Lebesgue-Stieltjes integral representations

$$\hat{S}_{n,\alpha}(g_x, x) = \int_0^{\infty} g_x(t)K_{n,\alpha,2}(x,t)\,\mathrm{d}t. \tag{8}$$

Decompose the integeral of (8) into four parts, as

$$\int_0^{\infty} g_x(t)K_{n,\alpha,2}(x,t)\,\mathrm{d}t = \Delta_{1,n}(g_x,x)+\Delta_{2,n}(g_x,x)+\Delta_{3,n}(g_x,x)+\Delta_{4,n}(g_x,x),$$

respectively, where

$$\Delta_{1,n}(g_x,x) = \int_0^{x-\frac{x}{\sqrt{n}}} g_x(t)K_{n,\alpha,2}(x,t)\,\mathrm{d}t,\quad \Delta_{2,n}(g_x,x) = \int_{x-\frac{x}{\sqrt{n}}}^{x+\frac{x}{\sqrt{n}}} g_x(t)K_{n,\alpha,2}(x,t)\,\mathrm{d}t,$$

$$\Delta_{3,n}(g_x,x) = \int_{x+\frac{x}{\sqrt{n}}}^{2x} g_x(t)K_{n,\alpha,2}(x,t)\,\mathrm{d}t,\quad \Delta_{4,n}(g_x,x) = \int_{2x}^{\infty} g_x(t)K_{n,\alpha,2}(x,t)\,\mathrm{d}t.$$

We shall evaluate $\Delta_{1,n}(g_x,x)$, $\Delta_{2,n}(g_x,x)$, $\Delta_{3,n}(g_x,x)$, $\Delta_{4,n}(g_x,x)$.

First we estimate $\Delta_{2,n}(g_x, x)$, note that $g_x(x) = 0$, and $\int_{x-\frac{x}{\sqrt{n}}}^{x+\frac{x}{\sqrt{n}}} K_{n,\alpha,2}(x,t)\,\mathrm{d}t \le 1$ hence, for $t \in \left[x - \frac{x}{\sqrt{n}},\ x + \frac{x}{\sqrt{n}}\right]$, we have

$$\left|\Delta_{2,n}(g_x, x)\right| \le \int_{x-\frac{x}{\sqrt{n}}}^{x+\frac{x}{\sqrt{n}}} \left|g_x(t) - g_x(x)\right| K_{n,\alpha,2}(x,t)\,\mathrm{d}t$$

$$\le \Omega_x\left(g_x, \frac{x}{\sqrt{n}}\right) \le \frac{x}{nx}\sum_{k=1}^{n}\Omega_x\left(g_x, \frac{x}{\sqrt{k}}\right). \tag{9}$$

Next, we estimate $\Delta_{1,n}(g_x, x)$, let $u = x - \frac{x}{\sqrt{n}}$, $\lambda_{n,\alpha,2}(x,u) = \int_0^u K_{n,\alpha,2}(x,t)\,\mathrm{d}t \le \left(\frac{\alpha x}{n} + \frac{\alpha}{n^2}\right)\frac{1}{(x-u)^2}$ in lemma 1, using partial Lebesgue-Stieltjes integration, we have

$$\left|\Delta_{1,n}(g_x, x)\right| = \left|\int_0^u g_x(t) K_{n,\alpha,2}(x,t)\,\mathrm{d}t\right| \le \int_0^u \Omega_x(g_x, x-t)\,\mathrm{d}_t\lambda_{n,\alpha,2}(x,t)$$

$$= \Omega_x(g_x, x-u)\lambda_{n,\alpha,2}(x,u+) + \int_0^u K_{n,\alpha,2}(x,t)\,\mathrm{d}_t(-\Omega_x(g_x, x-t))$$

$$\le \Omega_x(g_x, x-u)\left(\frac{\alpha x}{n} + \frac{\alpha}{n^2}\right)\frac{1}{(x-u)^2} + \left(\frac{\alpha x}{n} + \frac{\alpha}{n^2}\right)\int_0^u \frac{\mathrm{d}_t(-\Omega_x(g_x, x-t))}{(x-t)^2}$$

$$\le \Omega_x(g_x, x-u)\left(\frac{\alpha x}{n} + \frac{\alpha}{n^2}\right)\frac{1}{(x-u)^2} + \left(\frac{\alpha x}{n} + \frac{\alpha}{n^2}\right)\cdot$$

$$\left[\frac{1}{(x-t)^2}(-\Omega_x(g_x, x-t))\Big|_0^{u+} + 2\int_0^u \Omega_x(g_x, x-t)\frac{\mathrm{d}t}{(x-t)^3}\right]$$

$$\le \left(\frac{\alpha x}{n} + \frac{\alpha}{n^2}\right)\frac{1}{x^2}\Omega_x(g_x, x) + 2\left(\frac{\alpha x}{n} + \frac{\alpha}{n^2}\right)\int_0^u \Omega_x(g_x, x-t)\frac{\mathrm{d}t}{(x-t)^3}$$

$$= \left(\frac{\alpha x}{n} + \frac{\alpha}{n^2}\right)\frac{1}{x^2}\Omega_x(g_x, x) + \left(\frac{\alpha x}{n} + \frac{\alpha}{n^2}\right)\int_1^n \Omega_x\left(g_x, \frac{x}{\sqrt{v}}\right)\mathrm{d}v$$

$$\le \left(\frac{2\alpha}{nx} + \frac{2\alpha}{n^2x^2}\right)\sum_{k=1}^{n}\Omega_x\left(g_x, \frac{x}{\sqrt{k}}\right). \tag{10}$$

Using the similar method estimate $\Delta_{3,n}(g_x, x)$, we get

$$\left|\Delta_{3,n}(g_x, x)\right| \le \left(\frac{2\alpha}{nx} + \frac{2\alpha}{n^2x^2}\right)\sum_{k=1}^{n}\Omega_x\left(g_x, \frac{x}{\sqrt{k}}\right). \tag{11}$$

From estimations (9)-(11), we have

$$\left|\int_0^{2x} g_x(t)\,\mathrm{d}_tK_{n,\alpha,2}(x,t)\right| \le \left(\frac{4\alpha + x}{nx} + \frac{4\alpha}{n^2x^2}\right)\sum_{k=1}^{n}\Omega_x\left(g_x, \frac{x}{\sqrt{k}}\right). \tag{12}$$

Finally by the assumption that $f(t) = O(t^{\beta t})$ $(\beta > 0)$ as $t \to +\infty$, by lemma 2, then for a positive constant M and n sufficiently large. We have

$$\left|\Delta_{4,n}(g_x, x)\right| = \sum_{k=[2nx]+1}^{+\infty}\frac{Q_{nk}^{(\alpha)}(x)}{\int_{I_k}\mathrm{d}t}\int_{I_k} g_x(t)\,\mathrm{d}t \le M\alpha\sum_{k=[2nx]+1}^{+\infty} nq_{nk}(x)\int_{\frac{k}{n}}^{\frac{k+1}{n}} t^{\beta t}\,\mathrm{d}t$$

$$\le M\alpha\sum_{k=[2nx]+1}^{+\infty} q_{nk}(x)\max\left\{\left(\frac{k+1}{n}\right)^{\beta\left(\frac{k+1}{n}\right)}, \left(\frac{k}{n}\right)^{\beta\left(\frac{k}{n}\right)}\right\}$$

$$\leq 3M\alpha\left(1+\frac{(2x+1)^{(2x+1)\beta}}{\sqrt{nx}}\right)\left(\frac{e}{4}\right)^{nx}. \tag{13}$$

From lemma 2 of [1] and lemma of [4], we get

Lemma 4 Let $q_{nk}(x)=\frac{(nx)^k}{k!}e^{-nx}$ be Szász basis function, for $x\in(0,+\infty)$ and $Z(n,x)=\frac{0.8\sqrt{3x+1}}{\sqrt{nx}}$, we have

$$\left|\sum_{k>nx}q_{nk}(x)-\frac{1}{2}\right|\leq Z(n,x), \tag{14}$$

$$q_{nk}(x)<\frac{1}{\sqrt{2enx}},\ \forall k\in N. \tag{15}$$

3 Proof of Theorem 1

For any $f\in I_{l\alpha B}$, we decomepose $f(t)$ into four parts as

$$f(t)=\frac{1}{2^\alpha}f(x+)+\left(1-\frac{1}{2^\alpha}\right)f(x-)+g_x(t)+\frac{f(x+)-f(x-)}{2^\alpha}\hat{\operatorname{sgn}}(t-x)$$
$$+\delta_x(t)\left[f(x)-\frac{1}{2^\alpha}f(x+)-\left(1-\frac{1}{2^\alpha}\right)f(x-)\right],$$

where

$$\hat{\operatorname{sgn}}(t)=\begin{cases}2^\alpha-1, & t>0\\ 0, & t=0\\ -1, & t<0\end{cases},\qquad \delta_x(t)=\begin{cases}1, & t=x\\ 0, & t\neq x\end{cases}. \tag{16}$$

Obviously,

$$\hat{S}_{n,\alpha}(\delta_x,x)=0.$$

It follows that

$$\left|\hat{S}_{n,\alpha}(f,x)-\frac{f(x+)+(2^\alpha-1)f(x-)}{2^\alpha}\right|$$
$$\leq\left|\hat{S}_{n,\alpha}(g_x,x)\right|+\left|\frac{f(x+)-f(x-)}{2^\alpha}\right|\hat{S}_{n,\alpha}(\hat{\operatorname{sgn}}(t-x),x) \tag{17}$$

Below we estimate $\hat{S}_{n,\alpha}(\hat{\operatorname{sgn}}(t-x),x)$, let $x\in\left[\frac{k'}{n},\frac{k'+1}{n}\right]$, $k'=[nx]$. We have

$$\hat{S}_{n,\alpha}(\hat{\operatorname{sgn}}(t-x),x)=\sum_{k=0}^{k'}(-1)Q_{nk}^{(\alpha)}(x)+\frac{Q_{nk'}^{(\alpha)}(x)}{\int_{\frac{k'}{n}}^{\frac{k'+1}{n}}\mathrm{d}t}\int_{\frac{k'}{n}}^{x}(-1)\mathrm{d}t$$
$$+\frac{Q_{nk'}^{(\alpha)}(x)}{\int_{\frac{k'}{n}}^{\frac{k'+1}{n}}\mathrm{d}t}\int_x^{\frac{k'+1}{n}}(2^\alpha-1)\mathrm{d}t+\sum_{k=k'+1}^{+\infty}(2^\alpha-1)Q_{nk}^{(\alpha)}(x)$$
$$=\sum_{k=k'+1}^{+\infty}2^\alpha Q_{nk}^{(\alpha)}(x)+\frac{Q_{nk'}^{(\alpha)}(x)}{\int_{\frac{k'}{n}}^{\frac{k'+1}{n}}\mathrm{d}t}\int_x^{\frac{k'+1}{n}}2^\alpha\mathrm{d}t-1,$$

therefore,

$$\left|\hat{S}_{n,\alpha}(\widehat{\mathrm{sgn}}(t-x),x)\right| = \left|\sum_{k=k'+1}^{+\infty} 2^{\alpha} Q_{nk}^{(\alpha)}(x) + \frac{Q_{nk'}^{(\alpha)}(x)}{\int_{k'} \mathrm{d}t}\int_{x}^{\frac{k'+1}{n}} 2^{\alpha}\mathrm{d}t - 1\right|$$

$$\leq \left|\sum_{k=k'}^{+\infty} 2^{\alpha} Q_{nk}^{(\alpha)}(x) - 1\right| + 2^{\alpha} Q_{nk'}^{(\alpha)}(x) = 2^{\alpha}\left|\sum_{k=k'}^{+\infty} Q_{nk}^{(\alpha)}(x) - \frac{1}{2^{\alpha}}\right| + 2^{\alpha} Q_{nk'}^{(\alpha)}$$

$$\leq \alpha 2^{\alpha}\left|\sum_{k=k'}^{+\infty} q_{nk}(x) - \frac{1}{2}\right| + \alpha 2^{\alpha} q_{nk}(x) \leq \alpha 2^{\alpha}\left[Z(n,x) + \frac{1}{2enx}\right]. \quad (18)$$

Hence, from lemma 3, (17) and (18), by direct calculation, we have

$$\left|\hat{S}_{n,\alpha}(f,x) - \frac{f(x+) + (2^{\alpha}-1)f(x-)}{2^{\alpha}}\right| \leq \left(\frac{4\alpha + x}{nx} + \frac{4\alpha}{n^2x^2}\right)\sum_{k=1}^{n}\Omega_x\left(g_x, \frac{x}{\sqrt{k}}\right)$$

$$+ O(1)\frac{\alpha(2x+1)^{(2x+1)\beta}}{1+\sqrt{nx}}\left(\frac{e}{4}\right)^{nx} + \alpha\left[Z(n,x) + \frac{1}{2enx}\right]\left|f(x+) - f(x-)\right|.$$

The proof of Theorem 1 is complete.

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对于局部有界函数的积分型 Szász-Bézier 算子的逼近估计

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摘　要　引入一种积分型的 Szász-Bézier 算子, 并研究其逼近性质, 得到了此类算子对局部有界函数的逼近阶估计公式.

关键词　逼近度; Szász-Bézier 算子; 局部有界函数; Lebesgue-Stieltjes 积分