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关于 Prandtl 方程组整体弱解的存在性

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摘要:考虑非定常 Prandtl 方程组整体弱解的存在性.引进 Crocco 变换,将 Prandtl 方程组(1)、(2)转化成一个初边值问题(3).通过对问题(3)的正则化问题作一致估计,类似文 Xin-Zhang 或 Zhang-Zhao 的方法可证明整体弱解的存在性.这改进已有的 Prandtl 方程组存在整体 BV 弱解的条件.

关键词:Prandtl 方程组;整体解;存在性

中图分类号:O 175.2 **文献标识码**:A

考虑研究非定常边界层问题的 Prandtl 方程组

$$\begin{cases} u_t + uu_x + vu_y = \nu u_{yy} - P_x, \\ u_x + v_y = 0. \end{cases} \tag{1}$$

$$\begin{cases} u(0,x,y) = u_0(x,y), u(t,0,y) = u_1(t,y), \\ \quad v(t,x,0) = v_0(t,x), \\ u(t,x,0) = 0, \lim\limits_{y\to\infty} u(t,x,y) = U(t,x). \end{cases} \tag{2}$$

其中$(t,x,y) \in D \equiv \{0 < t < T, 0 < x < L, 0 < y < \infty\}$,$\nu$是粘性常数,$P$与$U$满足:

$$P_x + UU_x + U_t = 0.$$

引进 Crocco 变换,

$$\tau = t, \xi = x, \eta = \frac{u}{U}; w(\tau,\xi,\eta) = \frac{u_y(t,x,y)}{U(t,x)}.$$

并假设 $\nu = 1$,则式(1)、(2)可化为

$$\begin{cases} w_\tau + \eta U w_\xi + A w_\eta + B w - w^2 w_{\eta\eta} = 0, \\ ww_\eta - v_0 w + (U_\xi + U^{-1} U_\tau) = 0, \eta = 0, \\ w(0,\xi,\eta) = w_0(\xi,\eta), w(\tau,0,\eta) = w_1(\tau,\eta), \\ w(\tau,\xi,1) = 0. \end{cases} \tag{3}$$

其中$(\tau,\xi,\eta) \in \Omega \equiv \{0 < \tau < T, 0 < \xi < L, 0 < \eta < 1\}$,且

$$A = (1-\eta^2) U_\xi + (1-\eta) U^{-1} U_\tau,$$
$$B = \eta U_\xi + U^{-1} U_\tau.$$

$$w_0(\xi,\eta) \equiv \frac{u_{0y}}{U(0,x)}, \quad w_1(\tau,\eta) \equiv \frac{u_{1y}}{U(t,0)}.$$

文献[1]证明了非定常牛顿流及拟可塑流边界层问题局部古典解的存在性;文献[2]在假设(H) $U > 0$, $U_\xi + U^{-1}U_\tau \geq 0$; $v_0 \leq 0$条件下证明了非定常牛顿流边界层问题整体 BV 解的存在性;Zhang-Zhao 的文章①在类似文献[2]的假设下证明非定常拟可塑流边界层问题整体 BV 解及可膨胀流边界层问题局部BV解的存在性及惟一性.在上述文章中为了保证问题(3)的解有所要求的界,假设(H)起到关键作用.

在本文中,我们考虑对文献[2]和 Zhang-Zhao 的文章①的假设(H)进行弱化,即考虑允许 $-P_x = UU_x + U_t$ 变号的情形.

为此,考虑问题(3)的正则化问题(4):

$$\begin{cases} w_\tau + \eta U w_\xi + A w_\eta + B w - (w^2 + \varepsilon) w_{\eta\eta} - \\ \quad \varepsilon (w^2 + \varepsilon) w_{\xi\xi} = 0, \\ ww_\eta - v_0 w + (U_\xi + U^{-1} U_\tau) = 0, \eta = 0, \\ w(0,\xi,\eta) = w_{0\varepsilon}(\xi,\eta), \\ w(\tau,0,\eta) = w_{1\varepsilon}(\tau,\eta), \\ w(\tau,L,\eta) = w_{2\varepsilon}(\tau,\eta), w(\tau,\xi,1) = \varepsilon. \end{cases} \tag{4}$$

根据抛物方程经典理论[3]可知,问题任何可能存在的解如果有先验的界,则问题(4)存在古典解

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① Zhang Jianwen, Zhao Junning. On the global exilstence and uniqueness of solutions of nonstationary boundary layer system, to appear.

w_ε.

下面将分别对几种情形对 w_ε 作一致估计.

引理 1 假设:

(i) $C_0^{-1}(1+\varepsilon-\eta) \le w_{0\varepsilon}, w_{1\varepsilon}, w_{2\varepsilon} \le C_0(1+\varepsilon-\eta)$,

(ii) $v_0 \le 0$, $U\xi + U^{-1}U_\tau$ 在 $(0,T)\times(0,L)$ 上变号,

成立,则问题(4) 的解 w_ε 满足

$$\min(v_0^{-1}(U\xi + U^{-1}U_\tau)e^{\alpha\tau}) \le w_\varepsilon \le C(1+\varepsilon-\eta)e^{\beta\tau}.$$

其中 C,α,β 是与 ε 无关的正常数.

证明 先证明不等式的左端(为了方便起见省略 ε). 令 $\overline{w} = we^{-\alpha\tau}$,则 $\overline{w}$ 满足:

$$\overline{w}_\tau + \eta U\overline{w}_\xi + A\overline{w}_\eta + (B+\alpha)\overline{w} - (w^2+\varepsilon)\overline{w}_{\eta\eta} - \varepsilon(w^2+\varepsilon)\overline{w}_{\xi\xi} = 0 \tag{5}$$

$$w\overline{w}_\eta - v_0\overline{w} + (U\xi + U^{-1}U_\tau)e^{-\alpha\tau} = 0,\quad \eta = 0 \tag{6}$$

选取 α 充分大使得 $B+\alpha > 0$,则由假设(i),式(5) 及极值原理可知,$\overline{w}$ 不能在 $\tau = 0$, $\xi = 0$, $\xi = L$, $\eta = 1$ 及 Ω 内部取得负的极小. 若 $\overline{w}$ 在 $\eta = 0$ 取得负的极小,则此时 $\overline{w}_\eta \ge 0$. 由式(6) 及(ii),在 $\overline{w}$ 的极小值点有:

$$v_0\overline{w} \le (U\xi + U^{-1}U_\tau)e^{-\alpha\tau},$$

由此得出

$$w \ge \min(v_0^{-1}(U\xi + U^{-1}U_\tau)e^{\alpha\tau}).$$

现证明不等式的右端,为此令

$$V = C(1+\varepsilon-\eta)e^{\beta\tau},$$

选取 β, C 充分大,则有

$$\begin{aligned}L(V) &\equiv V_\tau + \eta UV_\xi + AV_\eta + BV - (V^2+\varepsilon)V_{\eta\eta} - \varepsilon(V^2+\varepsilon)V_{\xi\xi} = \\ &\beta C(1+\varepsilon-\eta)e^{\beta\tau} - ACe^{\beta\tau} + BC(1+\varepsilon-\eta)e^{\beta\tau} = \\ &Ce^{\beta\tau}(1+\varepsilon-\eta)\left(\beta - \frac{A}{1+\varepsilon-\eta} + B\right) > 0,\end{aligned}$$

$$\begin{aligned}l(V) &\equiv VV_\eta - v_0V + (U\xi + U^{-1}U_\tau) = \\ &-(Ce^{\beta\tau})^2(1+\varepsilon) - Ce^{\beta\tau}(1+\varepsilon)v_0 + (U\xi + U^{-1}U_\tau) < 0.\end{aligned}$$

令 $Z = V - w$,则 Z 满足:

$$Z_\tau + \eta UZ_\xi + AZ_\eta + BZ - (w^2+\varepsilon)Z_{\eta\eta} - \varepsilon(w^2+\varepsilon)Z_{\xi\xi} > 0,$$

$$VZ_\eta + (w_\eta - v_0)Z < 0,\ \eta = 0.$$

再令 $Z = Pe^{-\gamma_1\eta+\gamma_2\tau}$,则 P 满足:

$$P_\tau + \eta UP_\xi + [2\gamma_1(w^2+\varepsilon) + A]P_\eta + [\gamma_2 + B - \gamma_1A - \gamma_1^2(w^2+\varepsilon)]P - (w^2+\varepsilon)(P_{\eta\eta} + \varepsilon P_{\xi\xi}) > 0,$$

$$VP_\eta - (\gamma_1V - w_\eta + v_0)P < 0,\ \eta = 0.$$

选取 $\gamma_1(\varepsilon)$, $\gamma_2(\varepsilon)$ 充分大,使得:

$$\gamma_1V - w_\eta + v_0 > 0,\ \eta = 0.$$

$$\gamma_2 + B - \gamma_1A - \gamma_1^2(w^2+\varepsilon) > 0.$$

注意到,若 C 选取足够大,则 $P|_{\tau=0}$, $P|_{\xi=0,L}$, $P|_{\eta=1} \ge 0$. 因此由极值原理可得到 $P \ge 0$ 在 $\overline{\Omega}$ 中,即有 $Z \ge 0$, $w_\varepsilon \le C(1+\varepsilon-\eta)e^{\beta\tau}$ 在 $\overline{\Omega}$ 中.

引理 2 假设引理 1 的条件成立,则存在 $C,\alpha,\beta > 0$,使得当

$$U\xi + U^{-1}U_\tau \ge -C^2e^{-2\beta\tau}(\alpha - 1) + Cv_0e^{-\beta\tau},$$

$$Ce^{\alpha} \le C_0^{-1},$$

时,问题(4) 的解 w_ε 满足:

$$w_\varepsilon \ge Ce^{\alpha\eta-\beta\tau}(1+\varepsilon-\eta),$$

其中 C_0 为引理 1(i) 中的常数.

证明 为证明引理 2,令

$$V = C(1+\varepsilon-\eta)e^{\alpha\eta-\beta\tau},$$

显然

$$V|_{\tau=0} \le w_{0\varepsilon},\ V|_{\xi=0} \le w_{1\varepsilon},$$

$$V|_{\xi=L} \le w_{2\varepsilon},\ V|_{\eta=1} \le \varepsilon.$$

选取 β 适当大,$\alpha > (1+\varepsilon)^{-1}$,则

$$\begin{aligned}L(V) &\equiv V_\tau + \eta UV_\xi + AV_\eta + BV - (V^2+\varepsilon)V_{\eta\eta} - \varepsilon(V^2+\varepsilon)V_{\xi\xi} = \\ &C(1+\varepsilon-\eta)e^{\alpha\eta-\beta\tau}\left[-\beta + \alpha A - \frac{A}{1+\varepsilon-\eta} + B\right] - (V^2+\varepsilon)[\alpha^2(1+\varepsilon-\eta) - 2\alpha]Ce^{\alpha\eta-\beta\tau} \le \\ &C(1+\varepsilon-\eta)e^{\alpha\eta-\beta\tau}\left\{-\beta + \alpha A - \frac{A}{1+\varepsilon-\eta} + B - [\alpha^2(1+\varepsilon-\eta) - 2\alpha]\times\frac{V^2+\varepsilon}{1+\varepsilon-\eta}\right\} < 0,\end{aligned}$$

$$\begin{aligned}l(V)|_{\eta=0} &\equiv VV_\eta - v_0V + (U\xi + U^{-1}U_\tau) = \\ &(1+\varepsilon)(Ce^{-\beta\tau})^2[\alpha(1+\varepsilon) - 1] - v_0C(1+\varepsilon)e^{-\beta\tau} + (U\xi + U^{-1}U_\tau) > 0\end{aligned} \tag{7}$$

令 $Z = V - w$,则有

$$Z_\tau + \eta UZ_\xi + AZ_\eta + [B - (w+V)V_m]Z - (w^2+\varepsilon)(Z_{\eta\eta} + \varepsilon Z_{\xi\xi}) < 0,$$

$$l(V) - l(w) = VZ_\eta + (w_\eta - v_0)Z > 0,\ \eta = 0.$$

类似引理 1 的证明即可证明 $Z \le 0$. 因此,

$$w \ge Ce^{\alpha\eta-\beta\tau}(1+\varepsilon-\eta).$$

引理 3　假设引理 1 中的条件(i) 成立, $U_\xi + U^{-1}U_\tau > 0$ 且允许 v_0 变号,则问题(4) 的解 w_ε 满足:

$$C^{-1}(T)(1+\varepsilon-\eta) \le w_\varepsilon \le C(T)(1+\varepsilon-\eta).$$

证明　只需注意到在下界估计中,当 v_0 变号, $U_\xi + U^{-1}U_\tau > 0$ 时,若选取 $C > 0$ 充分小,则仍然有式(7) 成立. 因此类似引理 1,2 的证明可证明引理 3.

利用引理 2,3 类似 Zhang-Zhao 的文章①的证明,得到:

定理 1　在引理 2,3 的假设下,问题(3) 存在惟一的整体 BV 弱解.

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On the Global Existence of Weak Solutions to the Prandtl's System

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Abstract: In this paper, we consider the global existence of weak solutions to the non-stationary Prandtl's system (1)、(2). However, we consider the initial-boundary problem (3) instead of the Prandtl's system via Crocco transformation and obtain some uniform estimates of the solutions to the regularized problem of (3) under some assumptions. Similar to the argument in the paper by Xin-Zhang or the paper by Zhang-Zhao, we can prove the global existence of weak solutions and the conditions for the global existence have been improved.

Key words: Prandtl's system; global solution; existence