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Regularity of Solutions for the Evolution p-Laplacian Equations

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Abstract Consider the Cauchy problem for the evolution p-Laplacian equation

$$\begin{cases} \frac{\partial u}{\partial t} = \operatorname{div}(|\nabla u|^{p-2}\nabla u), & (x,t) \in Q_T = R^n \times (0,T), \\ u(x,0) = u_0(x) \in L^\infty(R^N), & x \in R^N. \end{cases}$$

In terms of the uniform estimates to the regulized solutions of the problem above, we prove that $u_{x_j} \in C_{loc}^{\beta, \beta/(1+\beta)}(Q_T)$, where the Hölder exponent with respect to t is great than $\frac{\beta}{2}$.

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In this note we consider the following Cauchy problem:

$$\begin{cases} \frac{\partial u}{\partial t} = \operatorname{div}(|\nabla u|^{p-2}\nabla u), & (x,t) \in Q_T = R^n \times (0,T), \\ u(x,0) = u_0(x) \in L^\infty(R^N), & x \in R^N, \end{cases} \tag{1}$$

where $p > 2$ is a constant. It is well known that there exists a solution $u \in C_{loc}^{\alpha}(Q_T) \cap L^\infty(Q_T)$ to (1), with $u_{x_j} \in C_{loc}^{\beta,\beta/2}(Q_T)$, $j = 1, 2, \cdots, N$ (see [1], [2]). The proofs of $u_{x_j} \in C_{loc}^{\beta,\beta/2}(Q_T)$ are very complicated and difficult. In this note we use another approach to prove the Hölder continuity. We prove $u_{x_j} \in C_{loc}^{\beta,\beta/(1+\beta)}(Q_T)$, where the Hölder exponent with respect to t is great than $\frac{\beta}{2}$.

Definition *A function $u(x,t)$ defined in Q_T is called a weak solution of (1), if $u \in C_{loc}^{\alpha}(Q_T) \cap L^p(0, T: W_{loc}^{1,p}(R^N)) \cap L^\infty(Q_T)$, $\alpha \in (0,1)$, and for any $\varphi(x,t) \in C^1(\overline{Q}_T)$ with $\varphi = 0$ when $|x|$ is large enough, it holds that*

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$$\int_{R^N} u(x,t)\varphi(x,t)\,dx + \int_0^t\int_{R^N}[-u\varphi_t + |\nabla u|^{p-2}\nabla u\cdot\nabla\varphi]\,dx\,dt$$
$$= \int_{R^N} u_0(x)\varphi(x,0)\,dx. \tag{2}$$

We prove the following theorem.

Theorem *Let $u_0(x)\in L^\infty(R^N)$, $u_0\geq 0$. Then there exist constants $C>0$ and $\beta\in(0,1)$ such that the solution of (1) satisfies*

$$|u_t(x,t)| \leq \frac{C}{(p-2)t},\quad u_{x_j}\in C_{loc}^{\beta,\beta/(1+\beta)}(Q_T). \tag{3}$$

Proof Acoording to [3], the solution u of (1) is the limit of the solutions of the following boundary value problems

$$\begin{cases} \dfrac{\partial u}{\partial t} = \mathrm{div}((|\nabla u|^{p-2})\nabla u), & (x,t)\in B_n\times(0,T), \\ u(x,t) = 0, & (x,t)\in\partial B_n\times(0,T), \\ u(x,0) = u_{0n}(x), & x\in B_n, \end{cases} \tag{4}$$

where $B_n=\{x;|x|<n\}$, $u_{0n}\in C_0^1(B_n)$ satisfy $\|u_{0n}\|_{L^\infty}\leq\|u_0\|_{L^\infty}$ and

$$\lim_{n\to\infty}u_{0n} = u_0 \qquad \text{a.e. on } R^N.$$

Let u_n be the solutions of (4). Set

$$v_n(x,t) = \lambda^\gamma u_n(x,\lambda t),\quad \lambda>1,\quad \gamma=\frac{1}{p-2}.$$

Then v_n satisfies

$$\begin{cases} \dfrac{\partial v}{\partial t} = \mathrm{div}(|\nabla v|^{p-2}\nabla v), & (x,t)\in B_n\times(0,T), \\ v(x,t) = 0, & (x,t)\in\partial B_n\times(0,T), \\ v(x,0) = \lambda^\gamma u_{0n}(x), & x\in B_n. \end{cases} \tag{5}$$

Set $w=v_n-u_n$. Then by Comparison Principle $w\geq 0$ and

$$\int_{B_n} w(x,t)\varphi(x,t)\,dx - \int_0^t\int_{B_n} w\varphi_t\,dx\,d\tau$$
$$+\int_0^t\int_{B_n}(|\nabla v_n|^{p-2}\nabla v_n - |\nabla u_n|^{p-2}\nabla u_n)\cdot\nabla\varphi\,dx\,d\tau$$
$$= \int_{B_n}(\lambda^\gamma-1)u_{0n}(x)\varphi(x,0)\,dx, \tag{6}$$

for any $\varphi\in C^1(\overline{B_n\times(0,T)})$, with $\varphi=0$ near ∂B_n. In (6) by an approximate process, we can take

$$\varphi=(w-k)_+,\qquad k=(\lambda^\gamma-1)\|u_0\|_{L^\infty}.$$

Then we get

$$\int_{B_n}(w-k-\epsilon)_+^2\,dx + 2\int_0^t\int_{B_n\cap\{w>k\}}(|\nabla v_n|^{p-2}\nabla v_n$$
$$-|\nabla u_n|^{p-2}\nabla u_n)(\nabla v_n-\nabla u_n)\,dx\,d\tau = 0.$$

This implies $w\leq k$ a.e. on $B_n\times(0,T)$. Thus

$$0\leq\lambda^\gamma u_n(x,\lambda t) - u_n(x,t)\leq(\lambda^\gamma-1)\|u_0\|_{L^\infty}. \tag{7}$$

Divide (7) by $\lambda - 1$ and let $\lambda \to 1^+$ we get

$$\left|\gamma u_n(x,t) + t u_{nt}(x,t)\right| \le \gamma \| u_0 \|_{L^\infty}.$$

This inequality implies

$$\left| u_t(x,t) \right| \le \frac{C}{t}.$$

Notice that for fixed $t \in (0, T)$, $u(x,t)$ is a solution of the elliptic equation

$$\operatorname{div}(|\nabla u|^{p-2} \nabla u) = u_t(x,t), \quad x \in R^N.$$

By [4], there exist constants $\beta \in (0,1)$, $C > 0$ depending only on $|u_t|_{L^\infty}$, $|u|_{L^\infty}$ such that

$$\left| \nabla u(x_1,t) - \nabla u(x_2,t) \right| \le C |x_1 - x_2|^\beta. \tag{8}$$

We now prove that ∇u is Hölder continuous in t. For convernience, we assume that u is a smooth solution; otherwise, by the uniqueness of solution we can consider the regularized problem. Take the x_j-derivative in (1) to obtain

$$\frac{\partial u_{x_j}}{\partial t} = (\operatorname{div}(|\nabla u|^{p-2} \nabla u))_{x_j}. \tag{9}$$

Let $x_0 \in R^N$, $0 < t_1 \le t_2$, $\Delta t = t_2 - t_1$, $B(\Delta t) = B_{(\Delta t)^\delta}(x_0)$. Integrating (9) by parts over $B(\Delta t) \times (t_1, t_2)$, we get

$$\int_{B(\Delta t)} (u_{x_j}(x, t_2) - u_{x_j}(x, t_1))\,dx = \int_{t_1}^{t_2} \int_{B(\Delta t)} (\operatorname{div}(|\nabla u|^{p-2} \nabla u))_{x_j}\,dx\,dt$$

$$= \int_{t_1}^{t_2} \int_{\partial B(\Delta t)} \operatorname{div}(|\nabla u|^{p-2} \nabla u)\nu_j\,d\sigma dt = \int_{t_1}^{t_2} \int_{\partial B(\Delta t)} u_t \nu_j\,d\sigma dt. \tag{10}$$

where $\nu = (\nu_1, \nu_2, \cdots, \nu_N)$ is the outward normal unit vector of $\partial B(\Delta t)$. By the mean value theorem, there exists $x^* \in B(\Delta t)$ such that

$$\left| u_{x_j}(x^*, t_2) - u_{x_j}(x^*, t_1) \right| \le C(\Delta t)^{1-\delta}. \tag{11}$$

Combining (8) with (11) and taking $\delta = \dfrac{1}{1+\beta}$, we get

$$\begin{aligned} &\left| u_{x_j}(x_0, t_2) - u_{x_j}(x_0, t_1) \right| \\ \le &\left| u_{x_j}(x_0, t_2) - u_{x_j}(x^*, t_2) \right| + \left| u_{x_j}(x^*, t_2) - u_{x_j}(x^*, t_1) \right| \\ &+ \left| u_{x_j}(x^*, t_1) - u_{x_j}(x_0, t_1) \right| \le C(\Delta t)^{\beta/(1+\beta)}. \end{aligned}$$

Therefore $u_{x_j} \in C_{loc}^{\beta, \beta/(1+\beta)}(R^N)$ and the theorem is proved.

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