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INITIAL TRACE OF SOLUTIONS FOR A DOUBLY NONLINEAR DEGENERATE PARABOLIC EQUATIONS*

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Abstract In this note, we study the existence of an initial trace of nonnegative solutions for the following problem

$$u_t - \operatorname{div}(|\bigtriangledown u^m|^{p-2}\bigtriangledown u^m) + u^q = 0 \quad \text{in} \quad Q_T = \Omega \times (0,T).$$

We prove that the initial trace is an outer regular Borel measure, which may not be locally bounded for some values of parameters p, q, and m. We also study the corresponding Cauchy problems with a given generalized Borel measure as initial data.

Key words Doubly degenerate; Initial trace; Borel measure

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1 Introduction

Let Ω be a domain in R^N ($N \ge 1$), possibly unbounded. The aim of this article is to investigate the initial trace problem for degenerate parabolic equation:

$$u_t - \operatorname{div}(|\bigtriangledown u^m|^{p-2}\bigtriangledown u^m) + u^q = 0 \quad \text{in} \quad Q_T = \Omega \times (0,T), \tag{1.1}$$

where $q \ge 0, p > 1$, and $m > 0$. We prove the existence of an initial trace in the class $\mathcal{B}^+_{\rm reg}(\Omega)$ of outer regular positive Borel measure in Ω, not necessarily locally bounded. Moreover, we study also Cauchy problem for (1.1) with initial data $\nu \in \mathcal{B}^+_{\rm reg}(R^N)$.

(1.1) was suggested as a mathematical model for a variety of physical problems [1, 2], which is also called polypropic filtration equation. The evolution p-Laplacian equation ((1.1) when $m = 1$) and the porous medium equation (Equ.(1.1) when $p = 2$) are the special cases of (1.1) and analogous problems were considered in [3–5].

Definition 1.1 A nonnegative function u is said to be a weak solution of (1.1) in Q_T, if

$$u \in L^1_{\rm loc}(Q_T), \qquad u^q \in L^1_{\rm loc}(Q_T), \qquad u^m \in L^p_{\rm loc}((0,T); W^{1,p}_{\rm loc}(\Omega))$$

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and

$$\int_0^T \int_\Omega (-u\partial_t\varphi + |\bigtriangledown u^m|^{p-2} \bigtriangledown u^m \bigtriangledown \varphi + u^q\varphi)\mathrm{d}x\mathrm{d}t = 0 \tag{1.2}$$

for any $\varphi \in C_0^\infty(Q_T)$.

By Steklov averaging process, it follows from the definition of solution that for any function $h \in C_b(R)\bigcap W^{1,\infty}(R)$ and $\varphi \in C_0^\infty(\Omega \times [0,T])$, we have

$$\begin{aligned}&\int_t^\theta \int_\Omega \Big(-\int_0^{u(x,t)} h(s^m)\mathrm{d}s\partial_t\varphi + |\bigtriangledown u^m|^{p-2} \bigtriangledown u^m \cdot \bigtriangledown(h(u^m)\varphi\Big) + u^q h(u^m)\varphi)\mathrm{d}x\mathrm{d}t \\ &= \int_\Omega \int_0^{u(x,t)} h(s^m)\mathrm{d}s\varphi(x,t)\mathrm{d}x - \int_\Omega \int_0^{u(x,\theta)} h(s^m)\mathrm{d}s\varphi(x,\theta)\mathrm{d}x\end{aligned} \tag{1.3}$$

for any $0 < t < \theta < T$.

It is well known that if $q > 1$, (1.1) admits a particular solution in $R^N \times (0,\infty)$,

$$W(x,t) = \Big(\frac{1}{t(q-1)}\Big)^{\frac{1}{q-1}},$$

which is called the flat solution. This particular solution play an important role because it dominates any nonnegative solution of (1.1) that is locally bounded in $R^N \times (0,\infty)$. The flat solution W shows that the initial trace of solution of (1.1) can not be Radon measure.

Our main results are as follows:

Theorem 1 Assume that $q > m(p-1)$ or $q \le m(p-1),\ m(p-1) > 1$ and that u is a nonnegative weak solution of (1.1) in Q_T. Then, for any $y \in \Omega$, the following alternative occurs:

(i) either for any open subset $U \subset \Omega$ containing y

$$\lim_{t\to 0} \int_U u(x,t)\mathrm{d}x = \infty, \tag{1.4}$$

or (ii) there exists an open neighborhood $U^* \subset \Omega$ of y and a nonnegative Radon measure $\ell_{U^*} \in \mathcal{M}^+$ such that for any $\xi \in C_0(U^*)$,

$$\lim_{t\to 0} \int_{U^*} u(x,t)\xi(x)\mathrm{d}x = \ell_{U^*}(\xi) \tag{1.5}$$

and in any open set $U \subset\subset U^*$

$$\int_0^\theta \int_U u^\sigma \mathrm{d}x\mathrm{d}t < \infty, \qquad \text{for any } \ \sigma \in (0, m(p-1) + p/N), \tag{1.6}$$

$$\int_0^\theta \int_U |\bigtriangledown u^m|^r \mathrm{d}x\mathrm{d}t < \infty, \quad \text{for any } \ r \in \Big(0, p - \frac{mN}{mN+1}\Big). \tag{1.7}$$

Owing to Theorem 1, we can define a set $\mathcal{R}$ by

$$\mathcal{R} = \Big\{y \in \Omega : \exists \text{ open set } U \subset \Omega,\ y \in U,\ \overline{\lim_{t\to 0}} \int_U u(x,t)\mathrm{d}x < \infty\Big\}. \tag{1.8}$$

Clearly, $\mathcal{R}$ is an open subset of Ω and by Theorem 1, there exists a unique Radon measure $\mu \in \mathcal{M}^+(\mathcal{R})$ such that

$$\lim_{t\to 0} \int_{\mathcal{R}} u(x,t)\xi(x)\mathrm{d}x = \int_{\mathcal{R}} \xi(x)\mathrm{d}\mu(x) \quad \forall \xi \in C_0(\mathcal{R}), \tag{1.9}$$

where u satisfies

$$\int_0^\theta \int_{\mathcal{R}} (-u\partial_t\varphi + |\bigtriangledown u^m|^{p-2} \bigtriangledown u^m \bigtriangledown \varphi + u^q\varphi)\mathrm{d}x\mathrm{d}t$$
$$= \int_{\mathcal{R}} \varphi(x,0)\mathrm{d}\mu - \int_{\mathcal{R}} \varphi(x,\theta)u(x,\theta)\mathrm{d}x \tag{1.10}$$

for any $0 < \theta < T$ and $\varphi \in C_0^\infty(\mathcal{R} \times [0,T))$ and (1.6),(1.7) hold in any open set $U \subset\subset \mathcal{R}$.

Definition 1.2 Let u be a nonnegative weak solution of (1.1) in Q_T. A point $y \in \Omega$ is called a regular point if $y \in \mathcal{R}$. Otherwise, it is called a singular point. The set of singular points is denoted by $\mathcal{S} = \Omega - \mathcal{R}$; it is a relatively closed subset of Ω. Denote

$$\mathrm{tr}_\Omega(u) = (\mathcal{S}, \mu),$$

where μ is the Radon measure in (1.9). $\mathrm{tr}_\Omega(u)$ is called the initial trace of u at $t = 0$.

Remark 1.1 By Definition 1.2, Theorem 1 can be rewritten as: the solution of (1.1) has initial trace

$$\nu = (\mathcal{S}, \mu) \in \mathcal{B}^+_{\mathrm{reg}}(\Omega).$$

Theorem 2 Assume that u is a nonnegative weak solution of (1.1) in Q_T and that $0 < q \le 1,\ m(p-1) < 1$, or $q \le 1 < m(p-1),\ \Omega = R^N$. Then, there exists a Radon measure $\mu \in \mathcal{M}^+(\Omega)$, such that

$$\lim_{t\to 0} \int_\Omega u(x,t)\xi(x)\mathrm{d}x = \int_\Omega \xi(x)\mathrm{d}\mu \qquad \forall \xi \in C_0(\Omega),$$

that is, the singular set $\mathcal{S}$ is empty.

Theorem 3 Let $\mu \in \mathcal{M}^+(R^N)$. Assume that

$$p > \frac{(m+1)N}{mN+1} \ \ (\text{or}\ \ p > 1\ \ \text{if}\ \mu \in L^1_{\mathrm{loc}}(R^N)),\ \ 0 < q < m(p-1) + \frac{p}{N}$$

and that either $m(p-1) < 1$ or $m(p-1) < q$. Then, the Cauchy problem

$$\begin{cases} \dfrac{\partial u}{\partial t} = \nabla\cdot(|\nabla u^m|^{p-2}\nabla u^m) + u^q & \text{in}\ \ Q_\infty = R^N \times (0,\infty) \\ u(x,0) = \mu & \text{on}\ \ R^N \end{cases} \tag{1.11}$$

has a solution.

Remark 1.2 In Theorem 3, the growth condition of μ has not been required.

Theorem 4 Let

$$\max\{1, m(p-1)\} < q \le m(p-1) + \frac{p}{N}.$$

Then, for any $\nu \in \mathcal{B}^+_{\mathrm{reg}}(R^N)$, there exists at least one solution to Cauchy problem (1.1) with initial trace ν.

2 Main Estimates

Proposition 2.1 Let $\alpha < 0,\ \alpha \ne -1, 0 < t < \theta < T$ and let u be a nonnegative weak solution of (1.1) in Q_T. Then, for any nonnegative function $\xi \in C_0^\infty(\Omega)$ and any $\tau > p$,

$$\int_\Omega \int_0^{u(x,t)} (1+s^m)^\alpha \mathrm{d}s\xi^\tau\mathrm{d}x + \frac{|\alpha|}{2}\int_t^\theta \int_\Omega (1+u^m)^{\alpha-1}\xi^\tau|\bigtriangledown u^m|^p\mathrm{d}x\mathrm{d}t$$

$$\le \int_\Omega \int_0^{u(x,\theta)} (1+s^m)^\alpha \mathrm{d}s\xi^\tau \mathrm{d}x + C\int_t^\theta \int_\Omega (1+u^m)^\alpha \xi^\tau u^q \mathrm{d}x\mathrm{d}t$$
$$+C\int_t^\theta \int_\Omega (1+u^m)^{\alpha+p-1}\xi^{\tau-p}|\bigtriangledown \xi|^p \mathrm{d}x\mathrm{d}t, \tag{2.1}$$

$$\begin{aligned}\int_\Omega (1+u(x,t))\xi^\tau \mathrm{d}x \le & \int_\Omega (1+u(x,\theta))\xi^\tau \mathrm{d}x + C\int_t^\theta \int_\Omega u^q \xi^\tau \mathrm{d}x\mathrm{d}t \\ & +C\int_t^\theta \int_\Omega (1+u^m)^{\alpha-1}\xi^\tau |\bigtriangledown u^m|^p \mathrm{d}x\mathrm{d}t \\ & +C\int_t^\theta \int_\Omega (1+u^m)^{(1-\alpha)(p-1)}\xi^{\tau-p} \mathrm{d}x\mathrm{d}t,\end{aligned} \tag{2.2}$$

$$\int_\Omega u(x,\theta)\xi^\tau \mathrm{d}x + \int_t^\theta \int_\Omega u^q \xi^\tau \mathrm{d}x\mathrm{d}t \le \int_\Omega u(x,t)\xi^\tau \mathrm{d}x + \tau\int_t^\theta \int_\Omega |\nabla u^m|^{p-1}|\nabla \xi^\tau|\mathrm{d}x\mathrm{d}t, \tag{2.3}$$

where $C=C(\alpha,p,q,\tau)$.

Proof Taking $h(s)=(1+s^m)^\alpha,\ \phi=\xi^\tau$ in (1.3), where $\alpha\le 0,\ \alpha\ne -1$, it yields

$$\int_\Omega \int_0^{u(x,t)} (1+s^m)^\alpha \mathrm{d}s\xi^\tau \mathrm{d}x + |\alpha|\int_t^\theta \int_\Omega (1+u^m)^{\alpha-1}|\bigtriangledown u^m|^p \xi^\tau \mathrm{d}x\mathrm{d}t$$
$$= \int_\Omega \int_0^{u(x,\theta)} (1+s^m)^\alpha)\mathrm{d}s\xi^\tau \mathrm{d}x + \int_t^\theta \int_\Omega u^q \xi^\tau (1+u^m)^\alpha$$
$$+\tau\int_t^\theta \int_\Omega (1+u^m)^\alpha \xi^{\tau-1}|\bigtriangledown u^m|^{p-2}\bigtriangledown u^m\cdot \bigtriangledown\xi \mathrm{d}x\mathrm{d}t. \tag{2.4}$$

Using Young's inequality

$$\tau\int_t^\theta \int_\Omega (1+u^m)^\alpha \xi^{\tau-1}|\bigtriangledown u^m|^{p-2}\bigtriangledown u^m\cdot\bigtriangledown\xi \mathrm{d}x\mathrm{d}t$$
$$\le \frac{|\alpha|}{2}\int_t^\theta \int_\Omega (1+u^m)^{\alpha-1}\xi^\tau|\bigtriangledown u^m|^p \mathrm{d}x\mathrm{d}t + C\int_t^\theta \int_\Omega (1+u^m)^{\alpha+p-1}\xi^{\tau-p}|\bigtriangledown\xi|^p \mathrm{d}x\mathrm{d}t. \tag{2.5}$$

Hence, (2.1) follows from (2.4) and (2.5).

As a particular case of (2.4) (with $\alpha=0$),

$$\begin{aligned}\int_\Omega (1+u(x,t))\xi^\tau \mathrm{d}x = & \int_\Omega (1+u(x,\theta))\xi^\tau \mathrm{d}x + \int_t^\theta \int_\Omega u^q\xi^\tau \mathrm{d}x\mathrm{d}t \\ & +\tau\int_t^\theta \int_\Omega \xi^{\tau-1}|\bigtriangledown u^m|^{p-2}\bigtriangledown u^m\cdot\bigtriangledown\xi \mathrm{d}x\mathrm{d}t.\end{aligned} \tag{2.6}$$

Thus, (2.3) holds. Using Young's inequality, for any $\alpha<0$,

$$\begin{aligned}\int_t^\theta \int_\Omega \xi^{\tau-1}|\bigtriangledown u^m|^{p-1}|\bigtriangledown\xi|\mathrm{d}x\mathrm{d}t \le & \int_t^\theta \int_\Omega \xi^\tau |\bigtriangledown u^m|^p (1+u^m)^{\alpha-1}\mathrm{d}x\mathrm{d}t \\ & +\int_t^\theta \int_\Omega \xi^{\tau-p}|\bigtriangledown\xi|^p(1+u^m)^{(1-\alpha)(p-1)}\mathrm{d}x\mathrm{d}t.\end{aligned} \tag{2.7}$$

Hence, (2.2) follows from (2.6) and (2.7).

Proposition 2.2 Let u be a nonnegative solution of (1.1) in Q_T and let $0<\theta<T$. For any open set $U\subset\subset\Omega$, let

$$\sup_{t\in(0,\theta]}\int_U u(x,t)\mathrm{d}x<\infty. \tag{2.8}$$

Then, for any $\xi\in C_0^1(U),\ \alpha\le 0,\ \alpha\ne -1$,

$$\begin{aligned}&\int_0^\theta\int_U \xi^{p-1+\alpha}(1+u^m)^{p-1+\alpha+\frac{p}{mN}}\mathrm{d}x\mathrm{d}t\\ \le{}& C\int_0^\theta\int_U\xi^p(1+u^m)^{\alpha-1}|\bigtriangledown u^m|^p\mathrm{d}x\mathrm{d}t+C\int_0^\theta\int_U(1+u^m)^{\alpha-1+p}|\nabla\xi|^p\mathrm{d}x\mathrm{d}t.\end{aligned} \tag{2.9}$$

Proof Let $\alpha\in(1-p,0),\ \alpha\ne-1$ be fixed and $\beta=\frac{p-1+\alpha}{p}$. Using Gagliardo-Nirenberg-Sobolev inequality and Hölder inequality, we obtain

$$\begin{aligned}&\int_0^\theta\int_U \xi^{p-1+\alpha}(1+u^m)^{p-1+\alpha+\frac{p}{mN}}\mathrm{d}x\mathrm{d}t\\ \le{}&\int_0^\theta\int_U[\xi(1+u^m)]^{p\beta}(u^m+1)^{\frac{p}{mN}}\mathrm{d}x\mathrm{d}t\\ \le{}&\int_0^\theta\left(\int_U[\xi(u^m+1)]^{\beta\frac{pN}{N-p}}\mathrm{d}x\right)^{\frac{N-p}{N}}\left(\int_U(u^m+1)^{1/m}\mathrm{d}x\right)^{p/N}\mathrm{d}t\\ \le{}& C\int_0^\theta\int_U\xi^p(1+u^m)^{\beta-1}|\bigtriangledown u^m|^p\mathrm{d}x\mathrm{d}t+C\int_0^\theta\int_U|\nabla\xi|^p(1+u^m)^{p\beta},\end{aligned}$$

and (2.9) is proved.

Proposition 2.3 Let u be a nonnegative solution of (1.1) in Q_T and let $0<\theta<T$. For any open set $U\subset\subset\Omega$, let (2.8) hold and

$$\int_0^\theta\int_U(u^{m(p-1)}+u^q)\mathrm{d}x\mathrm{d}t<\infty. \tag{2.10}$$

Then,

$$\int_0^\theta\int_U u^\sigma\mathrm{d}x\mathrm{d}t<\infty, \tag{2.11}$$

$$\int_0^\theta\int_U|\nabla u^m|^r\mathrm{d}x\mathrm{d}t<\infty, \tag{2.12}$$

where $\sigma\in(0,m(p-1)+\frac{p}{N})$ and $r\in(0,p-1+\frac{1}{mN+1})$. Finally, there exists a Radon measure $\ell\in\mu^+(\Omega)$ such that, for any $\xi\in C_0^\infty(\Omega)$,

$$\lim_{t\to0}\int_\Omega\xi u(x,t)\mathrm{d}x=\ell(\xi) \tag{2.13}$$

and u satisfies

$$\begin{aligned}&\int_0^\theta\int_\Omega(-u\partial_t\varphi+|\bigtriangledown u^m|^{p-2}\bigtriangledown u^m\bigtriangledown\varphi+u^q\varphi)\mathrm{d}x\mathrm{d}t\\ ={}&\int_\Omega\varphi(x,0)d\ell(x)-\int_\Omega u(x,\theta)\varphi(x,\theta)\mathrm{d}x,\end{aligned} \tag{2.14}$$

for any $0<\theta<T$ and $\varphi\in C_0^\infty(\Omega\times[0,T))$.

Proof Let $\alpha < 0$ be fixed. From (2.1), for any $0 < t < \theta$, we obtain

$$\begin{aligned}&\frac{|\alpha|}{2}\int_t^\theta\int_\Omega (1+u^m)^{\alpha-1}\xi^\tau|\bigtriangledown u^m|^p \mathrm{d}x\mathrm{d}t\\ \le& \int_\Omega\int_0^{u(x,\theta)}(1+s^m)^\alpha \mathrm{d}s\xi^\tau \mathrm{d}x + C\int_t^\theta\int_\Omega (1+u^m)^\alpha\xi^\tau u^q \mathrm{d}x\mathrm{d}t\\ &+C\int_t^\theta\int_\Omega (1+u^m)^{\alpha+p-1}\xi^{\tau-p}|\bigtriangledown \xi|^p \mathrm{d}x\mathrm{d}t. \end{aligned}\tag{2.15}$$

Because $(1+u^m)^\alpha u^q \le u^q$ and $(1+u^m)^{\alpha+p-1} \le (1+u^m)^{p-1}$, we find

$$\int_t^\theta\int_\Omega (1+u^m)^{\alpha-1}|\bigtriangledown u^m|^p\xi^\tau \mathrm{d}x\mathrm{d}t \le C,$$

hence,

$$\int_0^\theta\int_U (1+u^m)^{\alpha-1}|\bigtriangledown u^m|^p \mathrm{d}x\mathrm{d}t < \infty. \tag{2.16}$$

Using Proposition 2.2 and (2.16), we get (2.11).

Next, for any $0 < r < p$ and any $\alpha < 0$, we find

$$\begin{aligned}\int_0^\theta\int_U |\bigtriangledown u^m|^r \mathrm{d}x\mathrm{d}t \le& \left(\int_0^\theta\int_U (1+u^m)^{\alpha-1}|\bigtriangledown u^m|^p \mathrm{d}x\mathrm{d}t\right)^{r/p}\\ &\times\left(\int_0^\theta\int_U (1+u^m)^{\frac{(1-\alpha)r}{p-r}} \mathrm{d}x\mathrm{d}t\right)^{(p-r)/p}. \end{aligned}\tag{2.17}$$

Hence,

$$\int_0^\theta\int_U |\bigtriangledown u^m|^r \mathrm{d}x\mathrm{d}t \le C$$

if $0 < r < p-1+1/(mN+1)$. This proves (1.7), which implies (2.10) in particular. Now, from (1.3) with $h = 1$, for any $\xi \in C_0^\infty(\Omega)$ and any $0 < t < \theta < T$,

$$\int_\Omega u(x,t)\xi(x)\mathrm{d}x = \int_\Omega u(x,\theta)\xi(x)\mathrm{d}x + \int_t^\theta\int_U (|\bigtriangledown u^m|^{p-2}\bigtriangledown u^m\cdot\bigtriangledown\xi + u^q\xi)\mathrm{d}x\mathrm{d}t.$$

As the right-hand side of the above equation has a finite limit when $t \longrightarrow 0$, so does $\int_\Omega u(x,t)\xi(x)\mathrm{d}x$. Thus, the mapping $\xi \mapsto \lim_{t\to 0}\int_\Omega u(x,t)\xi(x)\mathrm{d}x$ is a positive linear functional over the space $C_0^\infty(\Omega)$. It can be extended in a unique way as a Radon measure ℓ on Ω, and (2.11) holds in Ω. Finally, let $0 < t < \theta$ be fixed. Taking $h = 1$, $\varphi \in C_0^\infty(\Omega\times[0,T))$ in (1.3), we obtain

$$\begin{aligned}&\int_t^\theta\int_\Omega (-u\partial_t\varphi + |\bigtriangledown u^m|^{p-2}\bigtriangledown u^m\cdot\bigtriangledown\varphi + u^q\varphi)\mathrm{d}x\mathrm{d}t\\ =& \int_\Omega u(x,t)\varphi(x,t)\mathrm{d}x - \int_\Omega u(x,\theta)\varphi(x,\theta)\mathrm{d}x. \end{aligned}\tag{2.18}$$

Letting t go to 0 in (2.18) and using (2.8),(2.10),(2.11), (2.12), and

$$\left|\int_\Omega u(x,t)(\varphi(x,t)-\varphi(x,0)\mathrm{d}x\right| \le Ct\int_U u(x,t)\mathrm{d}x \to 0 \quad \text{as } t\to 0,$$

we obtain

$$\int_\Omega u(x,t)\varphi(x,t)\mathrm{d}x \to \int_\Omega \varphi(x,0)d\ell(x).$$

This proves (2.13). (2.14) follows from (2.18).

3 Proof of Theorem 1

3.1 The Case $q > m(p-1) > 0$

We first prove the following lemma.

Lemma 3.1 Let $q > m(p-1) > 0$ and let u be a nonnegative solution of (1.1). Then, for any nonnegative function $\xi \in C_0^\infty(\Omega)$, the following dichotomy occurs:

(i) either $\int_0^T \int_\Omega u^q \xi^\tau \mathrm{d}x\mathrm{d}t < \infty$, then,

$$t \longmapsto \int_\Omega u(x,t)\xi^\tau \mathrm{d}x \quad \text{remains bounded near } \ t=0, \tag{3.1}$$

or (ii) $\int_0^T \int_\Omega u^q \xi^\tau \mathrm{d}x\mathrm{d}t = \infty$, then,

$$\lim_{t\to 0} \int_\Omega u(x,t)\xi^\tau \mathrm{d}x = \infty. \tag{3.2}$$

Proof Because $q > m(p-1)$, we choose α small enough such that

$$\int_t^\theta \int_\Omega (1+u^m)^{\alpha+p-1}\xi^{\tau-p}|\bigtriangledown \xi|^p \mathrm{d}x\mathrm{d}t \le C\int_t^\theta \int_\Omega (1+u^m)^\alpha u^q \xi^\tau \mathrm{d}x\mathrm{d}t + C \tag{3.3}$$

$$\int_t^\theta \int_\Omega \xi^{\tau-p}|\bigtriangledown \xi|^p (1+u^m)^{(1-\alpha)(p-1)}\mathrm{d}x\mathrm{d}t \le C\int_t^\theta \int_\Omega u^q\xi^\tau \mathrm{d}x\mathrm{d}t + C, \tag{3.4}$$

where $C = C(\xi,\alpha,\tau,p,q)$. Substituting (3.3) into (2.1), we obtain

$$\begin{aligned}&\frac{|\alpha|}{2}\int_t^\theta \int_\Omega (1+u^m)^{\alpha-1}\xi^\tau|\bigtriangledown u^m|^p \mathrm{d}x\mathrm{d}t\\ &\le \int_\Omega u(x,\theta)\xi^\tau \mathrm{d}x + C\int_t^\theta \int_\Omega (1+u^m)^\alpha u^q\xi^\tau \mathrm{d}x\mathrm{d}t + C.\end{aligned} \tag{3.5}$$

Combining (2.2), (3.4), and (3.5), it yields

$$\int_\Omega u(x,t)\xi^\tau \mathrm{d}x \le \int_\Omega u(x,\theta)\xi^\tau \mathrm{d}x + C\int_t^\theta \int_\Omega u^q\xi^\tau \mathrm{d}x\mathrm{d}t + C, \tag{3.6}$$

where $C = C(\tau,p,q,\alpha,\xi)$. Thus, if

$$\int_0^T \int_\Omega u^q\xi^\tau \mathrm{d}x\mathrm{d}t < \infty,$$

then, (3.1) holds.

We now consider the case

$$\int_0^T \int_\Omega u^q\xi^\tau \mathrm{d}x\mathrm{d}t = \infty.$$

Using Young's inequality and (3.5), for any $\varepsilon > 0$,

$$\begin{aligned}&\tau\int_t^\theta \int_\Omega \xi^{\tau-1}|\bigtriangledown u^m|^{p-1}|\bigtriangledown \xi|\mathrm{d}x\mathrm{d}t\\ &\le \epsilon\int_t^\theta \int_\Omega \xi^\tau|\bigtriangledown u^m|^p(1+u^m)^{\alpha-1}\mathrm{d}x\mathrm{d}t + C(\epsilon)\int_t^\theta \int_\Omega \xi^{\tau-p}|\bigtriangledown \xi|^p(1+u^m)^{(1-\alpha)(p-1)}\mathrm{d}x\mathrm{d}t\\ &\le \epsilon\int_t^\theta \int_\Omega (1+u^m)^{\alpha-1}|\bigtriangledown u^m|^p\xi^\tau \mathrm{d}x\mathrm{d}t + \epsilon\int_t^\theta \int_\Omega u^q\xi^\tau \mathrm{d}x\mathrm{d}t + C_1(\epsilon)\\ &\le C\varepsilon\int_\Omega u(x,\theta)\xi^\tau \mathrm{d}x + C\varepsilon\int_t^\theta \int_\Omega u^q\xi^\tau \mathrm{d}x\mathrm{d}t + C_1(\varepsilon).\end{aligned} \tag{3.7}$$

Choosing $\epsilon > 0$ small enough and combining (3.7) and (2.3), we obtain

$$\int_\Omega u(x,\theta)\xi^\tau \mathrm{d}x + \int_t^\theta \int_\Omega u^q \xi^\tau \mathrm{d}x\mathrm{d}t \le C \int_\Omega u(x,t)\xi^\tau \mathrm{d}x + C. \tag{3.8}$$

Thus, $\int_t^T \int_\Omega u^q\xi^\tau \mathrm{d}x\mathrm{d}t = \infty$ implies (3.2).

We now prove Theorem 1 for the case $q > m(p-1)$. We first assume that for any open subset U of Ω containing y and any nonnegative $\xi \in C_0^\infty(U)$, $\xi = 1$ in a neighborhood of y

$$\int_0^T \int_\Omega u^q \xi^\tau \mathrm{d}x\mathrm{d}t = \infty.$$

Then, (1.4) holds from Lemma 3.1.

Assume now that there exists an open neighborhood $\widetilde{U} \subset \Omega$ of y and a nonnegative function $\xi \in C_0^\infty(\widetilde{U}), \xi = 1$ in a neighborhood U^* of y such that

$$\int_0^T \int_\Omega u^q \xi^\tau \mathrm{d}x\mathrm{d}t < \infty.$$

Then,

$$t \longmapsto \int_{U^*} u(x,t)\mathrm{d}x$$

remains bounded near $t = 0$ from Lemma 3.1. Moreover, we have also

$$\int_0^T \int_{U^*} | \bigtriangledown u^m|^{p-1} \mathrm{d}x\mathrm{d}t < \infty. \tag{3.9}$$

Indeed, using Young's inequality and Hölder inequality, we have

$$\begin{aligned}\int_t^\theta \int_\Omega | \bigtriangledown u^m|^{p-1}\xi^\tau \mathrm{d}x\mathrm{d}t \le& \int_t^\theta \int_\Omega | \bigtriangledown u^m|^p (1+u^m)^{\alpha-1}\xi^\tau \mathrm{d}x\mathrm{d}t \\ &+ \int_t^\theta \int_\Omega (1+u^m)^{(1-\alpha)(p-1)}\xi^\tau \mathrm{d}x\mathrm{d}t,\end{aligned} \tag{3.10}$$

$$\begin{aligned}\int_t^\theta \int_\Omega (1+u^m)^{(1-\alpha)(p-1)}\xi^\tau \mathrm{d}x\mathrm{d}t \le& \int_t^\theta \int_\Omega (1+u^m)^{\frac{q}{m}}\xi^\tau \mathrm{d}x\mathrm{d}t + C \\ \le& C\int_t^\theta \int_\Omega u^q\xi^\tau \mathrm{d}x\mathrm{d}t + C,\end{aligned} \tag{3.11}$$

where $m(1-\alpha)(p-1) \le q$. Then, (3.9) follows from (3.5), (3.10), and (3.11).

3.2 The Case $q \le m(p-1)$, $m(p-1) > 1$

In the range of exponents, the proof of Theorem 1 is a consequence of the following lemma.

Lemma 3.2 Let $0 < q \le m(p-1)$and $m(p-1) > 1$. Assume that u is a nonnegative weak solution of (1.1) in Q_T and that for any open set $U \subset\subset \Omega$

$$t \mapsto \int_U u(x,t)\mathrm{d}x$$

remains bounded near $t = 0$. Then, for any $0 < \theta < T$,

$$\int_0^\theta \int_U u^{m(p-1)}(x,t)\mathrm{d}x\mathrm{d}t + \int_0^\theta \int_U | \bigtriangledown u^m|^{p-1}\mathrm{d}x\mathrm{d}t < \infty.$$

Proof Let $\alpha \in (1-p,0)$, $\alpha \neq -1$ be fixed and $\xi \in C_0^\infty(\Omega)$ as above. Combining (2.13), (2.8), and $0 < q \leq m(p-1)$, we obtain

$$\frac{|\alpha|}{2}\int_t^\theta \int_\Omega (1+u^m)^{\alpha-1}\xi^\tau |\bigtriangledown u^m|^p \mathrm{d}x\mathrm{d}t \leq C\int_t^\theta \int_\Omega (1+u^m)^{p-1+\alpha}\xi^{\tau-p}\mathrm{d}x\mathrm{d}t + C. \tag{3.12}$$

Let U, U^* be open sets with $U \subset\subset U^* \subset\subset \Omega$ and $\xi \in C_0^\infty(\Omega)$, $0 \leq \xi \leq 1$, $\xi = 1$ on U, and $\xi = 0$ outside of U^*. Using Proposition 2.2 and (3.12), we obtain

$$\int_0^\theta \int_U (1+u^m)^{p-1+\alpha+\frac{p}{mN}}\mathrm{d}x\mathrm{d}t \leq C + C\int_0^\theta \int_{U^*} (1+u^m)^{\alpha-1+p}\mathrm{d}x\mathrm{d}t. \tag{3.13}$$

Hence, any estimate of $(1+u^m)^{\alpha-1+p}$ in $L^1((0,\theta), L^1_{\mathrm{loc}}(\Omega))$ implies the same estimate for $(1+u^m)^{p-1+\alpha+\frac{p}{Nm}}$. We first take $\alpha_0 = 1+\frac{1}{m}-p$. From (3.13) and $\alpha_0+(p-1) = \frac{1}{m}$, we obtain

$$u^{m\sigma_1} \in L^1((0,\theta), L^1_{\mathrm{loc}}(\Omega))$$

with $\sigma_1 = \alpha_0 + p - 1 + \frac{p}{mN} = \frac{1}{m} + \frac{p}{mN}$.

Defining by induction

$$\alpha_{n+1} = \alpha_n + \frac{p}{mN}, \qquad \sigma_n = \alpha_n + p - 1, \quad \forall\, n \in \mathbb{N},$$

it yields

$$(1+u^m)^{\sigma_{n+1}} \in L^1((0,\theta), L^1_{\mathrm{loc}}(\Omega))$$

as long as $\alpha_n = \frac{np}{mN} + 1 - p + \frac{1}{m} < 0$. Let n_0 be the largest integer such that $\alpha_n < 0$. Then, $(1+u^m)^{\sigma_{n_0+1}} \in L^1((0,\theta), L^1_{\mathrm{loc}}(\Omega))$ and $\sigma_{n_0+1} \geq p-1$. In particular,

$$u^{m(p-1)} \in L^1((0,\theta), L^1_{\mathrm{loc}}(\Omega)).$$

Hence, from Proposition 2.3, we obtain $|\bigtriangledown u^m| \in L^r((0,\theta), L^1_{\mathrm{loc}}(\Omega))$ for any $r < p-1+\frac{1}{mN+1}$. In particular,

$$|\bigtriangledown u^m| \in L^{p-1}((0,\theta), L^1_{\mathrm{loc}}(\Omega)).$$

We now prove Theorem 1 for $0 < q \leq m(p-1)$ and $m(p-1) > 1$. Let $y \in \Omega$. Then, either statement (i) of Theorem 1 holds, or there exists an open subset $U^* \subset \Omega$ containing y such that $\int_{U^*} u(x,t)\mathrm{d}x$ is bounded near $t=0$. Hence, statement (ii) follows from Lemma 3.2 and Proposition 2.3.

4 Proof of Theorem 2

We first prove the following lemma.

Lemma 4.1 Let $0 < q \leq 1$, $m(p-1) < 1$ and let u be a nonnegative weak solution of (1.1) in Q_T. Then, there exists a Radon measure $\mu \in \mathcal{M}^+(\Omega)$ such that

$$\lim_{t\to 0}\int_\Omega u(x,t)\xi(x)\mathrm{d}x = \int_\Omega \xi(x)\mathrm{d}\mu(x), \quad \text{for } \forall \xi \in C_0(\Omega).$$

Proof Let $\alpha, t, \theta, \xi,$ *and* τ be as in Proposition 2.1. Using Proposition 2.1 and Young's inequality, it yields

$$\begin{aligned}&\frac{|\alpha|}{2}\int_t^\theta\int_\Omega(1+u^m)^{\alpha-1}\xi^\tau|\bigtriangledown u^m|^p\mathrm{d}x\mathrm{d}t\\ \leq&\int_\Omega\int_0^{u(x,\theta)}(1+s^m)^\alpha\mathrm{d}s\xi^\tau\mathrm{d}x+C\int_t^\theta\int_\Omega(1+u^m)^\alpha\xi^\tau u^q\mathrm{d}x\mathrm{d}t\\ &+C\int_t^\theta\int_\Omega(1+u^m)^{\alpha+p-1}\xi^{\tau-p}|\bigtriangledown\xi|^p\mathrm{d}x\mathrm{d}t\\ \leq&\int_\Omega(1+u(x,\theta))\xi^\tau\mathrm{d}x+C\int_t^\theta\int_\Omega(1+u(x,t))\xi^\tau\mathrm{d}x\mathrm{d}t+C.\end{aligned}\tag{4.1}$$

Combining (4.1), (2.2) and choosing α such that $m(p-1)(1-\alpha)\leq 1$, we obtain

$$\int_\Omega(1+u(x,t))\xi^\tau\mathrm{d}x\leq C\int_\Omega(1+u(x,\theta))\xi^\tau\mathrm{d}x+C\int_t^\theta\int_\Omega(1+u(x,t))\xi^\tau\mathrm{d}x\mathrm{d}t+C.$$

By Gronwall inequality, there exists $M>0$ such that

$$\int_t^\theta\int_\Omega(1+u(x,t))\xi^\tau\mathrm{d}x\mathrm{d}t<M\quad\int_\Omega(1+u(x,t))\xi^\tau\mathrm{d}x<M\tag{4.2}$$

for $t\in(0,\theta]$, which implies the claim of lemma.

Proof of Theorem 2 When $0<q\leq 1,\ m(p-1)<1$, (2.10) follows from $q\leq 1,\ m(p-1)<1$, (4.2) and Hölder inequality. (2.11)–(2.13) follow from (2.10) and Proposition 2.3.

We now consider the case $q\leq 1<m(p-1),\ \Omega=\mathbb{R}^N$. We show that, for any $b\in\mathbb{R}^N$, there exists $\rho>0$, such that

$$\limsup_{t\to 0}\int_{B_\rho(b)}u(x,t)\mathrm{d}x<\infty.\tag{4.3}$$

We argue by contradiction. Assume that (4.3) is false. Then, there exists some $b\in R^N$ such that, for any $\rho>0$, there exists a sequence $\{t_{n,\rho}\}$ converging to 0 with the property

$$\lim_{t_{n,\rho}\to 0}\int_{B_\rho(b)}u(x,t_{n,\rho})\mathrm{d}x=\infty.\tag{4.4}$$

Let $k>0$ be an integer. For any $\rho>0$, there exists N_ρ such that, for any $n_\rho\geq N_\rho$,

$$\int_{B_\rho(b)}u(x,t_{n_\rho})\mathrm{d}x\geq k.\tag{4.5}$$

By continuity of the integral with respect to the domain, there exists some $0<\widetilde{\rho}\leq\rho$ such that

$$\int_{B_{\widetilde{\rho}}(b)}u(x,t_{n_\rho})\mathrm{d}x=k.\tag{4.6}$$

Moreover, $\widetilde{\rho}$ is uniquely determined if we impose it to be the largest as possible. Clearly $t_{n_\rho}\to 0$ as $\rho\to 0$, because $t\mapsto u(.,t)$ is continuous from $(0,T)$ into $L^1_{\mathrm{loc}}(R^N)$. Let $w_{\rho k}$ be the solution of

$$\begin{cases}\partial_t w-\nabla\cdot(|\bigtriangledown w^m|^{p-2}\bigtriangledown w^m)+w^q=0 & \text{in }\mathbb{R}^N\times(0,\infty)\\ w(.,0)=u(.,t_{n_\rho})\chi_{B_{\widetilde{\rho}}(b)}, & \text{in }R^N.\end{cases}\tag{4.7}$$

Where $\chi_{B_{\widetilde{\rho}}(b)}$ is the characteristic function of $B_{\widetilde{\rho}}(b)$. As u is nonnegative, it follows by the comparison principle [6] that

$$u(x,t+t_{n_\rho,\rho}) \geq w_{\rho k}(x,t) \text{ in } R^N \times (0, T-t_{n_\rho}). \tag{4.8}$$

Notice that, when $\rho \to 0$, $w_{\rho k}$ converges to the solution w_k of the following problem

$$\begin{cases} \partial_t w_k - \nabla \cdot (|\bigtriangledown w_k^m|^{p-2} \bigtriangledown w_k^m) + w_k^q = 0 & \text{in } R^N \times (0,\infty) \\ w_k(.,0) = k\delta_b, & \text{in } R^N. \end{cases} \tag{4.9}$$

(4.8) implies

$$u(x,t) > w_k(x,t) \text{ in } R^N \times (0,\infty). \tag{4.10}$$

For $k_1 > k$, we require

$$\int_{B_{\widetilde{\rho}_1}(b)} u(x,t_{n_\rho})\mathrm{d}x = k_1$$

for some $\widetilde{\rho_1} > \widetilde{\rho}$. Let $w_{\rho_1 k_1}$ be the solution of

$$\begin{cases} \partial_t w - \nabla \cdot (|\bigtriangledown w^m|^{p-2} \bigtriangledown w^m) + w^q = 0 & \text{in } \mathbb{R}^N \times (0,\infty) \\ w(.,0) = u(.,t_{n_\rho})\chi_{B_{\widetilde{\rho}_1}(b)}, & \text{in } R^N. \end{cases}$$

By comparison principle,

$$w_{\rho_1 k_1}(x,t) \geq w_{\rho k}(x,t) \ \text{ in } R^N \times (0, T-t_{n_\rho}). \tag{4.11}$$

Let $w_{\rho_1 k_1} \to w_{k_1}$ as $\widetilde{\rho}_1 \to 0$. Then, w_{k_1} is the solution of the following problem

$$\begin{cases} \partial_t w_{k_1} - \nabla \cdot (|\bigtriangledown w_{k_1}^m|^{p-2} \bigtriangledown w_{k_1}^m) + w_{k_1}^q = 0 & \text{in } \mathbb{R}^N \times (0,\infty) \\ w_{k_1}(.,0) = k_1\delta_b, & \text{in } R^N. \end{cases}$$

(4.11) implies

$$w_{k_1}(x,t) \geq w_k(x,t).$$

Thus, $k \mapsto w_k$ is increasing. Let

$$w_\infty = \lim_{k\to\infty} w_k.$$

Then, w_∞ is a very singular solution and the convergence is uniformly to $t > t_0 > 0$ [7]. Notice that if w_k is a solution of (1.1), then,

$$N_\ell(w_k)(x,t) = \ell^{\frac{1}{q-1}} w_k(b + \ell^\gamma(x-b), \ell t)$$

with $\gamma = \frac{q-m(p-1)}{p(q-1)}$ and $\ell > 0$ is also a solution of (1.1) and

$$N_\ell(w_k)(x,t) = w_{k\ell^{1/(q-1)-\gamma N}}, \quad N_\ell(w_k)(x,0) = k\ell^{\frac{1}{q-1}-N\gamma}\delta_b. \tag{4.12}$$

Letting $k \to \infty$ in (4.12), it leads to the invariance property

$$N_\ell(w_\infty) = w_\infty, \ \ \forall \ell > 0. \tag{4.13}$$

By the uniform convergence of w_k, choosing $\ell = \frac{1}{t}$ in (4.12), we obtain

$$w_\infty(x,t) = t^{\frac{1}{1-q}} f(t^{-\gamma}(x-b)) \ \ \forall (x,t) \in \mathbb{R}^N \times (0,\infty). \tag{4.14}$$

This implies, in particular, that $f(0)$ is finite and

$$w_k(b,t) \le t^{\frac{1}{1-q}} f(0) \le u(b,t) \quad \forall t \in (0,T).$$

This contradicts the fact that $w_k(b,t) \to \infty$ when $t \to 0$, because $q < 1$. When $q = 1$ (and $m(p-1) \neq 1$ otherwise, the results is well known), (1.1) is invariant with respect to the transformation $M_\ell(w)$ defined (for $\ell > 0$) by

$$M_\ell(w)(x,t) = \ell w(b + \ell^{\frac{1-m(p-1)}{p}}(x-b), t),$$

which yields

$$M_\ell(w_k) = w_{k\ell^{1+N(m(p-1)-1)/p}}.$$

Let $k \to \infty$ to get

$$M_\ell(w_\infty) = w_\infty \ \ \forall \ell > 0.$$

This estimate implies

$$0 < M_k(b,T/2) \le w_\infty(b,T/2) = \ell w_\infty(b,T/2) \le u(b,T/2) \ \ \forall \ell > 0,$$

which is again a contradiction. Thus, (4.3) holds. (4.3) implies that for any bounded open set U

$$t \mapsto \int_U u(x,t)\mathrm{d}x$$

remains bounded near $t = 0$. Theorem 2 is proved.

5 Proofs of Theorems 3 and 4

Proof of Theorem 3 Let $\mu_n \in C_0^\infty(R^N)$ be nonnegative and converge to μ in weak sense. We consider the approximate problem

$$\begin{cases} \dfrac{\partial u}{\partial t} = \nabla \cdot (|\nabla u^m|^{p-2}\nabla u^m) - u^q & \text{in } \ Q_\infty = R^N \times (0,\infty) \\ u(x,0) = \mu_n & \text{on } \ R^N. \end{cases} \tag{5.1}$$

Problem (5.1) has a solution $u_n \in L^\infty(Q_\infty) \cap C([0,\infty) : L^1(R^N)), u_n^q \in C([0,\infty) : L^1(R^N))$, $\nabla u_n \in C([0,T) : L^p(R^N)), \frac{\partial u_n^{\frac{m+1}{2}}}{\partial t} \in L^2(Q_\infty)$ [6]. Moreover,

$$u_n \le \Big(\frac{1}{(q-1)t}\Big)^{\frac{1}{q-1}} \quad \text{if } \ q > 1,$$

by the comparison principle. Let $\xi \in C_0^2(B_{2\rho}), \xi = 1$ on $B_\rho, 0 \le \xi \le 1$, and $\tau > 0$ large enough, $0 < t < \theta$. Applying (2.3) to u_n and letting $t \to 0$, we obtain

$$\begin{aligned} &\int_{B_{2\rho}} u_n(x,\theta)\xi^\tau \mathrm{d}x + \int_0^\theta \int_{B_{2\rho}} u_n^q \xi^\tau \mathrm{d}x\mathrm{d}t \\ \le &\int_{B_{2\rho}} \mu_n \xi^\tau \mathrm{d}x + \tau \int_0^\theta \int_{B_{2\rho}} \xi^{\tau-1} |\nabla u_n^m|^{p-1} |\nabla \xi| \mathrm{d}x\mathrm{d}t. \end{aligned} \tag{5.2}$$

Similar to the proof of (3.8), if $q > m(p-1)$, we obtain

$$\int_{B_{2\rho}} u_n(x,\theta)\xi^\tau \mathrm{d}x + \int_0^\theta \int_{B_{2\rho}} u_n^q \xi^\tau \mathrm{d}x\mathrm{d}t \le C\int_{B_{2\rho}} \mu_n \xi^\tau \mathrm{d}x + C.$$

If $0 \le m(p-1) < 1$, using Young's inequality and (2.1), we obtain

$$\begin{aligned}
&\int_0^\theta \int_{B_{2\rho}} \xi^{\tau-1}|\nabla u_n^m|^{p-1}|\nabla\xi|\mathrm{d}x\mathrm{d}t \\
\le\; & \epsilon \int_0^\theta \int_{B_{2\rho}} \xi^\tau |\bigtriangledown u_n^m|^p (1+u_n^m)^{\alpha-1}\mathrm{d}x\mathrm{d}t + C_\epsilon \int_0^\theta \int_{B_{2\rho}} \xi^{\tau-p}(1+u_n^m)^{(1-\alpha)(p-1)}\mathrm{d}x\mathrm{d}t \\
\le\; & \epsilon \int_0^\theta \int_{B_{2\rho}} \xi^\tau |\bigtriangledown u_n^m|^p (1+u_n^m)^{\alpha-1}\mathrm{d}x\mathrm{d}t + \epsilon \int_0^\theta \int_{B_{2\rho}} \xi^\tau u_n \mathrm{d}x\mathrm{d}t + C_\epsilon \\
\le\; & \int_0^\theta \int_{B_{2\rho}} \xi^\tau u_n \mathrm{d}x\mathrm{d}t + C\epsilon \int_0^\theta \int_{B_{2\rho}} \xi^\tau u_n^q \mathrm{d}x\mathrm{d}t + C\epsilon \int_{B_{2\rho}} u_n(x,\theta)\xi^\tau \mathrm{d}x + +C_\epsilon, \qquad (5.3)
\end{aligned}$$

where $\alpha < 0, (1-\alpha)m(p-1) < 1$. Substituting (5.3) into (5.2) and using Gronwall's inequality, we obtain

$$\int_{B_{2\rho}} \xi^\tau u_n(x,\theta)\mathrm{d}x + \int_0^\theta \int_{B_{2\rho}} \xi^\tau u_n^q \mathrm{d}x\mathrm{d}t \le C.$$

In both cases, $\{u_n\}$ is uniformly bounded in $L^\infty((0,\infty); L^1(B_\rho))$ and $\{u_n^q\}$ is uniformly bounded in $L^1(B_\rho \times (0,\infty))$. Besides, $\{u_n^{m(p-1)}\}$ is bounded in $L^1(B_\rho \times (0,T))$, whenever $q > m(p-1)$ or $m(p-1) < 1$. Then, by Proposition 2.3,

$$\int_0^T \int_{B_\rho} |\nabla u_n^m|^r \mathrm{d}x\mathrm{d}t < M, \quad r \in \Big(0, p-1+\frac{1}{mN+1}\Big)$$

and

$$\int_0^T \int_{B_\rho} u_n^\sigma \mathrm{d}x\mathrm{d}t < M, \quad \sigma \in \Big(0, m(p-1)+\frac{p}{N}\Big),$$

where M is a constant independent of n. Similar to argument in [6], if $p > \frac{N(m+1)}{mN+1}$, there exists a subsequence of $\{u_n\}$ and $u \in L^1_{\mathrm{loc}}(R^N \times (0,T))$ such that

$$u_n \to u \quad \text{uniformly on any compact set of} \quad R^N \times (0,T)$$

and u is a weak solution of (1.1). Notice that for any $\xi \in C_0^\infty(R^N)$

$$\int_{R^N} u_n(x,t)\xi(x)\mathrm{d}x - \int_{R^N} \mu_n \xi(x)\mathrm{d}x = -\int_0^t \int_{R^N} (|\bigtriangledown u_n^m|^{p-2} \bigtriangledown u_n^m \cdot \bigtriangledown\xi + u_n^q \xi)\mathrm{d}x\mathrm{d}t.$$

Using proposition 2.3, $q < m(p-1) + \frac{p}{N}$, and Hölder's inequality, we can obtain $u(x,0) = \mu$ in weak sense.

To prove Theorem 4, consider the following Cauchy problem

$$\begin{cases} \partial_t w_k - \nabla(|\bigtriangledown w_k^m|^{p-1} \bigtriangledown w_k^m) + w_k^q = 0 & \text{in } \mathbb{R}^N \times (0,\infty) \\ w_k(.,0) = k\delta_b, & \text{in } \mathbb{R}^N. \end{cases} \qquad (5.4)$$

By Theorem 3, (5.4) has a singular solution w_k satisfying

$$w_k \le \Big(\frac{1}{(q-1)t}\Big)^{\frac{1}{q-1}}. \tag{5.5}$$

We require that w_k increases and converges to w_∞, which is a very singular solution of (1.1), that is, $w_\infty \in C(\overline{Q_\infty}\backslash\{0,0\}$ satisfies (1.1) and for any $\rho > 0$,

$$\lim_{t\to 0}\int_{B_\rho} w_\infty(x,t)\mathrm{d}x = \infty.$$

Similar to the argument of (4.14),

$$w_\infty(x,t) = t^{\frac{1}{1-q}} f(t^{-\gamma}(x-b)), \quad \forall (x,t) \in \mathbb{R}^N \times (0,\infty)$$

with $\gamma = \frac{q-m(p-1)}{p(q-1)}$.

Lemma 5.1 Assume that $\max(1, m(p-1)) < q < m(p-1)+\frac{p}{N}$ and let $u \in C(R^N\times(0,T))$ be a nonnegative weak solution of (1.1) with initial trace $\mathrm{tr}_{R^N}(u) = (\mathcal{S},\mu)$. If $y \in \mathcal{S}$, then,

$$u(x,t) \ge w_\infty(x-y,t) \quad \forall (x,t) \in R^N \times (0,T).$$

The proof of Lemma 5.1 is similar to the argument of (4.10).

Lemma 5.2 Let $\mu_1,\ \mu_2 \in \mathcal{M}^+$ with $\mu_1 \le \mu_2$. Assume that

$$p > \frac{N(m+1)}{mN+1},$$

and that

$$1 \le q < m(p-1) + \frac{p}{N} \ \text{ or } m(p-1) < 1.$$

Then, there exist solutions u_1 and u_2 with respective initial traces μ_1 μ_2, such that $u_1 \le u_2$ a.e. in Q_∞.

Proof Let $\mu_{1n},\ \mu_{2n} \in C_0^\infty(R^N),\ \ \mu_{1n} \le \mu_{2n}$, and

$$\mu_{1n} \rightharpoonup \mu_1, \quad \mu_{2n} \rightharpoonup \mu_2.$$

Then, there exist solutions u_{1n} and u_{2n} with respective initial traces μ_{1n} μ_{2n}. Then, by comparison principle, $u_{1n} \le u_{2n}$, hence $u_1 \le u_2$ a.e. in Q_∞.

Proof of Theorem 4 Suppose $\nu = (\mathcal{S},\mu)$ and let $\{a_k\}_{k=1}^{k=\infty}$ be a countable dense subset of $\mathcal{S}$. We define $\mu_k \in \mathcal{M}^+(R^N)$ by

$$\mu_k = \mu + k\sum_{j=1}^{j=k}\delta_{a_j}.$$

From Theorem 3 and Lemma 5.2, there exists a sequence $\{u_k\}$ of solutions of (1.11) with initial data μ_k such that

$$0 \le w_{a_j} \le u_k \le u_{k+1}, \quad \forall k > 0, \ \ j = 1,\cdots,k,$$

and u_k satisfies (5.5), where w_{a_j} is the solution of (1.1) with initial data $k\delta_{a_j}$. (5.5) implies that $\{u_k\}$ is uniformly bounded in $C^\alpha_{\rm loc}(Q_\infty)$ [8]. Thus, there exists a function $u \in C(Q_\infty)$, such that $u_k \to u$ uniformly in any compact set of Q_∞, as $k \to \infty$, and u is a weak solution of (1.1) in Q_∞. Notice that for $\forall \rho > 0$,

$$\lim_{t\to 0} \int_{B_\rho(a_j)} w_{a_j,\infty} \mathrm{d}x = \infty. \tag{5.6}$$

Because $\{a_j\}$ is dense in $\mathcal{S}$, the any point of $\mathcal{S}$ satisfies property (5.5). Thus, the initial trace of u in $\mathcal{S}$ is satisfied. In contrast, for any open sets $V \subset\subset V^* \subset\subset \mathcal{R} = R^N \backslash \mathcal{S}$, if we take a test function ξ with support in V^* in proof of Theorem 3, we verify that $\int_{R^N} u_k(x,t)\xi \mathrm{d}x$ is uniformly bounded to $t > 0$. They also hold for u, because μ and μ_n have the same restriction to $\mathcal{R}$. Finally, for any $\theta > 0$, letting $k \to \infty$ in the equation, we obtain

$$\begin{aligned}&\int_0^\theta \int_{V^*} (-u_k \partial_t \phi + |\nabla u_k^m|^{p-2} \nabla u^m \cdot \nabla \phi + u_k^q \phi) \mathrm{d}x\mathrm{d}t \\ = &\int_{V^*} \phi(x,0)\mathrm{d}\mu_k - \int_{V^*} u_k(x,\theta)\phi(x,\theta)\mathrm{d}x,\end{aligned}$$

where $\phi \in C_0^\infty(V^* \times [0,\infty))$, it implies that u satisfies (1.10) in V^*. This proves that the regular part of the initial trace of u is μ and consequently, $\mathrm{tr}_{R^N}(u) = \nu \in \mathcal{B}^+_{\mathrm{reg}}(R^N)$.

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