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关于 Bernstein-Bézier 算子对一类绝对连续函数的逼近

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摘要: Bernstein-Bézier 算子是一种重要的逼近算子, 在计算机辅助几何设计中也扮演了重要角色. 为了进一步了解它的理论及其逼近性质, 研究了它对一类绝对连续函数的逼近. 本文主要利用经典的 Bojanic-Cheng 分解方法, 结合分析技术, 分别讨论了 Bernstein-Bézier 算子在 $0<\alpha\leqslant 1$ 及 $\alpha\geqslant 1$ 时, 对这类绝对连续函数的逼近. 首先扩展了文献 Liu 的结果, 得到了 Bernstein-Bézier 算子的一阶中心绝对矩 $B_n^{(\alpha)}(|t-x|,x)$; 接着估计了另外一项 $B_n^{(\alpha)}(\int_x^t\varphi_x(u)\mathrm{d}u,x)$, 最后得到了比较精确的收敛价.

关键词: Bernstein-Bézier 算子; 逼近度; 绝对连续函数
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对于定义在区间 $[0,1]$ 的函数 f 及 $\alpha>0$, Bernstein-Bézier 算子定义为:

$$B_n^{(\alpha)}(f,x)=\sum_{k=0}^{n}f(k/n)Q_{nk}^{(\alpha)}(x) \tag{1}$$

其中 $Q_{nk}^{(\alpha)}(x)=J_{n,k}^{\alpha}(x)-J_{n,k+1}^{\alpha}(x)$,

$$J_{n,k}(x)=\sum_{j=k}^{n}P_{nj}(x),\ (J_{n,n+1}(x)\equiv 0),$$

$$P_{nk}(x)=\binom{n}{k}x^k(1-x)^{n-k}.$$

算子族 $B_n^{(\alpha)}$ 由 Chang G[1] 首先引进和研究. 后来, Li P 和 Gong Y H[2], Lin Z X[3], Zeng X M 和 Piriou A[4,5] 等对算子族 $B_n^{(\alpha)}$ 作了进一步的推广和深入研究.

本文讨论了 $B_n^{(\alpha)}(f,x)$ 对一类绝对连续函数的逼近性质, 首先给出一些基本记号及定义.

1 基本记号及定义

$BV[0,1]=\{f\mid f$ 是 $[0,1]$ 上的有界变差函数$\}$,

$\Phi_B=\{f\mid f$ 在 $[0,1]$ 上有界$\}$,

$\Phi_{DB}=\{f\mid f(x)=f(0)+\int_0^x h(t)\mathrm{d}t;\ 0\leqslant x\leqslant 1,\ h(t)\in\Phi_B\}$,

$\Omega_x(f,\lambda)=\sup_{t\in[x-\lambda,x+\lambda]}|f(t)-f(x)|$, 其中 $x\in[0,1]$ 固定, $\lambda\geqslant 0$.

易得

(i) $\Omega_x(f,\lambda)$ 对变量 λ 来说是单调非减函数,

(ii) 若 $f(t)$ 在 x 点连续, 那么有 $\lim_{\lambda\to 0}\Omega_x(f,\lambda)=0$,

(iii) 若 $f\in BV[a,b]$, $\bigvee_a^b(f)$ 表示 f 在 $[a,b]$ 上的全变差, 那么有 $\Omega_x(f,\lambda)\leqslant\bigvee_{x-\lambda}^{x+\lambda}(f)$.

$$\varphi_x(u)=\begin{cases}h(u)-h(x+), & x<u\leqslant 1\\ 0, & u=x,\\ h(u)-h(x-), & 0\leqslant u<x\end{cases}$$

$$K_{n,\alpha}(x,t)=\begin{cases}\sum_{k\leqslant nt}Q_{nk}^{(\alpha)}(x), & 0<t\leqslant 1\\ 0, & t=0\end{cases}.$$

由 Lebesgue-Stieltjes 积分形式[4,(21)], 算子 $B_n^{(\alpha)}(f,x)$ 可表示成:

$$B_n^{(\alpha)}(f,x)=\int_0^1 f(t)\mathrm{d}_tK_{n,\alpha}(x,t) \tag{2}$$

本文主要结果如下:

定理 对于 $f\in\Phi_{DB}$, 并且 $\forall x\in[0,1]$, $h(x+)$, $h(x-)$ 存在, 有:

(i) 当 $\alpha\geqslant 1$ 时,

$$|B_n^{(\alpha)}(f,x)-f(x)|\leqslant\frac{2\alpha+2}{n}\sum_{k=1}^{[\sqrt{n}]}\Omega_x(\varphi_x,1/k)+(|h(x+)|+|h(x-)|)M_\alpha n^{-1/2} \tag{3}$$

(ii) 当 $0<\alpha\leqslant 1$ 时,

$$|B_n^{(\alpha)}(f,x)-f(x)|\leqslant\frac{4+2A_\alpha x^\alpha(1-x)^{\alpha-1}}{n}\sum_{k=1}^{[\sqrt{n}]}\Omega_x(\varphi_x,1/k)+(|h(x+)|+|h(x-)|)M_\alpha n^{-1/2} \tag{4}$$

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其中 A_α, M_α 为与 α 有关的正常数.

2 基本引理

引理 1

(i) 当 $\alpha \geqslant 1$ 且 $0 \leqslant t < x$ 时,

$$K_{n,\alpha}(x,t) \leqslant \frac{\alpha x(1-x)}{n(x-t)^2},$$

(ii) 当 $\alpha \geqslant 1$ 且 $x < t \leqslant 1$ 时,

$$1-K_{n,\alpha}(x,t) \leqslant \frac{\alpha x(1-x)}{n(x-t)^2},$$

(iii) 当 $0 < \alpha \leqslant 1$ 且 $0 \leqslant t < x < 1$ 时,

$$K_{n,\alpha}(x,t) \leqslant K_{n,1}(x,t) \leqslant \frac{x(1-x)}{n(x-t)^2},$$

(iv) 当 $0 < \alpha \leqslant 1$ 且 $0 \leqslant x < t < 1$ 时,

$$1-K_{n,\alpha}(x,t) \leqslant \frac{A_\alpha\left(x(1-x)\right)^\alpha}{n(x-t)^2},$$

其中 A_α 为与 α 有关的正常数.

证明 (i), (ii) 见文献[4] 的引理 8, (iii), (iv) 见文献[5] 的引理 4 及引理 7.

引理 2 设 $\alpha > 0$, 则存在与 α 有关的正常数 M_α, 使得:

$$B_n^{(\alpha)}(|t-x|, x) \leqslant M_\alpha n^{-1/2} \tag{5}$$

证明 由文献[3] 引理 2 知:

$$B_n^{(\alpha)}(|t-x|, x) \leqslant \begin{cases} M_1\left(X/n\right)^{1/2}, & \frac{1}{2n} \leqslant x \leqslant 1 \\ M_2\left(X + X^\alpha n^{\alpha-1}\right), & 0 \leqslant x < \frac{1}{2n} \end{cases} \tag{6}$$

其中 $X = x(1-x)$, M_1, M_2 为与 α 有关的正常数.

由式(6) 得:

当 $x \geqslant \frac{1}{2n}$ 时, $B_n^{(\alpha)}(|t-x|, x) \leqslant M_1 n^{-1/2}$;

当 $x < \frac{1}{2n}$ 时,

$$B_n^{(\alpha)}(|t-x|, x) \leqslant M_2\left(\frac{1}{2n} + \frac{1}{2^\alpha n}\right) \leqslant \frac{3}{2}M_2 n^{-1/2},$$

取 $M_\alpha = \max\left(M_1; \frac{3}{2}M_2\right)$, 则式(5) 成立.

引理 3

(i) 设 $\alpha \geqslant 1$, 则有:

$$B_n^{(\alpha)}\left(\int_x^t \varphi_x(u)\mathrm{d}u, x\right) \leqslant \frac{2\alpha+2}{n}\sum_{k=1}^{[\sqrt{n}]}\Omega_x(\varphi_x, 1/k) \tag{7}$$

(ii) 设 $0 < \alpha \leqslant 1$, 则有:

$$B_n^{(\alpha)}\left(\int_x^t \varphi_x(u)\mathrm{d}u, x\right) \leqslant \frac{4+2A_\alpha x^\alpha(1-x)^{\alpha-1}}{n}\sum_{k=1}^{[\sqrt{n}]}\Omega_x(\varphi_x, 1/k) \tag{8}$$

其中 A_α 为与 α 有关的正常数.

证明 (i) 由式(2), 式(7) 左边可以写成:

$$B_n^{(\alpha)}\left(\int_x^t \varphi_x(u)\mathrm{d}u, x\right) = \int_0^1\left(\int_x^t \varphi_x(u)\mathrm{d}u\right)\mathrm{d}_t K_{n,\alpha}(x,t) = \int_0^x\left(\int_x^t \varphi_x(u)\mathrm{d}u\right)\mathrm{d}_t K_{n,\alpha}(x,t) + \int_x^1\left(\int_x^t \varphi_x(u)\mathrm{d}u\right)\mathrm{d}_t K_{n,\alpha}(x,t) = E_n(f,x) + F_n(f,x) \tag{9}$$

利用分部积分法, 注意到 $K_{n,\alpha}(x,0) = 0$, $\int_x^x \varphi_x(u)\mathrm{d}u = 0$,

$$|E_n(f,x)| = \left|\left[K_{n,\alpha}(x,t)\int_x^t \varphi_x(u)\mathrm{d}u\right]_0^x - \int_0^x K_{n,\alpha}(x,t)\varphi_x(t)\mathrm{d}t\right| = \left|\int_0^x K_{n,\alpha}(x,t)\varphi_x(t)\mathrm{d}t\right| \leqslant \int_0^x K_{n,\alpha}(x,t)\Omega_x(\varphi_x, x-t)\mathrm{d}t = \int_0^{x-x/\sqrt{n}} K_{n,\alpha}(x,t)\Omega_x(\varphi_x, x-t)\mathrm{d}t + \int_{x-x/\sqrt{n}}^x K_{n,\alpha}(x,t)\Omega_x(\varphi_x, x-t)\mathrm{d}t.$$

由引理 1(i), 且 $0 \leqslant K_{n,\alpha}(x,t) \leqslant 1$,

$$|E_n(f,x)| \leqslant \frac{\alpha x(1-x)}{n}\int_0^{x-x/\sqrt{n}}\frac{\Omega_x(\varphi_x, x-t)}{(x-t)^2}\mathrm{d}t + \frac{x}{\sqrt{n}}\Omega_x(\varphi_x, x/\sqrt{n}) \tag{10}$$

作变量替换 $t = x - \frac{x}{u}$, 则式(10) 右边第一项的积分可以化为:

$$\int_0^{x-x/\sqrt{n}}\frac{\Omega_x(\varphi_x, x-t)}{(x-t)^2}\mathrm{d}t = \frac{1}{x}\int_1^{\sqrt{n}}\Omega_x(\varphi_x, x/u)\mathrm{d}u \leqslant \frac{2}{x}\sum_{k=1}^{[\sqrt{n}]}\Omega_x(\varphi_x, 1/k) \tag{11}$$

由式(10), (11), 得:

$$|E_n(f,x)| \leqslant \frac{2\alpha(1-x)}{n}\sum_{k=1}^{[\sqrt{n}]}\Omega_x(\varphi_x, 1/k) + \frac{2x}{n}\sum_{k=1}^{[\sqrt{n}]}\Omega_x(\varphi_x, 1/k) = \frac{2\alpha(1-x)+2x}{n}\sum_{k=1}^{[\sqrt{n}]}\Omega_x(\varphi_x, 1/k) \tag{12}$$

由引理 1(ii), 方法同上, 可以得到:

$$|F_n(f,x)| \leqslant \frac{2\alpha x+2(1-x)}{n}\sum_{k=1}^{[\sqrt{n}]}\Omega_x(\varphi_x, 1/k) \tag{13}$$

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由式(9), (12), (13) 得式(7).

完全类似的方法, 可证得估计式(8).

3 定理的证明

由定理的条件, 得:

$$f(t)-f(x)=\int_x^t h(u)\mathrm{d}u \tag{14}$$

而根据 Bojanic-Cheng 分解[6], 有:

$$h(u)=\frac{h(x+)+h(x-)}{2}+\varphi_x(u)+\frac{h(x+)-h(x-)}{2}\operatorname{sgn}(u-x)+\delta_x(u)\left[h(x)-\frac{h(x+)+h(x-)}{2}\right] \tag{15}$$

其中,

$$\varphi_x(u)=\begin{cases}h(u)-h(x+), & x<u\leqslant 1\\ 0, & u=x\\ h(u)-h(x-), & 0\leqslant u<x\end{cases},$$

$$\delta_x(u)=\begin{cases}1, & u=x\\ 0, & u\neq x\end{cases}.$$

由式(14), (15), 并且 $\int_x^t \operatorname{sgn}(u-x)\,\mathrm{d}u=|t-x|$, $\int_x^t \delta_x(u)\,\mathrm{d}u=0$,

$$\begin{aligned}
|B_n^{(\alpha)}(f,x)-f(x)| &= |B_n^{(\alpha)}(f(t)-f(x),x)| = |B_n^{(\alpha)}(\int_x^t h(u)\mathrm{d}u,x)| = \\
&| \frac{h(x+)+h(x-)}{2}B_n^{(\alpha)}(t-x,x)+ \frac{h(x+)-h(x-)}{2}B_n^{(\alpha)}(|t-x|,x)+ B_n^{(\alpha)}(\int_x^t\varphi(u)\mathrm{d}u,x)| \leqslant \\
&|\frac{h(x+)+h(x-)}{2}|B_n^{(\alpha)}(|t-x|,x)+ |\frac{h(x+)-h(x-)}{2}|B_n^{(\alpha)}(|t-x|,x)+ |B_n^{(\alpha)}(\int_x^t\varphi_x(u)\mathrm{d}u,x)| \leqslant \\
&(|h(x+)|+|h(x-)|)B_n^{(\alpha)}(|t-x|,x)+ |B_n^{(\alpha)}(\int_x^t\varphi_x(u)\mathrm{d}u,x)|
\end{aligned} \tag{16}$$

式(3)可直接由式(5), (7), (16)得到; 式(4)可直接由式(5), (8), (16)得到.

参考文献:

[1] Chang G. Generalized Bernstein-Bézier polynomials[J]. J. Comput. Math., 1983, 1(4): 322-327.

[2] Li P, Gong Y H. The order of approximation by the generalized Bernstein-Bézier polynomials[J]. J. of China Univ. of Science and Technology, 1985, 15(1): 15-18.

[3] Liu Z X. Approximation of continuous functions by the generalized Bernstein-Bézier polynomials [J]. Approx. Theory. Appl., 1986, 2(4): 105-130.

[4] Zeng X M, Piriou A. On the rate of convergence of two Bernstein-Bézier type operators for bounded variation functions[J]. J. Approx. Theory, 1998, 95: 369-387.

[5] Zeng X M. On the rate of convergence of two Bernstein-Bézier type operators for bounded variation functions II [J]. J. Approx. Theory, 2000, 104: 330-344.

[6] Bojanic R, Cheng F. Rate of convergence of Bernstein polynomials for functions with derivatives of bounded variation[J]. J. Math. Anal. Appl., 1989, 141: 136-151.

On the Rate of Convergence of Bernstein-Bézier Operator for Some Absolutely Continuous Functions

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Abstract: The purpose of this paper is to investigate the rate of convergence of Bernstein-Bézier Operator for some absolutely continuous functions. The approximation properties were studied in the case of $0<\alpha\leqslant 1$ and $\alpha\geqslant 1$ respectively. By using Bojanic-Cheng's method and analysis techniques, the rate of convergence of Bernstein-Bézier Operator was derived. In the first, the authors extended the result of Liu Z X and got the first central absolute moment $B_n^{(\alpha)}(|t-x|,x)$. Later, the authors estimated the other part $B_n^{(\alpha)}(\int_x^t\varphi_x(u)\mathrm{d}u,x)$. Lastly, an asymptotically optimal estimate was obtained. The result is helpful to understand the properties of Bézier Operator well, thus further research can be done.

Key words: Bernstein-Bézier operator; rate of approximation; absolutely continuous functions

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