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# Semilocal modules and quasi-hereditary algebras

By

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**1. Main result.** Throughout this note all algebras are assumed to be finite-dimensional (associative) algebras with 1 over an algebraically closed field $k$ and all modules are finitely generated left modules. The composition of two morphisms $f\colon M_1 \longrightarrow M_2$ and $g\colon M_2 \longrightarrow M_3$ is denoted by $fg$.

Recall that a module $M$ is called local if the sum of all proper submodules of $M$ is a proper submodule of $M$. So a local module $M$ has a unique maximal submodule which is the radical of $M$. A module $M$ is called semilocal if $M$ is a direct sum of local modules. Finally, recall that an ideal $J$ of a given algebra $A$ with Jacobson radical $N := \operatorname{rad}(A)$ is said to be a heredity ideal if it satisfies (1) $J^2 = J$, (2) $JNJ = 0$, and (3) $J$ is a projective left $A$-module. The algebra $A$ is called quasi-hereditary provided there is a chain

$$0 = J_0 \subseteqq J_1 \subseteqq \cdots \subseteqq J_n = A$$

of ideals of $A$ such that $J_i/J_{i-1}$ is a heredity ideal of $A/J_{i-1}$ for $1 \leqq i \leqq n$. (Such a chain is called a heredity chain). Quasi-hereditary algebra was introduced by Cline, Parshall and Scott in order to describe the highest weight categories arising in the representation theory of complex semisimple Lie algebras and algebraic groups (cf. [2]). Some basic properties on quasi-hereditary algebras may be found in [3]. The aim of this note is to show the following

**Theorem.** *Let $A$ be a finite-dimensional $k$-algebra with the radical $N$ over an algebraically closed field $k$. If $M$ is a semilocal module with Loewy length $m$, then the endomorphism algebra* $\operatorname{End}_A\left(\bigoplus_{i=1}^{m} M/N^iM\right)$ *is quasi-hereditary.*

Note that in case $M = {}_AA$ the result was proved already by Dlab and Ringel in [4]. Thus the above result is a generalization of the main result in [4]. The proof of the theorem will cover the rest of this section.

Since the proof of Dlab and Ringel in [4] does not work in our case, we shall introduce first a partially order on the set of all non-isomorphic indecomposable direct summands of $\tilde{M} := \bigoplus_{i=1}^{m} M/N^iM$ which have the some Loewy length. Suppose $M = \bigoplus_{i=1}^{n} M_i$ be semilocal in which $M_i$ is local for $1 \leqq i \leqq n$. Let $m$ be the Loewy length of $M$. We denote by

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$\mathscr{F}$ the set of all pairwise non-isomorphic indecomposable direct summands of $\tilde{M}$ and by $\mathscr{F}_i$ the subset of $\mathscr{F}$ consisting of all modules with Loewy length $i$. Thus $\mathscr{F}$ is a disjoint union of $\mathscr{F}_i$, $1 \leqq i \leqq m$.

Let $\mathscr{F}_i = \{F_{ij} \mid 1 \leqq j \leqq n_i\}$. We define on $\mathscr{F}_i$ a relation $\leqq$ by setting $F_{is} \leqq F_{it}$ provided there exists a surjective morphism from $F_{is}$ onto $F_{it}$.

Since $F_{is}$ is a local module, it is easy to see that $(\mathscr{F}_i, \leqq)$ is a partially ordered set. So we can enumerate the elements of $\mathscr{F}_i$ in such a way $F_{i1}, \cdots, F_{in_i}$ that $F_{is} \leqq F_{it}$ implies that $s \leqq t$. Since $\operatorname{End}_A(\tilde{M})$ is Morita equivalent to $\operatorname{End}_A\left(\bigoplus_{X \in \mathscr{F}} X\right)$ and quasi-heredity is invariant under Morita equivalences, we shall prove that $E := \operatorname{End}_A\left(\bigoplus_{i=1}^{m} \bigoplus_{j=1}^{n_i} F_{ij}\right)$ is a quasi-hereditary algebra. To this end, let $J_{st}$ be the set of all endomorphisms in $E$ which factor through a module in add $\left(\bigoplus_{i=1}^{s-1} \bigoplus_{j=1}^{n_i} F_{ij} \bigoplus \bigoplus_{j=1}^{t} F_{sj}\right)$, we claim that the chain

$$0 = J_{00} \subset J_{11} \subset \cdots \subset J_{1n_1} \subset J_{21} \subset J_{22} \subset \cdots \subset J_{2n_2} \subset \cdots \subset J_{m1} \subset \cdots \subset J_{mn_m} = E$$

of ideals of $E$ is a heredity chain.

Given $F_{st} \in \mathscr{F}$, we denote by $e_{st}$ the endomorphism in $E$ which projects canonically $\oplus F_{ij}$ onto $F_{st}$. Thus, the elements $e_{st}$, $1 \leqq s \leqq m$, $1 \leqq t \leqq n_s$, form a set of pairwise orthogonal primitive idempotents of $E$. Note that for $F_{st}, F_{ij} \in \mathscr{F}$, we can identify $e_{st} E e_{ij}$ with $\operatorname{Hom}_A(F_{st}, F_{ij})$. The ideal $J_{st}$ is generated by $e_{ij}$ with $1 \leqq i < s$, $1 \leqq j \leqq n_i$ and $e_{sj}$ with $1 \leqq j \leqq t$. Let $\bar{E} = E/J_{s,t-1}$, the residue class of an element $\alpha \in E$ (or a subset $J \subseteqq E$) will be denoted by $\bar{\alpha}$ (or $\bar{J}$). We show that $\bar{J}_{st}$ is a heredity ideal of $\bar{E}$.

(1) It is obvious that $\bar{J}_{st}$ is an idempotent ideal of $\bar{E}$.

(2) Every non-invertible map $f: F_{st} \longrightarrow F_{st}$ factors through a module in add $\left(\bigoplus_{i=1}^{s} \bigoplus_{j=1}^{n_i} F_{ij} \bigoplus \bigoplus_{j=1}^{t-1} F_{sj}\right)$. In fact, $f$ can not be surjective. Thus the image of $f$ lies in the radical of $F_{st} = M_p/N^s M_p$ and $N^{s-1} \cdot \operatorname{Im}(f) = 0$. This means that $f$ factors through $F_{s-1,p} := M_p/N^{s-1} M_p$ and lies in $J_{s,t-1}$. Thus $\bar{J}_{st} \cdot \operatorname{rad}(\bar{E}) \cdot \bar{J}_{st} = 0$.

(3) It remains to show that $\bar{J}_{st}$ is a projective left $\bar{E}$-module. By [3] this is equivalent to showing that the multiplication map

$$\bar{E}\bar{e}_{st} \otimes_{\bar{e}_{st}\bar{E}\bar{e}_{st}} \bar{e}_{st}\bar{E} \longrightarrow \bar{E}\bar{e}_{st}\bar{E}$$

is bijective. Note that from (2) we have $\bar{e}_{st}\bar{E}\bar{e}_{st} \cong k$.

In fact, it is sufficient to show that the multiplication map

$$\mu: \bar{e}_{pq}\bar{E}\bar{e}_{st} \otimes_k \bar{e}_{st}\bar{E}\bar{e}_{vw} \longrightarrow \bar{e}_{pq}\bar{E}\bar{e}_{st}\bar{E}\bar{e}_{vw}$$

is injective for all $p, q, v, w$. Let $x = \sum_{l=1}^{c} \bar{\gamma}_l \otimes \bar{\delta}_l$ be in $\bar{e}_{pq}\bar{E}\bar{e}_{st} \otimes_k \bar{e}_{st}\bar{E}\bar{e}_{vw}$ such that $\sum \bar{\gamma}_l \bar{\delta}_l = 0$, where $\gamma_l \in \operatorname{Hom}_A(F_{pq}, F_{st})$ and $\delta_l \in \operatorname{Hom}_A(F_{st}, F_{vw})$. We may assume that $p \geqq s$ and $q \geqq t$ in case $p = s$, and that $v \geqq s$ and $w \geqq t$ in case $v = s$. Since if $N^{s-1} \cdot \operatorname{Im}(\gamma_l) = 0$ then $\gamma_l$ factors through $\bigoplus_{i=1}^{n_{s-1}} F_{s-1,i}$, we may assume that $N^{s-1} \cdot \operatorname{Im}(\gamma_l) \neq 0$ for all $l$. This implies that $\gamma_l$ is surjective since $F_{st}$ is local.

By definition we can write $F_{pq} = M_q/N^p M_q$ and $F_{st} = M_t/N^s M_t$. Let $i\colon N^s M_q/N^p M_q \longrightarrow M_q/N^p M_q$ be the canonical inclusion, then we have the following commutative diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & N^s M_q/N^p M_q & \xrightarrow{\ i\ } & M_q/N^p M_q & \xrightarrow{\ \pi\ } & M_q/N^s M_q & \longrightarrow & 0 \\ & & & & \downarrow{\scriptstyle \gamma_l} & & \downarrow{\scriptstyle \varphi_l} & & \\ & & & & M_t/N^s M_t & = & M_t/N^s M_t, & & \end{array}$$

where $\pi$ is the canonical projection and $\varphi_l$ is surjective. Since $M_q/N^s M_q, M_t/N^s M_t \in \mathscr{F}_s$, we may assume that $\varphi_l$ is an isomorphism. Otherwise, by the definition of $\leqq$ we have $\bar{\gamma}_l = 0$. Hence we can identify $M_q/N^s M_q$ with $M_t/N^s M_t$ and consider $\varphi_l$ as an endomorphism of $M_q/N^s M_q$. Thus, by (2), we have $\bar{\varphi}_l = \bar{\alpha}_l$ for some $\alpha_l \in k$. Let $\delta = \sum_{i=1}^{c} \alpha_l \delta_l$, we get $x = \bar{\pi} \otimes_k (\overline{\sum_l \alpha_l \delta_l}) = \bar{\pi} \otimes_k \bar{\delta}$ and $\bar{\pi}\bar{\delta} = 0$. Hence there is a module $T$ in add $\left(\bigoplus_{i=1}^{s-1} \bigoplus_{j=1}^{n_i} F_{ij} \bigoplus \bigoplus_{j=1}^{t-1} F_{sj}\right)$ such that $\pi\delta$ factors through the module $T$ and one obtains the following commutative diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & N^s M_q/N^p M_q & \xrightarrow{\ i\ } & M_q/N^p M_q & \xrightarrow{\ \pi\ } & M_q/N^s M_q & \longrightarrow & 0 \\ & & & & \downarrow{\scriptstyle f} & & \downarrow{\scriptstyle \delta} & & \\ & & & & T & \xrightarrow{\ g\ } & M_w/N^v M_w\,. & & \end{array}$$

Since $if = 0$, it follows that there exists a morphism $\psi\colon M_q/N^s M_q \longrightarrow T$ such that $f = \pi\psi$. So $\pi\delta = fg = \pi\psi g$, and $\delta = \psi g$ because $\pi$ is surjective. This means that $\bar{\delta} = 0$ and $x = \bar{\pi} \otimes_k \bar{\delta} = 0$. Hence $\mu$ is injective and the proof is completed.

Since ${}_A A$ is semilocal, we have the following

**Corollary** [4]. *Let $A$ be a finite-dimensional algebra with nilpotence index $n$. Then* $\mathrm{End}_A\left(\bigoplus_{i=1}^{n} A/N^i\right)$ *is quasi-hereditary.*

## 2. Examples.

Example 1. The following example shows that the chain given in [4] may not be a heredity chain, and that the refinement in the proof of the theorem is necessary. Let $A$ be given by the following quiver

$$2 \longleftarrow 1 \longrightarrow 3$$

We denote by $P(i)$ the indecomposable projective module corresponding to the vertex $i$. If we take modules $M_1 = P(1)/P(3)$ and $M_2 = P(1)$, and denote by $J_i'$ the ideal of $E$ in which every morphism factors through a module in $\mathscr{F}_i$, then $0 = J_0' \subset J_1' \subset E$ is the chain given in [4], it is easy to see that this chain is not a heredity chain for $E$.

E x a m p l e 2. The condition that $M$ is semilocal in the theorem can not be omitted. This can be seen by the following example. Let $A$ be the Kronecker algebra given by the quiver

$$2 \leftleftarrows 1.$$

We take the module

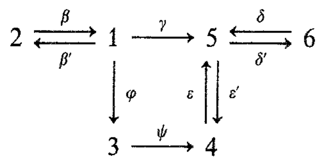
$$M := \left( k^2 \underset{\left[\begin{smallmatrix} 0 & 1 \\ 0 & 0 \end{smallmatrix}\right]}{\overset{\left[\begin{smallmatrix} 1 & 0 \\ 0 & 1 \end{smallmatrix}\right]}{\leftleftarrows}} k^2 \right).$$

Then $E = \mathrm{End}_A(M \oplus (M/NM))$ is Morita equivalent to the algebra given by the following quiver

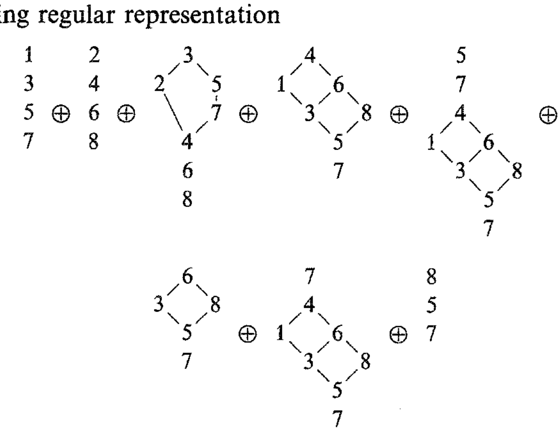
$$2 \underset{\gamma}{\overset{\beta}{\leftleftarrows}} 1 \circlearrowleft \alpha$$

with relations $\alpha^2 = \alpha\beta = 0$. Clearly, it is not a quasi-hereditary algebra.

E x a m p l e 3. Let $A$ be an algebra given by the quiver




with relations $\beta\beta' = \delta'\delta - \varepsilon'\varepsilon = \gamma\varepsilon' - \varphi\psi = \delta\delta' = \varepsilon\delta' = \beta'\beta\gamma = \beta'\beta\varphi = 0$. Let $P(i)$ denote the indecomposable projective $A$-module corresponding to the vertex $i$. If we take $M_1 = P(1)/N^4 P(1)$ and $M_2 = P(2)/\mathrm{Im}(f)$, where $f$ is a non-zero homomorphism from $P(4)$ to $P(2)$, and $M = M_1 \oplus M_2$, then $E = \mathrm{End}_A(\tilde{M}) = \mathrm{End}_A\left(\bigoplus_{i=1}^{4} \bigoplus_{j=1}^{2} M_j/N^i M_j\right)$ has the following regular representation



Auslander, Platzeck and Todorov have introduced in [1] the descending Loewy length condition on projective resolutions. Recall that an artin algebra $B$ satisfies the descending Loewy lenght condition on projective resolutions if for every module ${}_BX$ a minimal projective resolution $\cdots \longrightarrow P_i(X) \longrightarrow \cdots \longrightarrow P_0(X) \longrightarrow {}_BX \longrightarrow 0$ satisfies $LL(P_{i+1}(X)) < LL(P_i(X))$ for $i \geqq 1$, where $LL(X)$ stands for the Loewy length of the module $X$. Let $P_E(4)$ be the indecomposable projective $E$-module corresponding to the indecomposable $A$-module $M_2/N^2M_2$. If we take $X = P_E(4)/\mathrm{rad}(P_E(4))$, then $LL(P_2(X)) > LL(P_1(X))$. This shows that in general quasi-hereditary algebras of the form $\mathrm{End}_A(\tilde{M})$ with $M$ semilocal do not satisfy the descending Loewy length condition.

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