Strang-type Preconditioners for Solving Linear Systems from Neutral Delay Differential Equations

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Abstract

We study the solution of neutral delay differential equations (NDDEs) by using boundary value methods (BVMs). The BVMs require the solution of nonsymmetric, large and sparse linear systems. The GMRES method with the Strang-type block-circulant preconditioner is proposed to solve these linear systems. We show that if an A_{k_1,k_2}-stable BVM is used for solving an m-by-m system of NDDEs, then our preconditioner is invertible and the spectrum of the preconditioned system is clustered. It follows that when the GMRES method is applied to the preconditioned systems, the method could converge fast. Numerical results are given to show that our method is effective.

Keywords: NDDE, BVM, block-circulant preconditioner, GMRES method

AMS subject classifications: 65F10, 65N22, 65L05, 65F15, 15A18

1 Introduction

In this paper, we consider the solution of neutral delay differential equation (NDDE)

$$\begin{cases} \mathbf{y}'(t) = L_n\mathbf{y}'(t-\tau) + M_n\mathbf{y}(t) + N_n\mathbf{y}(t-\tau), & t \geq t_0, \\ \mathbf{y}(t) = \boldsymbol{\phi}(t), & t \leq t_0, \end{cases} \tag{1}$$

by boundary value methods (BVMs), where $\mathbf{y}(t)$, $\boldsymbol{\phi}(t) : \mathbb{R} \to \mathbb{R}^n$; L_n, M_n, $N_n \in \mathbb{R}^{n\times n}$, and $\tau > 0$ is a constant. Such kind of equations appear in many applications [11, 12]. BVMs that we used are relatively new numerical methods for solving ordinary differential equations (ODEs), which are based on the linear multistep formulae (LMF), see [3]. By applying a BVM, the discrete solution of (1) is given by the solution of a linear system

$$H\mathbf{y} = \mathbf{b}$$

where H depends on the LMF used. The advantage in using BVMs over classical initial value methods comes from the stability properties of BVMs, see for instance [3].

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Recently, Bertaccini in [1, 2] proposed to use BVMs with Krylov subspace methods [15], such as the GMRES method and the BiCGstab method, to solve initial value problems (IVPs) of ODEs. In order to speed up the convergence rate of Krylov subspace methods, he proposed two circulant matrices as preconditioners. The use of circulant preconditioners for solving Toeplitz systems has been studied extensively since 1986, see [5, 9]. It has been shown that they are good preconditioners for solving a large class of Toeplitz systems. In [6], Chan, Ng and Jin proposed a new preconditioner called the Strang-type block-circulant preconditioner for solving linear systems from IVPs. They proved that if an A_{k_1,k_2}-stable BVM is used to discretize IVP, then the Strang-type preconditioner is invertible and the spectrum of the preconditioned matrix is clustered. It follows that the GMRES method applied to the preconditioned system may converge fast. The Strang-type preconditioner was also used to solve linear systems from both differential-algebraic equations and delay differential equations, see [4, 10, 13, 14]. In this paper, we will use the Strang-type preconditioner for solving NDDE (1).

The paper is organized as follows. In §2, we recall BVMs and their stability properties. We introduce the Strang-type block-circulant preconditioner and prove its invertibility in §3. The spectral analysis of our method is given in §4 and numerical examples are given in §5.

2 BVMs and Their Stability Properties

For NDDE (1), in order to find a reasonable numerical solution, we require that the solution of (1) is asymptotically stable. Let $\sigma(\cdot)$ and $\mathrm{Re}(\cdot)$ denote the spectrum of the matrix and the real part of complex numbers respectively. We have the following lemma, see [8, 12].

Lemma 1 *Let L_n, M_n and N_n be any matrices with $\|L_n\|_2 < 1$. Then solution of (1) is asymptotically stable if $Re(\lambda_i) < 0$ for any i, where*

$$\lambda_i \in \sigma[(I_n - \eta L_n)^{-1}(M_n + \eta N_n)] \quad \textit{with} \quad |\eta| \le 1,$$

and I_n is the identity matrix.

Let $h = \tau/m$ be the step size where m is a positive integer. For (1), by using a BVM with (k_1, k_2)-boundary conditions, we have

$$\sum_{i=0}^{k} \alpha_i \mathbf{y}_{p+i-k_1} = \sum_{i=0}^{k} \alpha_i L_n \mathbf{y}_{p+i-k_1-m} + h \sum_{i=0}^{k} \beta_i (M_n \mathbf{y}_{p+i-k_1} + N_n \mathbf{y}_{p+i-k_1-m}), \tag{2}$$

for $p = k_1, \ldots, v-1$, where $k = k_1 + k_2$, and $\{\alpha_i\}$, $\{\beta_i\}$ are coefficients of the given BVM, see [3]. By providing the values

$$\mathbf{y}_{-m}, \ldots, \mathbf{y}_0, \quad \mathbf{y}_1, \ldots, \mathbf{y}_{k_1-1}, \quad \mathbf{y}_v, \ldots, \mathbf{y}_{v+k_2-1}, \tag{3}$$

the equation (2) can be written in a matrix form as

$$H\mathbf{y} = \mathbf{b}$$

where

$$H = A \otimes I_n - A^{(1)} \otimes L_n - hB \otimes M_n - hB^{(1)} \otimes N_n, \tag{4}$$

the vector $\mathbf{y}$ is defined by

$$\mathbf{y}^T = (\mathbf{y}_{k_1}^T, \mathbf{y}_{k_1+1}^T, \ldots, \mathbf{y}_{v-1}^T) \in \mathbb{R}^{n(v-k_1)},$$

and the vector $\mathbf{b} \in \mathbb{R}^{n(v-k_1)}$ depends on the boundary values and the coefficients of the method. In (4), A, $A^{(1)}$, B and $B^{(1)} \in \mathbb{R}^{(v-k_1)\times(v-k_1)}$ are Toeplitz matrices given as follows:

$$A = \begin{bmatrix} \alpha_{k_1} & \cdots & \alpha_k & & \\ \vdots & \ddots & \ddots & \ddots & \\ \alpha_0 & \ddots & \ddots & \ddots & \alpha_k \\ & \ddots & & \ddots & \vdots \\ & & \alpha_0 & \cdots & \alpha_{k_1} \end{bmatrix}, \qquad A^{(1)} = \begin{bmatrix} \mathbf{O} & & & & & \\ \alpha_k & \ddots & & & & \\ \vdots & \ddots & \ddots & & & \\ \alpha_0 & \cdots & \alpha_k & \ddots & & \\ & \ddots & & \ddots & \ddots & \\ & & \alpha_0 & \cdots & \alpha_k & \mathbf{O} \end{bmatrix},$$

and

$$B = \begin{bmatrix} \beta_{k_1} & \cdots & \beta_k & & \\ \vdots & \ddots & \ddots & \ddots & \\ \beta_0 & \ddots & \ddots & \ddots & \beta_k \\ & \ddots & & \ddots & \vdots \\ & & \beta_0 & \cdots & \beta_{k_1} \end{bmatrix} \qquad B^{(1)} = \begin{bmatrix} \mathbf{O} & & & & & \\ \beta_k & \ddots & & & & \\ \vdots & \ddots & \ddots & & & \\ \beta_0 & \cdots & \beta_k & \ddots & & \\ & \ddots & & \ddots & \ddots & \\ & & \beta_0 & \cdots & \beta_k & \mathbf{O} \end{bmatrix},$$

see [3]. We remark that the first column of $A^{(1)}$ is given by:

$$(\underbrace{0, \ldots, 0}_{m+k_1-k}, \alpha_k, \ldots, \alpha_0, \underbrace{0, \ldots, 0}_{v-m-2k_1-1})^T,$$

and the first column of $B^{(1)}$ is given by

$$(\underbrace{0, \ldots, 0}_{m+k_1-k}, \beta_k, \ldots, \beta_0, \underbrace{0, \ldots, 0}_{v-m-2k_1-1})^T.$$

Now we introduce the stability properties of the BVM. The characteristic polynomials $\rho(z)$ and $\sigma(z)$ of the BVM are defined by

$$\rho(z) \equiv \sum_{j=0}^{k} \alpha_j z^j \quad \text{and} \quad \sigma(z) \equiv \sum_{j=0}^{k} \beta_j z^j, \tag{5}$$

where $\{\alpha_i\}$ and $\{\beta_i\}$ are given by (2). The A_{k_1,k_2}-stability polynomial is defined by

$$\pi(z,q) \equiv \rho(z) - q\sigma(z) \tag{6}$$

where z, $q \in \mathbb{C}$. Let $\mathbb{C}^- \equiv \{q \in \mathbb{C} : \ \mathrm{Re}(q) < 0\}$.

Definition 1 **[3]** *The region*

$$\mathcal{D}_{k_1,k_2} = \{q \in \mathbb{C} : \pi(z,q) \text{ has } k_1 \text{ zeros inside } |z| = 1 \text{ and } k_2 \text{ zeros outside } |z| = 1\}$$

is called the region of A_{k_1,k_2}-stability of a given BVM with (k_1,k_2)-boundary conditions. Moreover, the BVM is said to be A_{k_1,k_2}-stable if $\mathbb{C}^- \subseteq \mathcal{D}_{k_1,k_2}$.

3 Strang-Type Preconditioner

In this section, we first introduce the Strang-type block-circulant preconditioner and then discuss its invertibility.

3.1 Strang-type block-circulant preconditioner

The Strang-type block-circulant preconditioner for (4) is defined as follows:

$$S \equiv s(A) \otimes I_n - s(A^{(1)}) \otimes L_n - hs(B) \otimes M_n - hs(B^{(1)}) \otimes N_n, \tag{7}$$

where $s(E)$ is Strang's preconditioner of Toeplitz matrix E, for $E = A,\, B,\, A^{(1)},\, B^{(1)}$ respectively. More precisely, for any given Toeplitz matrix

$$T_l = [t_{i-j}]_{i,j=1}^{l} = [t_q]_{q=-l+1}^{l-1},$$

Strang's preconditioner $s(T_l)$ is a circulant matrix with diagonals given by

$$[s(T_l)]_q = \begin{cases} t_q, & 0 \le q \le \lfloor l/2 \rfloor, \\ t_{q-l}, & \lfloor l/2 \rfloor < q < l, \\ [s(T_l)]_{l+q}, & 0 < -q < l, \end{cases}$$

see [5, 9]. Since any circulant matrix can be diagonalized by the Fourier matrix ([5, 9]), we obtain the following decomposition,

$$S = (F^* \otimes I_n)(\Lambda_A \otimes I_n - \Lambda_{A^{(1)}} \otimes L_n - h\Lambda_B \otimes M_n - h\Lambda_{B^{(1)}} \otimes N_n)(F \otimes I_n), \tag{8}$$

where Λ_E is the diagonal matrix given by $\Lambda_E = Fs(E)F^*$, for $E = A,\, B,\, A^{(1)},\, B^{(1)}$ respectively, and F is the Fourier matrix.

3.2 Invertibility of preconditioner

Now we discuss the invertibility of the Strang-type preconditioner. Note that the jth-block of

$$\Lambda_A \otimes I_n - \Lambda_{A^{(1)}} \otimes L_n - h\Lambda_B \otimes M_n - h\Lambda_{B^{(1)}} \otimes N_n$$

in (8) is given by

$$S_j = [\Lambda_A]_{jj} I_n - [\Lambda_{A^{(1)}}]_{jj} L_n - h[\Lambda_B]_{jj} M_n - h[\Lambda_{B^{(1)}}]_{jj} N_n,$$

for $j = 1, 2, \ldots, v - k_1$. Therefore, we need to prove that

$$S_j = [\Lambda_A]_{jj} I_n - [\Lambda_{A^{(1)}}]_{jj} L_n - h[\Lambda_B]_{jj} M_n - h[\Lambda_{B^{(1)}}]_{jj} N_n,$$

are invertible, for $j = 1, 2, \ldots, v - k_1$. Let $w_j = e^{\frac{2\pi i j}{v-k_1}}$ where $i \equiv \sqrt{-1}$. We have

$$[\Lambda_A]_{jj} = \rho(w_j)/w_j^{k_1}, \qquad [\Lambda_B]_{jj} = \sigma(w_j)/w_j^{k_1},$$

$$[\Lambda_{A^{(1)}}]_{jj} = \alpha_k w_j^{-m-k_1+k} + \cdots + \alpha_0 w_j^{-m-k_1} = \rho(w_j)/w_j^{m+k_1},$$

and

$$[\Lambda_{B^{(1)}}]_{jj} = \beta_k w_j^{-m-k_1+k} + \cdots + \beta_0 w_j^{-m-k_1} = \sigma(w_j)/w_j^{m+k_1},$$

where $\rho(z)$ and $\sigma(z)$ are defined as in (5), see [6]. Therefore,

$$S_j = \frac{1}{w_j^{m+k_1}} \Big[w_j^m (\rho(w_j) I_n - h\sigma(w_j) M_n) - \rho(w_j) L_n - h\sigma(w_j) N_n \Big].$$

In order to prove that S_j is invertible, we only need to show that

$$Q \equiv e^{im\theta}(\rho(e^{i\theta}) I_n - h\sigma(e^{i\theta}) M_n) - \rho(e^{i\theta}) L_n - h\sigma(e^{i\theta}) N_n$$

is invertible for any $\theta \in \mathbb{R}$. Note that the matrix Q can be rewritten as follows,

$$\begin{aligned} Q &= \rho(e^{i\theta})(e^{im\theta} I_n - L_n) - h(e^{im\theta} M_n + N_n)\sigma(e^{i\theta}) \\ &= e^{im\theta}(I_n - e^{-im\theta} L_n) T \end{aligned}$$

where

$$T \equiv \rho(e^{i\theta}) I_n - h(I_n - e^{-im\theta} L_n)^{-1}(M_n + e^{-im\theta} N_n)\sigma(e^{i\theta}).$$

Assume that $\|L_n\|_2 < 1$. It follows that the matrix $I_n - e^{-im\theta} L_n$ is invertible for any $\theta \in \mathbb{R}$. We therefore only need to prove that T is invertible on the unit circle $|z| = 1$. We have the following theorem.

Theorem 1 *If the BVM with (k_1, k_2)-boundary conditions is A_{k_1,k_2}-stable and the conditions in Lemma 1 hold, the matrix*

$$T \equiv \rho(z) I_n - h(I_n - e^{-im\theta} L_n)^{-1}(M_n + e^{-im\theta} N_n)\sigma(z)$$

is invertible on the unit circle $|z| = 1$. It follows that the Strang-type preconditioner S defined as in (7) is also invertible.

Proof: Let

$$U \equiv (I_n - e^{-im\theta} L_n)^{-1}(M_n + e^{-im\theta} N_n).$$

Then T can be written as

$$T = \rho(z) I_n - hU\sigma(z).$$

Note that the eigenvalues $\lambda_i(T)$ of T are given by

$$\lambda_i(T) = \rho(z) - h\lambda_i(U)\sigma(z), \qquad i = 1, \cdots, n,$$

where $\lambda_i(U)$ denote the eigenvalues of U. By Lemma 1, we know that

$$\mathrm{Re}[\lambda_i(U)] < 0, \qquad i = 1, \cdots, n.$$

It follows that $h\lambda_i(U) \in \mathbb{C}^-$. Since the BVM is A_{k_1,k_2}-stable, we have $h\lambda_i(U) \in \mathcal{D}_{k_1,k_2}$. Therefore, the A_{k_1,k_2}-stability polynomial defined by (6)

$$\pi[z, h\lambda_i(U)] \equiv \rho(z) - h\lambda_i(U)\sigma(z)$$

will have no root on the unit circle $|z| = 1$. Thus, for any $|z| = 1$, we have

$$\lambda_i(T) = \rho(z) - h\lambda_i(U)\sigma(z) \neq 0, \qquad i = 1, \cdots, n.$$

It follows that T is invertible. Therefore, the Strang-type preconditioner S defined as in (7) is also invertible. $\square$

4 Spectral Analysis

In this section, we study the convergence rate of the preconditioned GMRES method with the preconditioner S. It is well-known that the convergence rate of Krylov subspace methods is closely related to the spectrum of the preconditioned matrix $S^{-1}M$. By noting that

$$S^{-1}M = I + S^{-1}(M - S),$$

one can prove the following result easily. We therefore omit the proof.

Theorem 2 *Let H be given by (4) and S be given by (7). Then we have*

$$S^{-1}H = I_{n(v-k_1)} + L$$

where $I_{n(v-k_1)} \in \mathbb{R}^{n(v-k_1)\times n(v-k_1)}$ is the identity matrix and L is a matrix with

$$\mathrm{rank}(L) \leq 2(k + m + k_1 + 1)n.$$

Now, we discuss the convergence rate of the GMRES method for solving $S^{-1}H\mathbf{y} = S^{-1}\mathbf{b}$. It was proved in [9] that

Lemma 2 *Let A be invertible and can be decomposed as $A = I + L$ where I is the identity matrix. If the GMRES method is applied to the linear system $A\mathbf{x} = \mathbf{b}$, then the method will converge in at most $\mathrm{rank}(L) + 1$ iterations in exact arithmetic.*

By combining Theorem 2 and Lemma 2, we have

Corollary 1 *When the GMRES method is applied to the preconditioned system*

$$S^{-1}H\mathbf{y} = S^{-1}\mathbf{b},$$

the method will converge in at most $2(k + m + k_1 + 1)n + 1$ iterations in exact arithmetic.

We observe from Corollary 1 that if the step size $h = \tau/m$ is fixed, the number of iterations for convergence of the GMRES method, when applied to $S^{-1}H\mathbf{y} = S^{-1}\mathbf{b}$, will be independent of v (and therefore is independent of the length of the interval that we considered). We should emphasize that numerical examples in the next section show a much faster convergence rate than that predicted by Corollary 1.

5 Numerical Tests

In this section, we illustrate the efficiency of our preconditioner by solving the following problems. All the experiments were performed in MATLAB. We used the MATLAB-provided M-file "gmres" (see MATLAB on-line documentation) to solve the preconditioned systems. In our tests, the zero vector is the initial guess and the stopping criterion is

$$\frac{||\mathbf{r}_q||_2}{||\mathbf{r}_0||_2} < 10^{-6}$$

where $\mathbf{r}_q$ is the residual after the q-th iteration.

Example 1. Consider

$$\begin{cases} \mathbf{y}'(t) = L_n\mathbf{y}'(t-1) + M_n\mathbf{y}t + N_n\mathbf{y}(t-1), & t \geq 0, \\ \mathbf{y}(t) = (1, 1, \cdots, 1)^T, & t \leq 0, \end{cases}$$

where

$$L_n = \frac{1}{n}\begin{bmatrix} 2 & 1 & & \\ 1 & \ddots & \ddots & \\ & \ddots & \ddots & 1 \\ & & 1 & 2 \end{bmatrix}, \quad M_n = \begin{bmatrix} -8 & 2 & 1 & & \\ 2 & \ddots & \ddots & \ddots & \\ 1 & \ddots & \ddots & \ddots & 1 \\ & \ddots & \ddots & \ddots & 2 \\ & & 1 & 2 & -8 \end{bmatrix},$$

and

$$N_n = \frac{1}{n}\begin{bmatrix} 2 & -1 & & \\ -1 & \ddots & \ddots & \\ & \ddots & \ddots & -1 \\ & & -1 & 2 \end{bmatrix}.$$

Example 2. Consider

$$\begin{cases} \mathbf{y}'(t) = L_n\mathbf{y}'(t-1) + M_n\mathbf{y}t + N_n\mathbf{y}(t-1), & t \geq 0, \\ \mathbf{y}(t) = (1, 1, \cdots, 1)^T, & t \leq 0, \end{cases}$$

where

$$L_n = \frac{1}{n}\begin{bmatrix} 2 & 1 & & \\ 1 & \ddots & \ddots & \\ & \ddots & \ddots & 1 \\ & & 1 & 2 \end{bmatrix}, \quad M_n = \frac{1}{n}\begin{bmatrix} -10 & 2 & 1 & & \\ 2 & \ddots & \ddots & \ddots & \\ 1 & \ddots & \ddots & \ddots & 1 \\ & \ddots & \ddots & \ddots & 2 \\ & & 1 & 2 & -10 \end{bmatrix},$$

and

$$N_n = \frac{1}{n}\begin{bmatrix} 4 & -2 & -1 & & \\ -2 & \ddots & \ddots & \ddots & \\ -1 & \ddots & \ddots & \ddots & -1 \\ & \ddots & \ddots & \ddots & -2 \\ & & -1 & -2 & 4 \end{bmatrix}.$$

Example 1 is solved by using the fifth order generalized Adams method (GAM) and Example 2 is solved by using the third order generalized backward differentiation formulae (GBDF) for $t \in [0, 4]$. In practice, we do not have the boundary values $\mathbf{y}_1, \ldots, \mathbf{y}_{k_1-1}$ and $\mathbf{y}_v, \ldots, \mathbf{y}_{v+k_2-1}$ provided in (3). Instead of giving the above values, $k_1 - 1$ initial additional equations and k_2 final additional equations are given. The equations of the GAM and the GBDF with the corresponding additional equations can be found in [3]. We remark that after introducing the additional equations, the matrices A, B, $A^{(1)}$ and $B^{(1)}$ in (4) are Toeplitz matrices with small rank perturbations. By neglecting the small rank perturbations, we can also construct the Strang-type preconditioner (7).

Table 1 lists the number of iterations required for convergence of the GMRES method with different preconditioners. In the table, I means no preconditioner is used and S denotes the Strang-type block-circulant preconditioner defined as in (7). Besides, T and P denote the T. Chan's and Bertaccini's block-circulant preconditioners respectively. We remark that for a Toeplitz matrix $A = [t_{i-j}]_{i,j=1}^{l}$, the diagonals of T. Chan's circulant preconditioner $c(A)$ are defined by

$$[c(A)]_q = \left(1 - \frac{q}{l}\right) t_q + \frac{q}{l} t_{q-l}, \qquad q = 0, \ldots, l-1,$$

see [7]. Thus, T. Chan's block-circulant preconditioner for NDDEs is defined as

$$T \equiv c(A) \otimes I_n - c(A^{(1)}) \otimes L_n - hc(B) \otimes M_n - hc(B^{(1)}) \otimes N_n.$$

Similarly, the diagonals of Bertaccini's circulant preconditioner $p(A)$ for $A = [t_{i-j}]_{i,j=1}^{l}$ are defined as

$$[p(A)]_q = \left(1 + \frac{q}{l}\right) t_q + \frac{q}{l} t_{q-l}, \qquad q = 0, \ldots, l-1,$$

see [1], and therefore Bertaccini's block-circulant preconditioner for NDDEs is defined by

$$P \equiv p(A) \otimes I_n - p(A^{(1)}) \otimes L_n - hp(B) \otimes M_n - hp(B^{(1)}) \otimes N_n.$$

From Table 1, we note that the number of iterations for convergence with a block-circulant preconditioner is much less than that with no preconditioner. The performance of the Strang-type preconditioner is better than that of other preconditioners.

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n	m	I	S	T	P	n	m	I	S	T	P
12	10	41	9	9	10	12	10	212	38	41	40
	20	77	8	9	10		20	381	39	42	40
	40	151	8	9	9		40	*	41	42	40
	80	295	8	9	9		80	*	42	42	41
24	10	43	7	8	8	24	10	193	27	30	47
	20	83	7	7	8		20	348	27	30	49
	40	161	7	7	8		40	*	27	31	50
	80	*	7	7	8		80	*	27	33	50
48	10	44	6	7	7	48	10	184	20	23	31
	20	83	6	6	7		20	328	21	23	32
	40	163	6	6	7		40	*	22	23	33
	80	*	6	6	7		80	*	23	26	33

Table 1: Number of iterations for Example 1 (left), Example 2 (right).
'*' means out of memory.

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